An Introduction to Digital Image Processing with MATLAB

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To my dear wife, Felicity, for all her love, support and understanding while this book was written.

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Contents

Preface **1**

1 Introduction **11**

1.1 Images and pictures 11

1.2 What is image processing? 12

1.3 Image Acquisition and sampling 15

1.4 Images and digital images 20

1.5 Some applications 23

1.6 Aspects of image processing 23

1.7 An image processing task 24

1.8 Types of digital images 25

1.9 Image File Sizes 28

1.10 Image perception 29

Exercises 31

2 Images and MATLAB **33**

2.1 Greyscale images 33

2.2 RGB Images 35

2.3 Indexed colour images 37

2.4 Data types and conversions 41

2.5 Image Files and Formats 42

Exercises 54

3 Image Display **57**

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3.1 Introduction . . . 57
3.2 Basics of image display . . . 57
3.3 The `imshow` function . . . 60
3.4 Bit planes . . . 64
3.5 Spatial Resolution . . . 67
3.6 Quantization and dithering . . . 70
Exercises . . . 80

4 Point Processing **83**
4.1 Introduction . . . 83
4.2 Arithmetic operations . . . 84
4.3 Histograms . . . 88
4.4 Lookup tables . . . 102
Exercises . . . 104

5 Neighbourhood Processing **107**
5.1 Introduction . . . 107
5.2 Notation . . . 112
5.3 Filtering in MATLAB . . . 113
5.4 Frequencies; low and high pass filters . . . 119
5.5 Gaussian filters . . . 124
5.6 Edge sharpening . . . 128
5.7 Non-linear filters . . . 134
5.8 Region of interest processing . . . 137
Exercises . . . 139

6 Image Geometry **145**
6.1 Interpolation of data . . . 145
6.2 Image interpolation . . . 148
6.3 General interpolation . . . 152
6.4 Enlargement by spatial filtering . . . 155

NOT FOR CIRCULATION

6.5 Scaling smaller . . . 159
6.6 Rotation . . . 159
6.7 An example: anamorphosis . . . 164
Exercises . . . 166

7 The Fourier Transform 169
7.1 Introduction . . . 169
7.2 Background . . . 169
7.3 The one-dimensional discrete Fourier transform . . . 172
7.4 Properties of the one-dimensional DFT . . . 177
7.5 The two-dimensional DFT . . . 183
7.6 Fourier transforms in MATLAB . . . 190
7.7 Fourier transforms of images . . . 193
7.8 Filtering in the frequency domain . . . 198
7.9 Homomorphic filtering . . . 211
Exercises . . . 215

8 Image Restoration 219
8.1 Introduction . . . 219
8.2 Noise . . . 220
8.3 Cleaning salt and pepper noise . . . 224
8.4 Cleaning Gaussian noise . . . 230
8.5 Removal of periodic noise . . . 236
8.6 Inverse filtering . . . 239
8.7 Wiener filtering . . . 245
Exercises . . . 248

9 Image Segmentation 253
9.1 Introduction . . . 253
9.2 Thresholding . . . 253
9.3 Applications of thresholding . . . 257

NOT FOR CIRCULATION

9.4 Choosing an appropriate threshold value 258
9.5 Adaptive thresholding 264
9.6 Edge detection 267
9.7 Derivatives and edges 268
9.8 Second derivatives 275
9.9 The Canny edge detector 280
9.10 The Hough transform 285
9.11 Implementing the Hough transform in MATLAB 291
Exercises 299

10 Mathematical morphology 305

10.1 Introduction 305
10.2 Basic ideas 305
10.3 Dilation and erosion 307
10.4 Opening and closing 317
10.5 The hit-or-miss transform 324
10.6 Some morphological algorithms 327
10.7 A note on MATLAB's `bwmorph` function 335
10.8 Greyscale morphology 336
10.9 Applications of greyscale morphology 345
Exercises 347

11 Image Topology 351

11.1 Introduction 351
11.2 Neighbours and adjacency 352
11.3 Paths and components 352
11.4 Equivalence relations 354
11.5 Component labelling 355
11.6 Lookup tables 360
11.7 Distances and metrics 363

NOT FOR CIRCULATION

11.8 Skeletonization 376
Exercises 402

12 Shapes and boundaries **409**
12.1 Introduction 409
12.2 Chain codes and shape numbers 409
12.3 Fourier descriptors 428
Exercises 434

13 Colour processing **437**
13.1 What is colour? 437
13.2 Colour models 442
13.3 Colour images in MATLAB 448
13.4 Pseudocolouring 451
13.5 Processing of colour images 455
Exercises 463

14 Image coding and compression **467**
14.1 Lossless and lossy compression 467
14.2 Huffman coding 467
14.3 Run length encoding 471
14.4 The JPEG algorithm 478
Exercises 495

15 Wavelets **499**
15.1 Waves and wavelets 499
15.2 A simple wavelet: the Haar wavelet 505
15.3 Wavelets in MATLAB 511
15.4 The Daubechies wavelets 514
15.5 Image compression using wavelets 519
15.6 High pass filtering using wavelets 525

NOT FOR CIRCULATION

15.7 Denoising using wavelets 526
Exercises 528

16 Special effects **531**
16.1 Polar coordinates 531
16.2 Ripple effects 536
16.3 General distortion effects 539
16.4 Pixel effects 546
16.5 Colour images 550
Exercises 550

A Basic use of MATLAB **553**
A.1 Introduction 553
A.2 Basic use of MATLAB 555
A.3 Variables and the workspace 557
A.4 Dealing with matrices 559
A.5 Plots 574
A.6 Help in MATLAB 575
A.7 Programming in MATLAB 578
Exercises 581

B The fast Fourier transform **585**

Bibliography **595**

Index **598**

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Preface

Human beings are predominantly visual creatures, and our computing environments reflect this. We have the World Wide Web, filled with images of every possible type and provenance; and our own computers are crammed with images from the operating system, downloaded from elsewhere, or taken with our digital cameras. Then there are the vast applications which use digital images: remote sensing and satellite imaging; astronomy; medical imaging; microscopy; industrial inspection...

This book is an introduction to *digital image processing*, from a strictly elementary perspective. We have selected only those topics which can be introduced with simple mathematics; however these topics provide a very broad introduction to the discipline.

This book is based on some very successful image processing subjects which have been taught for the past few years at Victoria University of Technology, in Melbourne, Australia. The topics chosen, and their method of presentation, are the result of many hours talking to, and learning from, the hundreds of students who have taken these subjects.

There are a great many books on the subject of image processing, and this book differs from all of them in several important respects:

Not too much mathematics. Some mathematics is necessary for the explanation and discussion of image processing algorithms. But this book attempts to keep the mathematics at a level commensurate with elementary undergraduate computer science.

The level of mathematics required is about one year's study at a tertiary level, including calculus and linear algebra.

A discrete approach. Since digital images are discrete entities, we have adopted an approach

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using mainly discrete mathematics. Although calculus is required for the development of some image processing topics, we attempt to connect the calculus-based (continuous) theory with its discrete implementation.

A strong connection between theory and practice. Since we have used a mainly discrete approach to explain the material, it becomes much easier to extend the theory into practice. This becomes particularly significant when we develop out own MATLAB functions for implementing specific image processing algorithms.

Software based. There are image processing books which are based around programming languages: generally C or Java. The problem is that to use such software, a specialized image processing library must be used, and at least for C, there is no standard for this. And a problem with the Java image processing libraries is they are not really suitable for beginning students.

This book is based entirely around MATLAB and its Image Processing Toolbox. This provides a complete environment for image processing which is easy to use, easy to explain, and easy to extend.

Plenty of examples. *All* the example images in this text are accompanied by MATLAB commands. Thus if you work carefully through the book, you can create the images as given in the text.

Exercises. Most chapters finish off with a selection of exercises enabling the student to consolidate and extend the material. Some of the exercises are "pencil-and-paper"; designed for better understanding of the material; others are MATLAB based to explore the algorithms and methods of that chapter.

What is in the book

The first three chapters set the scene for much of the rest of the book; exploring the nature and use of digital images, and how they can be obtained, stored, and displayed. Chapter 1 provides a brief introduction to the field of image processing, and attempts to give some idea as to its scope and areas of practice. We also define some common terms. Chapter 2 shows how MATLAB handles images as matrices, and how the manipulation of these matrices forms the background of all of subsequent

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work. Chapter 3 investigates aspects of image display, and looks at resolution and quantization; and how these affect the appearance of the image. Appendix A provides a background in the use of MATLAB, and gives a brief introduction to MATLAB programming.

Chapter 4 looks at some of the simplest, yet most powerful and widely used, of all image processing algorithms. These are the *point operations*, where the value of a pixel (A single "dot" in a digital image) is changed according to a single function of its value.

Chapter 5 introduces spatial filtering. Spatial filtering can be used for a vast range of image processing operations: from removing unnecessary detail, to sharpening edges and removing noise.

Chapter 6 looks at the geometry of an image: its size and orientation. Resizing an image may be necessary for inclusion in a web page or printed text; we may need to reduce it to fit, or enlarge it.

Chapter 7 introduces the Fourier transform. This is possibly the single most important transform for image processing. To get a feeling for how the Fourier transform "works", and what information it provides, we need to spend some time exploring its mathematical foundations. This is the heaviest mathematics in this book, and requires some knowledge of complex numbers. In keeping with our philosophy, we use discrete mathematics only. We then show how images can be processed with great efficiency using the Fourier transform, and how various operations can be performed only using the Fourier transform.

Chapter 8 discusses the restoration of an image from different forms of degradation. Among these is the problem of *noise*, or "errors" in an image. Such errors are a natural consequence of electronic transmission of image signals, and although error correction of the signal can go a long way to ensuring that the image arrives "clean", we may still receive images with noise. We also look at the removal of blur.

Chapter 9 addresses the problems of thresholding, and of finding edges in an image. Edges are a vital aspect of object recognition: we can classify the size, shape and type of object by an analysis of its edges. As well, edges form a vital aspect of human visual interpretation, and so the sharpening of edges is often an important part of image enhancement.

Chapter 10 introduces *morphology* or *mathematical morphology*, which is an area of image processing very much wedded to set theory. Historically, morphology developed from the need for

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granulometry, or the measurement of grains in ore samples. It is now a very powerful method for investigating shapes and sizes of objects. Morphology is generally defined in terms of binary images (which is what we do here) and then can be extended to greyscale images. With the latter, we can also perform edge detection and some noise reduction.

Chapter 11 investigates the *topology* of digital images. This is concerned with the neighbourhoods of pixels, and how the exploration of different neighbourhoods leads to an understanding of the structure of image objects.

We continue the investigation of shapes in chapter 12, but from a more spatial viewpoint; we look at traversing the edges of an object, and how the traversal can be turned into descriptors of the size and shape of the object.

Chapter 13 looks at colour. Colour is one of most important aspects of human interpretation. We look at the definition of colour, from physical and digital perspectives, and how a colour image can be processed using the techniques we have developed so far.

Chapter 14 discusses some basic aspects of image compression. Image files tend to be large, and their compression can a matter of some concern, especially if there are many of them. We distinguish two types of compression: *lossless*, where there is no loss of information, and *lossy*, where higher compression rates can be obtained at the cost of losing some information.

Chapter 14 introduces wavelets. These have become a very hot topic in image processing; in some places they are replacing the use of the Fourier transform. Our treatment is introductory only; we show how wavelets and waves differ; how wavelets can be defined; how they can be applied to images; and the affects which can be obtained. In particular, we look at image compression, and show how wavelets can be used to obtain very high rates of lossy compression, with no apparent loss of image quality.

The final chapter is intended to be a bit more light-hearted than the others: here we look at some "special effects" on images. These are often provided with image editing programs—if you have a digital camera, chances are the accompanying software will allow this—our treatment attempts to provide an understanding into the nature of these algorithms.

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What this book is not

This book is *not* an introduction to either MATLAB or its Image Processing Toolbox. We have used only a small fraction of the many commands and functions available; only those useful for an elementary text such as this. There are an enormous number of books on MATLAB available; a fine general introduction is the text by Hanselman and Littlefield [8]. To really get to grips with the Image Processing Toolbox, you can either use its excellent manual, or browse through the comprehensive online documentation.

How to use this book

This book can be used for two separate streams of image processing; one very elementary, another a little more advanced. A first course consists of the following:

- chapter 1,
- chapter 2 except for section 2.5,
- chapter 3 except for section 3.6,
- chapter 4,
- chapter 5,
- chapter 7,
- chapter 8,
- chapter 9 except for sections 9.4, 9.5 and 9.9,
- chapter 10 except for sections 10.8 and 10.9,
- chapter 13
- chapter 14, sections 14.2 and 14.3 only.

A second course fills in the gaps:

- section 2.5,

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- section 3.6,
- section 6
- sections 9.4, 9.5 and 9.9,
- sections 10.8 and 10.9,
- chapter 11
- chapter 12
- section 14.4
- chapter 15

As taught at Victoria University of Technology, for the first course we concentrate on introducing students to the principles and practices of image processing, and don't worry about the implementation of the algorithms. In the second course, we spend much time discussing programming issues, and encourage the students to write their own programs, and edit programs already written. Both courses have been very popular since their inception.

Acknowledgments

This book began as set of notes written to accompany an undergraduate introductory course to image processing; these notes were then extended for a further course. Thanks must go to the many students who have taken these courses, and who have discussed the subject material and its presentation with me.

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A note on the images

Some of the images used in this book are the author's and may be used freely and without restriction. They are:

```
arch.tif
blocks.tif
buffalo.tif
cat.tif
emu.tif
engineer.tif
iguana.tif
newborn.tif
nicework.tif
pelicans.tif
twins.tif
wombats.tif
```

CRC Press has kindly allowed the use of the following images from "The Image Processing Handbook" by John Russ [28]:

```
bacteria.tif
circles.tif
flowers.tif
ic.tif
lily.tif
rice.tif
text.tif
tire.tif
```

The Mathworks has allowed the use of the images:

```
board.tif
```

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```
cameraman.tif
circbw.tif
coins.tif
nodules1.tif
paper.tif
pout.tif
spine.tif
```

The image

```
caribou.tif
```

has been cropped from the NOAA image `anim0614.jpg`, a photograph of a caribou (*Rangifer tarandus*) by Capt. Budd Christman of the NOAA Corps.

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Chapter 1

Introduction

1.1 Images and pictures

As we mentioned in the preface, human beings are predominantly visual creatures: we rely heavily on our vision to make sense of the world around us. We not only look at things to identify and classify them, but we can scan for differences, and obtain an overall rough "feeling" for a scene with a quick glance.

Humans have evolved very precise visual skills: we can identify a face in an instant; we can differentiate colours; we can process a large amount of visual information very quickly.

However, the world is in constant motion: stare at something for long enough and it will change in some way. Even a large solid structure, like a building or a mountain, will change its appearance depending on the time of day (day or night); amount of sunlight (clear or cloudy), or various shadows falling upon it.

We are concerned with single images: snapshots, if you like, of a visual scene. Although image processing can deal with changing scenes, we shall not discuss it in any detail in this text.

For our purposes, an *image* is a single picture which represents something. It may be a picture of a person, of people or animals, or of an outdoor scene, or a microphotograph of an electronic component, or the result of medical imaging. Even if the picture is not immediately recognizable, it will not be just a random blur.

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1.2 What is image processing?

Image processing involves changing the nature of an image in order to either

1. improve its pictorial information for human interpretation,
2. render it more suitable for autonomous machine perception.

We shall be concerned with *digital image processing*, which involves using a computer to change the nature of a *digital image* (see below). It is necessary to realize that these two aspects represent two separate but equally important aspects of image processing. A procedure which satisfies condition (1)—a procedure which makes an image "look better"—may be the very worst procedure for satisfying condition (2). Humans like their images to be sharp, clear and detailed; machines prefer their images to be simple and uncluttered.

Examples of (1) may include:

- Enhancing the edges of an image to make it appear sharper; an example is shown in figure 1.1. Note how the second image appears "cleaner"; it is a more pleasant image. Sharpening edges is a vital component of printing: in order for an image to appear "at its best" on the printed page; some sharpening is usually performed.
- Removing "noise" from an image; noise being random errors in the image. An example is given in figure 1.2. Noise is a very common problem in data transmission: all sorts of electronic components may affect data passing through them, and the results may be undesirable. As we shall see in chapter 8 noise may take many different forms;each type of noise requiring a different method of removal.
- Removing motion blur from an image. An example is given in figure 1.3. Note that in the deblurred image (b) it is easier to read the numberplate, and to see the spikes on the fence behind the car, as well as other details not at all clear in the original image (a). Motion blur may occur when the shutter speed of the camera is too long for the speed of the object. In photographs of fast moving objects: athletes, vehicles for example, the problem of blur may be considerable.

Examples of (2) may include:

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(a) The original image

(b) Result after "sharperning"

Figure 1.1: Image sharperning

(a) The original image

(b) After removing noise

Figure 1.2: Removing noise from an image

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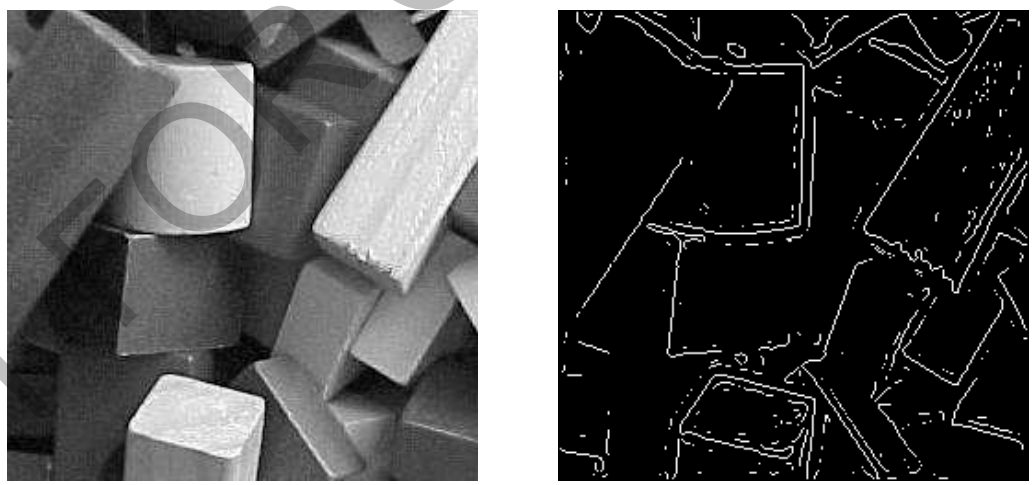
(a) The original image (b) After removing the blur

Figure 1.3: Image deblurring

- Obtaining the edges of an image. This may be necessary for the measurement of objects in an image; an example is shown in figures 1.4. Once we have the edges we can measure their spread, and the area contained within them. We can also use edge detection algorithms as a first step in edge enhancement, as we saw above.

  From the edge result, we see that it may be necessary to enhance the original image slightly, to make the edges clearer.



(a) The original image (b) Its edge image

Figure 1.4: Finding edges in an image

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- Removing detail from an image. For measurement or counting purposes, we may not be interested in all the detail in an image. For example, a machine inspected items on an assembly line, the only matters of interest may be shape, size or colour. For such cases, we might want to simplify the image. Figure 1.5 shows an example: in image (a) is a picture of an African buffalo, and image (b) shows a blurred version in which extraneous detail (like the logs of wood in the background) have been removed. Notice that in image (b) all the fine detail is gone; what remains is the coarse structure of the image. We could for example, measure the size and shape of the animal without being "distracted" by unnecessary detail.

(a) The original image (b) Blurring to remove detail

Figure 1.5: Blurring an image

1.3 Image Acquisition and sampling

Sampling refers to the process of digitizing a continuous function. For example, suppose we take the function

$$y = \sin(x) + \frac{1}{3}\sin(3x).$$

and sample it at ten evenly spaced values of x only. The resulting sample points are shown in figure 1.6. This shows an example of *undersampling*, where the number of points is not sufficient to reconstruct the function. Suppose we sample the function at 100 points, as shown in figure 1.7. We can clearly now reconstruct the function; all its properties can be determined from this sampling.

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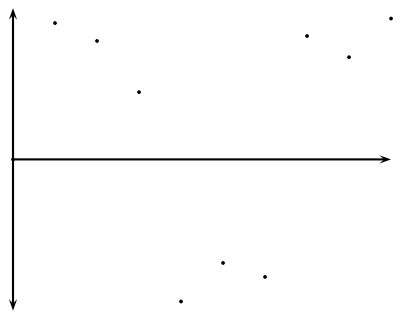

Figure 1.6: Sampling a function—undersampling

In order to ensure that we have enough sample points, we require that the sampling period is not

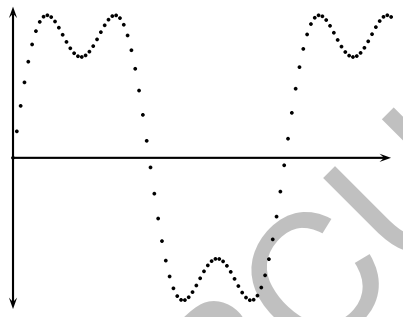

Figure 1.7: Sampling a function with more points

greater than one-half the finest detail in our function. This is known as the *Nyquist criterion*, and can be formulated more precisely in terms of "frequencies", which are discussed in chapter 7. The Nyquist criterion can be stated as the *sampling theorem*, which says, in effect, that a continuous function can be reconstructed from its samples provided that the sampling frequency is at least twice the maximum frequency in the function. A formal account of this theorem is provided by Castleman [4].

Sampling an image again requires that we consider the Nyquist criterion, when we consider an image as a continuous function of two variables, and we wish to sample it to produce a digital image.

An example is shown in figure 1.8 where an image is shown, and then with an undersampled version. The jagged edges in the undersampled image are examples of *aliasing*. The sampling rate will of course affect the final resolution of the image; we discuss this in chapter 3. In order to obtain a sampled (digital) image, we may start with a continuous representation of a scene. To view the scene, we record the energy reflected from it; we may use visible light, or some other energy source.

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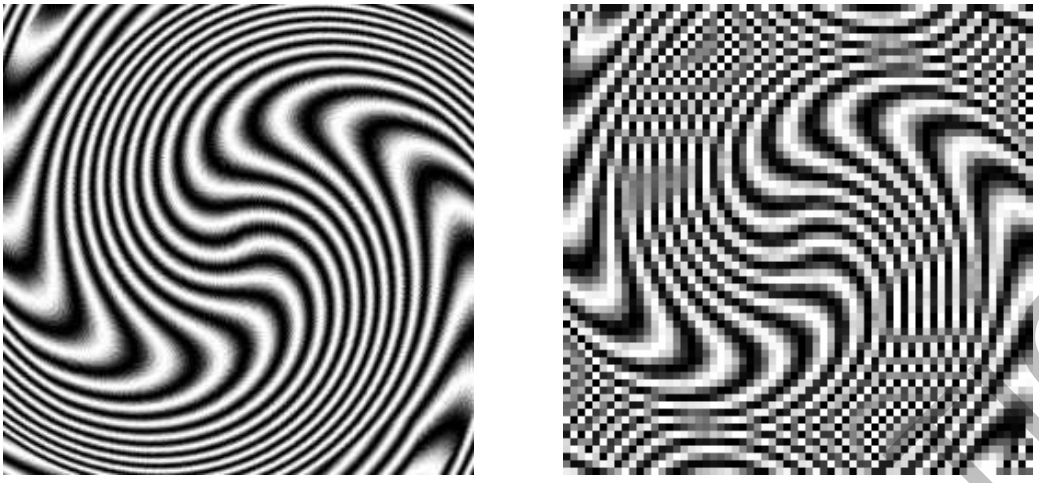

Correct sampling; no aliasing An undersampled version with aliasing

Figure 1.8: Effects of sampling

Using light

Light is the predominant energy source for images; simply because it is the energy source which human beings can observe directly. We are all familiar with photographs, which are a pictorial record of a visual scene.

Many digital images are captured using visible light as the energy source; this has the advantage of being safe, cheap, easily detected and readily processed with suitable hardware. Two very popular methods of producing a digital image are with a digital camera or a flat-bed scanner.

CCD camera. Such a camera has, in place of the usual film, an array of *photosites*; these are silicon electronic devices whose voltage output is proportional to the intensity of light falling on them.

For a camera attached to a computer, information from the photosites is then output to a suitable storage medium. Generally this is done on hardware, as being much faster and more efficient than software, using a *frame-grabbing card*. This allows a large number of images to be captured in a very short time—in the order of one ten-thousandth of a second each. The images can then be copied onto a permanent storage device at some later time.

This is shown schematically in figure 1.9.

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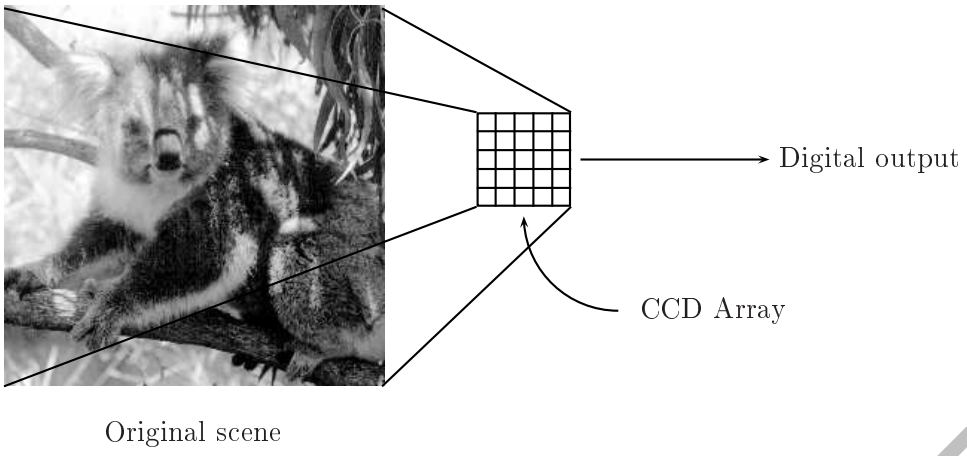

Figure 1.9: Capturing an image with a CCD array

The output will be an array of values; each representing a sampled point from the original scene. The elements of this array are called *picture elements*, or more simply *pixels*.

Digital still cameras use a range of devices, from floppy discs and CD's, to various specialized cards and "memory sticks". The information can then be downloaded from these devices to a computer hard disk.

Flat bed scanner. This works on a principle similar to the CCD camera. Instead of the entire image being captured at once on a large array, a single row of photosites is moved across the image, capturing it row-by-row as it moves. Tis is shown schematically in figure 1.10.

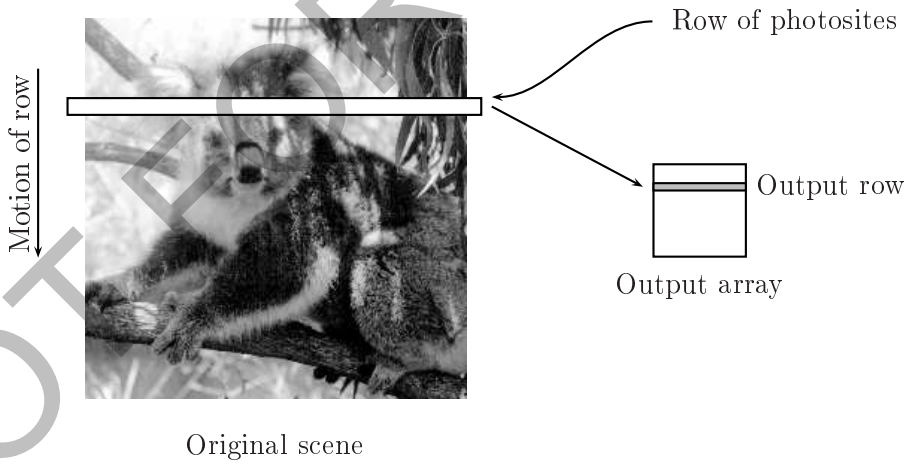

Figure 1.10: Capturing an image with a CCD scanner

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Since this is a much slower process than taking a picture with a camera, it is quite reasonable to allow all capture and storage to be processed by suitable software.

Other energy sources

Although light is popular and easy to use, other energy sources may be used to create a digital image. Visible light is part of the *electromagnetic spectrum*: radiation in which the energy takes the form of waves of varying wavelength. These range from cosmic rays of very short wavelength, to electric power, which has very long wavelength. Figure 1.11 illustrates this. For microscopy, we

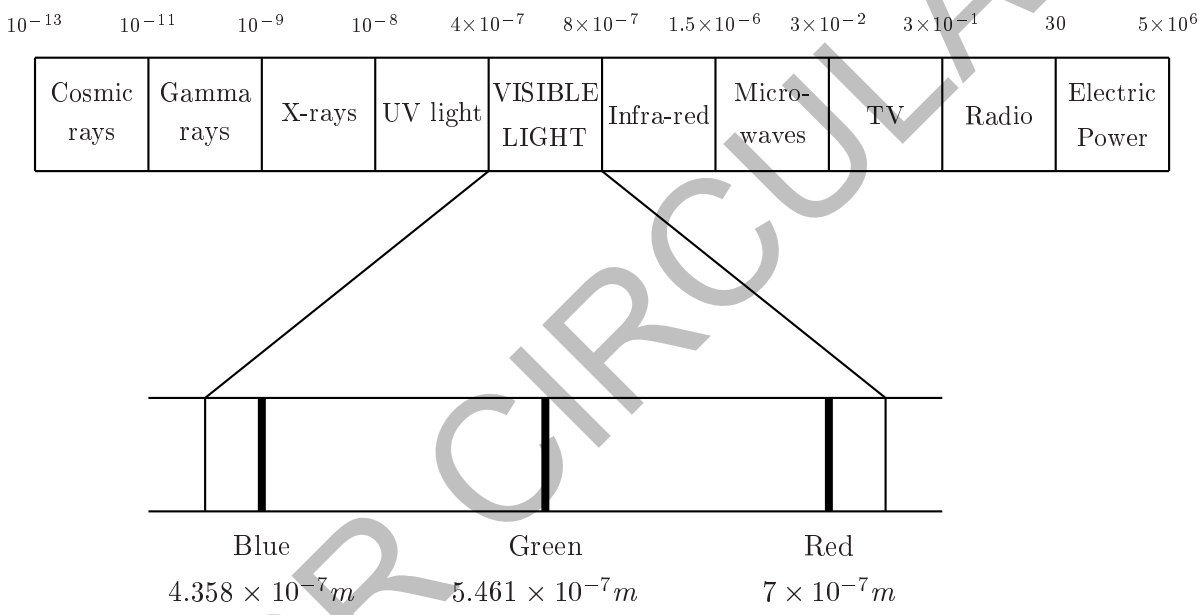

Figure 1.11: The electromagnetic spectrum

may use x-rays, or electron beams. As we can see from figure 1.11, x-rays have a shorter wavelength than visible light, and so can be used to resolve smaller objects than are possible with visible light. See Clark [5] for a good introduction to this. X-rays are of course also useful in determining the structure of objects usually hidden from view: such as bones.

A further method of obtaining images is by the use of *x-ray tomography*, where an object is encircled by an x-ray beam. As the beam is fired through the object, it is detected on the other side of the object, as shown in figure 1.12. As the beam moves around the object, an image of the object

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can be constructed; such an image is called a *tomogram*. In a CAT (Computed Axial Tomography) scan, the patient lies within a tube around which x-ray beams are fired. This enables a large number of tomographic "slices" to be formed, which can then be joined to produce a three-dimensional image. A good account of such systems (and others) is given by Siedband [31]

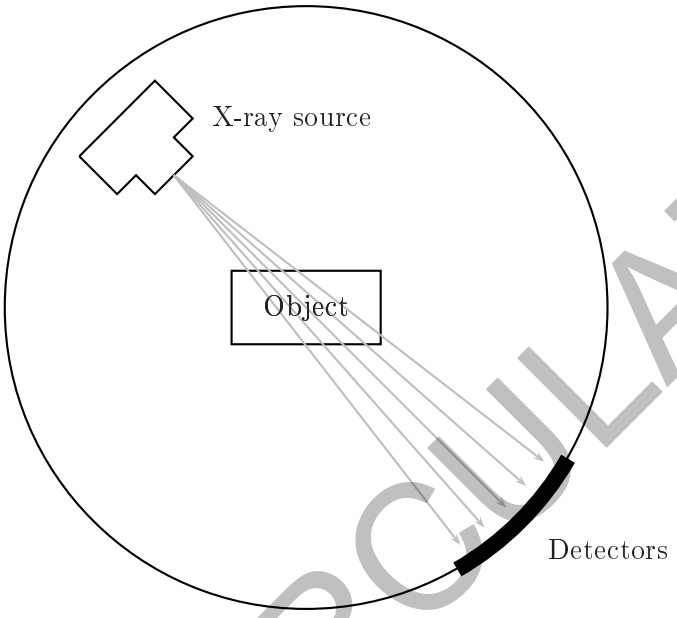

Figure 1.12: X-ray tomography

1.4 Images and digital images

Suppose we take an image, a photo, say. For the moment, lets make things easy and suppose the photo is monochromatic (that is, shades of grey only), so no colour. We may consider this image as being a two dimensional function, where the function values give the brightness of the image at any given point, as shown in figure 1.13. We may assume that in such an image brightness values can be any real numbers in the range 0.0 (black) to 1.0 (white). The ranges of x and y will clearly depend on the image, but they can take all real values between their minima and maxima.

Such a function can of course be plotted, as shown in figure 1.14. However, such a plot is of limited use to us in terms of image analysis. The concept of an image as a function, however, will be vital for the development and implementation of image processing techniques.

A *digital image* differs from a photo in that the x, y, and $f(x, y)$ values are all *discrete*. Usually

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Figure 1.13: An image as a function

they take on only integer values, so the image shown in figure 1.13 will have x and y ranging from 1 to 256 each, and the brightness values also ranging from 0 (black) to 255 (white). A digital image, as we have seen above, can be considered as a large array of sampled points from the continuous image, each of which has a particular quantized brightness; these points are the *pixels* which constitute the digital image. The pixels surrounding a given pixel constitute its *neighbourhood*. A neighbourhood can be characterized by its shape in the same way as a matrix: we can speak, for example, of a 3×3 neighbourhood, or of a 5×7 neighbourhood. Except in very special circumstances, neighbourhoods have odd numbers of rows and columns; this ensures that the current pixel is in the centre of the neighbourhood. An example of a neighbourhood is given in figure 1.15. If a neighbourhood has an even number of rows or columns (or both), it may be necessary to specify which pixel in the neighbourhood is the "current pixel".

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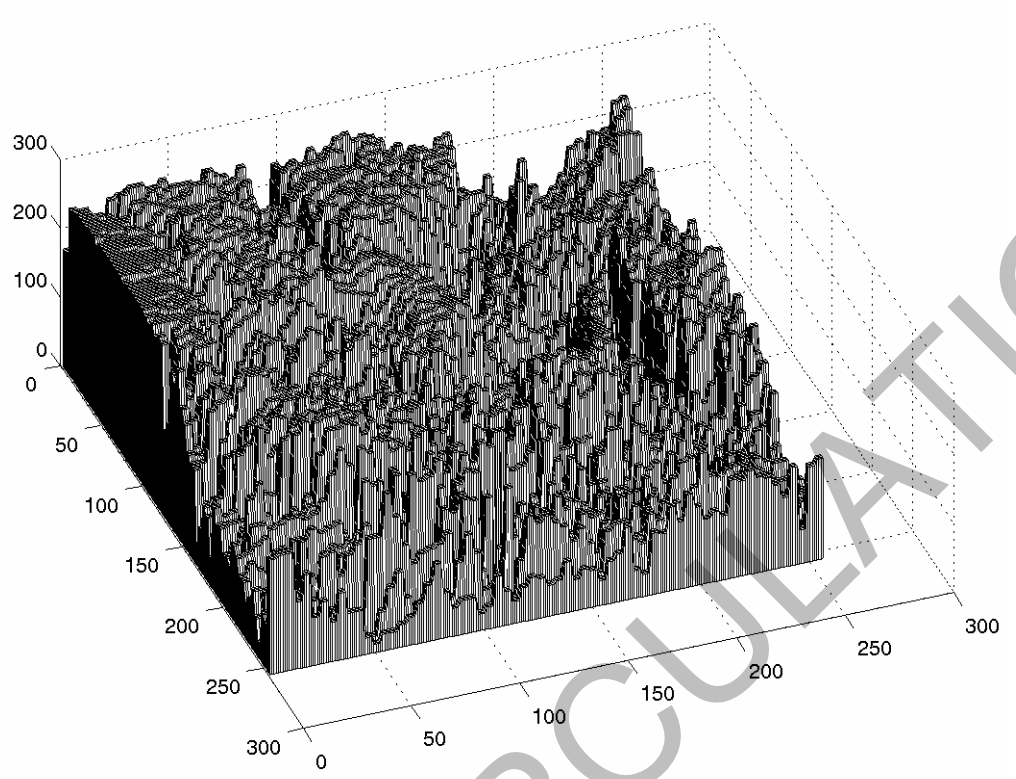

Figure 1.14: The image of figure 1.13 plotted as a function of two variables

48	219	168	145	244	188	120	58
49	218	87	94	133	35	17	148
174	151	74	179	224	3	252	194
77	127	87	139	44	228	149	135
138	229	136	113	250	51	108	163
38	210	185	177	69	76	131	53
178	164	79	158	64	169	85	97
96	209	214	203	223	73	110	200

Current pixel

3×5 neighbourhood

Figure 1.15: Pixels, with a neighbourhood

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1.5 Some applications

Image processing has an enormous range of applications; almost every area of science and technology can make use of image processing methods. Here is a short list just to give some indication of the range of image processing applications.

1. Medicine
   - Inspection and interpretation of images obtained from X-rays, MRI or CAT scans,
   - analysis of cell images, of chromosome karyotypes.
2. Agriculture
   - Satellite/aerial views of land, for example to determine how much land is being used for different purposes, or to investigate the suitability of different regions for different crops,
   - inspection of fruit and vegetables—distinguishing good and fresh produce from old.
3. Industry
   - Automatic inspection of items on a production line,
   - inspection of paper samples.
4. Law enforcement
   - Fingerprint analysis,
   - sharpening or de-blurring of speed-camera images.

1.6 Aspects of image processing

It is convenient to subdivide different image processing algorithms into broad subclasses. There are different algorithms for different tasks and problems, and often we would like to distinguish the nature of the task at hand.

Image enhancement. This refers to processing an image so that the result is more suitable for a particular application. Example include:

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- sharpening or de-blurring an out of focus image,
- highlighting edges,
- improving image contrast, or brightening an image,
- removing noise.

Image restoration. This may be considered as reversing the damage done to an image by a known cause, for example:

- removing of blur caused by linear motion,
- removal of optical distortions,
- removing periodic interference.

Image segmentation. This involves subdividing an image into constituent parts, or isolating certain aspects of an image:

- finding lines, circles, or particular shapes in an image,
- in an aerial photograph, identifying cars, trees, buildings, or roads.

These classes are not disjoint; a given algorithm may be used for both image enhancement or for image restoration. However, we should be able to decide what it is that we are trying to do with our image: simply make it look better (enhancement), or removing damage (restoration).

1.7 An image processing task

We will look in some detail at a particular real-world task, and see how the above classes may be used to describe the various stages in performing this task. The job is to obtain, by an automatic process, the postcodes from envelopes. Here is how this may be accomplished:

Acquiring the image. First we need to produce a digital image from a paper envelope. This an be done using either a CCD camera, or a scanner.

Preprocessing. This is the step taken before the "major" image processing task. The problem here is to perform some basic tasks in order to render the resulting image more suitable for the job

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to follow. In this case it may involve enhancing the contrast, removing noise, or identifying regions likely to contain the postcode.

Segmentation. Here is where we actually "get" the postcode; in other words we extract from the image that part of it which contains just the postcode.

Representation and description. These terms refer to extracting the particular features which allow us to differentiate between objects. Here we will be looking for curves, holes and corners which allow us to distinguish the different digits which constitute a postcode.

Recognition and interpretation. This means assigning labels to objects based on their descriptors (from the previous step), and assigning meanings to those labels. So we identify particular digits, and we interpret a string of four digits at the end of the address as the postcode.

1.8 Types of digital images

We shall consider four basic types of images:

Binary. Each pixel is just black or white. Since there are only two possible values for each pixel, we only need one bit per pixel. Such images can therefore be very efficient in terms of storage. Images for which a binary representation may be suitable include text (printed or handwriting), fingerprints, or architectural plans.

An example was the image shown in figure 1.4(b) above. In this image, we have only the two colours: white for the edges, and black for the background. See figure 1.16 below.

Greyscale. Each pixel is a shade of grey, normally from 0 (black) to 255 (white). This range means that each pixel can be represented by eight bits, or exactly one byte. This is a very natural range for image file handling. Other greyscale ranges are used, but generally they are a power of 2. Such images arise in medicine (X-rays), images of printed works, and indeed 256 different grey levels is sufficient for the recognition of most natural objects.

An example is the street scene shown in figure 1.1 above, and in figure 1.17 below.

True colour, or RGB. Here each pixel has a particular colour; that colour being described by the amount of red, green and blue in it. If each of these components has a range 0–255, this gives

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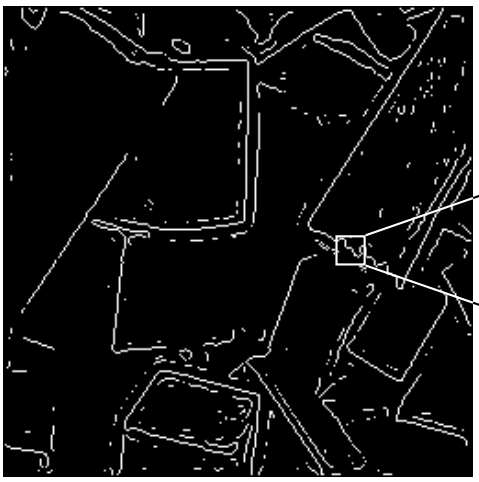

1	1	0	0	0	0
0	0	1	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0
0	0	0	1	1	0
0	0	0	0	0	1

Figure 1.16: A binary image

230	229	232	234	235	232	148
237	236	236	234	233	234	152
255	255	255	251	230	236	161
99	90	67	37	94	247	130
222	152	255	129	129	246	132
154	199	255	150	189	241	147
216	132	162	163	170	239	122

Figure 1.17: A greyscale image

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a total of $255^3 = 16,777,216$ different possible colours in the image. This is enough colours for any image. Since the total number of bits required for each pixel is 24, such images are also called 24-*bit colour images*.

Such an image may be considered as consisting of a "stack" of three matrices; representing the red, green and blue values for each pixel. This means that for every pixel there correspond three values.

An example is shown in figure 1.18.

49	55	56	57	52	53
58	60	60	58	55	57
58	58	54	53	55	56
83	78	72	69	68	69
88	91	91	84	83	82
69	76	83	78	76	75
61	69	73	78	76	76

Red

64	76	82	79	78	78
93	93	91	91	86	86
88	82	88	90	88	89
125	119	113	108	111	110
137	136	132	128	126	120
105	108	114	114	118	113
96	103	112	108	111	107

Green

66	80	77	80	87	77
81	93	96	99	86	85
83	83	91	94	92	88
135	128	126	112	107	106
141	129	129	117	115	101
95	99	109	108	112	109
84	93	107	101	105	102

Blue

Figure 1.18: A true colour image

Indexed. Most colour images only have a small subset of the more than sixteen million possible colours. For convenience of storage and file handling, the image has an associated *colour map*,

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or *colour palette*, which is simply a list of all the colours used in that image. Each pixel has a value which does not give its colour (as for an RGB image), but an *index* to the colour in the map.

It is convenient if an image has 256 colours or less, for then the index values will only require one byte each to store. Some image file formats (for example, Compuserve GIF), allow only 256 colours or fewer in each image, for precisely this reason.

Figure 1.19 shows an example. In this image the indices, rather then being the grey values of the pixels, are simply indices into the colour map. Without the colour map, the image would be very dark and colourless. In the figure, for example, pixels labelled `5` correspond to `0.2627 0.2588 0.2549`, which is a dark greyish colour.

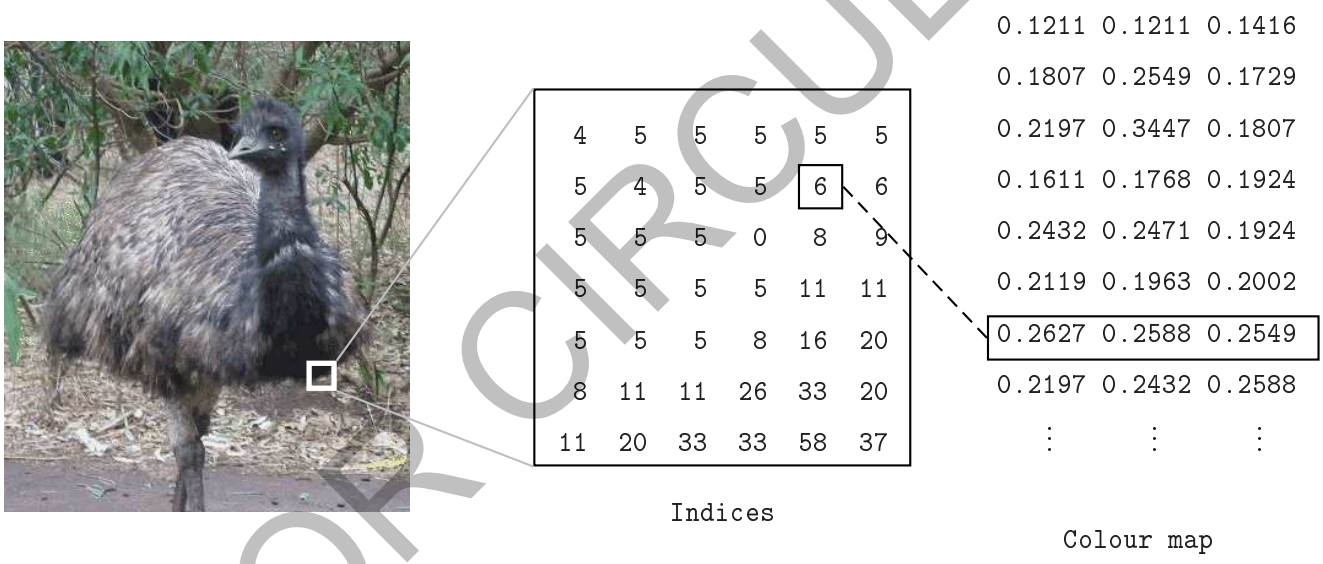

Figure 1.19: An indexed colour image

1.9 Image File Sizes

Image files tend to be large. We shall investigate the amount of information used in different image type of varying sizes. For example, suppose we consider a 512×512 binary image. The number of bits used in this image (assuming no compression, and neglecting, for the sake of discussion, any header information) is

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$$\begin{aligned} 512 \times 512 \times 1 &= 262,144 \\ &= 32768 \text{ bytes} \\ &= 32.768 \text{ Kb} \\ &\approx 0.033\text{Mb.} \end{aligned}$$

(Here we use the convention that a kilobyte is one thousand bytes, and a megabyte is one million bytes.)

A greyscale image of the same size requires:

$$\begin{aligned} 512 \times 512 \times 1 &= 262,144 \text{ bytes} \\ &= 262.14 \text{ Kb} \\ &\approx 0.262 \text{ Mb.} \end{aligned}$$

If we now turn our attention to colour images, each pixel is associated with 3 bytes of colour information. A 512×512 image thus requires

$$\begin{aligned} 521 \times 512 \times 3 &= 786,432 \text{ bytes} \\ &= 786.43 \text{ Kb} \\ &\approx 0.786 \text{ Mb.} \end{aligned}$$

Many images are of course such larger than this; satellite images may be of the order of several thousand pixels in each direction.

1.10 Image perception

Much of image processing is concerned with making an image appear "better" to human beings. We should therefore be aware of the limitations of the the human visual system. Image perception consists of two basic steps:

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1. capturing the image with the eye,

2. recognising and interpreting the image with the *visual cortex* in the brain.

The combination and immense variability of these steps influences the ways in we perceive the world around us.

There are a number of things to bear in mind:

1. *Observed intensities* vary as to the background. A single block of grey will appear darker if placed on a white background than if it were placed on a black background. That is, we don't perceive grey scales "as they are", but rather as they differ from their surroundings. In figure 1.20 a grey square is shown on two different backgrounds. Notice how much darker the square appears when it is surrounded by a light grey. However, the two central squares have exactly the same intensity.

Figure 1.20: A grey square on different backgrounds

2. We may observe non-existent intensities as bars in continuously varying grey levels. See for example figure 1.21. This image varies continuously from light to dark as we travel from left to right. However, it is impossible for our eyes not to see a few horizontal edges in this image.

3. Our visual system tends to undershoot or overshoot around the boundary of regions of different intensities. For example, suppose we had a light grey blob on a dark grey background. As our eye travels from the dark background to the light region, the boundary of the region

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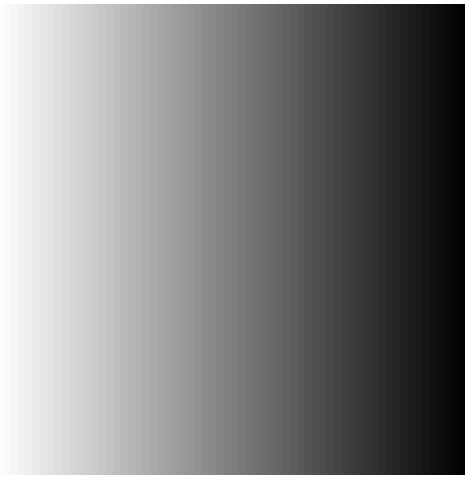

Figure 1.21: Continuously varying intensities

appears lighter than the rest of it. Conversely, going in the other direction, the boundary of the background appears *darker* than the rest of it.

Exercises

1. Watch the TV news, and see if you can observe any examples of image processing.
2. If your TV set allows it, turn down the colour as far as you can to produce a monochromatic display. How does this affect your viewing? Is there anything which is hard to recognize without colour?
3. Look through a collection of old photographs. How can they be enhanced, or restored?
4. For each of the following, list five ways in which image processing could be used:
   - medicine
   - astronomy
   - sport
   - music
   - agriculture
   - travel

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5. Image processing techniques have become a vital part of the modern movie production process. Next time you watch a film, take note of all the image processing involved.

6. If you have access to a scanner, scan in a photograph, and experiment with all the possible scanner settings.

   (a) What is the smallest sized file you can create which shows all the detail of your photograph?

   (b) What is the smallest sized file you can create in which the major parts of your image are still recognizable?

   (c) How do the colour settings affect the output?

7. If you have access to a digital camera, again photograph a fixed scene, using all possible camera settings.

   (a) What is the smallest file you can create?

   (b) How do the light settings effect the output?

8. Suppose you were to scan in a monochromatic photograph, and then print out the result. Then suppose you scanned in the printout, and printed out the result of that, and repeated this a few times. Would you expect any degradation of the image during this process? What aspects of the scanner and printer would minimize degradation?

9. Look up *ultrasonography*. How does it differ from the image acquisition methods discussed in this chapter? What is it used for? If you can, compare an ultrasound image with an x-ray image. How to they differ? In what ways are they similar?

10. If you have access to an image viewing program (other than MATLAB) on your computer, make a list of the image processing capabilities it offers. Can you find imaging tasks it is unable to do?

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Chapter 2

Images and MATLAB

We have seen in the previous chapter that matrices can be handled very efficiently in MATLAB. Images may be considered as matrices whose elements are the pixel values of the image. In this chapter we shall investigate how the matrix capabilities of MATLAB allow us to investigate images and their properties.

2.1 Greyscale images

Suppose you are sitting at your computer and have started MATLAB. You will have a MATLAB *command window* open, and in it the MATLAB prompt

```
>>
```

ready to receive commands. Type in the command

```
>> w=imread('wombats.tif');
```

This takes the grey values of all the pixels in the greyscale image `wombats.tif` and puts them all into a matrix `w`. This matrix `w` is now a MATLAB *variable*, and so we can perform various matrix operations on it. In general the `imread` function reads the pixel values from an image file, and returns a matrix of all the pixel values.

Two things to note about this command:

1. It ends in a *semicolon*; this has the effect of not displaying the results of the command to the

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screen. As the result of this particular command is a matrix of size 256×256, or with 65536 elements, we don't really want all its values displayed.

2. The name `wombats.tif` is given in single quote marks. Without them, MATLAB would assume that `wombats.tif` was the name of a variable, rather than the name of a file.

Now we can display this matrix as a greyscale image:

```
>> figure,imshow(w),pixval on
```

This is really three commands on the one line. MATLAB allows many commands to be entered on the same line; using commas to separate the different commands. The three commands we are using here are:

`figure,` which creates a *figure* on the screen. A figure is a window in which a graphics object can be placed. Objects may include images, or various types of graphs.

`imshow(g),` which displays the matrix `g` as an image.

`pixval on,` which turns on the pixel values in our figure. This is a display of the grey values of the pixels in the image. They appear at the bottom of the figure in the form

$$c \times r = p$$

where c is the column value of the given pixel; r its row value, and p its grey value. Since `wombats.tif` is an 8-bit greyscale image, the pixel values appear as integers in the range 0–255.

This is shown in figure 2.1.

If there are no figures open, then an `imshow` command, or any other command which generates a graphics object, will open a new figure for displaying the object. However, it is good practice to use the `figure` command whenever you wish to create a new figure.

We could display this image directly, without saving its grey values to a matrix, with the command

```
imshow('wombats.tif')
```

However, it is better to use a matrix, seeing as these are handled very efficiently in MATLAB.

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Figure 2.1: The wombats image with `pixval on`

2.2 RGB Images

As we shall discuss in chapter 13; we need to define colours in some standard way, usually as a subset of a three dimensional coordinate system; such a subset is called a *colour model*. There are in fact a number of different methods for describing colour, but for image display and storage a standard model is RGB, for which we may imagine all the colours sitting inside a "colour cube" of side 1 as shown in figure 2.2. The colours along the black-white diagonal, shown in the diagram as a dotted line, are the points of the space where all the R, G, B values are equal. They are the different intensities of grey. We may also think of the axes of the colour cube as being discretized to integers in the range 0–255.

RGB is the standard for the *display* of colours: on computer monitors; on TV sets. But it is not a very good way of *describing* colours. How, for example, would you define light brown using RGB? As we shall see also in chapter 13, there are some colours which are not realizable with the RGB model; in that they would require negative values of one or two of the RGB components. MATLAB handles 24-bit RGB images in much the same way as greyscale. We can save the colour values to a

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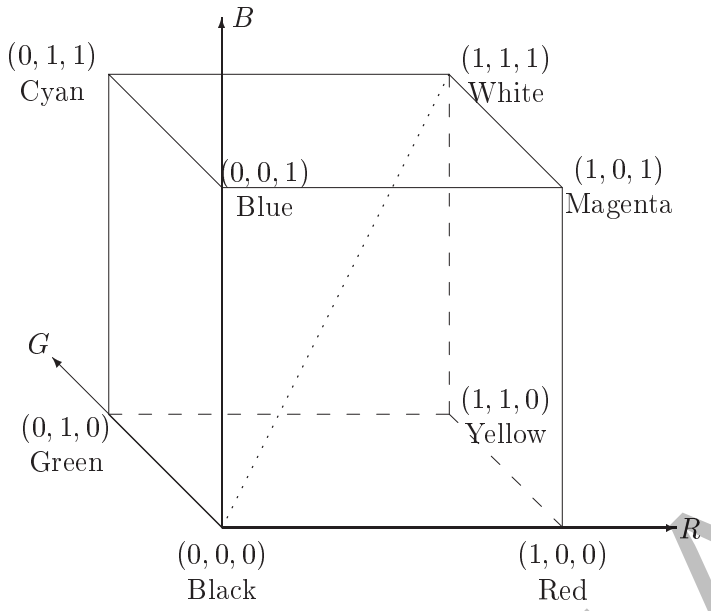

Figure 2.2: The colour cube for the RGB colour model

matrix and view the result:

```
>> a=imread('autumn.tif');
>> figure,imshow(a),pixval on
```

Note now that the pixel values now consist of a list of three values, giving the red, green and blue components of the colour of the given pixel.

An important difference between this type of image and a greyscale image can be seen by the command

```
>> size(a)
```

which returns *three* values: the number of rows, columns, and "pages" of `a`, which is a three-dimensional matrix, also called a *multidimensional array*. MATLAB can handle arrays of any dimension, and `a` is an example. We can think of `a` as being a stack of three matrices, each of the same size.

To obtain any of the RGB values at a given location, we use similar indexing methods to above. For example

```
>> a(100,200,2)
```

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returns the second colour value (green) at the pixel in row 100 and column 200. If we want all the colour values at that point, we can use

```
>> a(100,200,1:3)
```

However, MATLAB allows a convenient shortcut for listing all values along a particular dimension; just using a colon on its own:

```
>> a(100,200,:)
```

A useful function for obtaining RGB values is `impixel`; the command

```
>> impixel(a,200,100)
```

returns the red, green, and blue values of the pixel at column 200, row 100. Notice that the order of indexing is the same as that which is provided by the `pixval on` command. This is opposite to the row, column order for matrix indexing. This command also applies to greyscale images:

```
>> impixel(g,100,200)
```

will return three values, but since `g` is a single two-dimensional matrix, all three values will be the same.

2.3 Indexed colour images

The command

```
>> figure,imshow('emu.tif'),pixval on
```

produces a nice colour image of an emu. However, the pixel values, rather than being three integers as they were for the RGB image above, are three fractions between 0 and 1. What is going on here?

If we try saving to a matrix first and then displaying the result:

```
>> em=imread('emu.tif');
>> figure,imshow(em),pixval on
```

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we obtain a dark, barely distinguishable image, with single integer grey values, indicating that `em` is being interpreted as a single greyscale image.

In fact the image `emu.tif` is an example of an *indexed image*, consisting of two matrices: a *colour map*, and an *index* to the colour map. Assigning the image to a single matrix picks up only the index; we need to obtain the colour map as well:

```
>> [em,emap]=imread('emu.tif');
>> figure,imshow(em,emap),pixval on
```

MATLAB stores the RGB values of an indexed image as values of type `double`, with values between 0 and 1.

Information about your image

A great deal of information can be obtained with the `imfinfo` function. For example, suppose we take our indexed image `emu.tif` from above.

```
>> imfinfo('emu.tif')


ans =


                    Filename: 'emu.tif'
                 FileModDate: '26-Nov-2002 14:23:01'
                    FileSize: 119804
                      Format: 'tif'
               FormatVersion: []
                       Width: 331
                      Height: 384
                    BitDepth: 8
                   ColorType: 'indexed'
             FormatSignature: [73 73 42 0]
                   ByteOrder: 'little-endian'
```

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```
              NewSubfileType: 0
               BitsPerSample: 8
                 Compression: 'PackBits'
    PhotometricInterpretation: 'RGB Palette'
                StripOffsets: [16x1 double]
             SamplesPerPixel: 1
                RowsPerStrip: 24
             StripByteCounts: [16x1 double]
                 XResolution: 72
                 YResolution: 72
              ResolutionUnit: 'Inch'
                    Colormap: [256x3 double]
         PlanarConfiguration: 'Chunky'
                  TileWidth: []
                 TileLength: []
                 TileOffsets: []
              TileByteCounts: []
                 Orientation: 1
                   FillOrder: 1
             GrayResponseUnit: 0.0100
              MaxSampleValue: 255
              MinSampleValue: 0
                Thresholding: 1
```

Much of this information is not useful to us; but we can see the size of the image in pixels, the size of the file (in bytes), the number of bits per pixel (this is given by `BitDepth`), and the colour type (in this case "indexed").

For comparison, let's look at the output of a true colour file (showing only the first few lines of the output):

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```
>> imfinfo('flowers.tif')


ans =


                    Filename: [1x57 char]
                 FileModDate: '26-Oct-1996 02:11:09'
                    FileSize: 543962
                      Format: 'tif'
               FormatVersion: []
                       Width: 500
                      Height: 362
                    BitDepth: 24
                   ColorType: 'truecolor'
```

Now we shall test this function on a binary image:

```
>> imfinfo('text.tif')


ans =


                    Filename: [1x54 char]
                 FileModDate: '26-Oct-1996 02:12:23'
                    FileSize: 3474
                      Format: 'tif'
               FormatVersion: []
                       Width: 256
                      Height: 256
                    BitDepth: 1
                   ColorType: 'grayscale'
```

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What is going on here? We have a binary image, and yet the colour type is given as "grayscale". The fact is that MATLAB does not distinguish between greyscale and binary images: a binary image is just a special case of a greyscale image which has only two intensities. However, we can see that `text.tif` is a binary image since the number of bits per pixel is only one.

2.4 Data types and conversions

Elements in MATLAB matrices may have a number of different numeric data types; the most common are listed in table 2.1. There are others, but those listed will be sufficient for all our work with

Data type	Description	Range
`int8`	8-bit integer	−128 — 127
`uint8`	8-bit unsigned integer	0 — 255
`int16`	16-bit integer	−32768 — 32767
`uint16`	16-bit unsigned integer	0 — 65535
`double`	Double precision real number	Machine specific

Table 2.1: Data types in MATLAB

images. These data types are also functions, we can convert from one type to another. For example:

```
>>  a=23;
>> b=uint8(a);
>> b


b =


    23


>> whos a b
  Name      Size            Bytes  Class


  a         1x1                 8  double array
  b         1x1                 1  uint8 array
```

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Even though the variables `a` and `b` have the same numeric value, they are of different data types. An important consideration (of which we shall more) is that arithmetic operations are not permitted with the data types `int8`, `int16`, `uint8` and `uint16`.

A greyscale image may consist of pixels whose values are of data type `uint8`. These images are thus reasonably efficient in terms of storage space, since each pixel requires only one byte. However, arithmetic operations are not permitted on this data type; a `uint8` image must be converted to `double` before any arithmetic is attempted.

We can convert images from one image type to another. Table 2.2 lists all of MATLAB's functions for converting between different image types. Note that the `gray2rgb` function, does not create a

Function	**Use**	**Format**
`ind2gray`	Indexed to Greyscale	`y=ind2gray(x,map);`
`gray2ind`	Greyscale to indexed	`[y,map]=gray2ind(x);`
`rgb2gray`	RGB to greyscale	`y=rgb2gray(x);`
`gray2rgb`	Greyscale to RGB	`y=gray2rgb(x);`
`rgb2ind`	RGB to indexed	`[y,map]=rgb2ind;`
`ind2rgb`	Indexed to RGB	`y=ind2rgb(x,map);`

Table 2.2: Converting images in MATLAB

colour image, but an image all of whose pixel colours were the same as before. This is done by simply replicating the grey values of each pixel: greys in an RGB image are obtained by equality of the red, green and blue values.

2.5 Image Files and Formats

We have seen in section 1.8 that images may be classified into four distinct types: binary, greyscale, coloured, and indexed. In this section we consider some of the different image file formats; their advantages and disadvantages. You can use MATLAB for image processing very happily without ever really knowing the difference between GIF, TIFF, PNG and all the other formats. However, some knowledge of the different graphics formats can be extremely useful, in order to be able to make a reasoned decision as to which file type to use, and when.

There are a great many different formats for storing image data. Some have been designed to fulfill a particular need (for example, to transmit image data over a network), others have been

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designed around a particular operations system or environment.

As well as the grey values or colour values of the pixels, an image file will contain some *header information*. This will, at the very least, include the size of the image in pixels (height and width); it may also include the colour map, compression used, and a description of the image. MATLAB recognizes many standard formats, and can read image data from them, and write image data to them. The examples above have all been images with extension `.tif`, indicating that they are TIFF (Tagged Image File Format) images. This is a particularly general format, as it allows for binary, greyscale, RGB and indexed colour images, as well as allowing different amounts of compression. TIFF is thus a good format for transmitting images between different operating systems and environments. TIFF also allows for more than one image per file; the particular image can be read into MATLAB by using an optional numeric argument to the `imread` function.

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The `imread` and `imwrite` functions of MATLAB currently support the following formats:

JPEG Images created using the Joint Photographics Experts Group compression method. We will discuss this more in chapter 14.

TIFF "Tagged Image File Format": a very general format which supports different compression methods, multiple images per file, and binary, greyscale, truecolour and indexed images.

GIF "Graphics Interchange Format". This venerable format was designed for data transfer. It is still popular and well supported, but is somewhat restricted in the image types it can handle.

BMP Microsoft Bitmap. This format has become very popular, with its use by Microsoft operating systems.

PNG "Portable Network Graphics". This is designed to overcome some of the disadvantages of GIF, and to become a replacement for GIF.

HDF "Hierarchical Data Format". This is an extensible, versatile format designed principally for use with scientific images.

PCX This was originally designed for use with the MS-DOS based software PC Paintbrush. It is also used by some Microsoft products.

XWD "X Window Dump". This is used to store images created by screen dumps from the X window system. This is the standard windowing system used by Unix operating systems.

ICO This is the format used to display icons in Microsoft Windows operating systems. It allows multiple images per file.

CUR This is the format used to display the mouse cursor in Microsoft Windows operating systems.

We will discuss some of these briefly below.

A hexadecimal dump function

To explore binary files, we need a simple function which will enable us to list the contents of the file as hexadecimal values. If we try to list the contents of a binary file directly, for example with the `type` function (which prints the file contents to the screen), we will see masses of garbage. The

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trouble is that the contents are being interpreted as ASCII characters, and this means that most of these will either be unprintable, or nonsensical as values.

A simple hexadecimal dump function is shown in figure 2.3. The first three lines read in the

```
function dumphex(filename, n)
%
% DUMPHEX(FILENAME,N) prints the first 16*n bytes of the file FILENAME
% in hex and ASCII.  For example:
%
%   dumphex('picture.bmp',4)
%
fid = fopen(filename, 'r');
if fid==-1
  error('File does not exist or is not in your Matlab path');
end;
a=fread(fid,16*n,'uchar');
idx=find(a>=32 & a<=126);
ah=dec2hex(a);
b=repmat([' '],16*n,3);
b2=repmat('.',16,n);
b2(idx)=char(a(idx));
b(:,1:2)=ah;
[reshape(b',48,n)' repmat(' ',n,2) reshape(b2,16,n)']
```

Figure 2.3: A function for producing hexadecimal dumps

required amount of information from the file, and format it as hexadecimal values. The next three lines put those values into a matrix for display. For example:

```
>> dumphex('dataread.mexglx',4)


ans =


7F 45 4C 46 01 01 01 00 00 00 00 00 00 00 00 00   .ELF............
03 00 03 00 01 00 00 00 10 11 00 00 34 00 00 00   ............4...
E4 66 00 00 00 00 00 00 34 00 20 00 03 00 28 00   .f......4. ...(.
1B 00 18 00 01 00 00 00 00 00 00 00 00 00 00 00   ................
```

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2.5.1 Vector versus raster images

We may store image information in two different ways: as a collection of lines or vectors, or as a collection of dots. We refer to the former as *vector* images; the latter as *raster* images. The great advantage of vector images is that they can be magnified to any desired size without losing any sharpness. The disadvantage is that are not very good for the representation of natural scenes, in which lines may be scarce. The standard vector format is Adobe PostScript; this is an international standard for page layout. PostScript is the format of choice for images consisting mostly of lines and mathematically described curves: architectural and industrial plans; font information, and mathematical figures. The reference manual [14] provides all necessary information about PostScript.

The great bulk of image file formats store images as raster information; that is, as a list of the grey or colour intensities of each pixel. Images captured by digital means: digital cameras or scanners, will be stored in raster format.

2.5.2 A simple raster format

As well as containing all pixel information, an image file must contain some *header information*; this must include the size of the image, but may also include some documentation, a colour map, and compression used. To show the workings of a raster image file, we shall briefly describe the ASCII PGM format. This was designed to be a generic format used for conversion between other formats. Thus, to create conversion routines between, say, 40 different formats, rather than have $40 \times 39 = 1560$ different conversion routines, all we need is the $40 \times 2 = 80$ conversion routines between the formats and PGM.

```
P2
# CREATOR: The GIMP's PNM Filter Version 1.0
256 256
255
 41  53  53  53  53  49  49  53  53  56  56  49  41  46  53  53  53
 53  41  46  56  56  56  53  53  46  53  41  41  53  56  49  39  46
```

Figure 2.4: The start of a PGM file

Figure 2.4 shows the beginning of a PGM file. The file begins with `P2`; this indicates that the

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file is an ASCII PGM file. The next line gives some information about the file: any line beginning with a hash symbol is treated as a comment line. The next line gives the number of columns and rows, and the following line the number of greyscales. Finally we have all the pixel information, starting at the top left of the image, and working across and down. Spaces and carriage returns are delimiters, so the pixel information could be written in one very long line, or one very long column.

Note that this format has the advantage of being very easy to write to and to read from; it has the disadvantage of producing very large files. Some space can be saved by using "raw" PGM; the only difference is that the header number is `P3`, and the pixel values are stored one per byte. There are corresponding formats for binary and coloured images (PBM and PPM respectively); coloured images are stored as three matrices; one for each of red, green and blue; either as ASCII or raw. The format does not support colour maps.

Binary, greyscale or colour images using these formats are collectively called PNM images. MATLAB does not support PNM images directly; however functions for reading and writing them can be found at the Mathworks Central File Exchange[1].

2.5.3 Microsoft BMP

The Microsoft Windows BMP image format is a fairly simple example of a binary image format. Like the PGM format above, it consists of a header followed by the image information. The header is divided into two parts: the first 14 bytes (bytes number 0 to 13), is the "File Header", and the following 40 bytes (bytes 14 to 53), is the "Information Header". The header is arranged as follows:

[1] `http://www.mathworks.com/matlabcentral/fileexchange`, in the `External Interface→Image and Movie Formats` subdirectory.

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Bytes	Information	Description
0–1	Signature	"BM" in ASCII = `42 4D` in hexadecimal.
2–5	FileSize	The size of the file in bytes.
6–9	Reserved	All zeros.
10–13	DataOffset	File offset to the raster data.
14–17	Size	Size of the information header, = 40 bytes.
18–21	Width	Width of the image in pixels.
22–25	Height	Height of the image in pixels.
26–27	Planes	Number of image planes (= 1).
28–29	BitCount	Number of bits per pixel: 1: Binary images; two colours, 4: $2^4 = 16$ colours (indexed), 8: $2^8 = 256$ colours (indexed), 16: 16 bit RGB; $2^{16} = 65536$ colours, 24: 24 bit RGB; $2^{24} = 17,222,216$ colours.
30–33	Compression	Type of compression used: 0: no compression (most common), 1: 8 bit RLE encoding (rarely used), 2: 4 bit RLE encoding (rarely used).
34–37	ImageSize	Size of the image. If compression is 0, then this value may be 0.
38–41	HorizontalRes	The horizontal resolution in pixels per metre.
42–45	VerticalRes	The vertical resolution in pixels per metre.
46–49	ColoursUsed	The number of colours used in the image. If this value is zero, then the number of colours is the maximum obtainable with the bits per pixel, that is 2^{BitCount}.
50–53	ImportantColours	The number of important colours in the image. If all the colours are important, then this value is set to zero.

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After the header comes the Colour Table, which is only used if BitCount is less than or equal to 8. The total number of bytes used here is $4 \times$ ColoursUsed. This format uses the Intel "least endian" convention for bytes ordering, where in each word of four bytes; the *least* valued byte comes *first*. To see an example of this, consider a simple example:

```
>> dumphex('blocksets.bmp',4)


ans =


42 4D 6E 18 00 00 00 00 00 00 36 00 00 00 28 00   BMn.......6...(.
00 00 42 00 00 00 1F 00 00 00 01 00 18 00 00 00   ..B............
00 00 38 18 00 00 C4 0E 00 00 C4 0E 00 00 00 00   ..8............
00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00   ................
```

The image width is given by bytes 18–21; they are in the second row:

```
00 00 00
```

To find the actual width; we re-order these bytes back to front:

```
00 00 42
```

Now we can convert to decimal:

$$(4 \times 16^1) + (2 \times 16^0) = 66$$

which is the image width in pixels. We can do the same thing with the image height; bits 22–25:

```
00 00 00
```

Re-ordering and converting to hexadecimal:

$$(1 \times 16^1) + (\mathtt{F} \times 16^0) = 16 + 15 = 31.$$

2.5.4 GIF and PNG

Compuserve GIF (pronounced "jif") is a venerable image format which was first proposed in the late 1980's, as a means for distributing images over networks. Like PGM, it is a raster format, but it has the following properties:

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1. Colours are stored using a colour map; the GIF specification allows a maximum of 256 colours per image.

2. GIF doesn't allow for binary or greyscale images; except as can be produced with red, green and blue values.

3. The pixel data is compressed using LZW (Lempel-Ziv-Welch) compression. This works by constructing a "codebook" of the data: the first time a pattern is found, it is placed in the codebook; subsequent times the encoder will output the code for that pattern. LZW compression can be used on any data; however it is a patented algorithm, and legal use must be accompanied by a license from Unisys.

4. The GIF format allows for multiple images per file; this aspect can be used to create "animated GIFs".

A GIF file will contain a header including the image size (in pixels), the colour map, the colour *resolution* (number of bits per pixel), a flag indicating whether or not the colour map is ordered; and the colour map size.

The GIF format is greatly used; it has become one of the standard formats supported by the World Wide Web, and by the Java programming language. Full descriptions of the GIF format can be found in [2] or [18].

The PNG (pronounced "ping") format has been more recently designed to replace GIF, and to overcome some of its disadvantages. Specifically, PNG was not to rely on any patented algorithms, and it was to support more image types than GIF. PNG supports greyscale, true-colour and indexed images. Moreover, its compression utility, zlib, *always* results in genuine compression. This is not the case with LZW compression; it can happen that the result of an LZW compression is larger than the original data. PNG also includes support for *alpha channels*, which are ways of associating variable transparencies with an image, and *gamma correction*, which associates different numbers with different computer display systems, so as to ensure that a given image will appear the same independently of the system.

PNG is described in detail by [25]. PNG is certainly to be preferred to GIF, but being newer, it is not yet as well supported.

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2.5.5 JPEG

The compression methods used by GIF and PNG are *lossless*: the original information can be recovered completely. The JPEG (Joint Photographics Experts Group) algorithm uses *lossy* compression, in which not all the original data can be recovered. Such methods result in much higher compression rates, and JPEG images are in general much smaller than GIF or PNG images. Compression of JPEG images works by breaking the image into 8×8 blocks, applying the discrete cosine transform (DCT) to each block, and removing small values. JPEG images are best used for the representation of natural scenes, in which case they are to be preferred.

For data with any legal significance, or scientific data, JPEG is less suitable, for the very reason that not all data is preserved. However, the mechanics of the JPEG transform ensures that a JPEG image; when restored from its compression routine, will *look* the same as the original image. The differences are in general too small to be visible to the human eye. JPEG images are thus excellent for display.

We shall investigate the JPEG algorithm in section 14.4. More detailed accounts can be found in [7] and [24].

A JPEG image then contains the compression data with a small header providing the image size and file identification. We can see a header by using our `dumphex` function:

```
>> dumphex('ngc6543a.jpg',4)


ans =


FF D8 FF E0 00 10 4A 46 49 46 00 01 01 00 00 01   ......JFIF......
00 01 00 00 FF FE 00 47 43 52 45 41 54 4F 52 3A   .......GCREATOR:
20 58 56 20 56 65 72 73 69 6F 6E 20 33 2E 30 30    XV Version 3.00
62 20 20 52 65 76 3A 20 36 2F 31 35 2F 39 34 20   b  Rev: 6/15/94
```

An image file containing JPEG compressed data is usually just called a "JPEG image". But this is not quite correct; such an image should be called a "JFIF image"; where JFIF stands for "JPEG File Interchange Format". The JFIF definition allows for the file to contain a *thumbnail* version of

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the image; this is reflected in the header information:

Bytes	Information	Description
0–1	Start of image marker	Always `FF D8`.
2–3	Application Marker	Always `FF E0`.
4–5	Length of segment	
5–10	`JFIF\ 0`	ASCII "JFIF ".
11–12	JFIF Version	In our example above `01 01`, or version 1.1.
13	Units	Values are 0: Arbitrary units; 1: pixels/inch; 2: pixels/centimetre.
14–15	Horizontal pixel density	
16–17	Vertical pixel density	
18	Thumbnail Width	If this is 0, there is no thumbnail.
19	Thumbnail Height	If this is 0, there is no thumbnail.

After this would come the thumbnail information (stored as 24-bit RGB values), and further information required for decompressing the image data. See [2] or [18] for further information.

2.5.6 TIFF

The *Tagged Image File Format*, or TIFF, is one of the most comprehensive image formats. It can store multiple images per file. It allows different compression routines (none at all, LZW, JPEG, Huffman, RLE); different byte orderings (little-endian, as used in BMP, or big-endian, in which the bytes retain their order within words); it allows binary, greyscale, truecolour or indexed images; it allows for opacity or transparency.

For that reason, it requires skillful programming to write image reading software which will read all possible TIFF images. But TIFF is an excellent format for data exchange.

The TIFF header is in fact very simple; it consists of just eight bytes:

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Bytes	Information	Description
0–1	Byte order	Either `4D 4D`: ASCII "MM" for big-endian, or `49 49`: ASCII "II" for little endian.
2–3	TIFF version	Always `00 2A` or `2A 00` (depending on the byte order) = 42.
4–8	Image offset	Pointer to the position in the file of the data for the first image.

We can see this with:

```
>> dumphex('newborn.tif',4)


ans =


49 49 2A 00 E0 01 01 00 32 7C 5B 2D 23 19 0E 15   II*.....2|[-#...
10 0E 0D 0F 10 0F 0E 10 11 11 0E 12 13 12 10 17   ................
10 1D 70 8E 99 A0 AE B5 BB BA C2 C6 C6 CB D3 D0   ..p.............
D2 D1 CA DB DE DE E1 E5 E6 DF E4 E9 FE EB 0B ED   ................
```

This particular image uses the little-endian byte ordering. The first image in this file (which is in fact the only image), begins at byte

```
E0 01 01 00
```

Since this is a little-endian file, we reverse the order of the bytes: `00 01 01 E0`; this works out to 66016.

As well as the previously cited references, Baxes [1] provides a good introduction to TIFF.

2.5.7 DICOM

The DICOM (Digital Imaging and Communications in Medicine) 3.0 standard defines an image format which, like GIF, may hold multiple image files. However the files may be considered as "slices" or "frames" of a three dimensional object. Thus a DICOM file, in contradistinction to the

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image formats described above, may describe a three dimensional image. DICOM has become a standard for medical digital imaging modalities; more recently it has been used as much in pathology as in diagnosis.

A DICOM file contains a header, which contains image information (size, number of slices) as well as the modality used (CAT, MRI, PET etc), patient information, and compression used. Image data in a DICOM file may be compressed with either lossy or lossless methods; the former uses the DCT as does the JPEG format; the latter either a lossless version of the DCT, or *run-length encoding*, where a string of repeated characters is stored simply as the character plus the number of times it occurs.

The DICOM specification is huge and complex; drafts have been published on the world wide web, or information can be obtained from the National Electrical Manufacturers Association, who are one of the creators of this format.

Files in MATLAB

An image matrix may be written to an image file with the `imwrite` function; its general form is

```
imwrite(X,map,'filename','fmt')
```

which writes the image stored in matrix `X`, with colour map `map` (if appropriate), to file `filename`, with format `fmt`. Without the `map` argument, the image data is supposed to be greyscale or RGB. For example, suppose we have read an image as follows:

```
a=imread('autumn.tif');
```

We can write this RGB matrix `a` to an PNG (Portable Network Graphics) file with:

```
imwrite(a,'autumn.png','png');
```

Exercises

1. Type in the command

```
>> help imdemos
```

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This will give you a list of, amongst other things, all the sample TIFF images which come with the Image Processing Toolbox. Make a list of these sample images, and for each image

(a) determine its type (binary, greyscale, true colour or indexed colour),

(b) determine its size (in pixels)

(c) give a brief description of the picture (what it looks like; what it seems to be a picture of)

2. Pick a greyscale image, say `cameraman.tif` or `wombats.tif`. Using the `imwrite` function, write it to files of type JPEG, PNG and BMP.

   What are the sizes of those files?

3. Repeat the above question with

   (a) a binary image,

   (b) an indexed colour image,

   (c) a true colour image.

4. The following shows the hexadecimal dump of a BMP file:

```
42 4D 7E 05 02 00 00 00 00 00 36 04 00 00 28 00
00 00 23 01 00 00 C2 01 00 00 01 00 08 00 00 00
00 00 48 01 02 00 00 00 00 00 00 00 00 00 00 01
00 00 00 00 00 00 00 00 00 00 01 01 01 00 02 02
```

   Determine the height and width of this image (in pixels), and whether it is a greyscale or colour image.

5. Repeat the above question with

```
42 4D 36 00 09 00 00 00 00 00 36 00 00 00 28 00
00 00 00 02 00 00 80 01 00 00 01 00 18 00 00 00
00 00 00 00 09 00 00 00 00 00 00 00 00 00 00 00
00 00 00 00 00 00 00 00 7D 02 01 7F 05 04 84 07
```

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Chapter 3

Image Display

3.1 Introduction

We have touched briefly in chapter 2 on image display. In this chapter, we investigate this matter in more detail. We look more deeply at the use of the `imshow` function, and show how spatial resolution and quantization can affect the display and appearance of an image. In particular, we look at image quality, and how that may be affected by various image attributes. Quality is of course a highly subjective matter: no two people will agree precisely as to the quality of different images. However, for human vision in general, images are preferred to be sharp and detailed. This is a consequence of two properties of an image: its spatial resolution, and its quantization.

3.2 Basics of image display

An image may be represented as a matrix of the grey values of its pixels. The problem here is to display that matrix on the computer screen. There are many factors which will effect the display; they include:

1. ambient lighting,
2. the monitor type and settings,
3. the graphics card,
4. monitor resolution.

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The same image may appear very different when viewed on a dull CRT monitor or on a bright LCD monitor. The resolution can also affect the display of an image; a higher resolution may result in the image taking up less physical area on the screen, but this may be counteracted by a loss in the colour depth: the monitor may be only to display 24-bit colour at low resolutions. If the monitor is bathed in bright light (sunlight, for example), the display of the image may be compromised. Furthermore, the individual's own visual system will affect the appearance of an image: the same image, viewed by two people, may appear to have different characteristics to each person. For our purpose, we shall assume that the computer set up is as optimal as is possible, and the monitor is able to accurately reproduce the necessary grey values or colours in any image.

A very basic MATLAB function for image display is `image`. This function simply displays a matrix as an image. However, it may not give necessarily very good results. For example:

```
>> c=imread('cameraman.tif');
>> image(c)
```

will certainly display the cameraman, but possibly in an odd mixture of colours, and with some stretching. The strange colours come from the fact that the `image` command uses the current colour map to assign colours to the matrix elements. The default colour map is called `jet`, and consists of 64 very bright colours, which is inappropriate for the display of a greyscale image.

To display the image properly, we need to add several extra commands to the `image` line.

1. `truesize`, which displays one matrix element (in this case an image pixel) for each screen pixel. More formally, we may use `truesize([256 256])` where the vector components give the number of screen pixels vertically and horizontally to use in the display. If the vector is not specified, it defaults to the image size.

2. `axis off` which turns off the axis labelling,

3. `colormap(gray(247))`, which adjusts the image colour map to use shades of grey only. We can find the number of grey levels used by the cameraman image with

```
>> size(unique(c))
```

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```
ans =


   247     1
```

Since the cameraman image thus uses 247 different grey levels, we only need that number of greys in the colour map.

Thus a complete command for viewing this image will be

```
>> image(c),truesize,axis off, colormap(gray(247))
```

We may to adjust the colour map to use less or more colours; however this can have a dramatic effect on the result. The command

```
>> image(c),truesize,axis off, colormap(gray(512))
```

will produce a dark image. This happens because only the first 247 elements of the colour map will be used by the image for display, and these will all be in the first half of the colour map; thus all dark greys. On the other hand,

```
>> image(c),truesize,axis off, colormap(gray(128))
```

will produce a very light image, because any pixel with grey level higher than 128 will simply pick that highest grey value (which is white) from the colour map.

The `image` command works well for an indexed colour image, as long as we remember to use `imread` to pick up the colour map as well:

```
>> [x,map]=imread('cat.tif');
>> image(x),truesize,axis off,colormap(map)
```

For true colour images, the image data will be read (by `imread`) as a three dimensional array. In such a case, `image` will ignore the current colour map, and assign colours to the display based on the values in the array. So

```
>> t=imread('twins.tif');
>> image(t),truesize,axis off
```

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will produce the correct twins image.

In general the `image` function can be used to display any image or matrix. However, there is a command which is more convenient, and does most of the work of colour mapping for us; we discuss this in the next section.

3.3 The `imshow` function

Greyscale images

We have seen that if `x` is a matrix of type `uint8`, then the command

```
imshow(x)
```

will display `x` as an image. This is reasonable, since the data type `uint8` restricts values to be integers between 0 and 255. However, not all image matrices come so nicely bundled up into this data type, and lots of MATLAB image processing commands produces output matrices which are of type `double`. We have two choices with a matrix of this type:

1. convert to type `uint8` and then display,
2. display the matrix directly.

The second option is possible because `imshow` will display a matrix of type `double` as a greyscale image as long as the matrix elements are between 0 and 1. Suppose we take an image and convert it to type `double`:

```
>> c=imread('caribou.tif');
>> cd=double(c);
>> imshow(c),figure,imshow(cd)
```

The results are shown in figure 3.1.

However, as you can see, figure 3.1(b) doesn't look much like the original picture at all! This is because for a matrix of type `double`, the `imshow` function expects the values to be between 0 and 1, where 0 is displayed as black, and 1 is displayed as white. A value v with $0 < v < 1$ is displayed as grey scale $\lfloor 255v \rfloor$. Conversely, values greater than 1 will be displayed as 1 (white) and values less

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(a) The original image (b) After conversion to type `double`

Figure 3.1: An attempt at data type conversion

than 0 will be displayed as zero (black). In the caribou image, every pixel has value greater than or equal to 1 (in fact the minimum value is 21), so that every pixel will be displayed as white. To display the matrix `cd`, we need to scale it to the range 0—1. This is easily done simply by dividing all values by 255:

```
>> imshow(cd/255)
```

and the result will be the caribou image as shown in figure 3.1(a).

We can vary the display by changing the scaling of the matrix. Results of the commands:

```
>> imshow(cd/512)
>> imshow(cd/128)
```

are shown in figures 3.2.

Dividing by 512 darkens the image, as all matrix values are now between 0 and 0.5, so that the brightest pixel in the image is a mid-grey. Dividing by 128 means that the range is 0—2, and all pixels in the range 1—2 will be displayed as white. Thus the image has an over-exposed, washed-out appearance.

The display of the result of a command whose output is a matrix of type `double` can be greatly affected by a judicious choice of a scaling factor.

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(a) The matrix `cd` divided by 512

(b) The matrix `cd` divided by 128

Figure 3.2: Scaling by dividing an image matrix by a scalar

We can convert the original image to `double` more properly using the function `im2double`. This applies correct scaling so that the output values are between 0 and 1. So the commands

```
>> cd=im2double(c);
>> imshow(cd)
```

will produce a correct image. It is important to make the distinction between the two functions `double` and `im2double`: `double` changes the data type but does not change the numeric values; `im2double` changes both the numeric data type and the values. The exception of course is if the original image is of type `double`, in which case `im2double` does nothing. Although the command `double` is not of much use for direct image display, it can be very useful for image arithmetic. We have seen examples of this above with scaling.

Corresponding to the functions `double` and `im2double` are the functions `uint8` and `im2uint8`. If we take our image `cd` of type `double`, properly scaled so that all elements are between 0 and 1, we can convert it back to an image of type `uint8` in two ways:

```
>> c2=uint8(255*cd);
>> c3=im2uint8(cd);
```

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Use of `im2uint8` is to be preferred; it takes other data types as input, and always returns a correct result.

Binary images

Recall that a binary image will have only two values: 0 and 1. MATLAB does not have a `binary` data type as such, but it does have a `logical` flag, where `uint8` values as 0 and 1 can be interpreted as logical data. The logical flag will be set by the use of relational operations such as `==`, `<` or `>` or any other operations which provide a yes/no answer. For example, suppose we take the caribou matrix and create a new matrix with

```
>> cl=c>120;
```

(we will see more of this type of operation in chapter 4.) If we now check all of our variables with `whos`, the output will include the line:

```
  cl      256x256        65536  uint8 array (logical)
```

This means that the command

```
>> imshow(cl)
```

will display the matrix as a binary image; the result is shown in figure 3.3.

Suppose we remove the logical flag from `cl`; this can be done by a simple command:

```
>> cl = +cl;
```

Now the output of `whos` will include the line:

```
  cl      256x256        65536  uint8 array
```

If we now try to display this matrix with `imshow`, we obtain the result shown in figure 3.3(b). A very disappointing image! But this is to be expected; in a matrix of type `uint8`, white is 255, 0 is black, and 1 is a very dark grey–indistinguishable from black.

To get back to a viewable image, we can either turn the logical flag back on, and the view the result:

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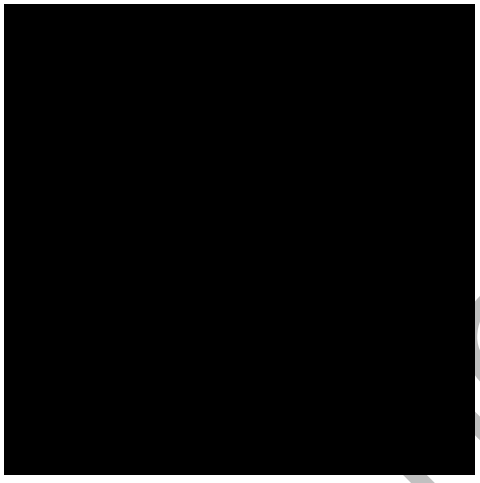
(a) The caribou image turned binary



(b) After conversion to type `uint8`

Figure 3.3: Making the image binary

```
>> imshow(logical(cl))
```

or simply convert to type `double`:

```
>> imshow(double(cl))
```

Both these commands will produce the image seen in figure 3.3.

3.4 Bit planes

Greyscale images can be transformed into a sequence of binary images by breaking them up into their *bit-planes*. If we consider the grey value of each pixel of an 8-bit image as an 8-bit binary word, then the 0th bit plane consists of the last bit of each grey value. Since this bit has the least effect in terms of the magnitude of the value, it is called the *least significant bit*, and the plane consisting of those bits the *least significant bit plane*. Similarly the 7th bit plane consists of the first bit in each value. This bit has the greatest effect in terms of the magnitude of the value, so it is called the *most significant bit*, and the plane consisting of those bits the *most significant bit plane*.

If we take a greyscale image, we start by making it a matrix of type `double`; this means we can perform arithmetic on the values.

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```
>> c=imread('cameraman.tif');
>> cd=double(c);
```

We now isolate the bit planes by simply dividing the matrix `cd` by successive powers of 2, throwing away the remainder, and seeing if the final bit is 0 or 1. We can do this with the `mod` function.

```
>> c0=mod(cd,2);
>> c1=mod(floor(cd/2),2);
>> c2=mod(floor(cd/4),2);
>> c3=mod(floor(cd/8),2);
>> c4=mod(floor(cd/16),2);
>> c5=mod(floor(cd/32),2);
>> c6=mod(floor(cd/64),2);
>> c7=mod(floor(cd/128),2);
```

These are all shown in figure 3.4. Note that the least significant bit plane, `c0`, is to all intents and purposes a random array, and that as the index value of the bit plane increases, more of the image appears. The most significant bit plane, `c7`, is actually a *threshold* of the image at level 127:

```
>> ct=c>127;
>> all(c7(:)==ct(:))

ans =

     1
```

We shall discuss thresholding in chapter 9.

We can recover and display the original image with

```
>> cc=2*(2*(2*(2*(2*(2*(2*c7+c6)+c5)+c4)+c3)+c2)+c1)+c0;
>> imshow(uint8(cc))
```

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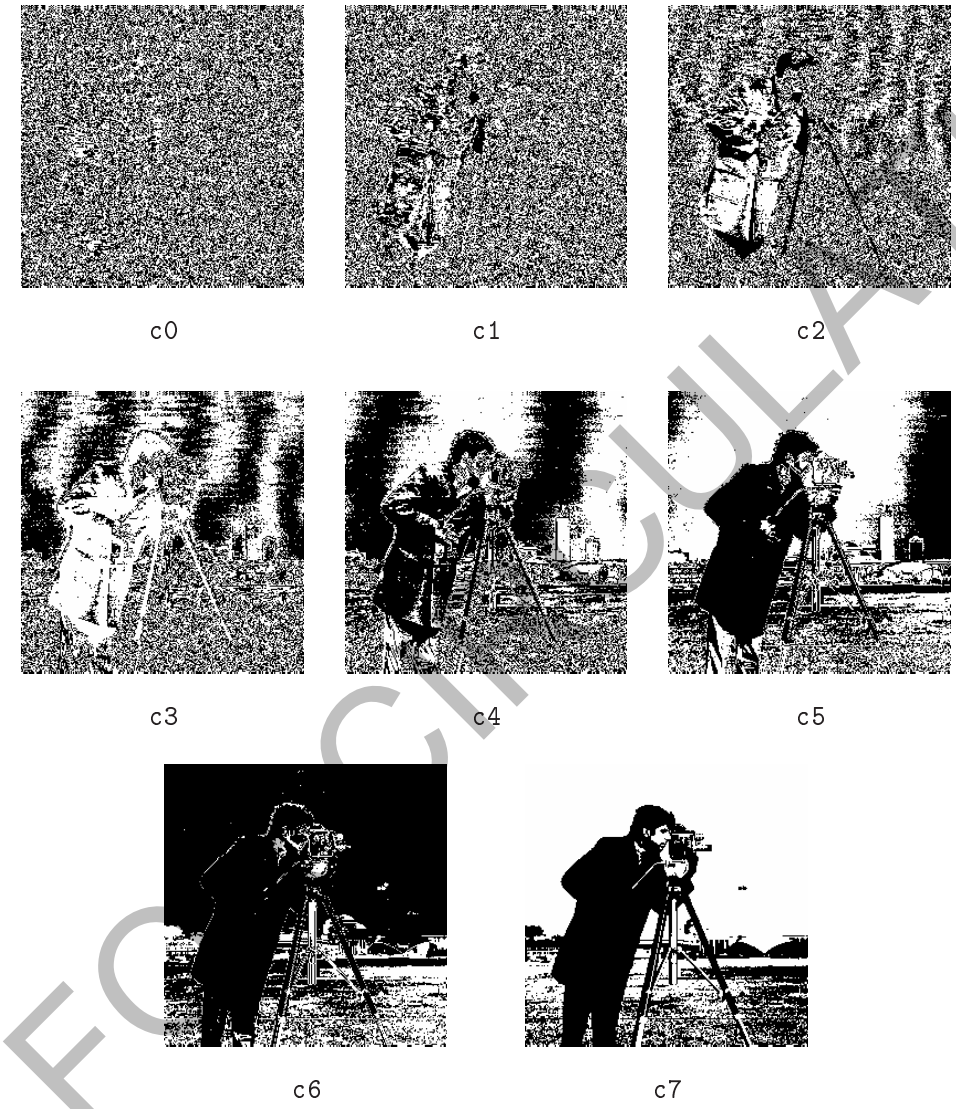

Figure 3.4: The bit planes of an 8-bit greyscale image

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3.5 Spatial Resolution

Spatial resolution is the density of pixels over the image: the greater the spatial resolution, the more pixels are used to display the image. We can experiment with spatial resolution with MATLAB's `imresize` function. Suppose we have an 256×256 8-bit greyscale image saved to the matrix `x`. Then the command

```
imresize(x,1/2);
```

will halve the size of the image. It does this by taking out every other row and every other column, thus leaving only those matrix elements whose row and column indices are even:

$$\begin{matrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & \cdots \\ x_{21} & \boxed{x_{22}} & x_{23} & \boxed{x_{24}} & x_{25} & \boxed{x_{26}} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} & \cdots \\ x_{41} & \boxed{x_{42}} & x_{43} & \boxed{x_{44}} & x_{45} & \boxed{x_{46}} & \cdots \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} & \cdots \\ x_{61} & \boxed{x_{62}} & x_{63} & \boxed{x_{64}} & x_{65} & \boxed{x_{66}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix} \longrightarrow \texttt{imresize(x,1/2)} \longrightarrow \begin{matrix} x_{22} & x_{24} & x_{26} & \cdots \\ x_{42} & x_{44} & x_{46} & \cdots \\ x_{62} & x_{64} & x_{66} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

If we apply `imresize` to the result with the parameter `2` rather than `1/2`, all the pixels are repeated to produce an image with the same size as the original, but with half the resolution in each direction:

$$\begin{matrix} \boxed{\begin{matrix} x_{22} & x_{22} \\ x_{22} & x_{22} \end{matrix}} & \boxed{\begin{matrix} x_{24} & x_{24} \\ x_{24} & x_{24} \end{matrix}} & \boxed{\begin{matrix} x_{26} & x_{26} \\ x_{26} & x_{26} \end{matrix}} & \cdots \\ \boxed{\begin{matrix} x_{42} & x_{42} \\ x_{42} & x_{42} \end{matrix}} & \boxed{\begin{matrix} x_{44} & x_{44} \\ x_{44} & x_{44} \end{matrix}} & \boxed{\begin{matrix} x_{46} & x_{46} \\ x_{46} & x_{46} \end{matrix}} & \cdots \\ \boxed{\begin{matrix} x_{62} & x_{62} \\ x_{62} & x_{62} \end{matrix}} & \boxed{\begin{matrix} x_{64} & x_{64} \\ x_{64} & x_{64} \end{matrix}} & \boxed{\begin{matrix} x_{66} & x_{66} \\ x_{66} & x_{66} \end{matrix}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

The effective resolution of this new image is only 128×128. We can do all this in one line:

```
x2=imresize(imresize(x,1/2),2);
```

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By changing the parameters of `imresize`, we can change the effective resolution of the image to smaller amounts:

Command	Effective resolution
`imresize(imresize(x,1/4),4);`	64×64
`imresize(imresize(x,1/8),8);`	32×32
`imresize(imresize(x,1/16),16);`	16×16
`imresize(imresize(x,1/32),32);`	8×8

To see the effects of these commands, suppose we apply them to the image `newborn.tif`:

```
x=imread('newborn.tif');
```

(a) The original image (b) at 128×128 resolution

Figure 3.5: Reducing resolution of an image

The effects of increasing blockiness or *pixelization* become quite pronounced as the resolution decreases; even at 128×128 resolution fine detail, such as the edges of the baby's fingers, are less clear, and at 64×64 all edges are now quite blocky. At 32×32 the image is barely recognizable, and at 16×16 and 8×8 the image becomes unrecognizable.

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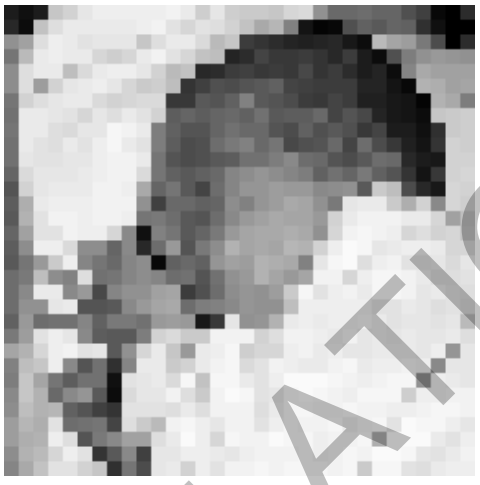
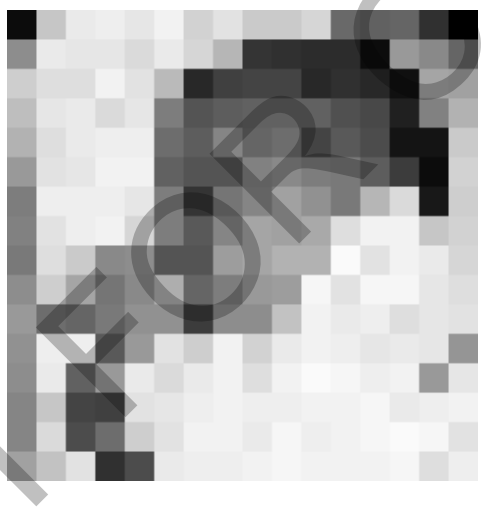
(a) At 64×64 resolution

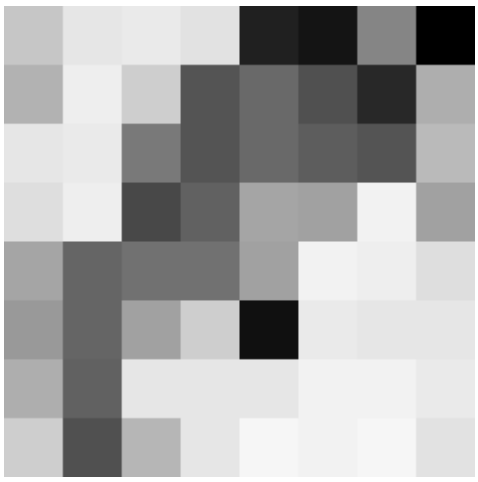


(b) At 32×32 resolution

Figure 3.6: Further reducing the resolution of an image



(a) At 16×16 resolution



(b) at 8×8 resolution

Figure 3.7: Even more reducing the resolution of an image

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3.6 Quantization and dithering

Quantization refers to the number of greyscales used to represent the image. As we have seen, most images will have 256 greyscales, which is more than enough for the needs of human vision. However, there are circumstances in which it may be more practical to represent the image with fewer greyscales. One simple way to do this is by *uniform quantization*: to represent an image with only n greyscales, we divide the range of greyscales into n equal (or nearly equal) ranges, and map the ranges to the values 0 to $n-1$. For example, if $n = 4$, we map greyscales to output values as follows:

Original values	Output value
0–63	0
64–127	1
128–191	2
192–255	3

and the values 0, 1, 2, 3 may need to be scaled for display. This mapping can be shown graphically, as in figure 3.8.

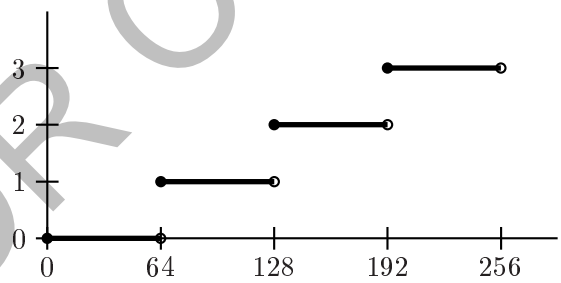

Figure 3.8: A mapping for uniform quantization

To perform such a mapping in MATLAB, we can perform the following operations, supposing `x` to be a matrix of type `uint8`:

```
f=floor(double(x)/64);

q=uint8(f*64);
```

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Since the greyscales are in the range 0–255, the first command simply applies the function shown in figure 3.8. The second command scales the values to more appropriate values for display. These commands can be done as one, with different values used to change the number of output greyscales:

Command	Number of greyscales
`uint8(floor(double(x)/2)*2)`	128
`uint8(floor(double(x)/4)*4)`	64
`uint8(floor(double(x)/8)*8)`	32
`uint8(floor(double(x)/16)*16)`	16
`uint8(floor(double(x)/32)*32)`	8
`uint8(floor(double(x)/64)*64)`	4
`uint8(floor(double(x)/128)*128)`	2

There is, however, a more elegant method of reducing the greyscales in an image, and it involves using the `grayslice` function. Given an image matrix `x` and an integer `n` the command `grayslice(x,n)` produces a matrix whose values have been reduced to the values $0, 1, \ldots, n-1$. So for example

```
>> grayslice(x,4)
```

will produce a `uint8` version of our image with values 0, 1, 2 and 3. We can't view this directly, as it will appear completely black: the four values are too close to zero to be distinguishable. We need to treat this matrix as the indices to a colour map, and the colour map we shall use is `gray(4)`, which produces a colour map of four evenly spaces grey values between 0 (black) and 1.0 (white). Thus, for display, we can implement the above commands as:

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Command	Number of greyscales
`imshow(grayslice(x,128),gray(128))`	128
`imshow(grayslice(x,64),gray(64))`	64
`imshow(grayslice(x,32),gray(32))`	32
`imshow(grayslice(x,16),gray(16))`	16
`imshow(grayslice(x,8),gray(8))`	8
`imshow(grayslice(x,4),gray(4))`	4
`imshow(grayslice(x,2),gray(2))`	2

If we apply these commands to our baby image we obtain the results shown in figures 3.9 to 3.12.

(a) The image quantized to 128 greyscales

(b) The image quantized to 64 greyscales

Figure 3.9: Quantization (1)

One immediate consequence of uniform quantization is that of "false contours", most noticeable with fewer greyscales. For example, in figure 3.10(b), we can see on the sleeves and on the baby's face that the gray texture is no longer smooth; there are observable discontinuities between different gray values. This effect is even more obvious in figures 3.11(a) and (b). We may expect that if fewer greyscales are used, and the jumps between consecutive greyscales becomes larger, that such false contours will occur.

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(a) The image quantized to 32 greyscales

(b) The image quantized to 16 greyscales

Figure 3.10: Quantization (2)

(a) The image quantized to 8 greyscales

(b) The image quantized to 4 greyscales

Figure 3.11: Quantization (3)

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Figure 3.12: The image quantized to 2 greyscales

Dithering

Dithering, in general terms, refers to the process of reducing the number of colours in an image. For the moment, we shall be concerned only with greyscale dithering. Dithering is necessary sometimes for display, if the image must be displayed on equipment with a limited number of colours, or for printing. Newsprint, in particular, only has two scales of grey: black and white. Representing an image with only two tones is also known as *halftoning*.

One method of dealing with such false contours involves adding random values to the image before quantization. Equivalently, for quantization to 2 greyscales, we may compare the image to a random matrix r. The trick is to devise a suitable matrix so that greyscales are represented evenly in the result. For example, an area containing mid-way grey level (around 127), would have a checkerboard pattern; a darker area will have a pattern containing more black than white, and a light area will have a pattern containing more white than black. Figure 3.13 illustrates this. One standard matrix is

$$D = \begin{bmatrix} 0 & 128 \\ 192 & 64 \end{bmatrix}$$

which is repeated until it is as big as the image matrix, when the two are compared. Suppose $d(i, j)$

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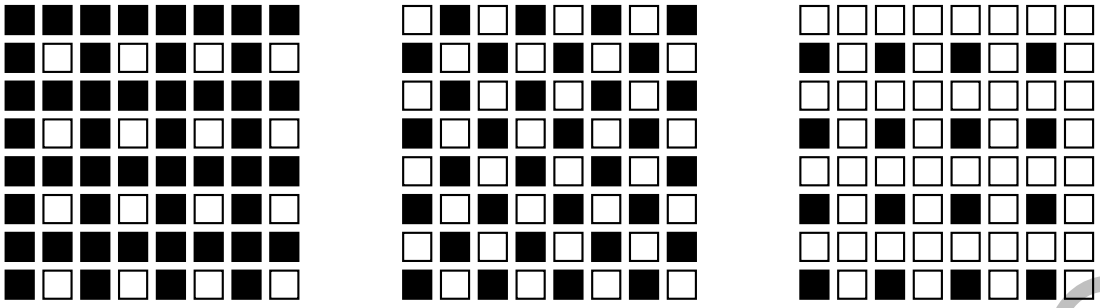

Dark grey Mid grey Light grey

Figure 3.13: Patterns for dithering output

is the matrix obtain by replicating D. Thus an output pixel $p(i,j)$ is defined by

$$p(i,j) = \begin{cases} 1 & \text{if } x(i,j) > d(i,j) \\ 0 & \text{if } x(i,j) \leq d(i,j) \end{cases}$$

This approach to quantization is called *dithering*, and the matrix D is an example of a *dither matrix*. Another dither matrix is given by

$$D_2 = \begin{bmatrix} 0 & 128 & 32 & 160 \\ 192 & 64 & 224 & 96 \\ 48 & 176 & 16 & 144 \\ 240 & 112 & 208 & 80 \end{bmatrix}$$

We can apply the matrices to our newborn image matrix `x` by using the following commands:

```
>> D=[0 128;192 64]
>> r=repmat(D,128,128);
>> x2=x>r;imshow(x2)
>> D2=[0 128 32 160;192 64 224 96;48 176 16 144;240 112 208 80];
>> r2=repmat(D2,64,64);
>> x4=x>r2;imshow(x4)
```

The results are shown in figure 3.14. The dithered images are an improvement on the uniformly quantized image shown in figure 3.12. General dither matrices are provided by Hawley [10] .

Dithering can be easily extended to more than two output grey values. Suppose for example we wish to quantize to four output levels 0, 1, 2 and 3. Since $255/3 = 85$, we first quantize by dividing

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(a) The baby image dithered using D

(b) The baby image dithered using D_2

Figure 3.14: Examples of dithering

the grey value $x(i,j)$ by 85:

$$q(i,j) = [x(i,j)/85].$$

This will produce only the values 0, 1 and 2, except for when $x(i,j) = 255$. Suppose now that our replicated dither matrix $d(i,j)$ is scaled so that its values are in the range $0 \ldots 85$. The final value $p(i,j)$ is then defined by

$$p(i,j) = q(i,j) + \begin{cases} 1 & \text{if } x(i,j) - 85q(i,j) > d(i,j) \\ 0 & \text{if } x(i,j) - 85q(i,j) \leq d(i,j) \end{cases}$$

This can be easily implemented in a few commands, modifying `D` slightly from above:

```
>> D=[0 56;84 28]
>> r=repmat(D,128,128);
>> x=double(x);
>> q=floor(x/85);
>> x4=q+(x-85*q>r);
>> imshow(uint8(85*x4))
```

and the result is shown in figure 3.15(a). We can dither to eight grey levels by using $255/7 = 37$ (we round the result up to ensure that our output values stay within range) instead of 85 above;

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our starting dither matrix will be

$$D = \begin{bmatrix} 0 & 24 \\ 36 & 12 \end{bmatrix}$$

and the result is shown in figure 3.15(b). Note how much better these images look than the corresponding uniformly quantized images in figures 3.11(b) and 3.11(a). The eight-level result, in particular, is almost indistinguishable from the original, in spite of the quantization.

(a) Dithering to 4 output greyscales (b) Dithering to 8 output greyscales

Figure 3.15: Dithering to more than two greyscales

Error diffusion

A different approach to quantization from dithering is that of *error diffusion*. The image is quantized at two levels, but for each pixel we take into account the *error* between its grey value and its quantized value. Since we are quantizing to grey values 0 and 255, pixels close to these values will have little error. However, pixels close to the centre of the range: 128, will have a large error. The idea is to spread this error over neighbouring pixels. A popular method, developed by Floyd and Steinberg, works by moving through the image pixel by pixel, starting at the top left, and working across each row in turn. For each pixel $p(i, j)$ in the image we perform the following sequence of steps:

1. Perform the quantization.

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2. Calculate the quantization error. This is defined as:

$$E = \begin{cases} p(i,j) & \text{if } p(i,j) < 128 \\ p(i,j) - 255 & \text{if } p(i,j) >= 128 \end{cases}$$

3. Spread this error E over pixels to the right and below according to this table:

		$+\frac{7}{16}E$
$+\frac{3}{16}E$	$+\frac{5}{16}E$	$+\frac{1}{16}E$

There are several points to note about this algorithm:

- The error is spread to pixels *before* quantization is performed on them. Thus the error diffusion will affect the quantization level of those pixels.

- Once a pixel has been quantized, its value will never be affected. This is because the error diffusion only affects pixels to the right and below, and we are working from the left and above.

- To implement this algorithm, we need to embed the image in a larger array of zeros, so that the indices do not go outside the bounds of our array.

The `dither` function, when applied to a greyscale image, actually implements Floyd-Steinberg error diffusion. However, it is instructive to write a simple MATLAB function to implement it ourselves, one possibility is shown in figure 3.16.

The result of applying this function to our newborn image is shown in figure 3.17. Note that the result is very pleasing; so much so that it's hard to believe that every pixel in the image is either back or white, and so that we have a binary image.

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```
function y = fl_stein(x)
%
% FL_STEIN applies Floyd-Steinberg error diffusion to an image x, which is
% assumed to be of type uint8.
%
height=size(x,1);
width=size(x,2);
y=uint8(zeros(height,width));
z=zeros(height+2,width+2);
z(2:height+1,2:width+1)=x;
for i=2:height+1,
  for j=2:width+1,
    if z(i,j) < 128
      y(i-1,j-1) = 0;
      e = z(i,j);
    else
      y(i-1,j-1) = 255;
      e = z(i,j)-255;
    end
    z(i,j+1)=z(i,j+1)+7*e/16;
    z(i+1,j-1)=z(i+1,j-1)+3*e/16;
    z(i+1,j)=z(i+1,j)+5*e/16;
    z(i+1,j+1)=z(i+1,j+1)+e/16;
  end
end
```

Figure 3.16: A MATLAB function for applying Floyd-Steinberg error diffusion to a greyscale image

Figure 3.17: The baby image after Floyd-Steinberg error diffusion

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Other error diffusion schemes are possible; two such: Jarvis-Judice-Ninke, and Stucki, have these error diffusion schemes:

			$+\frac{7}{48}E$	$+\frac{5}{48}E$
$+\frac{3}{48}E$	$+\frac{5}{48}E$	$+\frac{7}{48}E$	$+\frac{5}{48}E$	$+\frac{3}{48}E$
$+\frac{1}{48}E$	$+\frac{3}{48}E$	$+\frac{5}{48}E$	$+\frac{3}{48}E$	$+\frac{1}{48}E$

			$+\frac{8}{42}E$	$+\frac{4}{42}E$
$+\frac{2}{42}E$	$+\frac{4}{42}E$	$+\frac{8}{42}E$	$+\frac{4}{42}E$	$+\frac{2}{42}E$
$+\frac{1}{42}E$	$+\frac{2}{42}E$	$+\frac{4}{42}E$	$+\frac{2}{42}E$	$+\frac{1}{42}E$

They can be applied by modifying the Floyd-Steinberg function shown in figure 3.16. The results of applying these two error diffusion methods are shown in figure 3.18.

(a) Result of Jarvis-Judice-Ninke error diffusion (b) Result of Stucki error diffusion

Figure 3.18: Using other error-diffusion schemes

A good introduction to error diffusion (and dithering) is given by Schumacher [29].

Exercises

1. Open the greyscale image `cameraman.tif` and view it. What data type is it?

2. Enter the following commands:

```
>> em,map]=imread('emu.tif');
```

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```
>> e=ind2gray(em,map);
```

These will produce a greyscale image of type `double`. View this image.

3. Enter the command

```
>> e2=im2uint8(e);
```

and view the output.

What does the function `im2uint8` do? What affect does it have on

(a) the appearance of the image?

(b) the elements of the image matrix?

4. What happens if you apply `im2uint8` to the cameraman image?

5. Experiment with reducing spatial resolution of the following images:

(a) `cameraman.tif`

(b) The greyscale emu image

(c) `blocks.tif`

(d) `buffalo.tif`

In each case note the point at which the image becomes unrecognizable.

6. Experiment with reducing the quantization levels of the images in the previous question. Note the point at which the image becomes seriously degraded. Is this the same for all images, or can some images stand lower levels of quantization than others?

Check your hypothesis with some other greyscale images.

7. Look at a greyscale photograph in a newspaper with a magnifying glass. Describe the colours you see.

8. Show that the 2×2 dither matrix D provides appropriate results on areas of unchanging grey. Find the results of $D > G$ when G is 1 2×2 matrix of values (a) 50, (b) 100, (c) 150, (d) 200.

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9. What are the necessary properties of D to obtain the appropriate patterns for the different input grey levels?

10. How do quantization levels effect the result of dithering? Use `gray2ind` to display a greyscale image with fewer greyscales, and apply dithering to the result.

11. Apply each of Floyd-Steinberg, Jarvis-Judice-Ninke and Stucki error diffusion to the images in question 5. Which of the images looks best? Which error-diffusion method seems to produce the best results?

    Can you isolate what aspects of an image will render it most suitable for error-diffusion?

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Chapter 4

Point Processing

4.1 Introduction

Any image processing operation transforms the grey values of the pixels. However, image processing operations may be divided into into three classes based on the information required to perform the transformation. From the most complex to the simplest, they are:

1. **Transforms.** A "transform" represents the pixel values in some other, but equivalent form. Transforms allow for some very efficient and powerful algorithms, as we shall see later on. We may consider that in using a transform, the entire image is processed as a single large block. This may be illustrated by the diagram shown in figure 4.1.

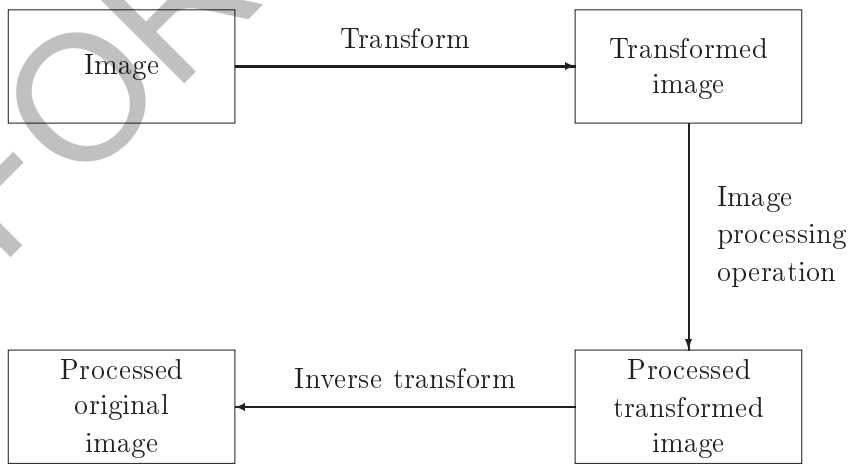

Figure 4.1: Schema for transform processing

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2. **Neighbourhood processing.** To change the grey level of a given pixel we need only know the value of the grey levels in a small neighbourhood of pixels around the given pixel.

3. **Point operations.** A pixel's grey value is changed without any knowledge of its surrounds.

Although point operations are the simplest, they contain some of the most powerful and widely used of all image processing operations. They are especially useful in image *pre-processing*, where an image is required to be modified before the main job is attempted.

4.2 Arithmetic operations

These operations act by applying a simple function

$$y = f(x)$$

to each grey value in the image. Thus $f(x)$ is a function which maps the range $0 \ldots 255$ onto itself. Simple functions include adding or subtract a constant value to each pixel:

$$y = x \pm C$$

or multiplying each pixel by a constant:

$$y = Cx.$$

In each case we may have to fiddle the output slightly in order to ensure that the results are integers in the $0 \ldots 255$ range. We can do this by first rounding the result (if necessary) to obtain an integer, and then "clipping" the values by setting:

$$y \leftarrow \begin{cases} 255 & \text{if } y > 255, \\ 0 & \text{if } y < 0. \end{cases}$$

We can obtain an understanding of how these operations affect an image by plotting $y = f(x)$. Figure 4.2 shows the result of adding or subtracting 128 from each pixel in the image. Notice that when we add 128, all grey values of 127 or greater will be mapped to 255. And when we subtract 128, all grey values of 128 or less will be mapped to 0. By looking at these graphs, we observe that in general adding a constant will lighten an image, and subtracting a constant will darken it.

We can test this on the "blocks" image `blocks.tif`, which we have seen in figure 1.4. We start by reading the image in:

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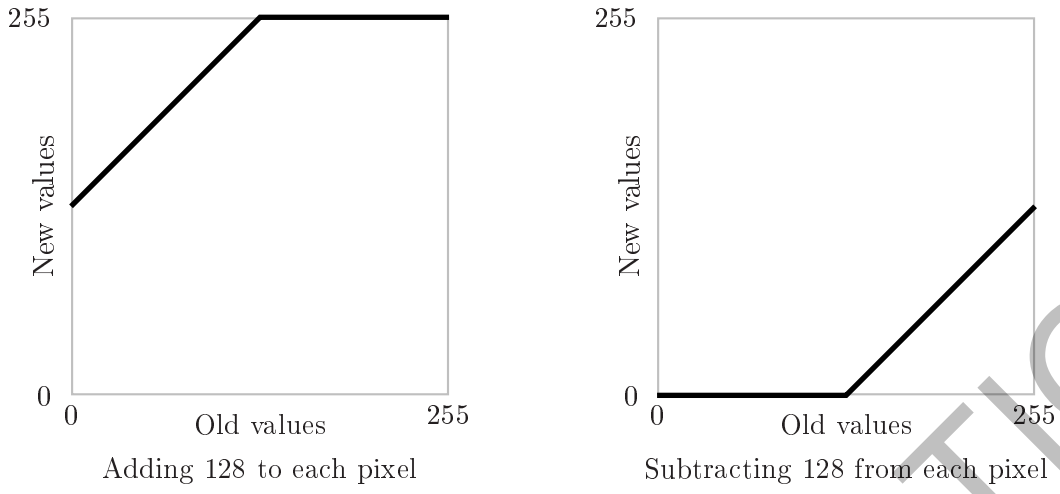

Figure 4.2: Adding and subtracting a constant

```
>> b=imread('blocks.tif');
>> whos b
  Name      Size            Bytes  Class


  b       256x256           65536  uint8 array
```

The point of the second command was to find the numeric data type of `b`; it is `uint8`. The `unit8` data type is used for data *storage* only; we can't perform arithmetic operations. If we try, we just get an error message:

```
>> b1=b+128
??? Error using ==> +
Function '+' not defined for variables of class 'uint8'.
```

We can get round this in two ways. We can first turn `b` into a matrix of type `double`, add the 128, and then turn back to `uint8` for display:

```
>> b1=uint8(double(b)+128);
```

A second, and more elegant way, is to use the MATLAB function `imadd` which is designed precisely to do this:

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```
>> b1=imadd(b,128);
```

Subtraction is similar; we can transform out matrix in and out of `double`, or use the `imsubtract` function:

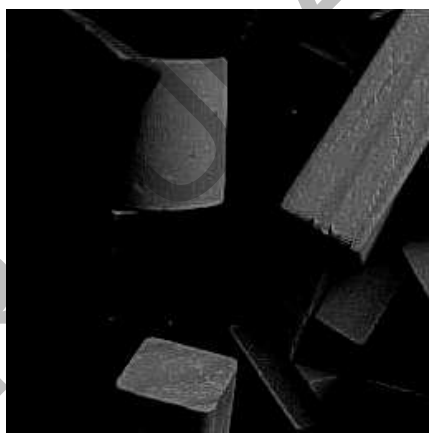
```
>> b2=imsubtract(b,128);
```

And now we can view them:

```
>> imshow(b1),figure,imshow(b2)
```

and the results are seen in figure 4.3.

`b1`: Adding 128



`b2`: Subtracting 128

Figure 4.3: Arithmetic operations on an image: adding or subtracting a constant

We can also perform lightening or darkening of an image by multiplication; figure 4.4 shows some examples of functions which will have these effects. To implement these functions, we use the `immultiply` function. Table 4.1 shows the particular commands required to implement the functions of figure 4.4. All these images can be viewed with `imshow`; they are shown in figure 4.5.

$y = x/2$	`b3=immultiply(b,0.5);` or `b3=imdivide(b,2)`
$y = 2x$	`b4=immultiply(b,2);`
$y = x/2 + 128$	`b5=imadd(immultiply(b,0.5),128);` or `b5=imadd(imdivide(b,2),128);`

Table 4.1: Implementing pixel multiplication by MATLAB commands

Compare the results of darkening `b2` and `b3`. Note that `b3`, although darker than the original, is

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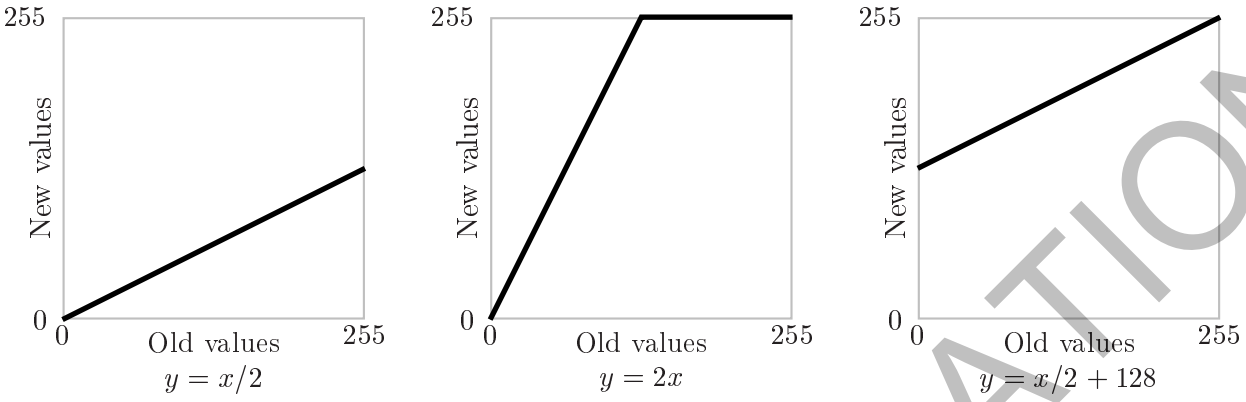
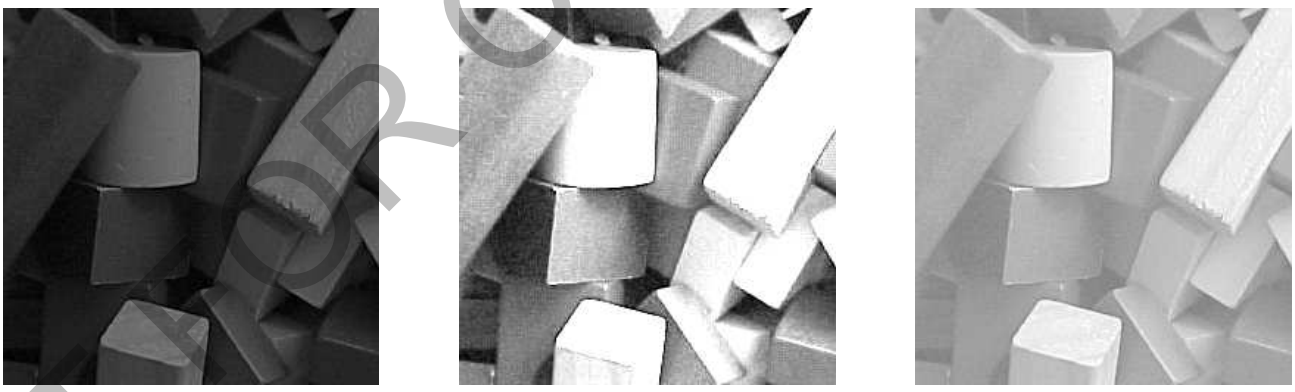


Figure 4.4: Using multiplication and division



`b3`: $y = x/2$ `b4`: $y = 2x$ `b5`: $y = x/2 + 128$

Figure 4.5: Arithmetic operations on an image: multiplication and division

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still quite clear, whereas a lot of information has been lost by the subtraction process, as can be seen in image `b2`. This is because in image `b2` all pixels with grey values 128 or less have become zero.

A similar loss of information has occurred in the images `b1` and `b4`. Note in particular the edges of the light coloured block in the bottom centre; in both `b1` and `b4` the right hand edge has disappeared. However, the edge is quite visible in image `b5`.

Complements

The *complement* of a greyscale image is its photographic negative. If an image matrix `m` is of type `double` and so its grey values are in the range 0.0 to 1.0, we can obtain its negative with the command

```
>> 1-m
```

If the image is binary, we can use

```
>> ~m
```

If the image is of type `uint8`, the best approach is the `imcomplement` function. Figure 4.6 shows the complement function $y = 255 - x$, and the result of the commands

```
>> bc=imcomplement(b);
>> imshow(bc)
```

Interesting special effects can be obtained by complementing only *part* of the image; for example by taking the complement of pixels of grey value 128 or less, and leaving other pixels untouched. Or we could take the complement of pixels which are 128 or greater, and leave other pixels untouched. Figure 4.7 shows these functions. The effect of these functions is called *solarization*, and will be discussed further in chapter 16.

4.3 Histograms

Given a greyscale image, its *histogram* consists of the histogram of its grey levels; that is, a graph indicating the number of times each grey level occurs in the image. We can infer a great deal about

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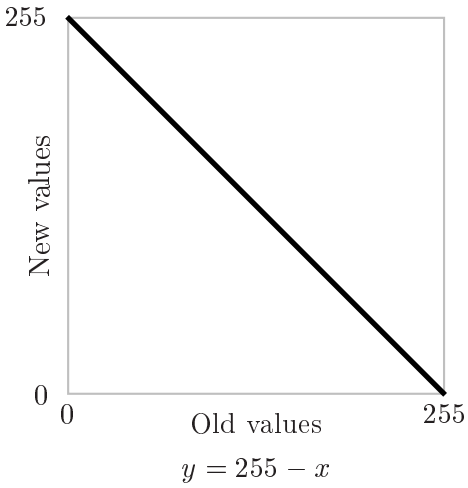

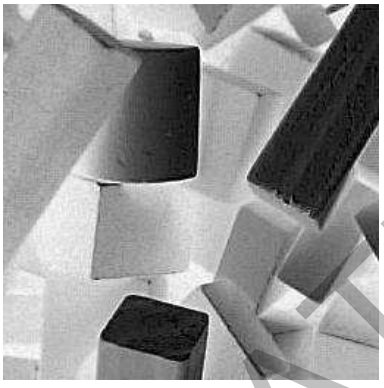

Figure 4.6: Image complementation

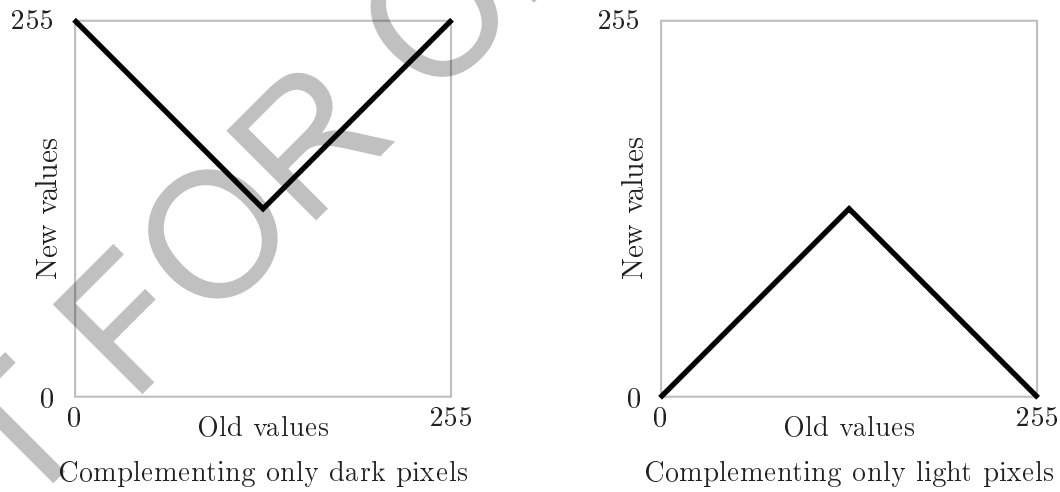

Figure 4.7: Part complementation

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the appearance of an image from its histogram, as the following examples indicate:

- In a dark image, the grey levels (and hence the histogram) would be clustered at the lower end:
- In a uniformly bright image, the grey levels would be clustered at the upper end:
- In a well contrasted image, the grey levels would be well spread out over much of the range:

We can view the histogram of an image in MATLAB by using the `imhist` function:

```
>> p=imread('pout.tif');
>> imshow(p),figure,imhist(p),axis tight
```

(the `axis tight` command ensures the axes of the histogram are automatically scaled to fit all the values in). The result is shown in figure 4.8. Since the grey values are all clustered together in the

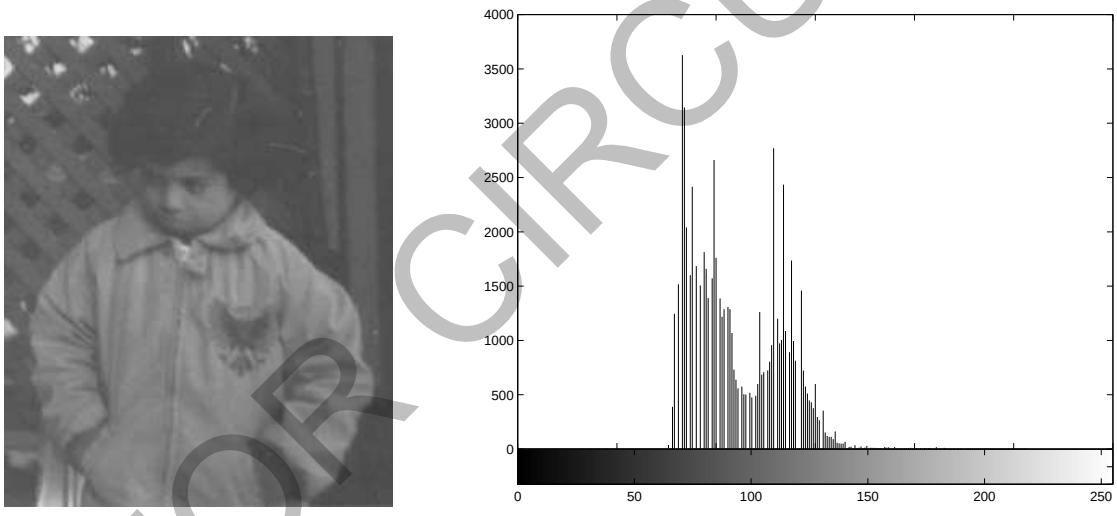

Figure 4.8: The image `pout.tif` and its histogram

centre of the histogram, we would expect the image to be poorly contrasted, as indeed it is.

Given a poorly contrasted image, we would like to enhance its contrast, by spreading out its histogram. There are two ways of doing this.

4.3.1 Histogram stretching (Contrast stretching)

Suppose we have an image with the histogram shown in figure 4.9, associated with a table of the

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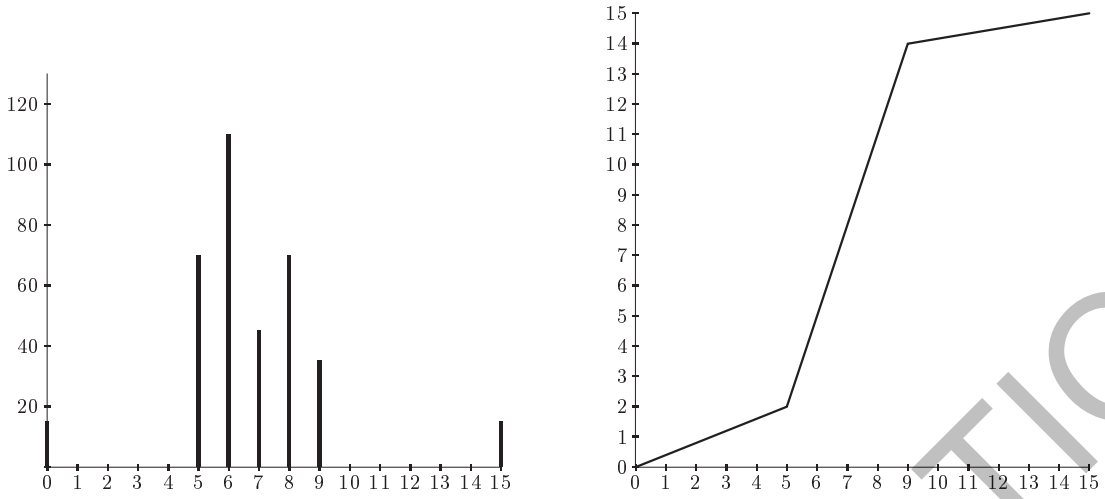

Figure 4.9: A histogram of a poorly contrasted image, and a stretching function

numbers n_i of grey values:

Grey level i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n_i	15	0	0	0	0	70	110	45	70	35	0	0	0	0	0	15

(with $n = 360$, as before.) We can stretch the grey levels in the centre of the range out by applying the piecewise linear function shown at the right in figure 4.9. This function has the effect of stretching the grey levels 5–9 to grey levels 2–14 according to the equation:

$$j = \frac{14-2}{9-5}(i-5)+2$$

where i is the original grey level and j its result after the transformation. Grey levels outside this range are either left alone (as in this case) or transformed according to the linear functions at the ends of the graph above. This yields:

i	5	6	7	8	9
j	2	5	8	11	14

and the corresponding histogram:

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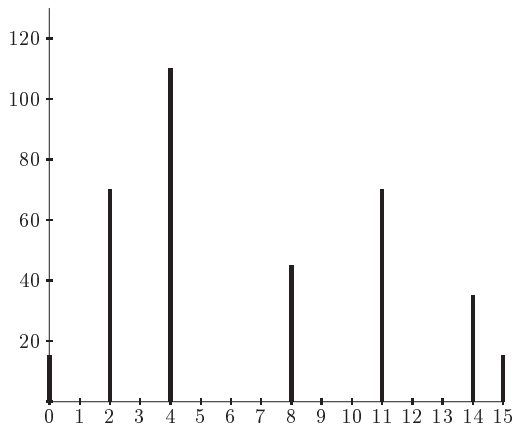

which indicates an image with greater contrast than the original.

Use of `imadjust`

To perform histogram stretching in MATLAB the `imadjust` function may be used. In its simplest incarnation, the command

```
imadjust(im,[a,b],[c,d])
```

stretches the image according to the function shown in figure 4.10. Since `imadjust` is designed to

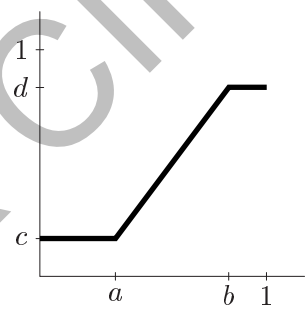

Figure 4.10: The stretching function given by `imadjust`

work equally well on images of type `double`, `uint8` or `uint16` the values of a, b, c and d must be between 0 and 1; the function automatically converts the image (if needed) to be of type `double`.

Note that `imadjust` does not work quite in the same way as shown in figure 4.9. Pixel values less than a are all converted to c, and pixel values greater than b are all converted to d. If either of `[a,b]` or `[c,d]` are chosen to be `[0,1]`, the abbreviation `[]` may be used. Thus, for example, the command

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```
>> imadjust(im,[],[])
```

does nothing, and the command

```
>> imadjust(im,[],[1,0])
```

inverts the grey values of the image, to produce a result similar to a photographic negative.

The `imadjust` function has one other optional parameter: the *gamma* value, which describes the shape of the function between the coordinates (a, c) and (b, d). If `gamma` is equal to 1, which is the default, then a linear mapping is used, as shown above in figure 4.10. However, values less than one produce a function which is concave downward, as shown on the left in figure 4.11, and values greater than one produce a figure which is concave upward, as shown on the right in figure 4.11.

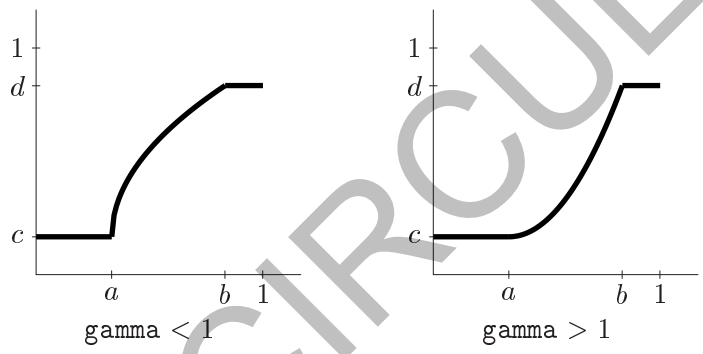

Figure 4.11: The `imadjust` function with `gamma` not equal to 1

The function used is a slight variation on the standard line between two points:

$$y = \left(\frac{x-a}{b-a}\right)^{\gamma}(d-c) + c.$$

Use of the gamma value alone can be enough to substantially change the appearance of the image. For example:

```
>> t=imread('tire.tif');
>> th=imadjust(t,[],[],0.5);
>> imshow(t),figure,imshow(th)
```

produces the result shown in figure 4.12.

We may view the `imadjust` stretching function with the `plot` function. For example,

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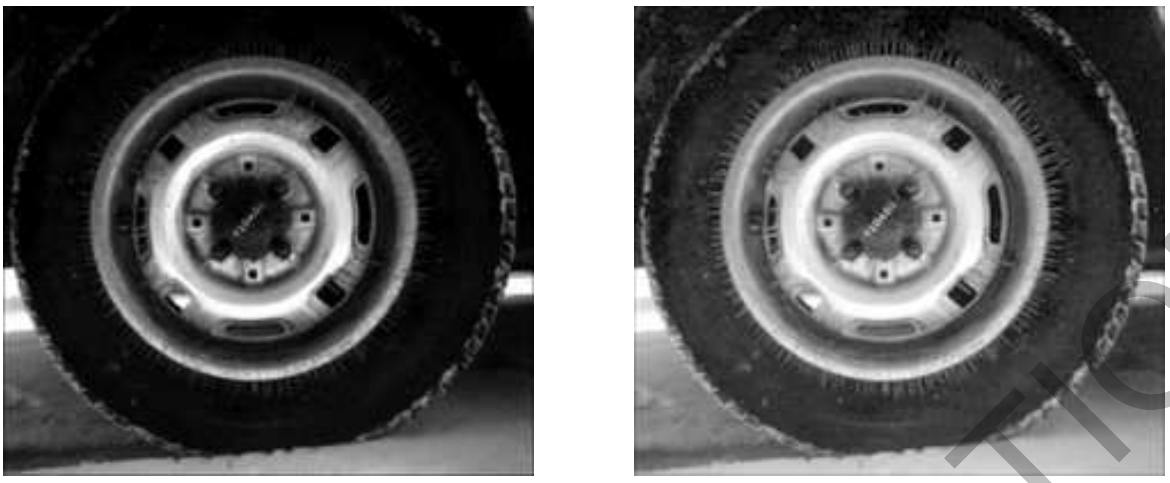

Figure 4.12: The tire image and after adjustment with the `gamma` value

```
>> plot(t,th,'.'),axis tight
```

produces the plot shown in figure 4.13. Since `p` and `ph` are matrices which contain the original

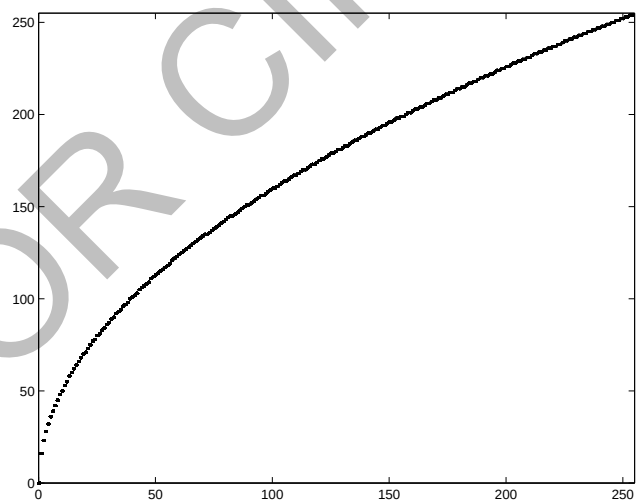

Figure 4.13: The function used in figure 4.12

values and the values after the `imadjust` function, the `plot` function simply plots them, using dots to do it.

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A piecewise linear stretching function

We can easily write our own function to perform piecewise linear stretching as shown in figure 4.14. To do this, we will make use of the `find` function, to find the pixel values in the image between a_i and a_{i+1}. Since the line between the coordinates (a_i, b_i) and (a_{i+1}, b_{i+1}) has the equation

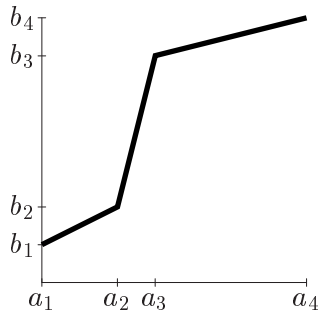

Figure 4.14: A piecewise linear stretching function

$$y = \frac{b_{i+1} - b_i}{a_{i+1} - a_i}(x - a_i) + b_i$$

the heart of our function will be the lines

```
pix=find(im >= a(i) & im < a(i+1));
out(pix)=(im(pix)-a(i))*(b(i+1)-b(i))/(a(i+1)-a(i))+b(i);
```

where `im` is the input image and `out` is the output image. A simple procedure which takes as inputs images of type `uint8` or `double` is shown in figure 4.15. As an example of the use of this function:

```
>> th=histpwl(t,[0 .25 .5 .75 1],[0 .75 .25 .5 1]);
>> imshow(th)
>> figure,plot(t,th,'.'),axis tight
```

produces the figures shown in figure 4.16.

4.3.2 Histogram equalization

The trouble with any of the above methods of histogram stretching is that they require user input. Sometimes a better approach is provided by *histogram equalization*, which is an entirely automatic procedure. The idea is to change the histogram to one which is uniform; that is that every bar on

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```
function out = histpwl(im,a,b)
%
% HISTPWL(IM,A,B) applies a piecewise linear transformation to the pixel values
% of image IM, where A and B are vectors containing the x and y coordinates
% of the ends of the line segments.  IM can be of type UINT8 or DOUBLE,
% and the values in A and B must be between 0 and 1.
%
% For example:
%
%   histpwl(x,[0,1],[1,0])
%
% simply inverts the pixel values.
%
classChanged = 0;
if ~isa(im, 'double'),
    classChanged = 1;
    im = im2double(im);
end

if length(a) ~= length (b)
  error('Vectors A and B must be of equal size');
end

N=length(a);
out=zeros(size(im));

for i=1:N-1
  pix=find(im>=a(i) & im<a(i+1));
  out(pix)=(im(pix)-a(i))*(b(i+1)-b(i))/(a(i+1)-a(i))+b(i);
end

pix=find(im==a(N));
out(pix)=b(N);

if classChanged==1
  out = uint8(255*out);
end
```

Figure 4.15: A MATLAB function for applying a piecewise linear stretching function

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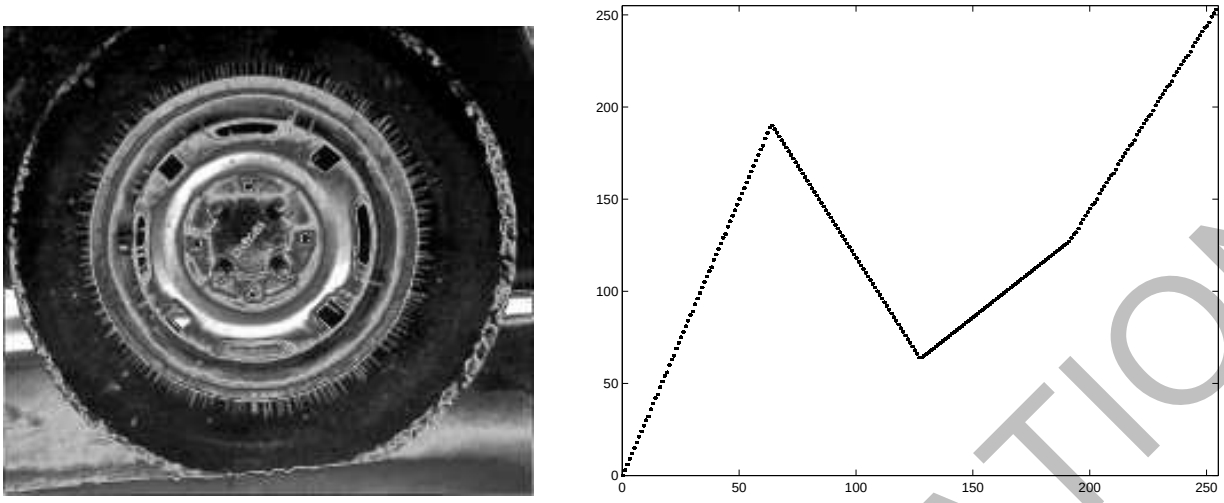

Figure 4.16: The tire image and after adjustment with the `gamma` value

the histogram is of the same height, or in other words that each grey level in the image occurs with the saem frequency. In practice this is generally not possible, although as we shall see the result of histogram equalization provides very good results.

Suppose our image has L different grey levels $0, 1, 2, \ldots L-1$, and that grey level i occurs n_i times in the image. Suppose also that the total number of pixels in the image is n (so that $n_0 + n_1 + n_2 + \cdots + n_{L-1} = n$. To transform the grey levels to obtain a better contrasted image, we change grey level i to

$$\left(\frac{n_0 + n_1 + \cdots + n_i}{n}\right)(L-1).$$

and this number is rounded to the nearest integer.

An example

Suppose a 4-bit greyscale image has the histogram shown in figure 4.17. associated with a table of the numbers n_i of grey values:

Grey level i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n_i	15	0	0	0	0	0	0	0	0	70	110	45	80	40	0	0

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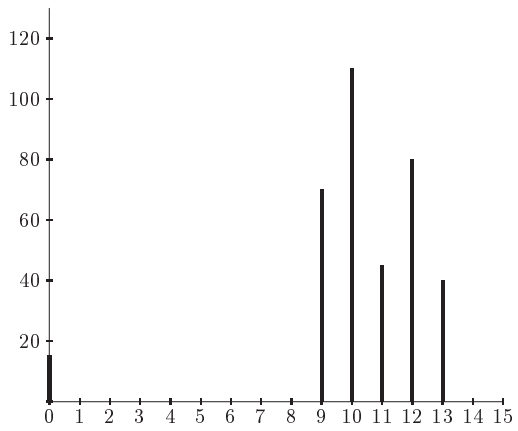

Figure 4.17: Another histogram indicating poor contrast

(with $n = 360$.) We would expect this image to be uniformly bright, with a few dark dots on it. To equalize this histogram, we form running totals of the n_i, and multiply each by $15/360 = 1/24$:

Grey level i	n_i	Σn_i	$(1/24)\Sigma n_i$	Rounded value
0	15	15	0.63	1
1	0	15	0.63	1
2	0	15	0.63	1
3	0	15	0.63	1
4	0	15	0.63	1
5	0	15	0.63	1
6	0	15	0.63	1
7	0	15	0.63	1
8	0	15	0.63	1
9	70	85	3.65	4
10	110	195	8.13	8
11	45	240	10	10
12	80	320	13.33	13
13	40	360	15	15
14	0	360	15	15
15	0	360	15	15

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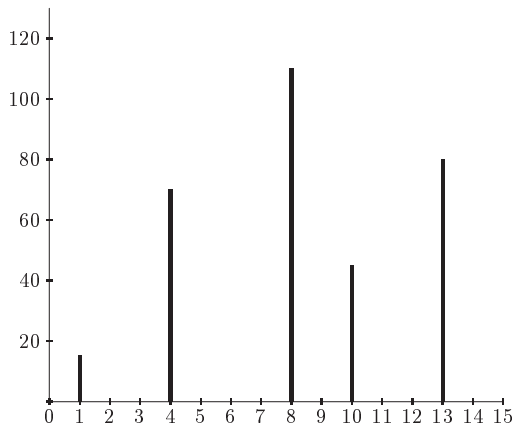

Figure 4.18: The histogram of figure 4.17 after equalization

We now have the following transformation of grey values, obtained by reading off the first and last columns in the above table:

Original grey level i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Final grey level j	1	1	1	1	1	1	1	1	1	4	8	10	13	15	15	15

and the histogram of the j values is shown in figure 4.18. This is far more spread out than the original histogram, and so the resulting image should exhibit greater contrast.

To apply histogram equalization in MATLAB, use the `histeq` function; for example:

```
>> p=imread('pout.tif');
>> ph=histeq(p);
>> imshow(ph),figure,imhist(ph),axis tight
```

applies histogram equalization to the `pout` image, and produces the resulting histogram. These results are shown in figure 4.19. Notice the far greater spread of the histogram. This corresponds to the greater increase of contrast in the image.

We give one more example, that of a very dark image. We can obtain a dark image by taking an image and using `imdivide`.

```
>> en=imread('engineer.tif');
>> e=imdivide(en,4);
```

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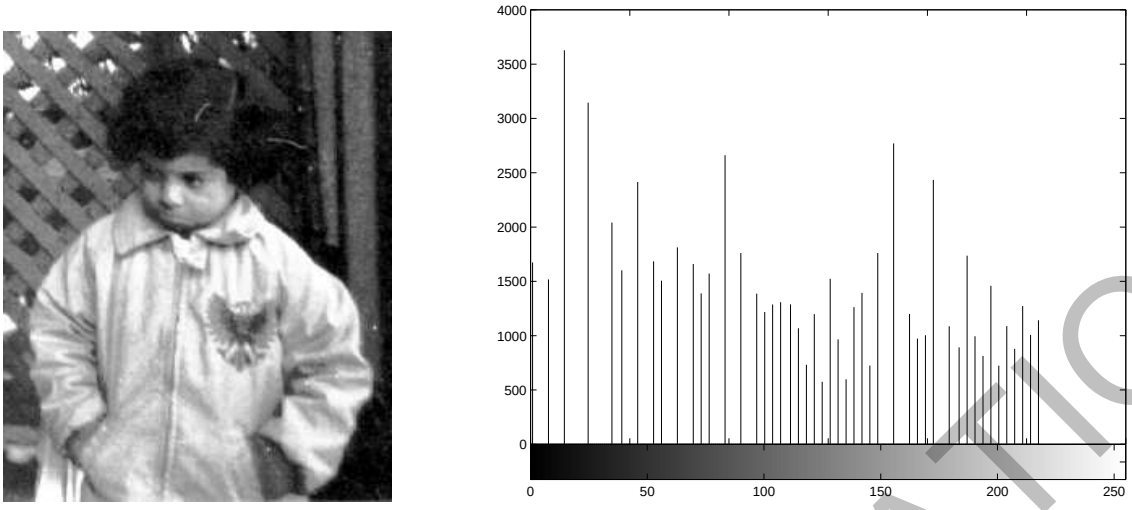

Figure 4.19: The histogram of figure 4.8 after equalization

Since the matrix `e` contains only low values it will appear very dark when displayed. We can display this matrix and its histogram with the usual commands:

```
>> imshow(e),figure,imhist(e),axis tight
```

and the results are shown in figure 4.20.

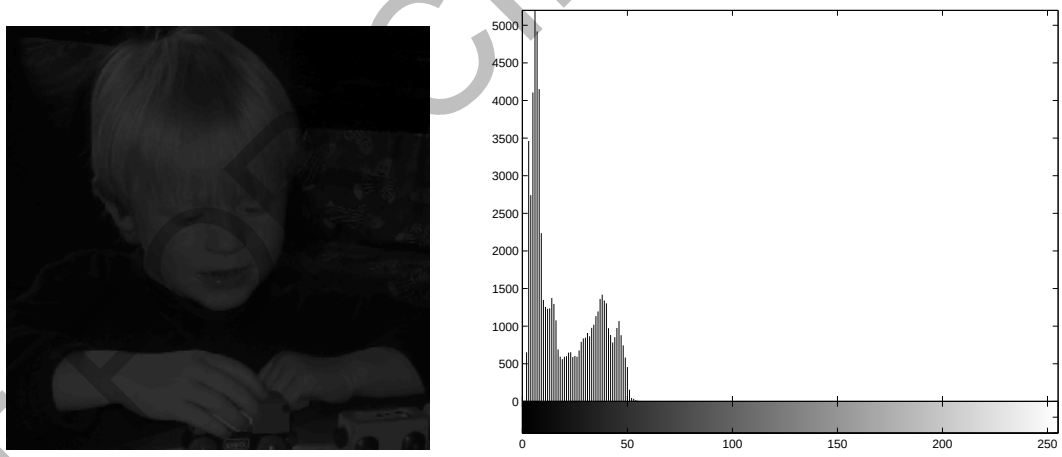

Figure 4.20: The darkened version of `engineer.tif` and its histogram

As you see, the very dark image has a corresponding histogram heavily clustered at the lower end of the scale.

But we can apply histogram equalization to this image, and display the results:

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```
>> eh=histeq(e);
>> imshow(eh),figure,imhist(eh),axis tight
```

and the results are shown in figure 4.21.

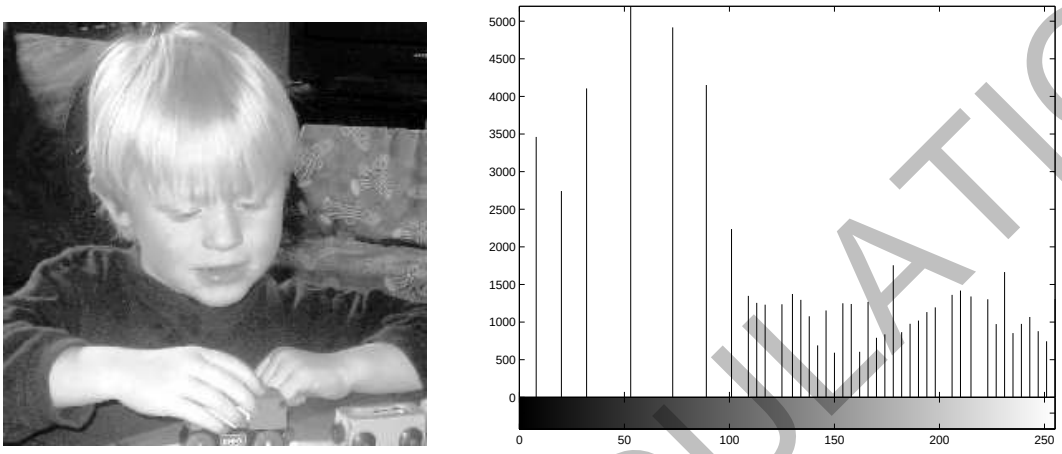

Figure 4.21: The image from 4.20 equalized and its histogram

Why it works

Consider the histogram in figure 4.17. To apply histogram stretching, we would need to stretch out the values between grey levels 9 and 13. Thus, we would need to apply a piecewise function similar to that shown in figure 4.9.

Let's consider the cumulative histogram, which is shown in figure 4.22. The dashed line is simply joining the top of the histogram bars. However, it can be interpreted as an appropriate histogram stretching function. To do this, we need to scale the y values so that they are between 0 and 15, rather than 0 and 360. But this is precisely the method described in section 4.3.2.

As we have seen, none of the example histograms, after equalization, are uniform. This is a result of the discrete nature of the image. If we were to treat the image as a continuous function $f(x, y)$, and the histogram as the area between different contours (see for example Castleman [4], then we can treat the histogram as a probability density function. But the corre'sponding cumulative density function will always have a uniform histogram; see for example Hogg and Craig [12].

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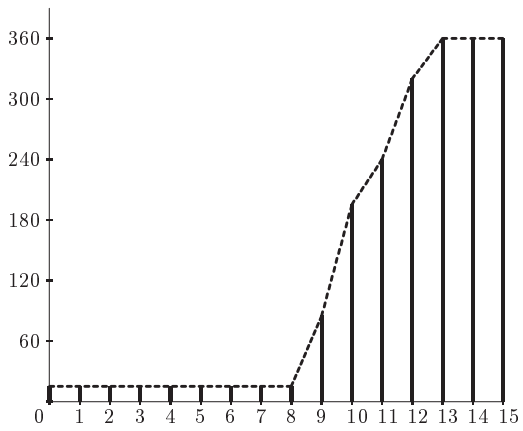

Figure 4.22: The cumulative histogram

4.4 Lookup tables

Point operations can be performed very effectively by the use of a *lookup table*, known more simply as an LUT. For operating on images of type `uint8`, such a table consists of a single array of 256 values, each value of which is an integer in the range $0 \ldots 255$. Then our operation can be implemented by replacing each pixel value p by the corresponding value t_p in the table.

For example, the LUT corresponding to division by 2 looks like:

Index:	0	1	2	3	4	5	...	250	251	252	253	254	255
LUT:	0	0	1	1	2	2	...	125	125	126	126	127	127

This means, for example, that a pixel with value 4 will be replaced with 2; a pixel with value 253 will be replaced with value 126.

If `T` is a lookup table in MATLAB, and `im` is our image, the the lookup table can be applied by the simple command

```
T(im)
```

For example, suppose we wish to apply the above lookup table to the blocks image. We can create the table with

```
>> T=uint8(floor(0:255)/2);
```

apply it to the blocks image `b` with

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```
>> b2=T(b);
```

The image `b2` is of type `uint8`, and so can be viewed directly with `imshow`.

As another example, suppose we wish to apply an LUT to implement the contrast stretching function shown in figure 4.23. Given the equation used in section 4.3.1, the equations of the three

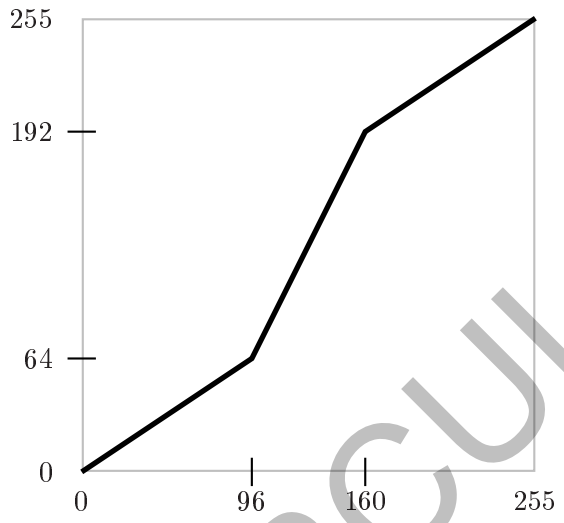

Figure 4.23: A piecewise linear contrast stretching function

lines used are:

$$
\begin{aligned}
y &= \frac{64}{96}x, \\
y &= \frac{192-64}{160-96}(x-96)+64, \\
y &= \frac{255-192}{255-160}(x-160)+192,
\end{aligned}
$$

and these equations can be written more simply as

$$
\begin{aligned}
y &= 0.6667x, \\
y &= 2x-128, \\
y &= 0.6632x+85.8947.
\end{aligned}
$$

We can then construct the LUT with the commands:

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```
>> t1=0.6667*[0:64];
>> t2=2*[65:160]-128;
>> t3=0.6632*[161:255]+85.8947;
>> T=uint8(floor([t1 t2 t3]));
```

Note that the commands for `t1`, `t2` and `t3` are direct translations of the line equations into MATLAB, except that in each case we are applying the equation only to its domain.

Exercises

Image Arithmetic

1. Describe lookup tables for

   (a) multiplication by 2,

   (b) image complements

2. Enter the following command on the blocks image `b`:

   ```
   >> b2=imdivide(b,64);
   >> bb2=immultiply(b2,64);
   >> imshow(bb2)
   ```

   Comment on the result. Why is the result not equivalent to the original image?

3. Replace the value 64 in the previous question with 32, and 16.

Histograms

4. Write informal code to calculate a histogram $h[f]$ of the grey values of an image $f[row][col]$.

5. The following table gives the number of pixels at each of the grey levels 0–7 in an image with those grey values only:

0	1	2	3	4	5	6	7
3244	3899	4559	2573	1428	530	101	50

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Draw the histogram corresponding to these grey levels, and then perform a histogram equalization and draw the resulting histogram.

6. The following tables give the number of pixels at each of the grey levels 0–15 in an image with those grey values only. In each case draw the histogram corresponding to these grey levels, and then perform a histogram equalization and draw the resulting histogram.

(a)

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
20	40	60	75	80	75	65	55	50	45	40	35	30	25	20	30

(b)

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	40	80	45	110	70	0	0	0	0	0	0	0	0	15

7. The following small image has grey values in the range 0 to 19. Compute the grey level histogram and the mapping that will equalize this histogram. Produce an 8×8 grid containing the grey values for the new histogram-equalized image.

```
12   6   5  13  14  14  16  15
11  10   8   5   8  11  14  14
 9   8   3   4   7  12  18  19
10   7   4   2  10  12  13  17
16   9  13  13  16  19  19  17
12  10  14  15  18  18  16  14
11   8  10  12  14  13  14  15
 8   6   3   7   9  11  12  12
```

8. Is the histogram equalization operation idempotent? That is, is performing histogram equalization *twice* the same as doing it just once?

9. Apply histogram equalization to the indices of the image `emu.tif`.

10. Create a dark image with

```
>> c=imread('cameraman.tif');
>> [x,map]=gray2ind(c);
```

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The matrix `x`, when viewed, will appear as a very dark version of the cameraman image. Apply histogram equalization to it, and compare the result with the original image.

11. Using `p` and `ph` from section 4.3.2, enter the command

```
>> figure,plot(p,ph,'.'),grid on
```

What are you seeing here?

12. Experiment with some other greyscale images.

13. Using LUTs, and following the example given in section 4.4, write a simpler function for performing piecewise stretching than the function described in section 4.3.1.

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Chapter 5

Neighbourhood Processing

5.1 Introduction

We have seen in chapter 4 that an image can be modified by applying a particular function to each pixel value. Neighbourhood processing may be considered as an extension of this, where a function is applied to a neighbourhood of each pixel.

The idea is to move a "mask": a rectangle (usually with sides of odd length) or other shape over the given image. As we do this, we create a new image whose pixels have grey values calculated from the grey values under the mask, as shown in figure 5.1. The combination of mask and function

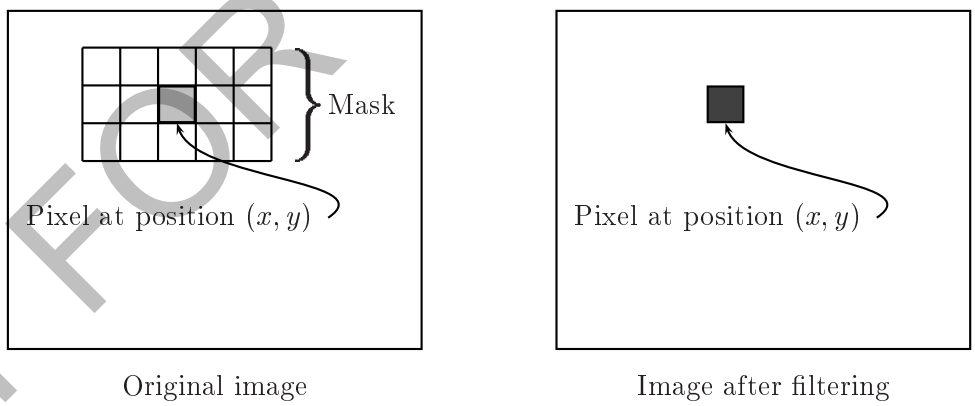

Figure 5.1: Using a spatial mask on an image

is called a *filter*. If the function by which the new grey value is calculated is a linear function of all the grey values in the mask, then the filter is called a *linear filter*.

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A linear filter can be implemented by multiplying all elements in the mask by corresponding elements in the neighbourhood, and adding up all these products. Suppose we have a 3×5 mask as illustrated in figure 5.1. Suppose that the mask values are given by:

$m(-1,-2)$	$m(-1,-1)$	$m(-1,0)$	$m(-1,1)$	$m(-1,2)$
$m(0,-2)$	$m(0,-1)$	$m(0,0)$	$m(0,1)$	$m(0,2)$
$m(1,-2)$	$m(1,-1)$	$m(1,0)$	$m(1,1)$	$m(1,2)$

and that corresponding pixel values are

$p(i-1,j-2)$	$p(i-1,j-1)$	$p(i-1,j)$	$p(i-1,j+1)$	$p(i-1,j+2)$
$p(i,j-2)$	$p(i,j-1)$	$p(i,j)$	$p(i,j+1)$	$p(i,j+2)$
$p(i+1,j-2)$	$p(i+1,j-1)$	$p(i+1,j)$	$p(i+1,j+1)$	$p(i+1,j+2)$

We now multiply and add:

$$\sum_{s=-1}^{1} \sum_{t=-2}^{2} m(s,t)p(i+s,j+t).$$

A diagram illustrating the process for performing spatial filtering is given in figure 5.2.

Spatial filtering thus requires three steps:

1. position the mask over the current pixel,

2. form all products of filter elements with the corresponding elements of the neighbourhood,

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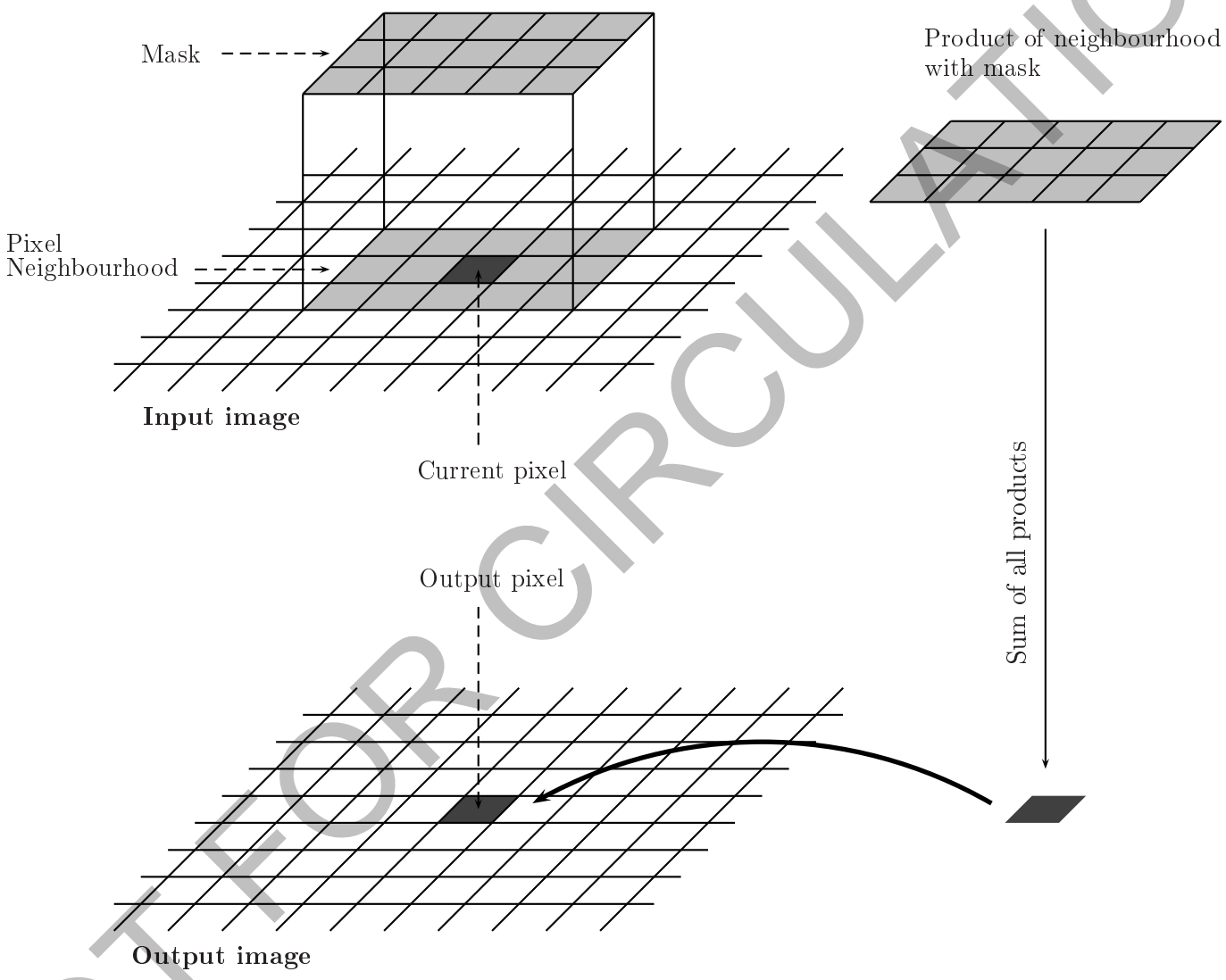

Figure 5.2: Performing spatial filtering

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3. add up all the products.

This must be repeated for every pixel in the image.

An example: One important linear filter is to use a 3×3 mask and take the average of all nine values within the mask. This value becomes the grey value of the corresponding pixel in the new image. This operation may be described as follows:

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & a & b & c \\ \hline & d & e & f \\ \hline & g & h & i \\ \hline & & & \\ \hline \end{array} \longrightarrow \frac{1}{9}(a+b+c+d+e+f+g+h+i)$$

where e is grey value of the current pixel in the original image, and the average is the grey value of the corresponding pixel in the new image.

To apply this to an image, consider the 5×5 "image" obtained by:

```
>> x=uint8(10*magic(5))


x =


   170   240    10    80   150
   230    50    70   140   160
    40    60   130   200   220
   100   120   190   210    30
   110   180   250    20    90
```

We may regard this array as being made of nine overlapping 3×3 neighbourhoods. The output of our working will thus consist only of nine values. We shall see later how to obtain 25 values in the output.

Consider the top left 3×3 neighbourhood of our image $\mathbf{x}$:

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170	240	10	80	150
230	50	70	140	160
40	60	130	200	220
100	120	190	210	30
110	180	250	20	90

Now we take the average of all these values:

```
>> mean2(x(1:3,1:3))

ans =

  111.1111
```

which can be rounded to 111.1. Now we can move to the second neighbourhood:

170	240	10	80	150
230	50	70	140	160
40	60	130	200	220
100	120	190	210	30
110	180	250	20	90

and take its average:

```
>> mean2(x(1:3,2:4))

ans =

  108.8889
```

and this can be rounded either down to 108, or to the nearest integer 109. If we continue in this manner, the following output is obtained:

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```
111.1111  108.8889  128.8889

110.0000  130.0000  150.0000

131.1111  151.1111  148.8889
```

This array is the result of filtering $\mathbf{x}$ with the 3×3 averaging filter.

5.2 Notation

It is convenient to describe a linear filter simply in terms of the coefficients of all the grey values of pixels within the mask. This can be written as a matrix.

The averaging filter above, for example, could have its output written as

$$\frac{1}{9}a + \frac{1}{9}b + \frac{1}{9}c + \frac{1}{9}d + \frac{1}{9}e + \frac{1}{9}f + \frac{1}{9}g + \frac{1}{9}h + \frac{1}{9}i$$

and so this filter can be described by the matrix

$$\begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

An example: The filter

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

would operate on grey values as

$$\begin{array}{|c|c|c|} \hline & & \\ \hline a & b & c \\ \hline d & \boxed{e} & f \\ \hline g & h & i \\ \hline & & \\ \hline \end{array} \longrightarrow a - 2b + c - 2d + 4e - 2d + g - 2h + i$$

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Edges of the image

There is an obvious problem in applying a filter—what happens at the edge of the image, where the mask partly falls outside the image? In such a case, as illustrated in figure 5.3 there will be a lack of grey values to use in the filter function.

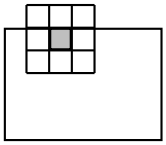

Figure 5.3: A mask at the edge of an image

There are a number of different approaches to dealing with this problem:

Ignore the edges. That is, the mask is only applied to those pixels in the image for with the mask will lie fully within the image. This means all pixels except for the edges, and results in an output image which is smaller than the original. If the mask is very large, a significant amount of information may be lost by this method.

We applied this method in our example above.

"Pad" with zeros. We assume that all necessary values outside the image are zero. This gives us all values to work with, and will return an output image of the same size as the original, but may have the effect of introducing unwanted artifacts (for example, edges) around the image.

5.3 Filtering in MATLAB

The `filter2` function does the job of linear filtering for us; its use is

```
filter2(filter,image,shape)
```

and the result is a matrix of data type `double`. The parameter `shape` is optional, it describes the method for dealing with the edges:

- `filter2(filter,image,'same')` is the default; it produces a matrix of equal size to the original `image` matrix. It uses zero padding:

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```
>> a=ones(3,3)/9

a =

    0.1111    0.1111    0.1111
    0.1111    0.1111    0.1111
    0.1111    0.1111    0.1111

>> filter2(a,x,'same')

ans =

   76.6667   85.5556   65.5556   67.7778   58.8889
   87.7778  111.1111  108.8889  128.8889  105.5556
   66.6667  110.0000  130.0000  150.0000  106.6667
   67.7778  131.1111  151.1111  148.8889   85.5556
   56.6667  105.5556  107.7778   87.7778   38.8889
```

- `filter2(filter,image,'valid')` applies the mask only to "inside" pixels. The result will always be smaller than the original:

```
>> filter2(a,x,'valid')

ans =

  111.1111  108.8889  128.8889
  110.0000  130.0000  150.0000
  131.1111  151.1111  148.8889
```

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The result of `'same'` above may also be obtained by padding with zeros and using `'valid'`:

```
>> x2=zeros(7,7);
>> x2(2:6,2:6)=x

x2 =

     0     0     0     0     0     0     0
     0   170   240    10    80   150     0
     0   230    50    70   140   160     0
     0    40    60   130   200   220     0
     0   100   120   190   210    30     0
     0   110   180   250    20    90     0
     0     0     0     0     0     0     0

>> filter2(a,x2,'valid')
```

- `filter2(filter,image,'full')` returns a result *larger* than the original; it does this by padding with zero, and applying the filter at all places on and around the image where the mask intersects the image matrix.

```
>> filter2(a,x,'full')

ans =

   18.8889   45.5556   46.6667   36.6667   26.6667   25.5556   16.6667
   44.4444   76.6667   85.5556   65.5556   67.7778   58.8889   34.4444
   48.8889   87.7778  111.1111  108.8889  128.8889  105.5556   58.8889
```

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```
    41.1111   66.6667  110.0000  130.0000  150.0000  106.6667   45.5556

    27.7778   67.7778  131.1111  151.1111  148.8889   85.5556   37.7778

    23.3333   56.6667  105.5556  107.7778   87.7778   38.8889   13.3333

    12.2222   32.2222   60.0000   50.0000   40.0000   12.2222   10.0000
```

The shape parameter, being optional, can be omitted; in which case the default value is `'same'`.

There is no single "best" approach; the method must be dictated by the problem at hand; by the filter being used, and by the result required.

We can create our filters by hand, or by using the `fspecial` function; this has many options which makes for easy creation of many different filters. We shall use the `average` option, which produces averaging filters of given size; thus

```
>> fspecial('average',[5,7])
```

will return an averaging filter of size 5×7; more simply

```
>> fspecial('average',11)
```

will return an averaging filter of size 11×11. If we leave out the final number or vector, the 3×3 averaging filter is returned.

For example, suppose we apply the 3×3 averaging filter to an image as follows:

```
>> c=imread('cameraman.tif');
>> f1=fspecial('average');
>> cf1=filter2(f1,c);
```

We now have a matrix of data type double. To display this, we can do any of the following:

- transform it to a matrix of type `uint8`, for use with `imshow`,
- divide its values by 255 to obtain a matrix with values in the 0.1–1.0 range, for use with `imshow`,
- use `mat2gray` to scale the result for display. We shall discuss the use of this function later.

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Using the second method:

```
>> figure,imshow(c),figure,imshow(cf1/255)
```

will produce the images shown in figures 5.4(a) and 5.4(b).

The averaging filter blurs the image; the edges in particular are less distinct than in the original. The image can be further blurred by using an averaging filter of larger size. This is shown in figure 5.4(c), where a 9×9 averaging filter has been used, and in figure 5.4(d), where a 25×25 averaging filter has been used.

Notice how the zero padding used at the edges has resulted in a dark border appearing around the image. This is especially noticeable when a large filter is being used. If this is an unwanted artifact of the filtering; if for example it changes the average brightness of the image, then it may be more appropriate to use the `'valid'` shape option.

The resulting image after these filters may appear to be much "worse" than the original. However, applying a blurring filter to reduce detail in an image may the perfect operation for autonomous machine recognition, or if we are only concentrating on the "gross" aspects of the image: numbers of objects; amount of dark and light areas. In such cases, too much detail may obscure the outcome.

Separable filters

Some filters can be implemented by the successive application of two simpler filters. For example, since

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

the 3×3 averaging filter can be implemented by first applying a 3×1 averaging filter, and then applying a 1×3 averaging filter to the result. The 3×3 averaging filter is thus *separable* into two smaller filters. Separability can result in great time savings. Suppose an $n \times n$ filter is separable into two filters of size $n \times 1$ and $1 \times n$. The application of an $n \times n$ filter requires n^2 multiplications, and $n^2 - 1$ additions for each pixel in the image. But the application of an $n \times 1$ filter only requires n multiplications and $n - 1$ additions. Since this must be done twice, the total number of

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3 x 3 1/9

(a) Original image

(b) Average filtering

(c) Using a 9×9 filter

9 $\boxed{1/81}$

(d) Using a 25×25 filter

$\boxed{1/25}$

Figure 5.4: Average filtering

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multiplications and additions are $2n$ and $2n-2$ respectively. If n is large the savings in efficiency can be dramatic.

All averaging filters are separable; another separable filter is the laplacian

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}.$$

Other examples will be considered below.

5.4 Frequencies; low and high pass filters

It will be convenient to have some standard terminology by which we can discuss the effects a filter will have on an image, and to be able to choose the most appropriate filter for a given image processing task. One important aspect of an image which enables us to do this is the notion of *frequencies*. Roughly speaking, the frequencies of an image are a measure of the amount by which grey values change with distance. This concept will be given a more formal setting in chapter 7. *High frequency components* are characterized by large changes in grey values over small distances; example of high frequency components are edges and noise. *Low frequency components*, on the other hand, are parts of the image characterized by little change in the grey values. These may include backgrounds, skin textures. We then say that a filter is a

high pass filter if it "passes over" the high frequency components, and reduces or eliminates low frequency components,

low pass filter if it "passes over" the low frequency components, and reduces or eliminates high frequency components,

For example, the 3×3 averaging filter is low pass filter, as it tends to blur edges. The filter

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

is a high pass filter.

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We note that the sum of the coefficients (that is, the sum of all e elements in the matrix), in the high pass filter is zero. This means that in a low frequency part of an image, where the grey values are similar, the result of using this filter is that the corresponding grey values in the new image will be close to zero. To see this, consider a 4×4 block of similar values pixels, and apply the above high pass filter to the central four:

150	152	148	149
147	152	151	150
152	148	149	151
151	149	150	148

$\longrightarrow$

11	6
-13	-5

The resulting values are close to zero, which is the expected result of applying a high pass filter to a low frequency component. We shall see how to deal with negative values below.

High pass filters are of particular value in edge detection and edge enhancement (of which we shall see more in chapter 9). But we can provide a sneak preview, using the cameraman image.

```
>> f=fspecial('laplacian')


f =


    0.1667    0.6667    0.1667
    0.6667   -3.3333    0.6667
    0.1667    0.6667    0.1667


>> cf=filter2(f,c);
>> imshow(cf/100)
>> f1=fspecial('log')


f1 =


    0.0448    0.0468    0.0564    0.0468    0.0448
    0.0468    0.3167    0.7146    0.3167    0.0468
```

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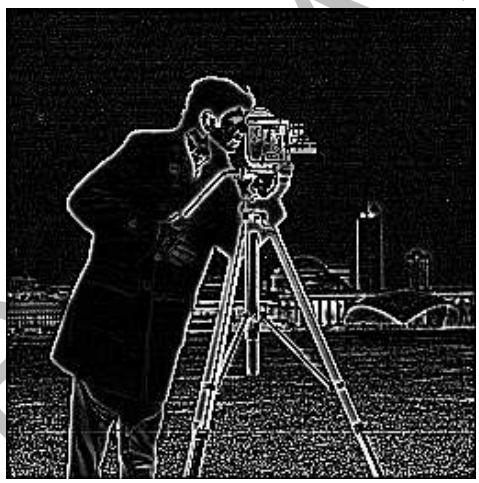
```
    0.0564    0.7146   -4.9048    0.7146    0.0564

    0.0468    0.3167    0.7146    0.3167    0.0468

    0.0448    0.0468    0.0564    0.0468    0.0448


>> cf1=filter2(f1,c);

>> figure,imshow(cf1/100)
```

The images are shown in figure 5.5. Image (a) is the result of the Laplacian filter; image (b) shows the result of the Laplacian of Gaussian ("log") filter. We discuss Gaussian filters in section 5.5.

(a) Laplacian filter



(b) Laplacian of Gaussian ("log") filtering

Figure 5.5: High pass filtering

In each case, the sum of all the filter elements is zero.

Values outside the range 0–255

We have seen that for image display, we would like the grey values of the pixels to lie between 0 and 255. However, the result of applying a linear filter may be values which lie outside this range. We may consider ways of dealing with values outside of this "displayable" range.

Make negative values positive. This will certainly deal with negative values, but not with values greater than 255. Hence, this can only be used in specific circumstances; for example, when there are only a few negative values, and when these values are themselves close to zero.

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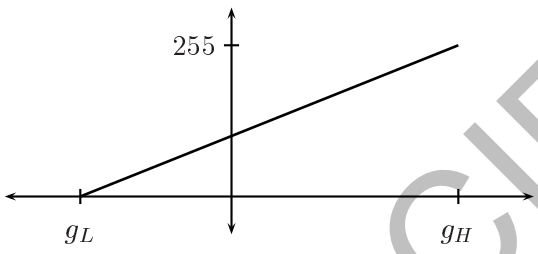
Clip values. We apply the following thresholding type operation to the grey values x produced by the filter to obtain a displayable value y:

$$y = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 255 \\ 255 & \text{if } x > 255 \end{cases}$$

This will produce an image with all pixel values in the required range, but is not suitable if there are many grey values outside the 0–255 range; in particular, if the grey values are equally spread over a larger range. In such a case this operation will tend to destroy the results of the filter.

Scaling transformation. Suppose the lowest grey value produced by the filter if g_L and the highest value is g_H. We can transform all values in the range g_L–g_H to the range 0–255 by the linear transformation illustrated below:



Since the gradient of the line is $255/(g_H - g_L)$ we can write the equation of the line as

$$y = 255\frac{x - g_L}{g_H - g_L}$$

and applying this transformation to all grey levels x produced by the filter will result (after any necessary rounding) in an image which can be displayed.

As an example, let's apply the high pass filter given in section 5.4 to the cameraman image:

```
>> f2=[1 -2 1;-2 4 -2;1 -2 1];
>> cf2=filter2(f2,c);
```

Now the maximum and minimum values of the matrix `cf2` are 593 and −541 respectively. The `mat2gray` function automatically scales the matrix elements to displayable values; for any matrix

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M, it applies a linear transformation to to its elements, with the lowest value mapping to 0.0, and the highest value mapping to 1.0. This means the output of `mat2gray` is always of type `double`. The function also requires that the input type is `double`.

```
>> figure,imshow(mat2gray(cf2));
```

To do this by hand, so to speak, applying the linear transformation above, we can use:

```
>> maxcf2=max(cf2(:));
>> mincf2=min(cf2(:));
>> cf2g=(cf2-mincf2)/(maxcf2-mncf2);
```

The result will be a matrix of type `double`, with entries in the range 0.0–1.0. This can be be viewed with `imshow`. We can make it a `uint8` image by multiplying by 255 first. The result can be seen in figure 5.6.

We can generally obtain a better result by dividing the result of the filtering by a constant before displaying it:

```
>> figure,imshow(cf2/60)
```

and this is also shown in figure 5.6.

Using `mat2gray` Dividing by a constant

Figure 5.6: Using a high pass filter and displaying the result

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High pass filters are often used for edge detection. These can be seen quite clearly in the right hand image of figure 5.6.

5.5 Gaussian filters

We have seen some examples of linear filters so far: the averaging filter, and a high pass filter. The `fspecial` function can produce many different filters for use with the `filter2` function; we shall look at a particularly important filter here.

Gaussian filters are a class of low-pass filters, all based on the Gaussian probability distribution function

$$f(x) = e^{-\frac{x^2}{2\sigma^2}}$$

where σ is the standard deviation: a large value of σ produces to a flatter curve, and a small value leads to a "pointier" curve. Figure 5.7 shows examples of such one dimensional gaussians. Gaussian filters are important for a number of reasons:

1. They are mathematically very "well behaved"; in particular the Fourier transform (see chapter 7) of a Gaussian filter is another Gaussian.

2. They are rotationally symmetric, so are very good starting points for some edge detection algorithms (see chapter 9),

3. They are *separable*; in that a Gaussian filter may be applied by first applying a one-dimension Gaussian in the x direction, followed by another in the y direction. This can lead to very fast implementations.

4. The convolution of two Gaussians is another Gaussian.

A two dimensional Gaussian function is given by

$$f(x, y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$$

The command `fspecial('gaussian')` produces a discrete version of this function. We can draw pictures of this with the `surf` function, and to ensure a nice smooth result, we shall create a large filter (size 50×50) with different standard deviations.

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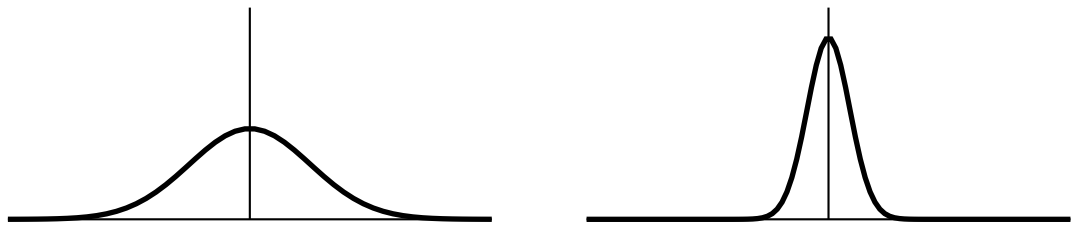
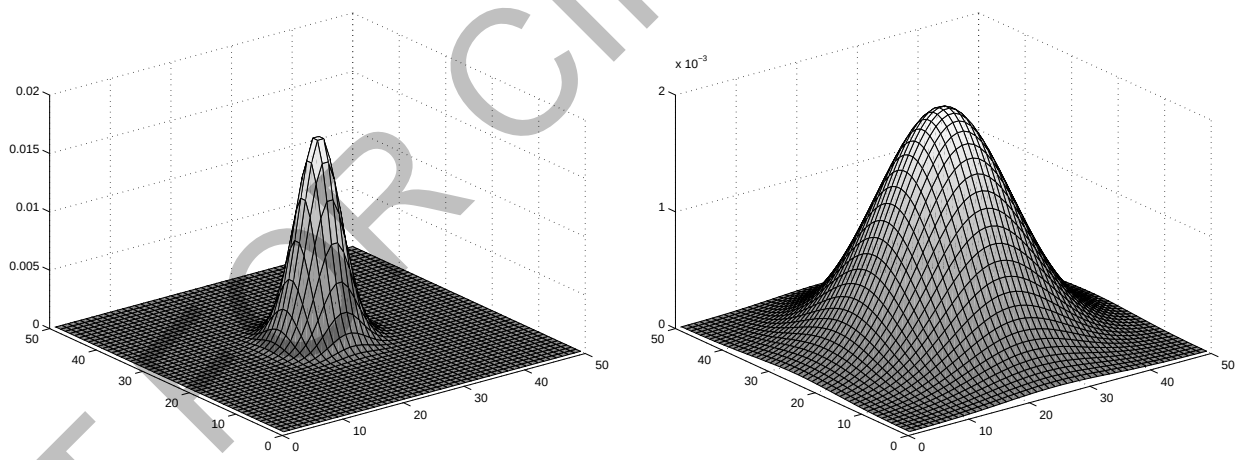


Large value of σ Small value of σ

Figure 5.7: One dimensional gaussians

```
>> a=50;s=3;
>> g=fspecial('gaussian',[a a],s);
>> surf(1:a,1:a,g)
>> s=9;
>> g2=fspecial('gaussian',[a a],s);
>> figure,surf(1:a,1:a,g2)
```

The surfaces are shown in figure 5.8.



$\sigma = 3$ $\sigma = 9$

Figure 5.8: Two dimensional gaussians

.

Gaussian filters have a blurring effect which looks very similar to that produced by neighbourhood averaging. Let's experiment with the cameraman image, and some different gaussian filters.

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```
>> g1=fspecial('gaussian',[5,5]);
>> g1=fspecial('gaussian',[5,5],2);
>> g1=fspecial('gaussian',[11,11],1);
>> g1=fspecial('gaussian',[11,11],5);
```

The final parameter is the standard deviation; which if not given defaults to 0.5. The second parameter (which is also optional), gives the size of the filter; the default is 3×3. If the filter is to be square, as in all the above examples, we can just give a single number in each case.

Now we can apply the filter to the cameraman image matrix `c` and vier the result.

```
>> imshow(filter2(g1,c)/256)
>> figure,imshow(filter2(g2,c)/256)
>> figure,imshow(filter2(g3,c)/256)
>> figure,imshow(filter2(g4,c)/256)
```

and the results are shown in figure 5.9. Thus to obtain a spread out blurring effect, we need a large standard deviation. In fact, if we let the standard deviation grow large without bound, we obtain the averaging filters as limiting values. For example:

```
>> fspecial('gaussian',3,100)


ans =


    0.1111    0.1111    0.1111
    0.1111    0.1111    0.1111
    0.1111    0.1111    0.1111
```

and we have the 3×3 averaging filter.

Although the results of Gaussian blurring and averaging look similar, the Gaussian filter has some elegant mathematical properties which make it particularly suitable for blurring.

Other filters will be discussed in future chapters; also check the documentation for `fspecial` for other filters.

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5×5, $\sigma = 0.5$

5×5, $\sigma = 2$

11×11, $\sigma = 1$

11×11, $\sigma = 5$

Figure 5.9: Effects of different gaussian filters on an image

.

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5.6 Edge sharpening

Spatial filtering can be used to make edges in an image slightly sharper and crisper, which generally results in an image more pleasing to the human eye. The operation is variously called "edge enhancement", "edge crispening", or "unsharp masking". This last term comes from the printing industry.

Unsharp masking

The idea of unsharp masking is to subtract a scaled "unsharp" version of the image from the original. In practice, we can achieve this affect by subtracting a scaled blurred image from the original. The schema for unsharp masking is shown in figure 5.10.

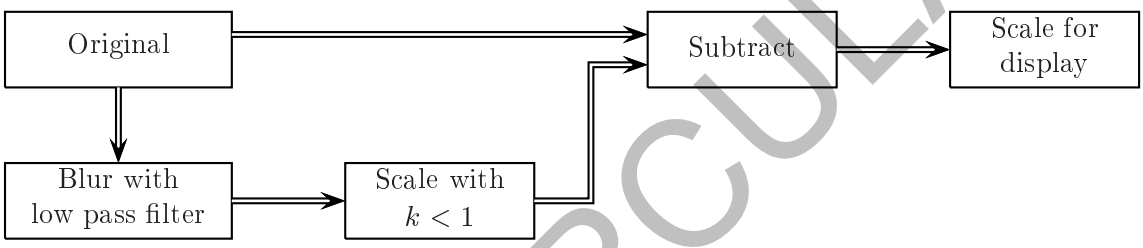

Figure 5.10: Schema for unsharp masking

Suppose an image `x` is of type `uint8`. The unsharp masking can be applied by the following sequence of commands:

```
>> f=fspecial('average');
>> xf=filter2(f,x);
>> xu=double(x)-xf/1.5
>> imshow(xu/70)
```

The last command scales the result so that `imshow` displays an appropriate image; the value may need to be adjusted according to the input image. Suppose that `x` is the image shown in figure 5.11(a), then the result of unsharp masking is given in figure 5.11(b). The result appears to be a better image than the original; the edges are crisper and more clearly defined.

To see why this works, we may consider the function of grey values as we travel across an edge, as shown in figure 5.12.

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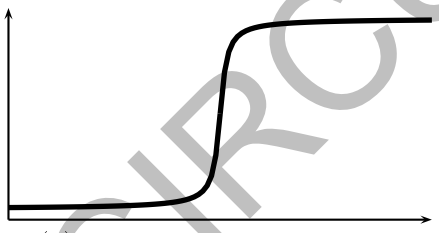
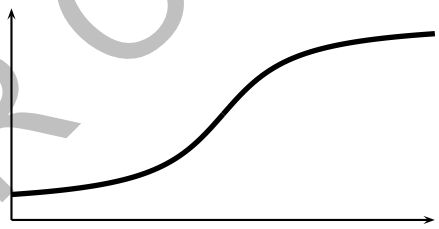
(a) Original image

(b) The image after unsharp masking

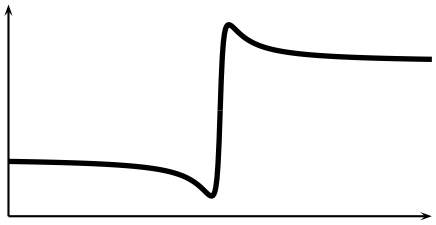
Figure 5.11: An example of unsharp masking



(a) Pixel values over an edge



(b) The edge blurred



(c) (a) $-\ k$(b)

Figure 5.12: Unsharp masking

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As a scaled blur is subtracted from the original, the result is that the edge is enhanced, as shown in graph (c) of figure 5.12.

We can in fact perform the filtering and subtracting operation in one command, using the linearity of the filter, and that the 3×3 filter

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the "identity filter".

Hence unsharp masking can be implemented by a filter of the form

$$f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{k} \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

where k is a constant chosen to provide the best result. Alternatively, the unsharp masking filter may be defined as

$$f = k \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

so that we are in effect subtracting a blur from a scaled version of the original; the scaling factor may also be split between the identity and blurring filters.

The `unsharp` option of `fspecial` produces such filters; the filter created has the form

$$\frac{1}{\alpha+1} \begin{bmatrix} -\alpha & \alpha-1 & -\alpha \\ \alpha-1 & \alpha+5 & \alpha-1 \\ -\alpha & \alpha-1 & -\alpha \end{bmatrix}$$

where α is an optional parameter which defaults to 0.2. If $\alpha = 0.5$ the filter is

$$\frac{1}{3} \begin{bmatrix} -1 & -1 & -1 \\ -1 & 11 & -1 \\ -1 & -1 & -1 \end{bmatrix} = 4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 3 \begin{bmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

Figure 5.13 was created using the MATLAB commands

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```
>> p=imread('pelicans.tif');
>> u=fspecial('unsharp',0.5);
>> pu=filter2(u,p);
>> imshow(p),figure,imshow(pu/255)
```

Figure 5.13(b), appears much sharper and "cleaner" than the original. Notice in particular the rocks and trees in the background, and the ripples on the water.

(a) The original (b) After unsharp masking

Figure 5.13: Edge enhancement with unsharp masking

Although we have used averaging filters above, we can in fact use any low pass filter for unsharp masking.

High boost filtering

Allied to unsharp masking filters are the *high boost* filters, which are obtained by

$$\text{high boost} = A(\text{original}) - (\text{low pass}).$$

where A is an "amplification factor". If $A = 1$, then the high boost filter becomes an ordinary high pass filter. If we take as the low pass filter the 3×3 averaging filter, then a high boost filter will

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have the form

$$\frac{1}{9}\begin{bmatrix} -1 & -1 & -1 \\ -1 & z & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

where $z > 8$. If we put $z = 11$, we obtain a filtering very similar to the unsharp filter above, except for a scaling factor. Thus the commands:

```
>> f=[-1 -1 -1;-1 11 -1;-1 -1 -1]/9;
>> xf=filter2(x,f);
>> imshow(xf/80)
```

will produce an image similar to that in figure 5.11. The value 80 was obtained by trial and error to produce an image with similar intensity to the original.

We can also write the high boost formula above as

$$\begin{aligned} \text{high boost} &= A(\text{original}) - (\text{low pass}) \\ &= A(\text{original}) - ((\text{original}) - (\text{high pass})) \\ &= (A-1)(\text{original}) + (\text{high pass}). \end{aligned}$$

Best results for high boost filtering are obtained if we multiply the equation by a factor w so that the filter values sum to 1; this requires

$$wA - w = 1$$

or

$$w = \frac{1}{A-1}.$$

So a general unsharp masking formula is

$$\frac{A}{A-1}(\text{original}) - \frac{1}{A-1}(\text{low pass}).$$

Another version of this formula is

$$\frac{A}{2A-1}(\text{original}) - \frac{1-A}{2A-1}(\text{low pass})$$

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where for best results A is taken so that

$$\frac{3}{5} \leq A \leq \frac{5}{6}.$$

If we take $A = 3/5$, the formula becomes

$$\frac{3/5}{2(3/5)-1}(\text{original}) - \frac{1-(3/5)}{2(3/5)-1}(\text{low pass}) = 3(\text{original}) - 2(\text{low pass})$$

If we take $A = 5/6$ we obtain

$$\frac{5}{4}(\text{original}) - \frac{1}{4}(\text{low pass})$$

Using the identity and averaging filters, we can obtain high boost filters by:

```
>> id=[0 0 0;0 1 0;0 0 0];
>> f=fspecial('average');
>> hb1=3*id-2*f


hb1 =


   -0.2222   -0.2222   -0.2222
   -0.2222    2.7778   -0.2222
   -0.2222   -0.2222   -0.2222


>> hb2=1.25*id-0.25*f


hb2 =


   -0.0278   -0.0278   -0.0278
   -0.0278    1.2222   -0.0278
   -0.0278   -0.0278   -0.0278
```

If each of the filters `hb1` and `hb2` are applied to an image with `filter2`, the result will have enhanced edges. The images in figure 5.14 show these results; figure 5.14(a) was obtained with

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```
>> x1=filter2(hb1,x);
>> imshow(x1/255)
```

and figure 5.14(b) similarly.

(a) High boost filtering with `hb1`

(b) High boost filtering with `hb2`

Figure 5.14: High boost filtering

Of the two filters, `hb1` appears to produce the best result; `hb2` produces an image not very much crisper than the original.

5.7 Non-linear filters

Linear filters, as we have seen in the previous sections, are easy to describe, and can be applied very quickly and efficiently by MATLAB.

A *non-linear filter* is obtained by a non-linear function of the greyscale values in the mask. Simple examples are the *maximum filter*, which has as its output the maximum value under the mask, and the corresponding *minimum filter*, which has as its output the minimum value under the mask.

Both the maximum and minimum filters are examples of *rank-order filters*. In such a filter, the elements under the mask are ordered, and a particular value returned as output. So if the values are given in increasing order, the minimum filter is a rank-order filter for which the *first* element is returned, and the maximum filter is a rank-order filter for which the *last* element is returned

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For implementing a general non-linear filter in MATLAB, the function to use is `nlfilter`, which applies a filter to an image according to a pre-defined function. If the function is not already defined, we have to create an m-file which defines it.

Here are some examples; first to implement a maximum filter over a 3×3 neighbourhood:

```
>> cmax=nlfilter(c,[3,3],'max(x(:))');
```

The `nlfilter` function requires three arguments: the image matrix, the size of the filter, and the function to be applied. The function must be a matrix function which returns a scalar value. The result of this operation is shown in figure 5.15(a).

A corresponding implementation of the minimum filter is:

```
>> cmin=nlfilter(c,[3,3],'min(x(:))');
```

and the result is shown in figure 5.15(b).

(a) Using a maximum filter (b) Using a minimum filter

Figure 5.15: Using non-linear filters

Note that in each case the image has lost some sharpness, and has been brightened by the maximum filter, and darkened by the minimum filter. The `nlfilter` function is very slow; in general there is little call for non-linear filters except for a few which are defined by their own commands. We shall investigate these in later chapters.

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Non-linear filtering using `nlfilter` can be very slow. A faster alternative is to use the `colfilt` function, which rearranges the image into columns first. For example, to apply the maximum filter to the cameraman image, we can use

```
>> cmax=colfilt(c,[3,3],'sliding',@max);
```

The parameter `sliding` indicates that overlapping neighbourhoods are being used (which of course is the case with filtering). This particular operation is almost instantaneous, as compared with the use of `nlfilter`.

To implement the maximum and minimum filters as rank-order filters, we may use the MATLAB function `ordfilt2`. This requires three inputs: the image, the index value of the ordered results to choose as output, and the definition of the mask. So to apply the maximum filter on a 3×3 mask, we use

```
>> cmax=ordfilt2(c,9,ones(3,3));
```

and the minimum filter can be applied with

```
>> cmin=ordfilt2(c,1,ones(3,3));
```

A very important rank-order filter is the *median filter*, which takes the *central* value of the ordered list. We could apply the median filter with

```
>> cmed=ordfilt2(c,5,ones(3,3));
```

However, the median filter has its own command, `medfilt2`, which we discuss in more detail in chapter 8.

Other non-linear filters are the *geometric mean filter*, which is defined as

$$\left(\prod_{(i,j)\in M} x(i,j)\right)^{(1/|M|)}$$

where M is the filter mask, and $|M|$ its size; and the *alpha-trimmed mean filter*, which first orders the values under the mask, trims off elements at either end of the ordered list, and takes the mean of the remainder. So, for example, if we have a 3×3 mask, and we order the elements as

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_9$$

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and trim off two elements at either end, the result of the filter will be

$$(x_3 + x_4 + x_5 + x_6 + x_7)/5.$$

Both of these filters have uses for image restoration; again see chapter 8.

Further examples of non-linear filters are given in chapter 16.

5.8 Region of interest processing

We may often not want to apply a filter to an entire image, but only to a small region within it. A non-linear filter, for example, may be too computationally expensive to apply to the entire image, or we may only be interested in the small region. Such small regions within an image are called *regions of interest* or *ROIs*, and their processing is called *region of interest processing*.

5.8.1 Regions of interest in MATLAB

Before we can process a ROI, we have to define it. There are two ways: by listing the coordinates of a polygonal region; or interactively, with the mouse. For example, suppose we take our image of an iguana (a sort of large lizard) peering over a branch:

```
>> ig=imread('iguana.tif');
```

and attempt to isolate the head of the animal. If view the image with `pixval on`, then the coordinates of a rectangle which enclose the head can be determined to be $(58, 406)$, $(58, 600)$, $(231, 406)$ and $(231, 600)$, as shown on in figure 5.16. We can then define a region of interest using the `roipoly` function:

```
>> roi=roipoly(ig,[406 600 600 406],[58 58 231 231]);
```

Note that the ROI is defined by two sets of coordinates: first the columns,and then the rows, taken in order as we traverse the ROI from vertex to vertex. We can view the resulting ROI mask as a binary image; it is shown on the right in figure 5.17. In general a ROI mask will be a binary image the same size as the original image, with 1's for the ROI, and 0's elsewhere. We can also use the function `roipoly` interactively:

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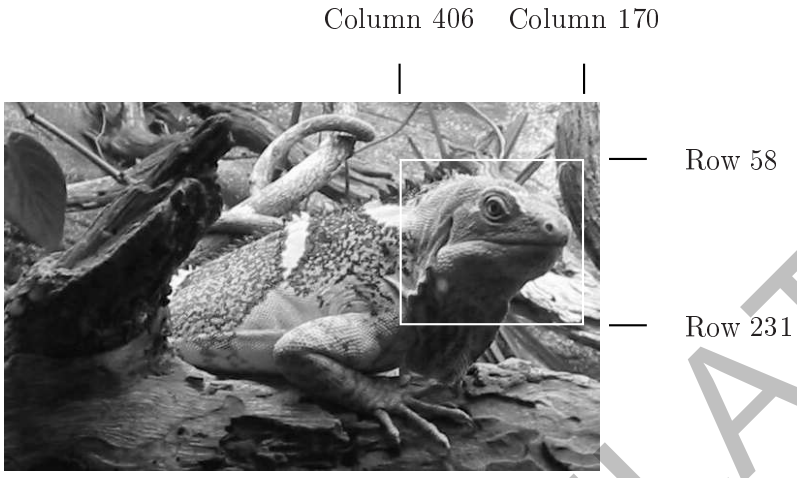

Figure 5.16: An image with a ROI

Figure 5.17: The mask corresponding to the ROI defined in figure 5.16

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```
>> roi=roipoly(ig);
```

This will bring up the iguana image (if it isn't shown already). Vertices of the ROI can be selected with the mouse: a left click selects a new vertex, backspace or delete removes the most recently chosen vertex, and a right click finishes the selection.

5.8.2 Region of interest filtering

One of the simplest operations on a ROI is spatial filtering; this is implemented with the function `roifilt2`. With the iguana image and the ROI found above, we can experiment:

```
>> a=fspecial('average',[15,15]);
>> iga=roifilt2(a,ig,roi);
>> imshow(iga)
>> u=fspecial('unsharp');
>> igu=roifilt2(u,ig,roi);
>> figure,imshow(igu)
>> l=fspecial('log');
>> igl=roifilt2(l,ig,roi);
>> figure,imshow(igl)
```

The images are shown in figure 5.18.

Exercises

1. The array below represents a small greyscale image. Compute the images that result when the image is convolved with each of the masks (a) to (h) shown. At the edge of the image use a restricted mask. (In other words, pad the image with zeroes.)

20	20	20	10	10	10	10	10	10
20	20	20	20	20	20	20	20	10
20	20	20	10	10	10	10	20	10

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`iga`: Average filtering

`igu`: Unsharp masking

`igl`: Laplacian of Gaussian

Figure 5.18: Examples of the use of `roifilt2`

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20	20	10	10	10	10	10	20	10
20	10	10	10	10	10	10	20	10
10	10	10	10	20	10	10	20	10
10	10	10	10	10	10	10	10	10
20	10	20	20	10	10	10	20	20
20	10	10	20	10	10	20	10	20

(a)
-1	-1	0
-1	0	1
0	1	1

(b)
0	-1	-1
1	0	-1
1	1	0

(c)
-1	-1	-1
2	2	2
-1	-1	-1

(d)
-1	2	-1
-1	2	-1
-1	2	-1

(e)
-1	-1	-1
-1	8	-1
-1	-1	-1

(f)
1	1	1
1	1	1
1	1	1

(g)
-1	0	1
-1	0	1
-1	0	1

(h)
-1	2	-1
-1	4	-1
0	-1	0

2. Check your answers to the previous question with MATLAB.

3. Describe what each of the masks in the previous question might be used for. If you can't do this, wait until question 5 below.

4. Devise a 3×3 mask for an "identity filter"; which causes no change in the image.

5. Obtain a greyscale image of a monkey (a *mandrill*) with the following commands:

```
>> load('mandrill.mat');
>> m=im2uint8(ind2gray(X,map));
```

   Apply all the filters listed in question 1 to this image. Can you now see what each filter does?

6. Apply larger and larger averaging filters to this image. What is the smallest sized filter for which the whiskers cannot be seen?

7. Repeat the previous question with Gaussian filters with the following parameters:

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Size	Standard deviation		
[3,3]	0.5	1	2
[7,7]	1	3	6
[11,11]	1	4	8
[21,21]	1	5	10

At what values do the whiskers disappear?

8. Can you see any observable difference in the results of average filtering and of using a Gaussian filter?

9. Read through the help page of the `fspecial` function, and apply some of the other filters to the cameraman image, and to the mandrill image.

10. Apply different laplacian filters to the mandrill and cameraman images. Which produces the best edge image?

11. Is the 3×3 median filter separable? That is, can this filter be implemented by a 3×1 filter followed by a 1×3 filter?

12. Repeat the above question for the maximum and minimum filters.

13. Apply a 3×3 averaging filter to the middle 9 values of the matrix

$$\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ u & v & w & x & y \end{bmatrix}$$

and then apply another 3×3 averaging filter to the result.

Using your answer, describe a 5×5 filter which has the effect of two averaging filters.

Is this filter separable?

14. MATLAB also has an `imfilter` function, which if `x` is an image matrix (of any type), and `f` is a filter, has the syntax

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```
imfilter(x,f);
```

It differs from `filter2` in the different parameters it takes (read its help file), and in that the output is always of the same class as the original image.

(a) Use `imfilter` on the mandrill image with the filters listed in question 1.

(b) Apply different sized averaging filters to the mandrill image using `imfilter`.

(c) Apply different laplacian filters to the mandrill image using `imfilter`. Compare the results with those obtained with `filter2`. Which do you think gives the best results?

15. Display the difference between the `cmax` and `cmin` images obtained in section 5.7. You can do this with

```
>> imshow(imsubtract(cmax,cmin))
```

What are you seeing here? Can you account for the output of these commands?

16. Using the `tic` and `toc` timer function, compare the use of `nlfilter` and `colfilt` functions.

17. Use `colfilt` to implement the geometric mean and alpha-trimmed mean filters.

18. Can unsharp masking be used to reverse the effects of blurring? Apply an unsharp masking filter after a 3×3 averaging filter, and describe the result.

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Chapter 6

Image Geometry

There are many situations in which we might want to change the shape, size or orientation of an image. We may wish to enlarge an image, to fit into a particular space, or for printing; we may wish also do reduce its size, say for inclusion on a web page. We might also wish to rotate it: maybe to adjust for an incorrect camera angle, or simply for affect. Rotation and scaling are examples of *affine transformations*, where lines are transformed to lines, and in particular parallel lines remain parallel after the transformation. Non-affine geometrical transformations include warping, which we will not consider.

6.1 Interpolation of data

We will start with a simple problem: suppose we have a collection of 4 values, which we wish to enlarge to 8. How do we do this? To start, we have our points x_1, x_2, x_3 and x_4 which we suppose to be evenly spaced, and we have the values at those points: $f(x_1)$, $f(x_2)$, $f(x_3)$ and $f(x_4)$. Along the line $x_1 \ldots x_4$ we wish to space eight points $x'_1, x'_2, \ldots, x'_8$. Figure 6.1 shows how this would be done.

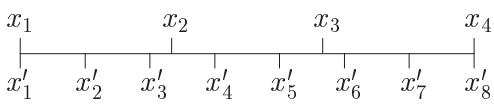

Figure 6.1: Replacing four points with eight

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Suppose that the distance between each of the x_i points is 1; thus the length of the line is 3. Thus, since there are seven increments from x'_i to x'_8, the distance between each two will be $3/7 \approx 0.4286$. To obtain a relationship between x and x' we draw figure 6.1 slightly differently as shown in figure 6.2. then

$$\begin{aligned} x' &= \frac{1}{3}(7x-4), \\ x &= \frac{1}{7}(3x'+4). \end{aligned}$$

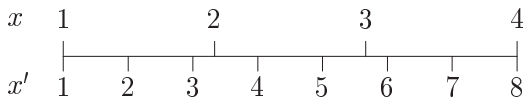

Figure 6.2: Figure 6.1 slightly redrawn

As you see from figure 6.1, none of the x'_i coincides exactly with an original x_j, except for the first and last. Thus we are going to have to "guess" at possible function values $f(x'_i)$. This guessing at function values is called *interpolation*. Figure 6.3 shows one way of doing this: we assign $f(x'_i) = f(x_j)$, where x_j is the original point closest to x'_i. This is called *nearest neighbour interpolation*.

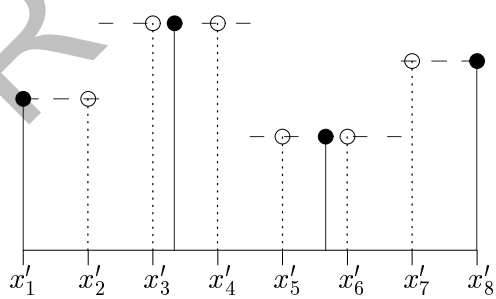

Figure 6.3: Nearest neighbour interpolation

The closed circles indicate the original function values $f(x_i)$; the open circles the interpolated values $f(x'_i)$.

Another way is to join the original function values by straight lines, and take our interpolated

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values as the values at those lines. Figure 6.4 shows this approach to interpolation; this is called *linear interpolation*.

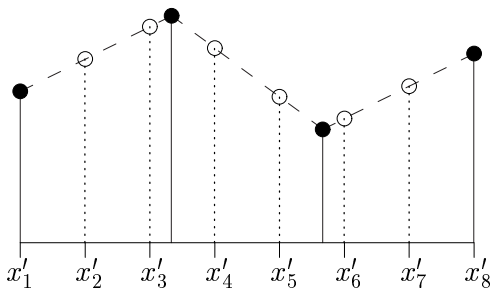

Figure 6.4: Linear interpolation

To calculate the values required for linear interpolation, consider the diagram shown in figure 6.5.

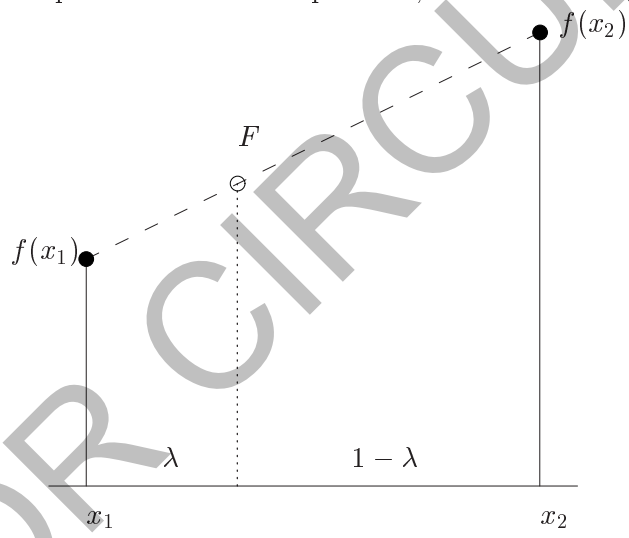

Figure 6.5: Calculating linearly interpolated values

In this figure we assume that $x_2 = x_1 + 1$, and that F is the value we require. By considering slopes:

$$\frac{F - f(x_1)}{\lambda} = \frac{f(x_2) - f(x_1)}{1}.$$

Solving this equation for F produces:

$$F = \lambda f(x_2) + (1 - \lambda) f(x_1). \tag{6.1}$$

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As an example of how to use this, suppose we have the values $f(x_1) = 2$, $f(x_2) = 3$, $f(x_3) = 1.5$ and $f(x_4) = 2.5$. Consider the point x'_4. This is between x_2 and x_3, and the corresponding value for λ is $2/7$. Thus

$$\begin{aligned} f(x'_4) &= \frac{2}{7}f(x_3) + \frac{5}{7}f(x_2) \\ &= \frac{2}{7}(1.5) + \frac{5}{7}(3) \\ &\approx 2.5714. \end{aligned}$$

For x'_7, we are between x_3 and x_4 with $\lambda = 4/7$. So:

$$\begin{aligned} f(x'_7) &= \frac{4}{7}f(x_4) + \frac{3}{7}f(x_3) \\ &= \frac{4}{7}(2.5) + \frac{3}{7}(1.5) \\ &\approx 2.0714. \end{aligned}$$

6.2 Image interpolation

The methods of the previous section can be applied to images. Figure 6.6 shows how a 4×4 image would be interpolated to produce an 8×8 image. Here the large open circles are the original points, and the smaller closed circles are the new points.

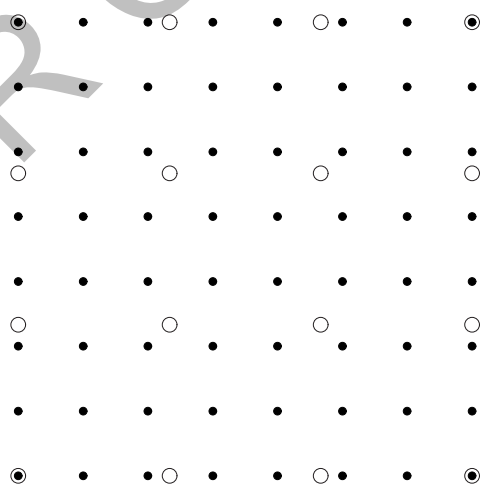

Figure 6.6: Interpolation on an image

To obtain function values for the interpolated points, consider the diagram shown in figure 6.7.

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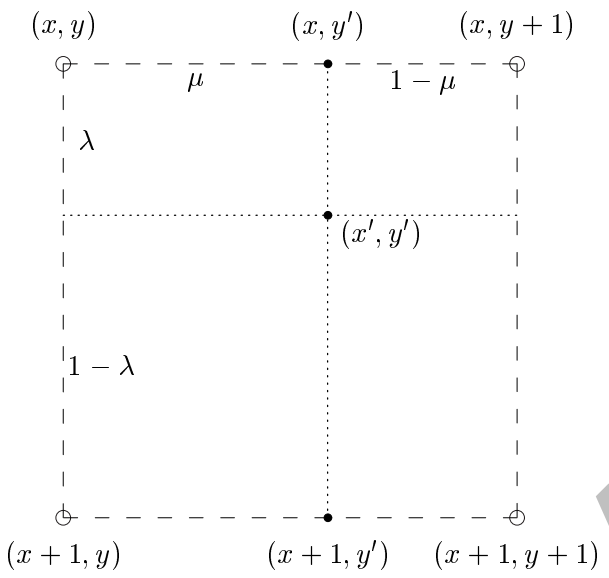

Figure 6.7: Interpolation between four image points

We can give a value to $f(x', y')$ by either of the methods above: by setting it equal to the function values of the closest image point, or by using linear interpolation. We can apply linear interpolation first along the top row to obtain a value for $f(x, y')$, and then along the bottom row to obtain a value for $f(x+1, y')$. Finally we can interpolate along the y' column between these new values to obtain $f(x', y')$. Using the formula given by 6.1, then

$$f(x, y') = \mu f(x, y+1) + (1-\mu) f(x, y)$$

and

$$f(x+1, y') = \mu f(x+1, y+1) + (1-\mu) f(x+1, y).$$

Along the y' column we have

$$f(x', y') = \lambda f(x+1, y') + (1-\lambda) f(x, y')$$

and substituting in the values just obtained produces

$$\begin{aligned} f(x', y') &= \lambda(\mu f(x+1, y+1) + (1-\mu) f(x+1, y)) + (1-\lambda)(\mu f(x, y+1) + (1-\mu) f(x, y)) \\ &= \lambda\mu f(x+1, y+1) + \lambda(1-\mu) f(x+1, y) + (1-\lambda)\mu f(x, y+1) \\ &\quad + (1-\lambda)(1-\mu) f(x, y) \end{aligned}$$

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This last equation is the formula for *bilinear interpolation*.

Now image scaling can be performed easily. Given our image, and either a scaling factor (or separate scaling factors for x and y directions), or a size to be scaled to, we first create an array of the required size. In our example above, we had a 4×4 image, given as an array (x, y), and a scale factor of two, resulting in an array (x', y') of size 8×8. Going right back to figures 6.1 and 6.2, the relationship between (x, y) and (x', y') is

$$\begin{aligned}(x', y') &= \left(\frac{1}{3}(7x-4), \frac{1}{3}(7y-4)\right), \\ (x, y) &= \left(\frac{1}{7}(3x'+4), \frac{1}{7}(3y'+4)\right).\end{aligned}$$

Given our (x', y') array, we can step through it point by point, and from the corresponding surrounding points from the (x, y) array calculate an interpolated value using either nearest neighbour or bilinear interpolation.

There's nothing in the above theory that requires the scaling factor to be greater than one. We can choose a scaling factor *less* than one, in which case the resulting image array will be *smaller* than the original. We can consider figure 6.6 in this light: the small closed circles are the *original* image points, and the large open circles the smaller array on which we are to find interpolated values.

MATLAB has the function `imresize` which does all this for us. It can be called with

```
resize(A,k,'method')
```

where `A` is an image of any type, `k` is a scaling factor, and `'method'` is either `'nearest'` or `'bilinear'` (or another method to be described later). Another way of using `imresize` is

```
resize(A,[m,n],'method')
```

where `[m,n]` provide the size of the scaled output. There is a further, optional parameter allowing you to choose either the size, or type of low pass filter to be applied to the image before reducing its size—see the help file for details.

Let's try a few examples. We shall start by taking the head of the cameraman, and enlarging it by a factor of four:

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```
>> c=imread('cameraman.tif');
>> head=c(33:96,90:153);
>> imshow(head)
>> head4n=imresize(head,4,'nearest');imshow(head4n)
>> head4b=imresize(head,4,'bilinear');imshow(head4b)
```

The head is shown in figure 6.8 and the results of the scaling are shown in figure 6.9.

Figure 6.8: The head

(a) Nearest neighbour scaling (b) Bilinear interpolation

Figure 6.9: Scaling by interpolation

Nearest neighbour interpolation gives an unacceptable blocky effect; edges in particular appear very jagged. Bilinear interpolation is much smoother, but the trade-off here is a certain blurriness to the result. This is unavoidable: interpolation can't predict values: we can't create data from nothing! All we can do is to guess at values which fit best with the original data.

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6.3 General interpolation

Although we have presented nearest neighbour and bilinear interpolation as two different methods, they are in fact two special cases of a more general approach. The idea is this: we wish to interpolate a value $f(x')$ for $x_1 \leq x' \leq x_2$, and suppose $x' - x_1 = \lambda$. We define an interpolation function $R(u)$, and set

$$f(x') = R(-\lambda)f(x_1) + R(1-\lambda)f(x_2). \tag{6.2}$$

Figure 6.10 shows how this works. The function $R(u)$ is centred at x', so x_1 corresponds with

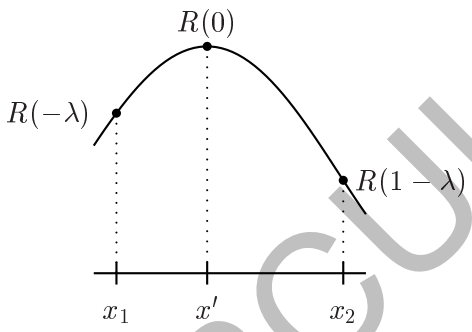

Figure 6.10: Using a general interpolation function

$u = -\lambda$, and x_2 with $u = 1 - \lambda$. Now consider the two functions $R_0(u)$ and $R_1(u)$ shown in figure 6.11. Both these functions are defined on the interval $-1 \leq u \leq 1$ only. Their formal

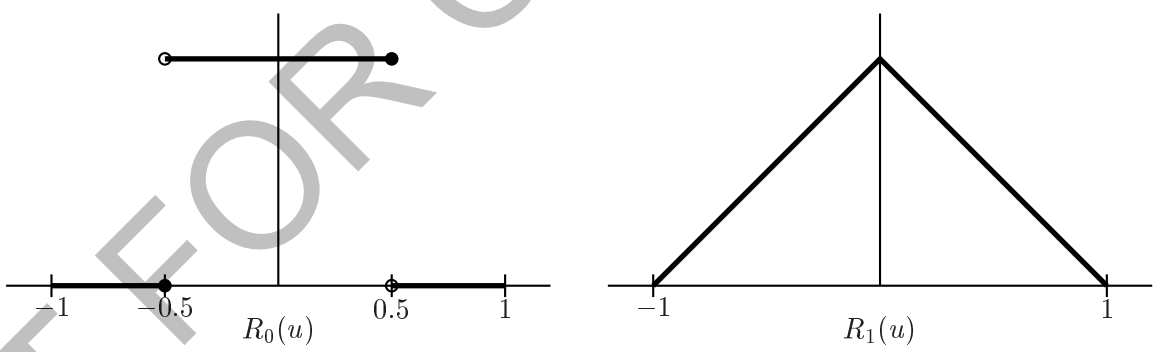

Figure 6.11: Two interpolation functions

definitions are:

$$R_0(u) = \begin{cases} 0 & \text{if } u \leq -0.5 \\ 1 & \text{if } -0.5 < u \leq 0.5 \\ 0 & \text{if } u > 0.5 \end{cases}$$

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and

$$R_1(u) = \begin{cases} 1+u & \text{if } u \leq 0 \\ 1-u & \text{if } u \geq 0 \end{cases}$$

The function $R_1(u)$ can also be written as $1-|x|$. Now substituting $R_0(u)$ for $R(u)$ in equation 6.2 will produce nearest neighbour interpolation. To see this, consider the two cases $\lambda < 0.5$ and $\lambda \geq 0.5$ separately. If $\lambda < 0.5$ then $R_0(-\lambda) = 1$ and $R_0(1-\lambda) = 0$. Then

$$f(x') = (1)f(x_1) + (0)f(x_2) = f(x_1).$$

If $\lambda \geq 0.5$ then $R_0(-\lambda) = 0$ and $R_0(1-\lambda) = 1$. Then

$$f(x') = (0)f(x_1) + (1)f(x_2) = f(x_2).$$

In each case $f(x')$ is set to the function value of the point closest to x'.

Similarly, substituting $R_1(u)$ for $R(u)$ in equation 6.2 will produce linear interpolation. We have

$$\begin{aligned} f(x') &= R_1(-\lambda)f(x_1) + R_1(1-\lambda)f(x_2) \\ &= (1-\lambda)f(x_1) + \lambda f(x_2) \end{aligned}$$

which is the correct equation.

The functions $R_0(u)$ and $R_1(u)$ are just two members of a family of possible interpolation functions. Another such function provides *cubic interpolation*; its definition is:

$$R_3(u) = \begin{cases} 1.5|u|^3 - 2.5|u|^2 + 1 & \text{if } |u| \leq 1, \\ -0.5|u|^3 + 2.5|u|^2 - 4|u| + 2 & \text{if } 1 < |u| \leq 2. \end{cases}$$

Its graph is shown in figure 6.12. This function is defined over the interval $-2 \leq u \leq 2$, and its use is slightly different from that of $R_0(u)$ and $R_1(u)$, in that as well as using the function values $f(x_1)$ and $f(x_2)$ for x_1 and x_2 on either side of x', we use values of x further away. In fact the formula we use, which extends equation 6.2, is:

$$f(x') = R_3(-1-\lambda)f(x_1) + R_3(-\lambda)f(x_2) + R_3(1-\lambda)f(x_3) + R_4(2-\lambda)f(x_4)$$

where x' is between x_2 and x_3, and $x - x_2 = \lambda$. Figure 6.13 illustrates this. To apply this interpolation to images, we use the sixteen known values around our point (x', y'). As for bilinear

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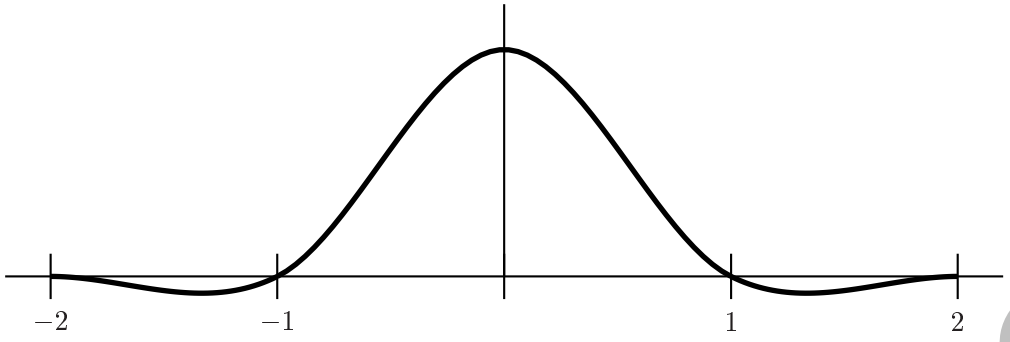

Figure 6.12: The cubic interpolation function $R_3(u)$

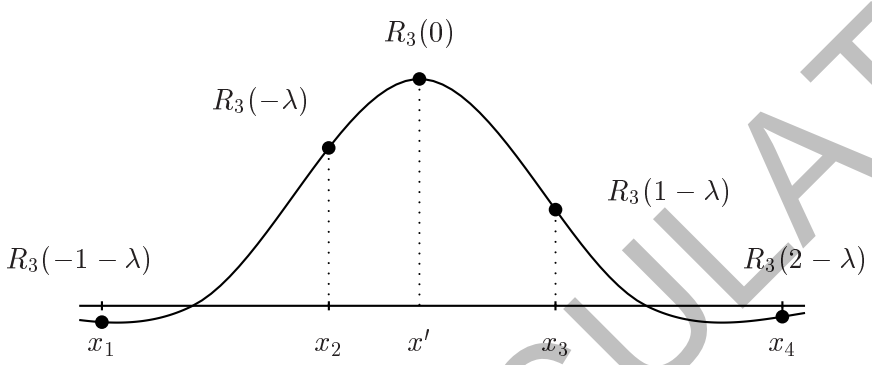

Figure 6.13: Using $R_3(u)$ for interpolation

interpolation, we first interpolate along the rows, and then finally down the columns, as shown in figure 6.14. Alternately we could first interpolate down the columns, and then across the row. This means of image interpolation by applying cubic interpolation in both directions is called *bicubic interpolation*. To perform bicubic interpolation on an image with MATLAB, we use the `'bicubic'`

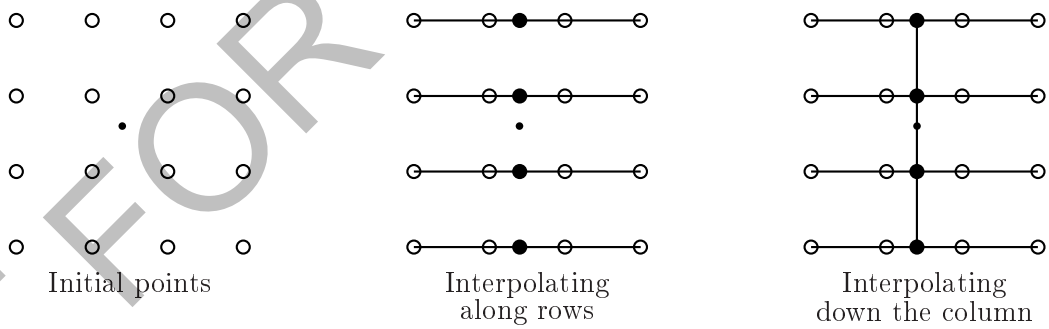

Figure 6.14: How to apply bicubic interpolation

method of the `imresize` function. To enlarge the cameraman's head, we enter:

```
>> head4c=imresize(head,4,'bicubic');imshow(head4c)
```

and the result is shown in figure 6.15.

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Figure 6.15: Enlargement using bicubic interpolation

6.4 Enlargement by spatial filtering

If we merely wish to enlarge an image by a power of 2, there is a quick and dirty method which uses linear filtering. We give an example. Suppose we take a simple 4×4 matrix:

```
>> m=magic(4)


m =


    16     2     3    13
     5    11    10     8
     9     7     6    12
     4    14    15     1
```

Our first step is to create a *zero-interleaved* version of this matrix. This is obtained by interleaving rows and columns of zeros between the rows and columns of the original matrix. Such a matrix will be double the size of the original, and will contain mostly zeros. If m_2 is the zero-interleaved version of m, then it is defined by:

$$m_2(i,j) = \begin{cases} m((i+1)/2, (j+1)/2) & \text{if } i \text{ and } j \text{ are both odd,} \\ 0 & \text{otherwise.} \end{cases}$$

This can be implemented with simple function; one such is shown in figure 6.16. Applied to our

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```
function out=zeroint(a)
%
% ZEROINT(A) produces a zero-interleaved version of the matrix A.
% For example:
%
%   a=[1 2 3;4 5 6];
%   zeroint(a)
%
%     1    0    2    0    3
%     0    0    0    0    0
%     4    0    5    0    6
%
[m,n]=size(a);
a2=reshape([a;zeros(m,n)],m,2*n);
out=reshape([a2';zeros(2*n,m)],2*n,2*m)';
```

Figure 6.16: A simple function for implementing zero-interleaving

matrix from above:

```
>> m2=zeroint(m)


m2 =


    16     0     2     0     3     0    13     0

     0     0     0     0     0     0     0     0

     5     0    11     0    10     0     8     0

     0     0     0     0     0     0     0     0

     9     0     7     0     6     0    12     0

     0     0     0     0     0     0     0     0

     4     0    14     0    15     0     1     0

     0     0     0     0     0     0     0     0
```

We can now replace the zeros by applying a spatial filter to this matrix. The spatial filters

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

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implement nearest neighbour interpolation and bilinear interpolation respectively. We can test this with a few commands:

```
>> filter2([1 1 0;1 1 0;0 0 0],m2)

ans =

    16    16     2     2     3     3    13    13
    16    16     2     2     3     3    13    13
     5     5    11    11    10    10     8     8
     5     5    11    11    10    10     8     8
     9     9     7     7     6     6    12    12
     9     9     7     7     6     6    12    12
     4     4    14    14    15    15     1     1
     4     4    14    14    15    15     1     1

>> filter2([1 2 1;2 4 2;1 2 1]/4,m2)

ans =

   16.0000    9.0000    2.0000    2.5000    3.0000    8.0000   13.0000    6.5000
   10.5000    8.5000    6.5000    6.5000    6.5000    8.5000   10.5000    5.2500
    5.0000    8.0000   11.0000   10.5000   10.0000    9.0000    8.0000    4.0000
    7.0000    8.0000    9.0000    8.5000    8.0000    9.0000   10.0000    5.0000
    9.0000    8.0000    7.0000    6.5000    6.0000    9.0000   12.0000    6.0000
    6.5000    8.5000   10.5000   10.5000   10.5000    8.5000    6.5000    3.2500
    4.0000    9.0000   14.0000   14.5000   15.0000    8.0000    1.0000    0.5000
    2.0000    4.5000    7.0000    7.2500    7.5000    4.0000    0.5000    0.2500
```

We can check these with the commands

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```
>> m2b=imresize(m,[8,8],'nearest');m2b


>> m2b=imresize(m,[7,7],'bilinear');m2b
```

In the second command we only scaled up to 7×7, to ensure that the interpolation points lie exactly half-way between the original data values. The filter

$$\frac{1}{64}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

can be used to approximate bicubic interpolation.

We can try all of these with the cameraman's head, doubling its size.

```
>> imshow(hz)
>> imshow(filter2([1 1 0;1 1 0;0 0 0],hz)/255)
>> imshow(filter2([1 2 1;2 4 2;1 2 1]/4,hz)/255)
>> bfilt=[1 4 6 4 1;4 16 24 16 4;6 24 36 24 6;4 16 24 16 4;1 4 6 4 1]/64;
>> imshow(filter2(bfilt,hz)/255)
```

and the results are shown in figure 6.17. We can enlarge more by simply taking the result of the

Zero interleaving Nearest neighbour Bilinear Bicubic

Figure 6.17: Enlargement by spatial filtering

filter, applying a zero-interleave to it, and then another filter.

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6.5 Scaling smaller

The business of making an image smaller is also called *image minimization*; one way is to take alternate pixels. If we wished to produce an image one sixteenth the size of the original we would take out only those pixels (i, j) for which i and j are both multiples of four. This method is called image *subsampling* and corresponds to the `nearest` option of `imresize`, and is very easy to implement.

However, it does not give very good results at high frequency components of an image. We shall give a simple example; we shall construct a large image consisting of a white square with a single circle on it.

```
>> t=zeros(1024,1024);
>> t=((255.5)^2<(i-512).^2+(j-512).^2) & ((i-512).^2+(j-512).^2<(256.5)^2);
>> t=~t;
```

Now we can resize it by taking out most pixels:

```
>> tr=imresize(t,0.25);
```

and this is shown in figure 6.18(a). Notice that because of the way that pixels were removed, the resulting circle contains gaps. If we were to use one of the other methods:

```
>> trc=imresize(t,0.25,'bicubic');
```

a low pass filter is applied to the image first. The result is shown in figure 6.18(b). The image in figure 6.18(b) can be made binary by thresholding (which will be discussed in greater detail in chapter 9). In this case

```
>> trc=imresize(t,0.25,'bicubic')>0.9;
```

does the trick.

6.6 Rotation

Having done the hard work of interpolation for scaling, we can easily apply the same theory to image rotation. First recall that the mapping of a point (x, y) to another (x', y') through a counter-

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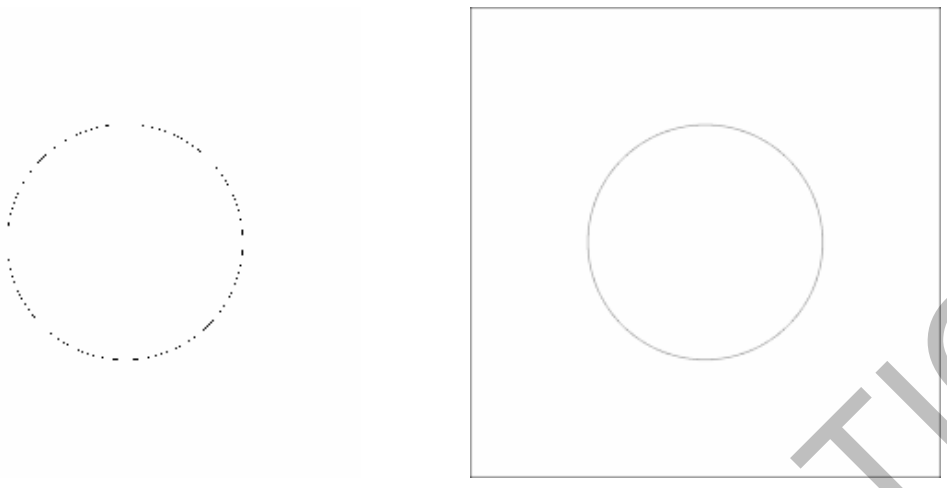

(a) Nearest neighbour minimization (b) Bicubic interpolation for minimization

Figure 6.18: Minimization

clockwise rotation of θ as shown in figure 6.19 is obtained by the matrix product

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

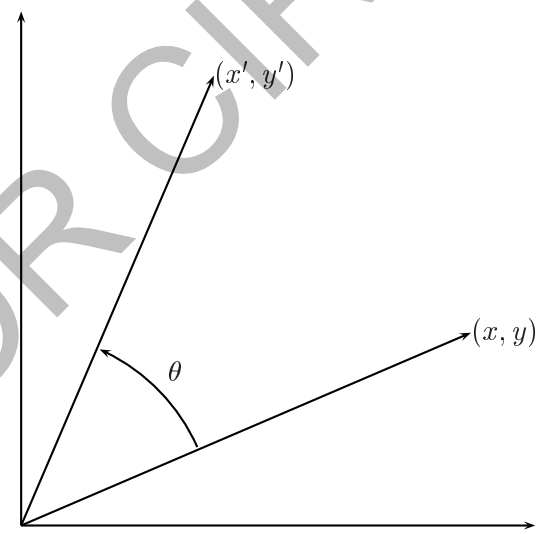

Figure 6.19: Rotating a point through angle θ

Similarly, since the matrix involved is orthogonal (its inverse is equal to its transpose), we have:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

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Now we can rotate an image by considering it as a large collection of points. Figure 6.20 illustrates the idea. In this figure the dark circles indicate the original position; the light points

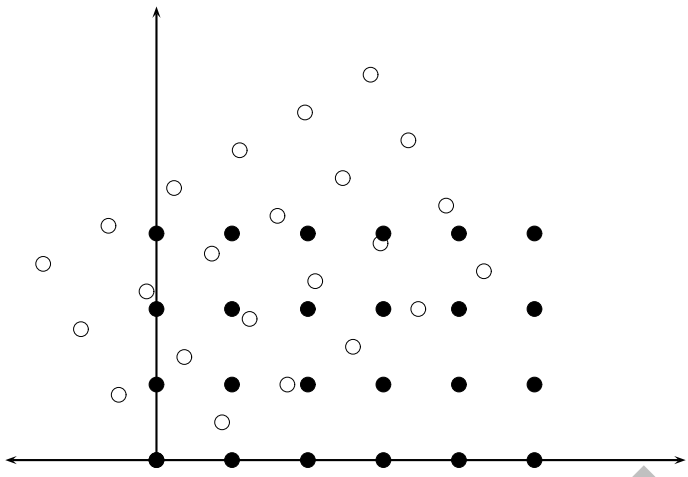

Figure 6.20: Rotating a rectangle

their positions after rotation. However, this approach won't work for images. Since an image grid can be considered as pixels forming a subset of the cartesian (integer valued) grid, we must ensure that even after rotation, the points remain in that grid. To do this we consider a rectangle which includes the rotated image, as shown in figure 6.21. Now consider all integer-valued points (x', y') in

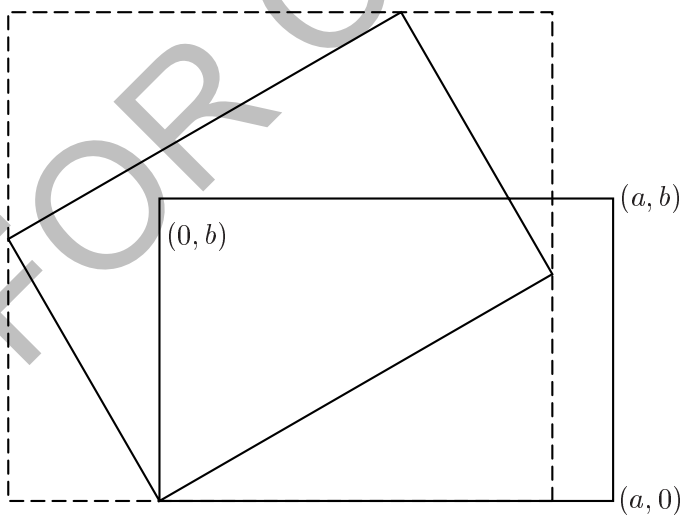

Figure 6.21: A rectangle surrounding a rotated image

the dashed rectangle. A point will be in the image if, when rotated back, it lies within the original

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image limits. That is, if

$$\begin{array}{rcrcl} 0 & \leq & x'\cos\theta + y'\sin\theta & \leq & a \\ 0 & \leq & -x'\sin\theta + y'\cos\theta & \leq & b \end{array}$$

If we consider the array of 6×4 points shown in figure 6.20, then the points after rotation by $30°$ are shown in figure 6.22 This gives us the position of the pixels in our rotated image, but what about

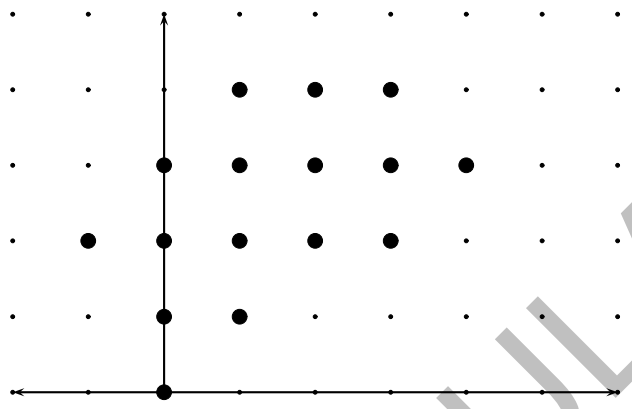

Figure 6.22: The points on a grid after rotation

their value? Take a point (x', y') in the rotated image, and rotate it back into the original image to produce a point (x'', y''), as shown in figure 6.23. Now the grey value at (x'', y'') can be found

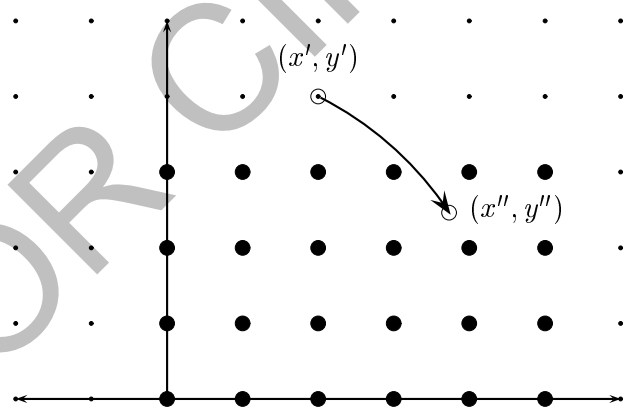

Figure 6.23: Rotating a point back into the original image

by interpolation; using surrounding grey values. This value is then the grey value for the pixel at (x', y') in the rotated image.

Image rotation in MATLAB is obtained using the command `imrotate`; it has the syntax

```
rotate(image,angle,'method')
```

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where `method`, as with `imresize`, can be any one of `nearest`, `bilinear` or `bicubic`. Also as with `imresize`, the `method` parameter may be omitted, in which case nearest neighbour interpolation is used.

For an example, let's take our old friend the cameraman, and rotate him by 60°. We shall do this twice; once with nearest neighbour interpolation, and once using bicubic interpolation.

```
>> cr=imrotate(c,60);
>> imshow(cr)
>> crc=imrotate(c,60,'bicubic');
>> imshow(crc)
```

The results are shown in figure 6.24. There's not a great deal of observable difference between the

(a) Nearest neighbour (b) Bicubic interpolation

Figure 6.24: Rotation with interpolation

two images; however nearest neighbour interpolation produces slightly more jagged edges.

Notice that for angles which are integer multiples of 90° image rotation can be accomplished far more efficiently with simple matrix transposition and reversing of the order of rows and columns. Using the commands

```
ipud
```

for flipping a matrix in the up/down direction, and

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```
iplr
```

for flipping a matrix in the left/right direction, we have the following commands for rotations of 90° multiples:

90°	`flipud(c');`
180°	`fliplr(flipud(c));`
270°	`fliplr(c');`

In fact `imrotate` uses these simpler commands for these particular angles.

6.7 An example: anamorphosis

Anamorphosis refers to the deliberate stretching or distorting the shape of an object for artistic or dramatic effect. It was popularly used by painters in the 16th and 17th centuries. Figure 6.25 shows a painting by Hans Holbein called "The Ambassadors".The odd shape at the bottom of the painting is in fact an anamorphic skull; this can only be seen properly when the painting is viewed from a certain angle. In MATLAB:

```
>> a=imread('AMBASSADORS.JPG');
>> a=rgb2gray(a);
```

By a judicious use of `pixval` we can isolate the skull:

```
>> skull=a(566:743,157:586);
```

and this is shown in figure 6.26. To "undo" the effects of the anamorphosis, we need to rotate the skull slightly clockwise, and then stretch it vertically. This can be done with one command:

```
>> skull2=imresize(imrotate(skull,-22,'bicubic'),[500,150],'bicubic');
```

The angle of rotation and size of the scaling were determined using trial and error. After much fiddling and experimentation, these seem to produce the best output.

Finally, we shall view the portion of `skull2` which contains the corrected skull:

```
>> imshow(skull2(200:350,:))
```

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Figure 6.25: "The Ambassadors" (1533) by Hans Holbein

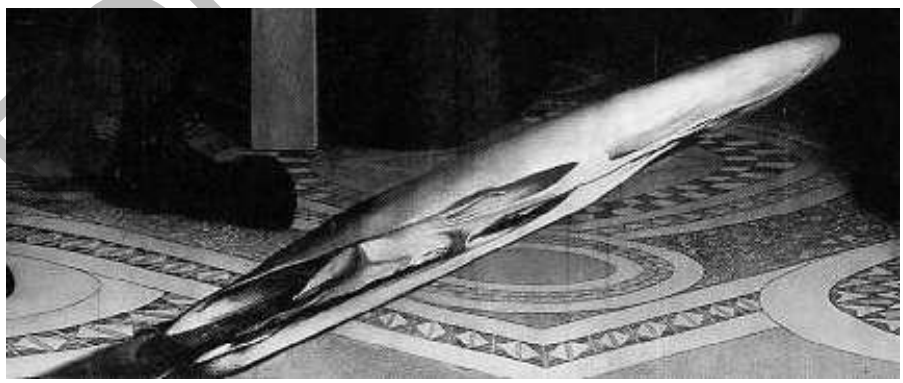

Figure 6.26: The skull alone

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and this is shown in figure 6.27.

Figure 6.27: The corrected skull

Exercises

1. By hand, enlarge the list

```
1 4 7 4 3 6
```

to lengths

(a) 9, (b) 11, (c) 2

by

(a) nearest neighbour interpolation,

(b) linear interpolation.

Check your answers with MATLAB.

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2. By hand, enlarge the matrix

$$\begin{bmatrix} 8 & 6 & 13 & 9 \\ 1 & 13 & 1 & 15 \\ 5 & 4 & 7 & 7 \\ 5 & 10 & 3 & 7 \end{bmatrix}$$

to sizes

(i) 7×7, (ii) 8×8, (iii) 10×10

by

(a) nearest neighbour interpolation,

(b) bilinear interpolation.

Check your answers with MATLAB.

3. Use zero-interleaving and spatial filtering to enlarge the cameraman's heads by a factor of four in each dimension, using the three filters given. Use the following sequence of commands:

```
>> head2=zeroint(head);
>> head2n=filter2(filt,head2);
>> head4=zeroint(head2n);
>> head4n=filter2(filt,head4);
>> imshow(head4n/255)
```

where `filt` is a filter. Compare your results to those given by `imresize`. Are there any observable differences?

4. Take another small part of an image: say the head of the girl in `pout.tif`. This can be obtained with:

```
>> p=imread('pout.tif');
>> ph=histeq(p);
>> head=ph(10:129,60:179);
```

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Enlarge the head to four times as big using both `imresize` with the different parameters, and the zero-interleave method with the different filters.

As above, compare the results.

5. Suppose an image is enlarged by some amount k, and the result decreased by the same amount. Should this result be exactly the same as the original? If not, why not?

6. What happens if the image is decreased first, and the result enlarged?

7. Create an image consisting of a white square with a black background. Rotate the image by 30° and 45°. Use (a) `imrotate` with the `nearest` option, and (b) `imrotate` with `bilinear`. Compare the results.

8. For the rotated squares in the previous question, rotate back to the original orientation. How close is the result to the original square?

9. In general, suppose an image is rotated, and then the result rotated back. Should this result be exactly the same as the original? If not, why not?

10. Write a MATLAB function to implement image enlargement using zero-interleaving and spatial filtering. The function should have the syntax

    `imenlarge(image,n,filt)`

    where `n` is the number of times the interleaving is to be done, and `filt` is the filter to use. So for example, the command

    ```
    >> imenlarge(head,2,bfilt);
    ```

    would enlarge an image to four times its size, using the 5×5 filter described in section 6.4.

11. Consider the image `ic.tif`. By trial and error, find an angle by which the image can be rotated so that its lines are all horizontal and vertical.

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Chapter 7

The Fourier Transform

7.1 Introduction

The Fourier Transform is of fundamental importance to image processing. It allows us to perform tasks which would be impossible to perform any other way; its efficiency allows us to perform other tasks more quickly. The Fourier Transform provides, among other things, a powerful alternative to linear spatial filtering; it is more efficient to use the Fourier transform than a spatial filter for a large filter. The Fourier Transform also allows us to isolate and process particular image "frequencies", and so perform low-pass and high-pass filtering with a great degree of precision.

Before we discuss the Fourier transform of images, we shall investigate the one-dimensional Fourier transform, and a few of its properties.

7.2 Background

Our starting place is the observation that a periodic function may be written as the sum of sines and cosines of varying amplitudes and frequencies. For example, in figure 7.1 we plot a function, and its decomposition into sine functions.

Some functions will require only a finite number of functions in their decomposition; others will require an infinite number. For example, a "square wave", such as is shown in figure 7.2, has the decomposition

$$f(x) = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \frac{1}{9}\sin 9x + \cdots \tag{7.1}$$

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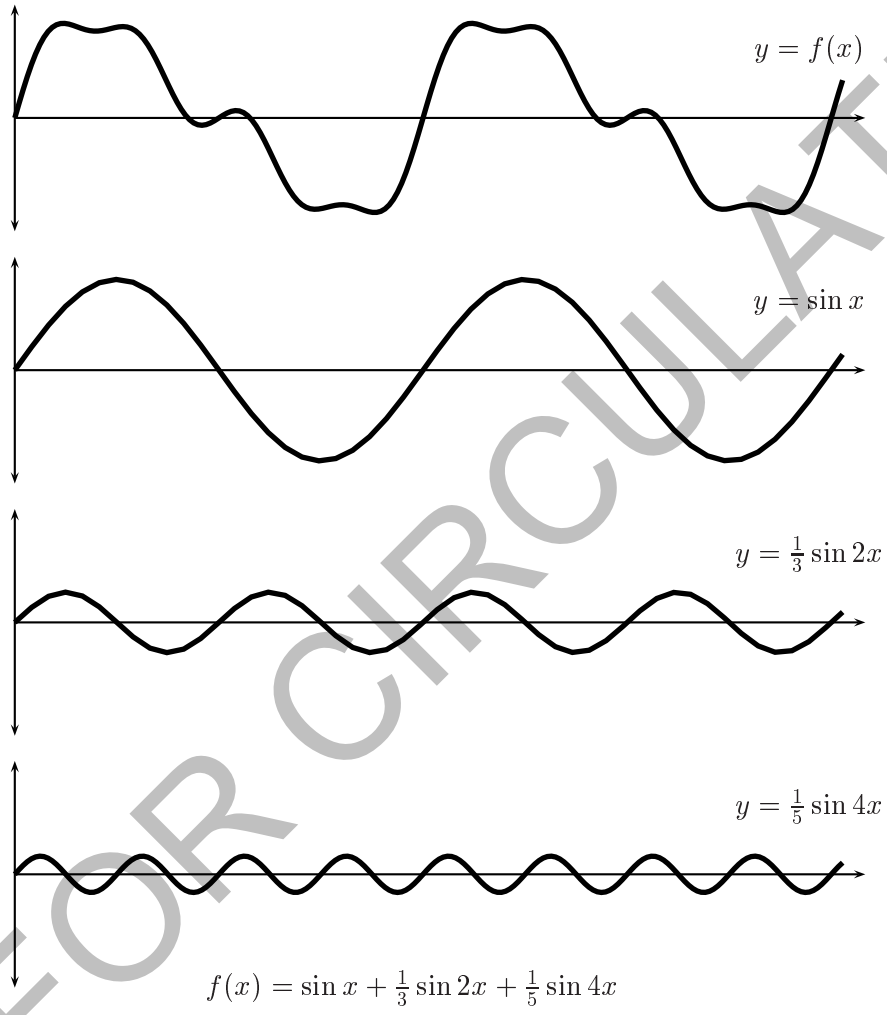

$$f(x) = \sin x + \tfrac{1}{3}\sin 2x + \tfrac{1}{5}\sin 4x$$

Figure 7.1: A function and its trigonometric decomposition

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In figure 7.2 we take the first four terms only to provide the approximation. The more terms of the series we take, the closer the sum will approach the original function. This can be formalized; if

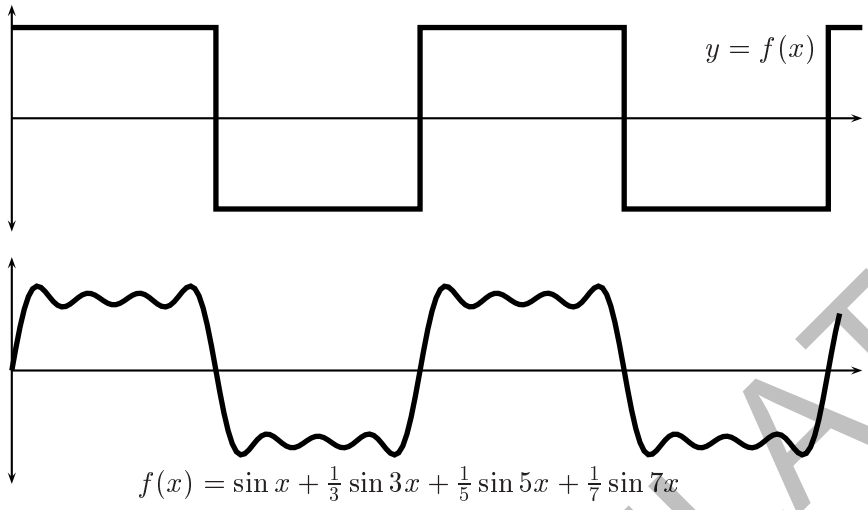

Figure 7.2: A square wave and its trigonometric approximation

$f(x)$ is a function of period $2T$, then we can write

$$f(x) = a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\Big)$$

where

$$\begin{aligned} a_0 &= \frac{1}{2T}\int_{-T}^{T} f(x)\,dx \\ a_n &= \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}\,dx, \quad n = 1, 2, 3, \ldots \\ b_n &= \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\,dx, \quad n = 1, 2, 3, \ldots \end{aligned}$$

These are the equations for the *Fourier series expansion* of $f(x)$, and they can be expressed in complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\Big(\frac{in\pi x}{T}\Big)dx$$

where

$$c_n = \frac{1}{2T}\int_{-T}^{T} f(x)\exp\Big(\frac{-in\pi x}{T}\Big)dx.$$

If the function is non-periodic, we can obtain similar results by letting $T \to \infty$, in which case

$$f(x) = \int_0^{\infty} [a(\omega)\cos\omega x + b(\omega)\sin\omega x]d\omega$$

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where

$$\begin{aligned} a(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx, \\ b(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx. \end{aligned}$$

These equations can be written again in complex form:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega, \\ F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx. \end{aligned}$$

In this last form the functions $f(x)$ and $F(\omega)$ form a *Fourier transform pair*. Further details can be found, for example, in James [16].

7.3 The one-dimensional discrete Fourier transform

When we deal with a *discrete* function, as we shall do for images, the situation from the previous section changes slightly. Since we only have to obtain a finite number of values, we only need a finite number of functions to do it.

Consider for example the discrete sequence

$$1, \quad 1, \quad 1, \quad 1, \quad -1, \quad -1, \quad -1, \quad -1$$

which we may take as a discrete approximation to the square wave of figure 7.2. This can be expressed as the sum of only *two* sine functions; this is shown in figure 7.3. We shall see below how to obtain those sequences.

The Fourier transform allows us to obtain those individual sine waves which compose a given function or sequence. Since we shall be concerned with discrete sequences, and of course images, we shall investigate only the *discrete Fourier transform*, abbreviated DFT.

7.3.1 Definition of the one dimensional DFT

Suppose

$$\mathbf{f} = [f_0, f_1, f_2, \ldots, f_{N-1}]$$

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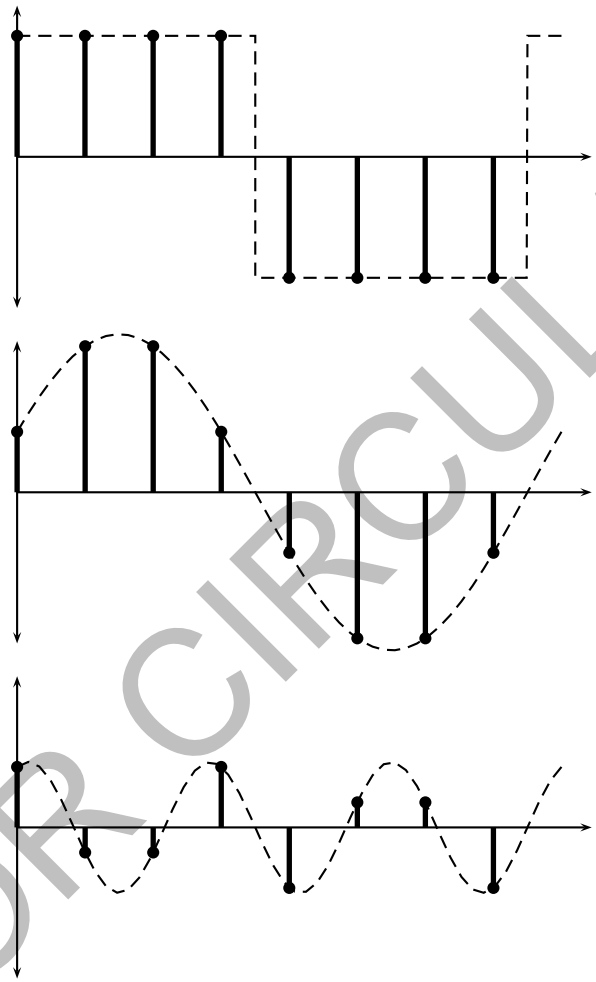

Figure 7.3: Expressing a discrete function as the sum of sines

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is a sequence of length N. We define its *discrete Fourier transform* to be the sequence

$$\mathbf{F} = [F_0, F_1, F_2, \ldots, F_{N-1}]$$

where

$$F_u = \frac{1}{N} \sum_{x=0}^{N-1} \exp\left[-2\pi i \frac{xu}{N}\right] f_x. \tag{7.2}$$

Note the similarity between this equation and the equations for the Fourier series expansion discussed in the previous section. Instead of an integral, we now have a finite sum. This definition can be expressed as a matrix multiplication:

$$F = \mathcal{F} f$$

where $\mathcal{F}$ is an $N \times N$ matrix defined by

$$\mathcal{F}_{m,n} = \frac{1}{N} \exp\left[-2\pi i \frac{mn}{N}\right].$$

Given N, we shall define

$$\omega = \exp\left[\frac{-2\pi i}{N}\right]$$

so that

$$\mathcal{F}_{m,n} = \frac{1}{N} \omega^{mn}.$$

Then we can write

$$\mathcal{F} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \cdots & \omega^{3(N-1)} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \cdots & \omega^{4(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \omega^{4(N-1)} & \cdots & \omega^{(N-1)^2} \end{bmatrix}.$$

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Example. Suppose $\mathbf{f} = [1, 2, 3, 4]$ so that $N = 4$. Then

$$\begin{aligned}
\omega &= \exp\left[\frac{-2\pi i}{4}\right] \\
&= \exp\left[-\frac{\pi i}{2}\right] \\
&= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) \\
&= -i.
\end{aligned}$$

Then we have

$$\mathcal{F} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

and so

$$\mathbf{F} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 10 \\ -2+2i \\ -2 \\ -2-2i \end{bmatrix}.$$

The inverse DFT

The formula for the inverse DFT is very similar to the forward transform:

$$x_u = \sum_{x=0}^{N-1} \exp\left[2\pi i \frac{xu}{N}\right] F_u. \tag{7.3}$$

If you compare this equation with equation 7.2 you will see that there are really only two differences:

1. there is no scaling factor $1/N$,

2. the sign inside the exponential function has been changed to positive.

As with the forward transform, we can express this as a matrix product:

$$f = \mathcal{F}^{-1} F$$

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with

$$\mathcal{F}^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \overline{\omega}^1 & \overline{\omega}^2 & \overline{\omega}^3 & \overline{\omega}^4 & \cdots & \overline{\omega}^{N-1} \\ 1 & \overline{\omega}^2 & \overline{\omega}^4 & \overline{\omega}^6 & \overline{\omega}^8 & \cdots & \overline{\omega}^{2(N-1)} \\ 1 & \overline{\omega}^3 & \overline{\omega}^6 & \overline{\omega}^9 & \overline{\omega}^{12} & \cdots & \overline{\omega}^{3(N-1)} \\ 1 & \overline{\omega}^4 & \overline{\omega}^8 & \overline{\omega}^{12} & \overline{\omega}^{16} & \cdots & \overline{\omega}^{4(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{\omega}^{N-1} & \overline{\omega}^{2(N-1)} & \overline{\omega}^{3(N-1)} & \overline{\omega}^{4(N-1)} & \cdots & \overline{\omega}^{(N-1)^2} \end{bmatrix}$$

where

$$\overline{\omega} = \frac{1}{\omega} = \exp\left[\frac{2\pi i}{N}\right].$$

In MATLAB, we can calculate the forward and inverse transforms with `fft` and `ifft`. Here `fft` stands for *Fast Fourier Transform*, which is a fast and efficient method of performing the DFT (see below for details). For example:

```
a =


     1     2     3     4     5     6


>> fft(a')


ans =


  21.0000
  -3.0000 + 5.1962i
  -3.0000 + 1.7321i
  -3.0000
  -3.0000 - 1.7321i
  -3.0000 - 5.1962i
```

We note that to apply a DFT to a single vector in MATLAB, we should use a column vector.

NOT FOR CIRCULATION

7.4 Properties of the one-dimensional DFT

The one-dimensional DFT satisfies many useful and important properties. We will investigate some of them here. A more complete list can be found in, for example [15].

Linearity. This is a direct consequence of the definition of the DFT as a matrix product. Suppose $\mathtt{f}$ and $\mathtt{g}$ are two vectors of equal length, and p and q are scalars, with $\mathtt{h} = p\mathtt{f} + q\mathtt{g}$. If $\mathtt{F}$, $\mathtt{G}$ and $\mathtt{H}$ are the DFT's of $\mathtt{f}$, $\mathtt{g}$ and $\mathtt{h}$ respectively, we have

$$\mathtt{H} = p\mathtt{F} + q\mathtt{G}.$$

This follows from the definitions of

$$\mathtt{F} = \mathcal{F}\mathtt{f}, \quad \mathtt{G} = \mathcal{F}\mathtt{g}, \quad \mathtt{H} = \mathcal{F}\mathtt{h}$$

and the linearity of the matrix product.

Shifting. Suppose we multiply each element x_n of a vector $\mathtt{x}$ by $(-1)^n$. In other words, we change the sign of every second element. Let the resulting vector be denoted $\mathtt{x}'$. The the DFT $\mathtt{X}'$ of $\mathtt{x}'$ is equal to the DFT $\mathtt{X}$ of $\mathtt{x}$ with the swapping of the left and right halves.

Let's do a quick MATLAB example:

```
>> x = [2 3 4 5 6 7 8 1];


>> x1=(-1).^[0:7].*x


x1 =


     2    -3     4    -5     6    -7     8    -1


>> X=fft(x')


  36.0000

  -9.6569 + 4.0000i
```

NOT FOR CIRCULATION

```
  -4.0000 - 4.0000i

   1.6569 - 4.0000i

   4.0000

   1.6569 + 4.0000i

  -4.0000 + 4.0000i

  -9.6569 - 4.0000i


>> X1=fft(x1')


X1 =


   4.0000

   1.6569 + 4.0000i

  -4.0000 + 4.0000i

  -9.6569 - 4.0000i

  36.0000

  -9.6569 + 4.0000i

  -4.0000 - 4.0000i

   1.6569 - 4.0000i
```

Notice that the first four elements of `X` are the last four elements of `X1`, and vice versa.

Conjugate symmetry If `x` is real, and of length N, then its DFT `X` satisfies the condition that

$$X_k = \overline{X_{N-k}},$$

where $\overline{X_{N-k}}$ is the complex conjugate of X_{N-k}, for all $k = 1, 2, 3, \ldots, N-1$. So in our example of length 8, we have

$$X_1 = \overline{X_7}, \quad X_2 = \overline{X_6}, \quad X_3 = \overline{X_5}.$$

In this case we also have $X_4 = \overline{X_4}$, which means X_4 must be real. In fact if N is even, then $X_{N/2}$ will be real. Examples can be seen above.

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Convolution. Suppose `x` and `y` are two vector of the same length N. Then we define their *convolution* (or more properly, their *circular convolution*) to be the vector

$$z = x * y$$

where

$$z_k = \frac{1}{n}\sum_{n=0}^{N-1} x_n y_{k-n}.$$

For example, if $N = 4$, then

$$\begin{aligned}
z_0 &= \frac{1}{4}(x_0y_0 + x_1y_{-1} + x_2y_{-2} + x_3y_{-3}) \\
z_1 &= \frac{1}{4}(x_0y_1 + x_1y_0 + x_2y_{-1} + x_3y_{-2}) \\
z_2 &= \frac{1}{4}(x_0y_2 + x_1y_1 + x_2y_0 + x_3y_{-1}) \\
z_3 &= \frac{1}{4}(x_0y_3 + x_2y_1 + x_2y_1 + x_3y_0)
\end{aligned}$$

The negative indices can be interpreted by imagining the `y` vector to be periodic, and can be indexed backwards from 0 as well as forwards:

$$\begin{array}{cccccccccc}
 & \cdots & y_0 & y_1 & y_2 & y_3 & y_0 & y_1 & y_2 & y_3 & \cdots \\
= & \cdots & y_0 & y_{-3} & y_{-2} & y_{-1} & y_0 & y_1 & y_2 & y_3 & \cdots
\end{array}$$

Thus $y_{-1} = y_3$, $y_{-2} = y_2$ and $y_{-3} = y_1$.

It can be checked from the definition that convolution is *commutative* (the order of the operands is irrelevant):

$$x * y = y * x.$$

Circular convolution as defined looks like a messy operation. However, it can be defined in terms of polynomial products. Suppose $p(u)$ is the polynomial in u whose coefficients are the elements of `x`. And let $q(u)$ be the polynomial whose coefficients are the elements of `y`. Form the product $p(u)q(u)(1 + u^N)$, and extract the coefficients of u^N to u^{2N-1}. These will be our required circular convolution.

For example, suppose we have:

$$x = [1, \quad 2, \quad 3, \quad 4], \qquad y = [5, \quad 6, \quad 7, \quad 8].$$

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Then we have

$$p(u) = 1 + 2u + 3u^2 + 4u^3$$

and

$$q(u) = 5 + 6u + 7u^2 + 8u^3.$$

The we expand

$$p(u)q(u)(1 + u^4) = 5 + 16x + 34u^2 + 60u^3 + 66u^4 + 68u^5 + 66u^6 + 60u^7 + 61u^8 + 52u^9 + 32u^{10}.$$

Extracting the coefficients of $u^4, u^5, \ldots, u^7$ we obtain

$$x * y = [66, \quad 68, \quad 66, \quad 60].$$

MATLAB has a `conv` function, which produces the coefficients of the polynomial $p(u)q(u)$ as defined above:

```
>> a=[1 2 3 4]

a =

     1     2     3     4

>> b=[5 6 7 8]

b =

     5     6     7     8

>> conv(a,b)

ans =

     5    16    34    60    61    52    32
```

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We can write a simple function `cconv` to perform circular convolution. We first notice that the polynomial $p(u)(1+u^N)$ can be obtained by simply repeating the coefficients of `x`. This leads to the function given in figure 7.4.

```
function out=cconv(a,b)

if length(a)~=length(b)
  error('Vectors must be the same length')
end;
la=length(a);
temp=conv([a a],b);
out=temp(la+1:2*la);
```

Figure 7.4: A function to calculate the circular convolution of two vectors

For example:

```
>> cconv(a,b)


ans =


    66    68    66    60
```

which is exactly what we obtained above.

The importance of convolution is the *convolution theorem*, which states:

> Suppose `x` and `y` are vectors of equal length. Then the DFT of their circular convolution $\mathtt{x} * \mathtt{y}$ is equal to the element-by-element product of the DFT's of `x` and `y`.

So if `Z`, `X`, `Y` are the DFT's of $\mathtt{z} = \mathtt{x} * \mathtt{y}$, `x` and `y` respectively, then

$$\mathtt{Z} = \mathtt{X}.\mathtt{Y}.$$

We can check this with our vectors above:

```
>> fft(cconv(a,b)')


ans =
```

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```
   1.0e+02 *


   2.6000
        0 - 0.0800i
   0.0400
        0 + 0.0800i


>> fft(a').*fft(b');


ans =


   1.0e+02 *


   2.6000
        0 - 0.0800i
   0.0400
        0 + 0.0800i
```

Note that in each case the results are the same. The convolution theorem thus provides us with another way of performing convolution: multiply the DFT's of our two vectors, and invert the result:

```
>> fft(a').*fft(b');
>> ifft(ans)'


ans =


    66    68    66    60
```

A formal proof of the convolution theorem for the DFT is given by Petrou [22].

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The Fast Fourier Transform. One of the many aspects which make the DFT so attractive for image processing is the existence of very fast algorithms to compute it. There are a number of extremely fast and efficient algorithms for computing a DFT; such an algorithm is called a *fast Fourier transform*, or FFT. The use of an FFT vastly reduces the time needed to compute a DFT.

One FFT method works recursively by dividing the original vector into two halves, computing the FFT of each half, and then putting the results together. This means that the FFT is most efficient when the vector length is a power of 2. This method is discussed in appendix B.

Table 7.1 shows that advantage gained by using the FFT algorithm as opposed to the direct arithmetic definition of equations 7.6 and 7.7 by comparing the number of multiplications required for each method. For a vector of length 2^n, the direct method takes $(2^n)^2 = 2^{2n}$ multiplications; the FFT only $n2^n$. The saving in time is thus of an order of $2^n/n$. Clearly the advantage of using an FFT algorithm becomes greater as the size of the vector increases.

Because of the this computational advantage, any implementation of the DFT will use an FFT algorithm.

2^n	Direct arithmetic	FFT	Increase in speed
4	16	8	2.0
8	84	24	2.67
16	256	64	4.0
32	1024	160	6.4
64	4096	384	10.67
128	16384	896	18.3
256	65536	2048	32.0
512	262144	4608	56.9
1024	1048576	10240	102.4

Table 7.1: Comparison of FFT and direct arithmetic

7.5 The two-dimensional DFT

In two dimensions, the DFT takes a matrix as input, and returns another matrix, of the same size, as output. If the original matrix values are $f(x, y)$, where x and y are the indices, then the output matrix values are $F(u, v)$. We call the matrix F the *Fourier transform of* f and write

$$F = \mathcal{F}(f).$$

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Then the original matrix f is the *inverse Fourier transform of* F, and we write

$$f = \mathcal{F}^{-1}(F).$$

We have seen that a (one-dimensional) function can be written as a sum of sines and cosines. Given that an image may be considered as a two-dimensional function $f(x, y)$, it seems reasonable to assume that f can be expressed as sums of "corrugation" functions which have the general form

$$z = a\sin(bx + cy).$$

A sample such function is shown in figure 7.5. And this is in fact exactly what the two-dimensional

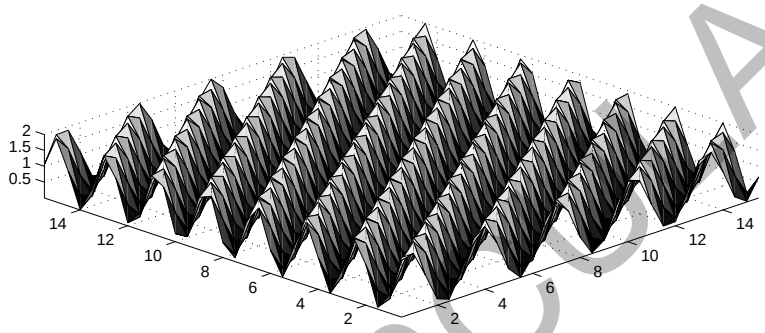

Figure 7.5: A "corrugation" function

Fourier transform does: it rewrites the original matrix in terms of sums of corrugations.

The definition of the two-dimensional discrete Fourier transform is very similar to that for one dimension. The forward and inverse transforms for an $M \times N$ matrix, where for notational convenience we assume that the x indices are from 0 to $M-1$ and the y indices are from 0 to $N-1$, are:

complex

$a+ib$

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \exp\left[-2\pi i \left(\frac{xu}{M} + \frac{yv}{N}\right)\right]. \tag{7.4}$$

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \exp\left[2\pi i \left(\frac{xu}{M} + \frac{yv}{N}\right)\right]. \tag{7.5}$$

and 0 min

Tan so'

These are horrendous looking formulas, but if we spend a bit of time pulling them apart, we shall see that they aren't as bad as they look.

Before we do this, we note that the formulas given in equations 7.4 and 7.5 are not used by all authors. The main change is the position of the scaling factor $1/MN$. Some people put it in front of

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the sums in the forward formula. Others put a factor of $1/\sqrt{MN}$ in front of both sums. The point is the sums by themselves would produce a result (after both forward and inverse transforms) which is too large by a factor of MN. So somewhere in the forward-inverse formulas a corresponding $1/MN$ must exist; it doesn't really matter where.

7.5.1 Some properties of the two dimensional Fourier transform

All the properties of the one-dimensional DFT transfer into two dimensions. But there are some further properties not previously mentioned, which are of particular use for image processing.

Similarity. First notice that the forward and inverse transforms are very similar, with the exception of the scale factor $1/MN$ in the inverse transform, and the negative sign in the exponent of the forward transform. This means that the same algorithm, only very slightly adjusted, can be used for both the forward an inverse transforms.

The DFT as a spatial filter. Note that the values

$$\exp\left[\pm 2\pi i\left(\frac{xu}{M}+\frac{yv}{N}\right)\right]$$

are independent of the values f or F. This means that they can be calculated in advance, and only then put into the formulas above. It also means that every value $F(u,v)$ is obtained by multiplying every value of $f(x,y)$ by a fixed value, and adding up all the results. But this is precisely what a linear spatial filter does: it multiplies all elements under a mask with fixed values, and adds them all up. Thus we can consider the DFT as a linear spatial filter which is as big as the image. To deal with the problem of edges, we assume that the image is tiled in all directions, so that the mask always has image values to use.

Separability. Notice that the Fourier transform "filter elements" can be expressed as products:

$$\exp\left[2\pi i\left(\frac{xu}{M}+\frac{yv}{N}\right)\right]=\exp\left[2\pi i\frac{xu}{M}\right]\exp\left[2\pi i\frac{yv}{N}\right].$$

The first product value

$$\exp\left[2\pi i\frac{xu}{M}\right]$$

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depends only on x and u, and is independent of y and v. Conversely, the second product value

$$\exp\left[2\pi i\frac{yv}{N}\right]$$

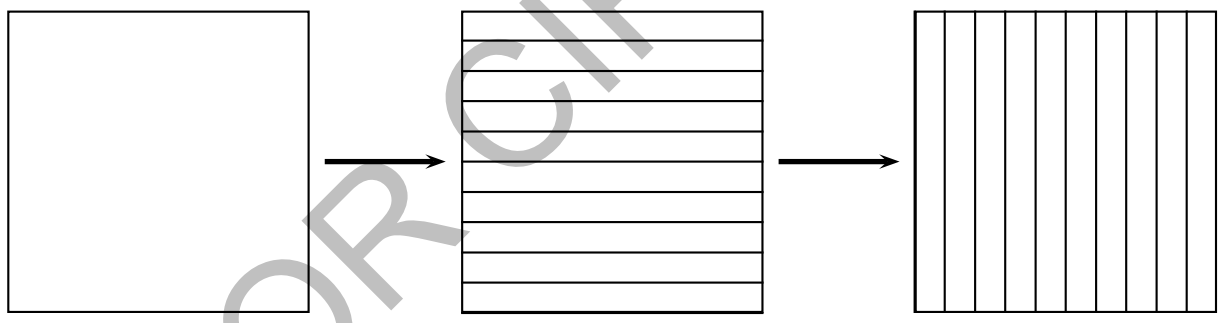
depends only on y and v, and is independent of x and u. This means that we can break down our formulas above to simpler formulas that work on single rows or columns:

$$F(u) = \sum_{x=0}^{M-1} f(x)\exp\left[-2\pi i\frac{xu}{M}\right], \tag{7.6}$$

$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u)\exp\left[2\pi i\frac{xu}{M}\right]. \tag{7.7}$$

If we replace x and u with y and v we obtain the corresponding formulas for the DFT of matrix columns. These formulas define the *one-dimensional DFT* of a vector, or simply the DFT.

The 2-D DFT can be calculated by using this property of "separability"; to obtain the 2-D DFT of a matrix, we first calculate the DFT of all the rows, and then calculate the DFT of all the columns of the result, as shown in figure 7.6. Since a product is independent of the order, we can equally well calculate a 2-D DFT by calculating the DFT of all the columns first, then calculating the DFT of all the rows of the result.



(a) Original image (b) DFT of each row of (a) (c) DFT of each column of (b)

Figure 7.6: Calculating a 2D DFT

Linearity An important property of the DFT is its linearity; the DFT of a sum is equal to the sum of the individual DFT's, and the same goes for scalar multiplication:

$$\mathcal{F}(f+g) = \mathcal{F}(f)+\mathcal{F}(g)$$

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$$\mathcal{F}(kf) \quad = \quad k\mathcal{F}(f)$$

where k is a scalar, and f and g are matrices. This follows directly from the definition given in equation 7.4.

This property is of great use in dealing with image degradation such as noise which can be modelled as a sum:

$$d = f + n$$

where f is the original image; n is the noise, and d is the degraded image. Since

$$\mathcal{F}(d) = \mathcal{F}(f) + \mathcal{F}(n)$$

we may be able to remove or reduce n by modifying the transform. As we shall see, some noise appears on the DFT in a way which makes it particularly easy to remove.

The convolution theorem. This result provides one of the most powerful advantages of using the DFT. Suppose we wish to convolve an image M with a spatial filter S. Our method has been place S over each pixel of M in turn, calculate the product of all corresponding grey values of M and elements of S, and add the results. The result is called the *digital convolution* of M and S, and is denoted

$$M * S.$$

This method of convolution can be very slow, especially if S is large. The *convolution theorem* states that the result $M * S$ can be obtained by the following sequence of steps:

1. Pad S with zeroes so that is the same size as M; denote this padded result by S'.

2. Form the DFT's of both M and S, to obtain $\mathcal{F}(M)$ and $\mathcal{F}(S')$.

3. Form the element-by-element product of these two transforms:

$$\mathcal{F}(M) \cdot \mathcal{F}(S').$$

4. Take the inverse transform of the result:

$$\mathcal{F}^{-1}(\mathcal{F}(M) \cdot \mathcal{F}(S')).$$

NOT FOR CIRCULATION

Put simply, the convolution theorem states:

$$M * S = \mathcal{F}^{-1}(\mathcal{F}(M) \cdot \mathcal{F}(S'))$$

or equivalently that

$$\mathcal{F}(M * S) = \mathcal{F}(M) \cdot \mathcal{F}(S').$$

Although this might seem like an unnecessarily clumsy and roundabout way of computing something so simple as a convolution, it can have enormous speed advantages if S is large.

For example, suppose we wish to convolve a 512×512 image with a 32×32 filter. To do this directly would require $32^2 = 1024$ multiplications for each pixel, of which there are $512 \times 512 = 262144$. Thus there will be a total of $1024 \times 262144 = 268,435,456$ multiplications needed. Now look at applying the DFT (using an FFT algorithm). Each row requires 4608 multiplications by table 7.1; there are 512 rows, so a total of $4608 \times 512 = 2359296$ multiplications; the same must be done again for the columns. Thus to obtain the DFT of the image requires 4718592 multiplications. We need the same amount to obtain the DFT of the filter, and for the inverse DFT. We also require 512×512 multiplications to perform the product of the two transforms.

Thus the total number of multiplications needed to perform convolution using the DFT is

$$4718592 \times 3 + 262144 = 14,417,920$$

which is an enormous saving compared to the direct method.

The DC coefficient. The value $F(0,0)$ of the DFT is called the *DC coefficient*. If we put $u = v = 0$ in the definition given in equation 7.4 then

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp(0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y).$$

That is, this term is equal to the sum of *all* terms in the original matrix.

Shifting. For purposes of display, it is convenient to have the DC coefficient in the centre of the matrix. This will happen if all elements $f(x,y)$ in the matrix are multiplied by $(-1)^{x+y}$ before the transform. Figure 7.7 demonstrates how the matrix is shifted by this method. In each diagram the DC coefficient is the top left hand element of submatrix A, and is shown as a black square.

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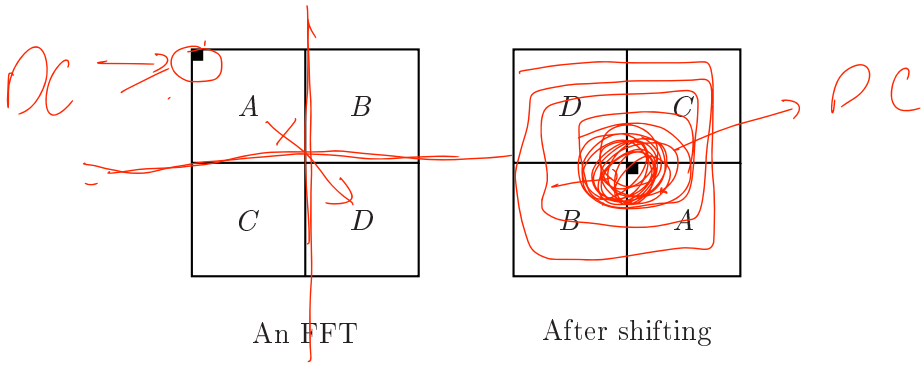

Figure 7.7: Shifting a DFT

Conjugate symmetry An analysis of the Fourier transform definition leads to a symmetry property; if we make the substitutions $u = -u$ and $v = -v$ in equation 7.4 then

$$\mathcal{F}(u, v) = \mathcal{F}^*(-u + pM, -v + qN)$$

for any integers p and q. This means that half of the transform is a mirror image of the conjugate of the other half. We can think of the top and bottom halves, or the left and right halves, being mirror images of the conjugates of each other.

Figure 7.8 demonstrates this symmetry in a shifted DFT. As with figure 7.7, the black square shows the position of the DC coefficient. The symmetry means that its information is given in just

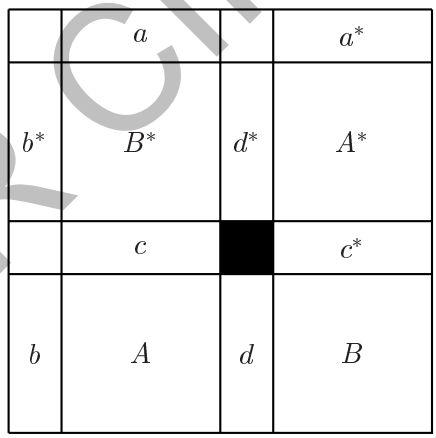

Figure 7.8: Conjugate symmetry in the DFT

half of a transform, and the other half is redundant.

Displaying transforms. Having obtained the Fourier transform $F(u, v)$ of an image $f(x, y)$, we would like to see what it looks like. As the elements $F(u, v)$ are complex numbers, we can't

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view them directly, but we can view their magnitude $|F(u,v)|$. Since these will be numbers of type `double`, generally with large range, we have two approaches

1. find the maximum value m of $|F(u,v)|$ (this will be the DC coefficient), and use `imshow` to view $|F(u,v)|/m$,

2. use `mat2gray` to view $|F(u,v)|$ directly.

One trouble is that the DC coefficient is generally very much larger than all other values. This has the effect of showing a transform as a single white dot surrounded by black. One way of stretching out the values is to take the logarithm of $|F(u,v)|$ and to display

$$\log(1 + |F(u,v)|).$$

The display of the magnitude of a Fourier transform is called the *spectrum* of the transform. We shall see some examples later on.

7.6 Fourier transforms in MATLAB

The relevant MATLAB functions for us are:

- `fft` which takes the DFT of a vector,
- `ifft` which takes the inverse DFT of a vector,
- `fft2` which takes the DFT of a matrix,
- `ifft2` which takes the inverse DFT of a matrix,
- `fftshift` which shifts a transform as shown in figure 7.7.

of which we have seen the first two above.

Before attacking a few images, let's take the Fourier transform of a few small matrices to get more of an idea what the DFT "does".

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Example 1. Suppose we take a constant matrix $f(x, y) = 1$. Going back to the idea of a sum of corrugations, then *no* corrugations are required to form a constant. Thus we would hope that the DFT consists of a DC coefficient and zeroes everywhere else. We will use the `ones` function, which produces an $n \times n$ matrix consisting of 1's, where n is an input to the function.

```
>> a=ones(8);
>> fft2(a)
```

The result is indeed as we expected:

```
ans =

    64     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
```

Biên độ của FFT số đó

Note that the DC coefficient is indeed the sum of all the matrix values.

Example 2. Now we'll take a matrix consisting of a single corrugation:

```
>> a = [100 200; 100 200];
>> a = repmat(a,4,4)

ans =

    100    200    100    200    100    200    100    200
    100    200    100    200    100    200    100    200
    100    200    100    200    100    200    100    200
    100    200    100    200    100    200    100    200
    100    200    100    200    100    200    100    200
```

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```
     100    200    100    200    100    200    100    200

     100    200    100    200    100    200    100    200

     100    200    100    200    100    200    100    200


>> af = fft2(a)


ans =


    9600      0      0      0  -3200      0      0      0

       0      0      0      0      0      0      0      0

       0      0      0      0      0      0      0      0

       0      0      0      0      0      0      0      0

       0      0      0      0      0      0      0      0

       0      0      0      0      0      0      0      0

       0      0      0      0      0      0      0      0

       0      0      0      0      0      0      0      0
```

What we have here is really the sum of two matrices: a constant matrix each element of which is 150, and a corrugation which alternates -50 and 50 from left to right. The constant matrix alone would produce (as in example 1), a DC coefficient alone of value $64 \times 150 = 9600$; the corrugation a single value. By linearity, the DFT will consist of just the two values.

Example 3. We will take here a single step edge:

```
>> a = [zeros(8,4) ones(8,4)]
  a =


       0      0      0      0      1      1      1      1

       0      0      0      0      1      1      1      1

       0      0      0      0      1      1      1      1

       0      0      0      0      1      1      1      1
```

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```
     0     0     0     0     1     1     1     1
     0     0     0     0     1     1     1     1
     0     0     0     0     1     1     1     1
     0     0     0     0     1     1     1     1
```

Now we shall perform the Fourier transform with a shift, to place the DC coefficient in the centre, and since it contains some complex values, for simplicity we shall just show the rounded absolute values:

```
>> af=fftshift(fft2(a));
>> round(abs(af))


ans =


     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     9     0    21    32    21     0     9
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
```

The DC coefficient is of course the sum of all values of `a`; the other values may be considered to be the coefficients of the necessary sine functions required to from an edge, as given in equation 7.1. The mirroring of values about the DC coefficient is a consequence of the symmetry of the DFT.

7.7 Fourier transforms of images

We shall create a few simple images, and see what the Fourier transform produces.

Example 1. We shall produce a simple image consisting of a single edge:

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```
>> a=[zeros(256,128) ones(256,128)];
```

This is displayed on the left in figure 7.10. Now we shall take its DFT, and shift it:

```
>> af=fftshift(fft2(a));
```

Now we'll view its spectrum; we have the choice of two commands:

1. `afl=log(1+abs(af));`

   `imshow(afl/afl(129,129))`

   This works because after shifting, the DC coefficient is at position $x = 129$, $y = 129$. We stretch the transform using `log`, and divide the result by the middle value to obtain matrix of type `double` with values in the range 0.0–1.0. This can then be viewed directly with `imshow`.

2. `imshow(mat2gray(log(1+abs(af))))`

   The `mat2gray` function automatically scales a matrix for display as an image, as we have seen in chapter 5

It is in fact convenient to write a small function for viewing transforms. One such is shown in figure 7.9. Then for example

```
>> fftshow(af,'log')
```

will show the logarithm of the absolute values of the transform, and

```
>> fftshow(af,'abs')
```

will show the absolute values of the transform without any scaling.

The result is shown on the right in figure 7.10. We observe immediately that the result is similar (although larger) to example 3 in the previous section.

Example 2. Now we'll create a box, and then its Fourier transform:

```
>> a=zeros(256,256);
>> a(78:178,78:178)=1;
```

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```
function fftshow(f,type)

% Usage:  FFTSHOW(F,TYPE)
%
% Displays the fft matrix F using imshow, where TYPE must be one of
% 'abs' or 'log'.  If TYPE='abs', then then abs(f) is displayed; if
% TYPE='log' then log(1+abs(f)) is displayed.  If TYPE is omitted, then
% 'log' is chosen as a default.
%
%  Example:
%    c=imread('cameraman.tif');
%    cf=fftshift(fft2(c));
%    fftshow(cf,'abs')
%

if nargin<2,
  type='log';
end

if (type=='log')
  fl = log(1+abs(f));
  fm = max(fl(:));
  imshow(im2uint8(fl/fm))
elseif (type=='abs')
  fa=abs(f);
  fm=max(fa(:));
  imshow(fa/fm)
else
  error('TYPE must be abs or log.');
end;
```

Figure 7.9: A function to display a Fourier transform

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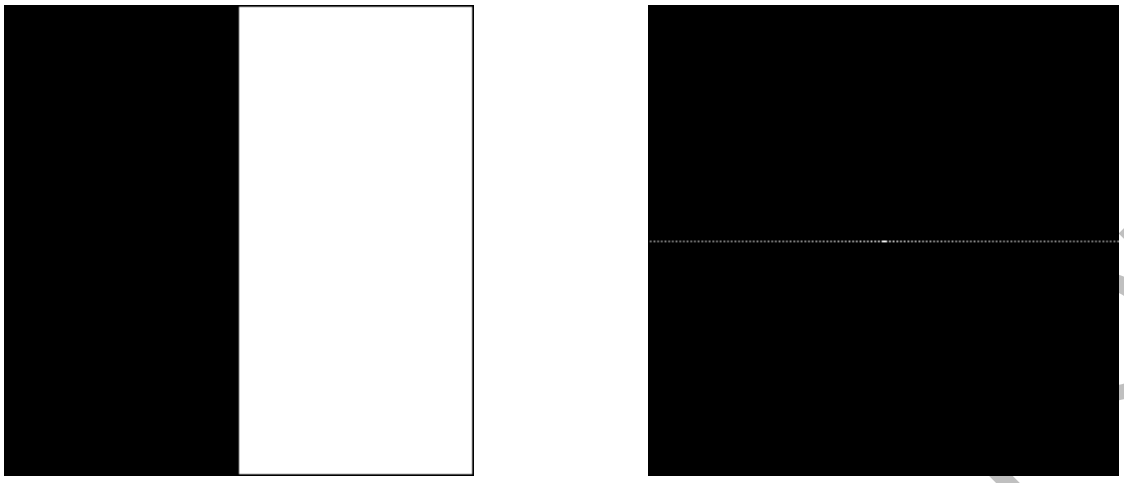

Figure 7.10: A single edge and its DFT

```
>> imshow(a)
>> af=fftshift(fft2(a));
>> figure,fftshow(af,'abs')
```

The box is shown on the left in figure 7.11, and its Fourier transform is is shown on the right.

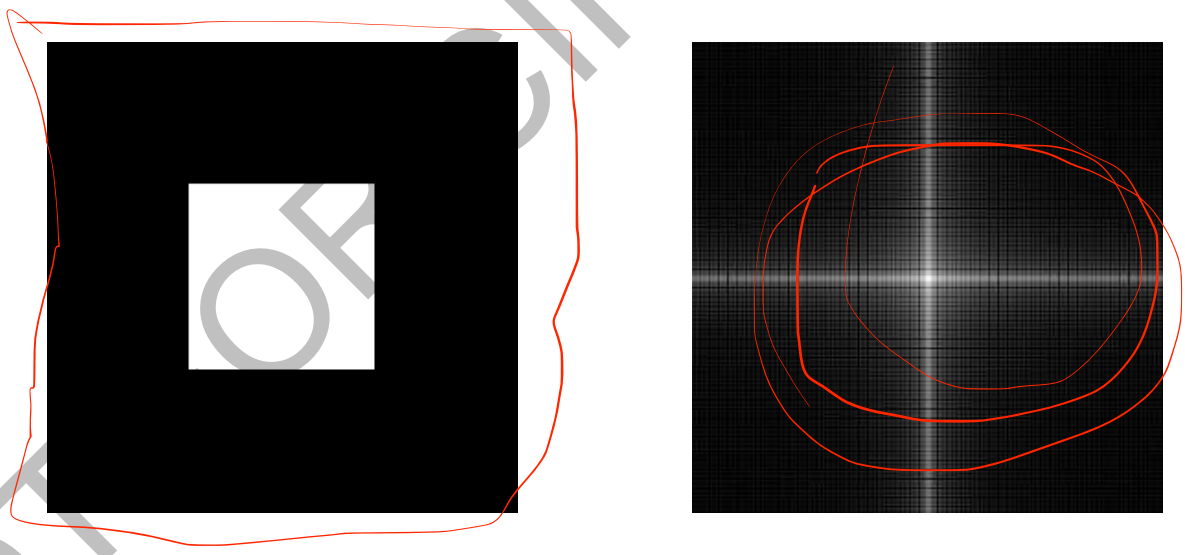

Figure 7.11: A box and its DFT

Example 3. Now we shall look at a box rotated 45°.

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```
>> [x,y]=meshgrid(1:256,1:256);
>> b=(x+y<329)&(x+y>182)&(x-y>-67)&(x-y<73);
>> imshow(b)
>> bf=fftshift(fft2(b));
>> figure,fftshow(bf)
```

The results are shown in figure 7.12. Note that the transform of the rotated box is the rotated

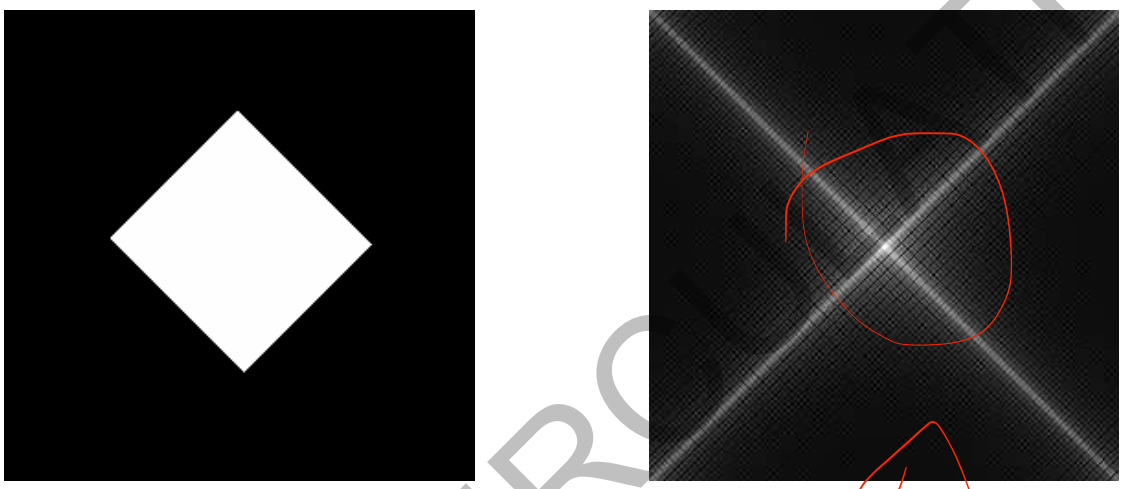

Figure 7.12: A rotated box and its DFT

transform of the original box.

Example 4. We will create a small circle, and then transform it:

```
>> [x,y]=meshgrid(-128:217,-128:127);
>> z=sqrt(x.^2+y.^2);
>> c=(z<15);
```

The result is shown on the left in figure 7.13. Now we will create its Fourier transform and display it:

```
>> cf=fft2shift(fft2(z));
>> fftshow(cf,'log')
```

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and this is shown on the right in figure 7.13. Note the "ringing" in the Fourier transform. This is an

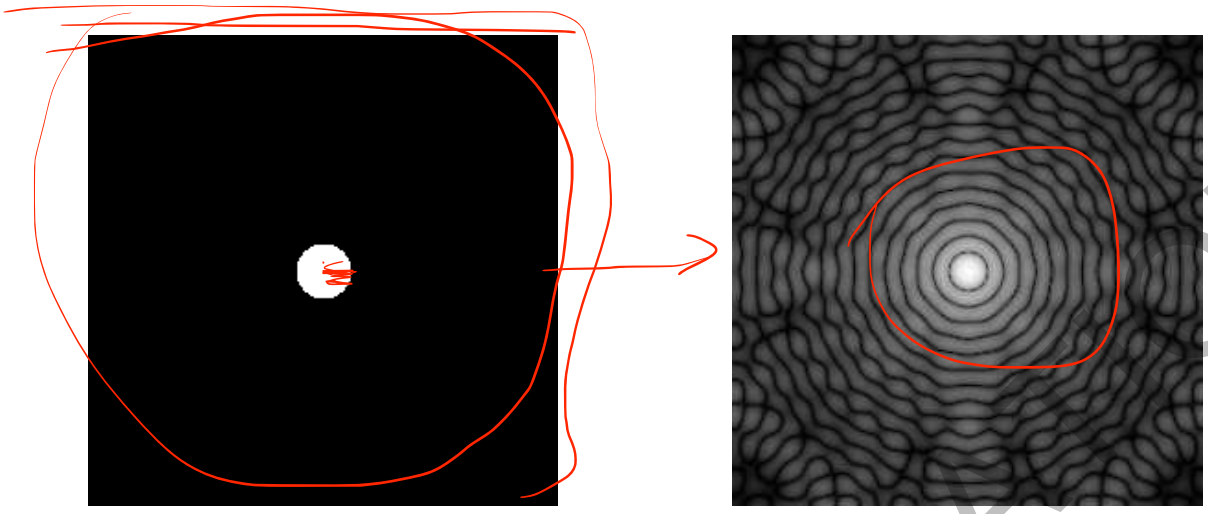

Figure 7.13: A circle and its DFT

artifact associated with the sharp cutoff of the circle. As we have seen from both the edge and box images in the previous examples, an edge appears in the transform as a line of values at right angles to the edge. We may consider the values on the line as being the coefficients of the appropriate corrugation functions which sum to the edge. With the circle, we have lines of values radiating out from the circle; these values appear as circles in the transform.

A circle with a gentle cutoff, so that its edge appears blurred, will have a transform with no ringing. Such a circle can be made with the command (given `z` above):

```
b=1./(1+(z./15).^2);
```

This image appears as a blurred circle, and its transform is very similar—check them out!

7.8 Filtering in the frequency domain

We have seen in section 7.5 that one of the reasons for the use of the Fourier transform in image processing is due to the convolution theorem: a spatial convolution can be performed by element-wise multiplication of the Fourier transform by a suitable "filter matrix". In this section we shall explore some filtering by this method.

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7.8.1 Ideal filtering

Low pass filtering

Suppose we have a Fourier transform matrix F, shifted so that the DC coefficient is in the centre. Since the low frequency components are towards the centre, we can perform low pass filtering by multiplying the transform by a matrix in such a way that centre values are maintained, and values away from the centre are either removed or minimized. One way to do this is to multiply by an *ideal low-pass matrix*, which is a binary matrix m defined by:

$$m(x,y) = \begin{cases} 1 & \text{if } (x,y) \text{ is closer to the center than some value } D, \\ 0 & \text{if } (x,y) \text{ is further from the center than } D. \end{cases}$$

The circle `c` displayed in figure 7.13 is just such a matrix, with $D = 15$. Then the inverse Fourier transform of the element-wise product of F and m is the result we require:

$$\mathcal{F}^{-1}(F \cdot m).$$

Let's see what happens if we apply this filter to an image. First we obtain an image and its DFT.

```
>> cm=imread('cameraman.tif');
>> cf=fftshift(fft2(cm));
>> figure,fftshow(cf,'log')
```

The cameraman image and its DFT are shown in figure 7.14. Now we can perform a low pass filter by multiplying the transform matrix by the circle matrix (recall that "dot asterisk" is the MATLAB syntax for element-wise multiplication of two matrices):

```
>> cfl=cf.*c;
>> figure,fftshow(cfl,'log')
```

and this is shown in figure 7.15(a). Now we can take the inverse transform and display the result:

```
>> cfli=ifft2(cfl);
>> figure,fftshow(cfli,'abs')
```

and this is shown in figure 7.15(b). Note that even though `cfli` is supposedly a matrix of real numbers, we are still using `fftshow` to display it. This is because the `fft2` and `fft2` functions,

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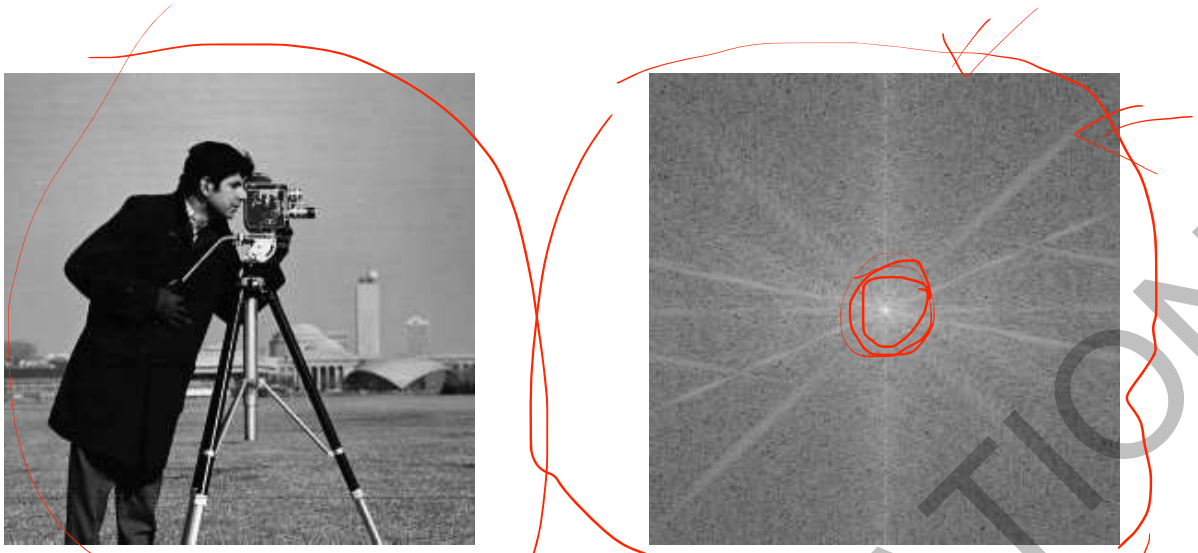
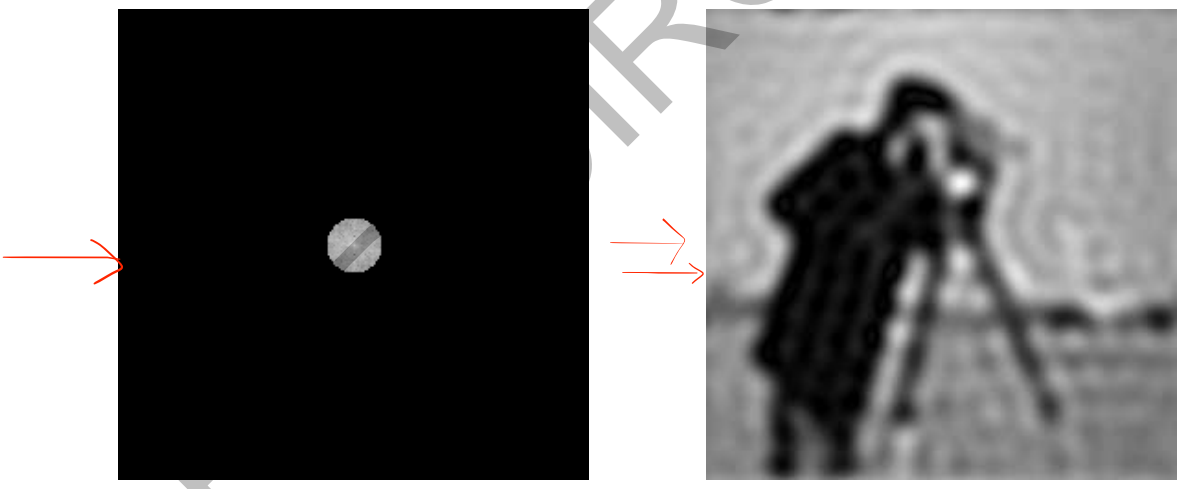


Figure 7.14: The "cameraman" image and its DFT

being numeric, will not produce mathematically perfect results, but rather very close numeric approximations. So using `fftshow` with the `'abs'` option rounds out any errors obtained during the transform and its inverse. Note the "ringing" about the edges in this image. This is a direct result



(a) Ideal filtering on the DFT (b) After inversion

Figure 7.15: Applying ideal low pass filtering

of the sharp cutoff of the circle. The ringing as shown in figure 7.13 is transferred to the image.

We would expect that the smaller the circle, the more blurred the image, and the larger the circle; the less blurred. Figure 7.16 demonstrates this, using cutoffs of 5 and 30. Notice that ringing

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(a) Cutoff of 5 (b) Cutoff of 30

Figure 7.16: Ideal low pass filtering with different cutoffs

is still present, and clearly visible in figure 7.16(b).

High pass filtering

Just as we can perform low pass filtering by keeping the centre values of the DFT and eliminating the others, so high pass filtering can be performed by the opposite: eliminating centre values and keeping the others. This can be done with a minor modification of the preceding method of low pass filtering. First we create the circle:

```
>> [x,y]=meshgrid(-128:127,-128:127);
>> z=sqrt(x.^2+y.^2);
>> c=(z>15);
```

and the multiply it by the DFT of the image:

```
>> cfh=cf.*c;
>> figure,fftshow(cfh,'log')
```

This is shown in figure 7.17(a). The inverse DFT can be easily produced and displayed:

```
>> cfhi=ifft2(cfh);
```

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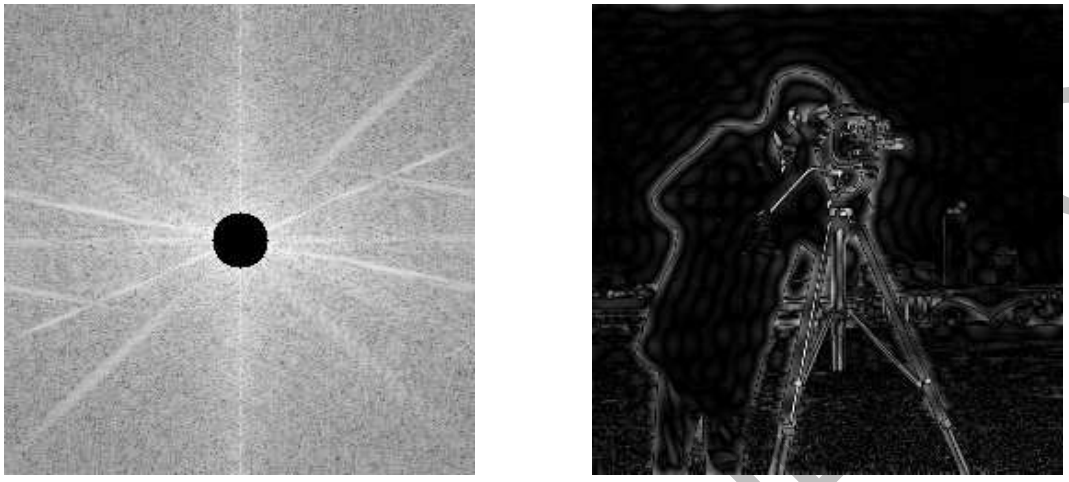
```
>> figure,fftshow(cfhi,'abs')
```

and this is shown in figure 7.17(b). As with low pass filtering, the size of the circle influences



(a) The DFT after high pass filtering (b) The resulting image

Figure 7.17: Applying an ideal high pass filter to an image

the information available to the inverse DFT, and hence the final result. Figure 7.18 shows some results of ideal high pass filtering with different cutoffs. If the cutoff is large, then more information is removed from the transform, leaving only the highest frequencies. This can be observed in figure 7.18(c) and (d); only the edges of the image remain. If we have small cutoff, such as in figure 7.18(a), we are only removing a small amount of the transform. We would thus expect that only the lowest frequencies of the image would be removed. And this is indeed true, as seen in figure 7.18(b); there is some greyscale detail in the final image, but large areas of low frequency are close to zero.

7.8.2 Butterworth filtering

Ideal filtering simply cuts off the Fourier transform at some distance from the centre. This is very easy to implement, as we have seen, but has the disadvantage of introducing unwanted artifacts: ringing, into the result. One way of avoiding this is to use as a filter matrix a circle with a less sharp cutoff. A popular choice is to use *Butterworth filters*.

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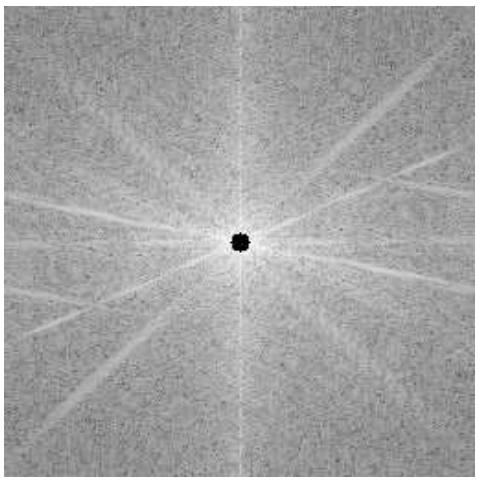
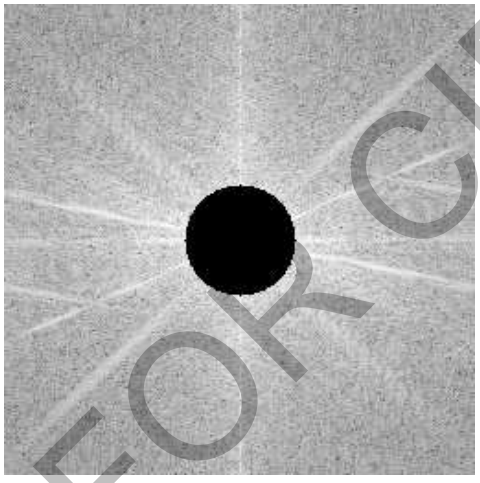


(a) Cutoff of 5

(b) The resulting image



(a) Cutoff of 30

(b) The resulting image

Figure 7.18: Ideal high pass filtering with different cutoffs

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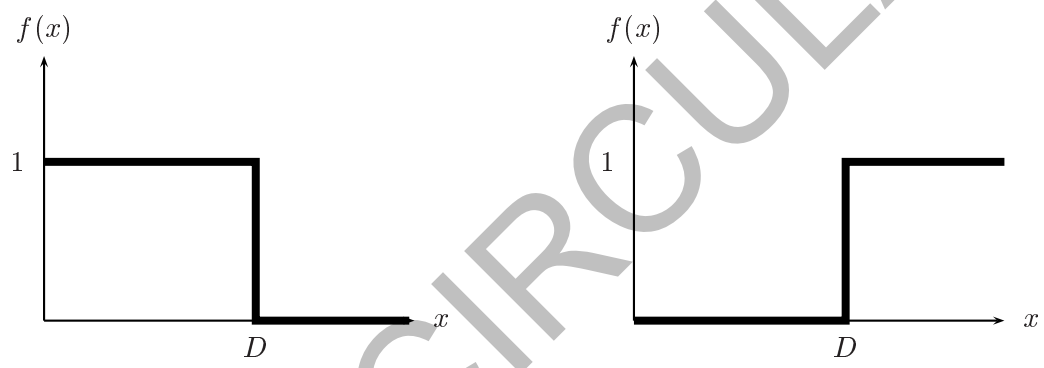
Before we describe these filters, we shall look again at the ideal filters. As these are radially symmetric about the centre of the transform, they can be simply described in terms of their cross sections. That is, we can describe the filter as a function of the distance x from the centre. For an ideal low pass filter, this function can be expressed as

$$f(x) = \begin{cases} 1 & \text{if } x < D, \\ 0 & \text{if } x \geq D \end{cases}$$

where D is the cutoff radius. Then the ideal high pass filters can be described similarly:

$$f(x) = \begin{cases} 1 & \text{if } x > D, \\ 0 & \text{if } x \leq D \end{cases}$$

These functions are illustrated in figure 7.19. Butterworth filter functions are based on the following



(a) Low pass (b) High pass

Figure 7.19: Ideal filter functions

functions for low pass filters:

$$f(x) = \frac{1}{1 + (x/D)^{2n}}$$

and for high pass filters:

$$f(x) = \frac{1}{1 + (D/x)^{2n}}$$

where in each case the parameter n is called the *order* of the filter. The size of n dictates the sharpness of the cutoff. These functions are illustrated in figures 7.20 and 7.21.

It is easy to implement these in MATLAB; here are the commands to produce a Butterworth low pass filter of size 256×256 with $D = 15$ and order $n = 2$:

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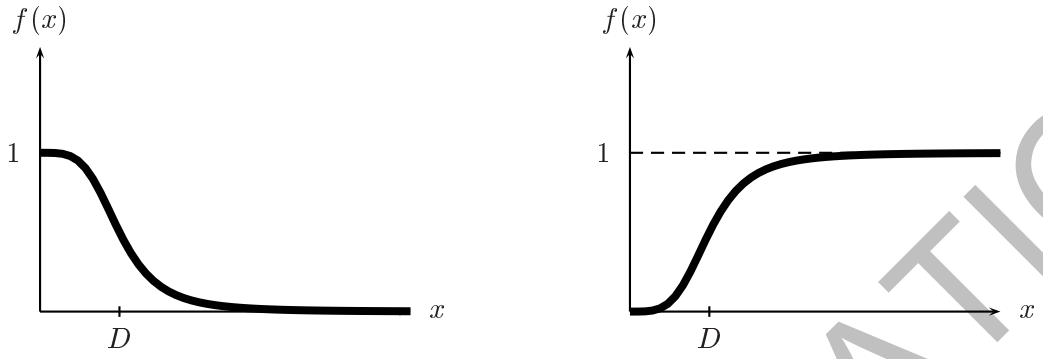
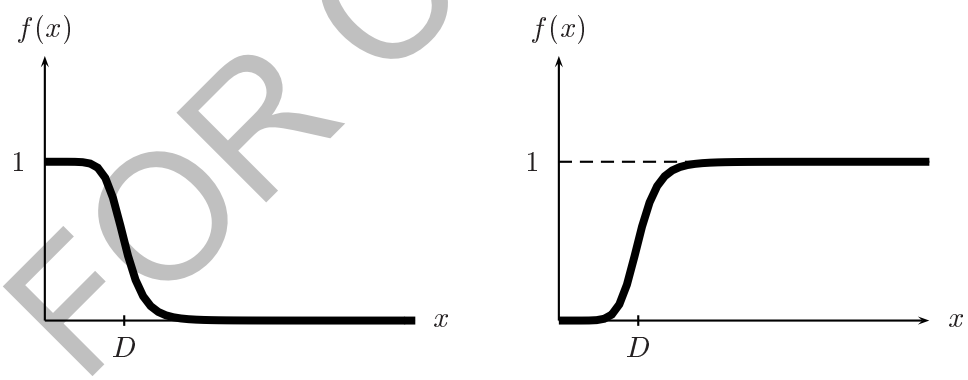


(a) Low pass (b) High pass

Figure 7.20: Butterworth filter functions with $n = 2$



(a) Low pass (b) High pass

Figure 7.21: Butterworth filter functions with $n = 4$

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```
>> [x,y]=meshgrid(-128:217,-128:127));

>> bl=1./(1+((x.^2+y.^2)/15).^2);
```

Since a Butterworth high pass filter can be obtained by subtracting a low pass filter from 1, we can write general MATLAB functions to generate Butterworth filters of general sizes. These are shown in figures 7.22 and 7.23.

```
function out=lbutter(im,d,n)
% LBUTTER(IM,D,N) creates a low-pass Butterworth filter
% of the same size as image IM, with cutoff D, and order N
%
% Use:
%   x=imread('cameraman.tif');
%   l=lbutter(x,25,2);
%
height=size(im,1);
width=size(im,2);
[x,y]=meshgrid(-floor(width/2):floor((width-1)/2),-floor(height/2): ...
       floor((height-1)/2));
out=1./(1+(sqrt(2)-1)*((x.^2+y.^2)/d^2).^n);
```

Figure 7.22: A function to generate a low pass Butterworth filter

```
function out=hbutter(im,d,n)
% HBUTTER(IM,D,N) creates a high-pass Butterworth filter
% of the same size as image IM, with cutoff D, and order N
%
% Use:
%   x=imread('cameraman.tif');
%   l=hbutter(x,25,2);
%

out=1-lbutter(im,d,n);
```

Figure 7.23: A function to generate a high pass Butterworth filter

So to apply a Butterworth low pass filter to the DFT of the cameraman image:

```
>> bl=lbutter(c,15,1);

>> cfbl=cf.*bl;

>> figure,fftshow(cfbl,'log')
```

and this is shown in figure 7.24(a). Note that there is no sharp cutoff as seen in figure 7.15; also that

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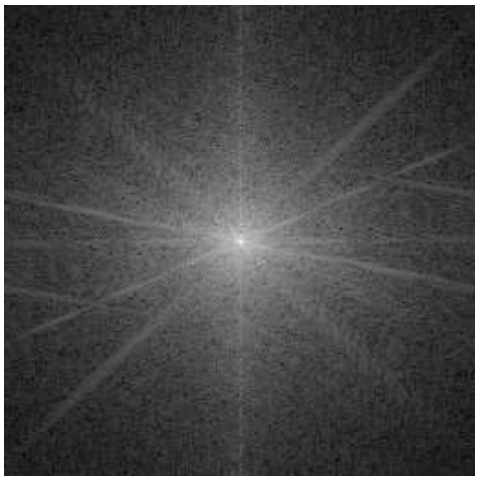
the outer parts of the transform are not equal to zero, although they are dimmed considerably. Performing the inverse transform and displaying it as we have done previously produces figure 7.24(b). This is certainly a blurred image, but the ringing seen in figure 7.15 is completely absent. Compare



(a) The DFT after Butterworth low pass filtering

(b) The resulting image

Figure 7.24: Butterworth low pass filtering

the transform after multiplying with a Butterworth filter (figure 7.24(a)) with the original transform (in figure 7.14). The Butterworth filter does cause an attenuation of values away from the centre, even if they don't become suddenly zero, as with the ideal low pass filter in figure 7.15.

We can apply a Butterworth high pass filter similarly, first by creating the filter and applying it to the image transform:

```
>> bh=hbutter(cm,15,1);
>> cfbh=cf.*bh;
>> figure,fftshow(cfbh,'log')
```

and then inverting and displaying the result:

```
>> cfbhi=ifft2(cfbh);
>> figure,fftshow(cfbhi,'abs')
```

The images are shown in figure 7.25

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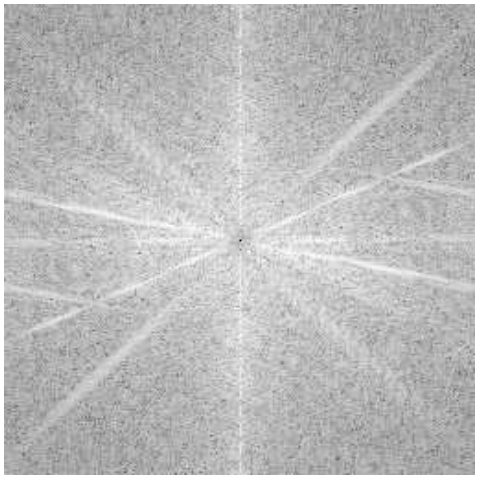

(a) The DFT after Butterworth high pass filtering

(b) The resulting image

Figure 7.25: Butterworth high pass filtering

7.8.3 Gaussian filtering

We have met Gaussian filters in chapter 5, and we saw that they could be used for low pass filtering. However, we can also use Gaussian filters in the frequency domain. As with ideal and Butterworth filters, the implementation is very simple: create a Gaussian filter, multiply it by the image transform, and invert the result. Since Gaussian filters have the very nice mathematical property that a Fourier transform of a Gaussian is a Gaussian, we should get exactly the same results as when using a linear Gaussian spatial filter.

Gaussian filters may be considered to be the most "smooth" of all the filters we have discussed so far, with ideal filters the least smooth, and Butterworth filters in the middle.

We can create Gaussian filters using the `fspecial` function, and apply them to our transform.

```
>> g1=mat2gray(fspecial('gaussian',256,10));
>> cg1=cf.*g1;
>> fftshow(cg1,'log')
>> g2=mat2gray(fspecial('gaussian',256,30));
>> cg2=cf.*g2;
>> figure,fftshow(cg2,'log')
```

Note the use of the `mat2gray` function. The `fspecial` function on its own produces a low pass

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Gaussian filter with a very small maximum:

```
>> g=fspecial('gaussian',256,10);
>> format long, max(g(:)), format

ans =

   0.00158757552679
```

The reason is that `fspecial` adjusts its output to keep the volume under the Gaussian function always 1. This means that a wider function, with a large standard deviation, will have a low maximum. So we need to scale the result so that the central value will be 1; and `mat2gray` does that automatically.

The transforms are shown in figure 7.26(a) and (c). In each case, the final parameter of the `fspecial` function is the standard deviation; it controls the width of the filter. Clearly, the larger the standard deviation, the wider the function, and so the greater amount of the transform is preserved.

The results of the transform on the original image can be produced using the usual sequence of commands:

```
>> cgi1=ifft2(cg1);
>> cgi2=ifft2(cg2);
>> fftshow(cgi1,'abs');
>> fftshow(cgi2,'abs');
```

and the results are shown in figure 7.26(b) and (d)

We can apply a high pass Gaussian filter easily; we create a high pass filter by subtracting a low pass filter from 1.

```
>> h1=1-g1;
>> h2=1-g2;
>> ch1=cf.*h1;
>> ch2=cf.*h2;
>> ch1i=ifft2(ch1);
```

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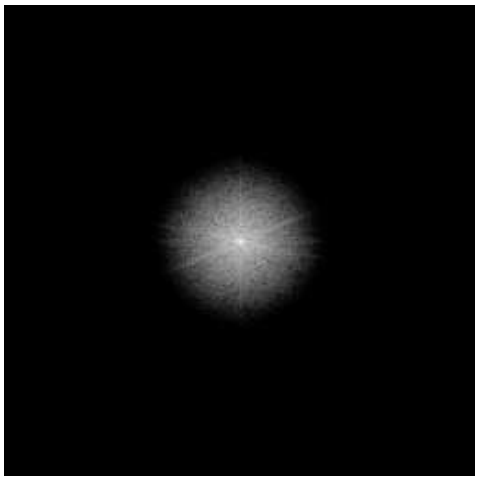

(a) $\sigma = 10$

(b) Resulting image

(c) $\sigma = 30$

(d) Resulting image

Figure 7.26: Applying a Gaussian low pass filter in the frequency domain

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```
>> chi1=ifft2(ch1);
>> chi2=ifft2(ch2);
>> fftshow(chi1,'abs')
>> figure,fftshow(chi2,'abs')
```

and the images are shown in figure 7.27. As with ideal and Butterworth filters, the wider the high

(a) Using $\sigma = 10$ (b) Using $\sigma = 30$

Figure 7.27: Applying a Gaussian high pass filter in the frequency domain

pass filter, the more of the transform we are reducing, and the less of the original image will appear in the result.

7.9 Homomorphic filtering

If we have an image which suffers from variable illumination (dark in some sections, light in others), we may wish to enhance the contrast locally, and in particular to enhance the dark regions. Such an image may be obtained if we are recording a scene with high intensity range (say, an outdoor scene on a sunny day which includes shadows), onto a medium with a smaller intensity range. The resulting image will contain very bright regions (those well illuminated); on the other hand regions in shadow may appear very dark indeed.

In such a case histogram equalization won't be of much help, as the image already has high

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contrast. What we need to do is to reduce the intensity range and at the same time increase the local contrast. We first note that the intensity of an object in an image may be considered to be a combination of two factors: the amount of light falling on it, and how much light is reflected by the object. In fact, if $f(x, y)$ is the intensity of a pixel at position (x, y) in our image, we may write:

$$f(x, y) = i(x, y)r(x, y)$$

where $i(x, y)$ is the *illumination* and $r(x, y)$ is the *reflectance*. These satisfy

$$0 < i(x, y) < \infty$$

and

$$0 < r(x, y) < 1.$$

There is no (theoretical) limit as to the amount of light which can fall on an object, but reflectance is strictly bounded.

To reduce the intensity range we need to reduce the illumination, and to increase the local contrast we need to increase the reflectance. However, this means we need to separate $i(x, y)$ and $r(x, y)$. We can't do this directly, as the image is formed from their product. However, if we take the logarithm of the image:

$$\log f(x, y) = \log i(x, y) + \log r(x, y)$$

we can then separate the logarithms of $i(x, y)$ and $r(x, y)$. The basis of homomorphic filtering is this working with the logarithm of the image, rather than with the image directly. A schema for homomorphic filtering is given in figure 7.28.

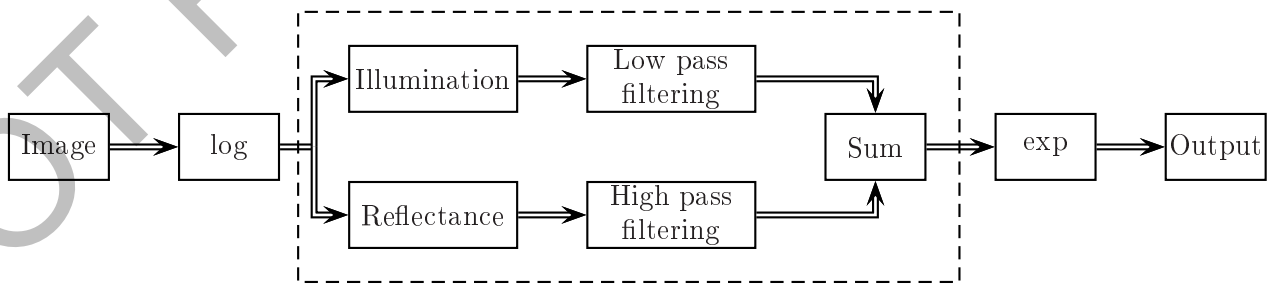

Figure 7.28: A schema for homomorphic filtering

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The "exp" in the second last box simply reverses the effect of the original logarithm. We assume that the logarithm of illumination will vary slowly, and the logarithm of reflectance will vary quickly, so that the filtering processes given in figure 7.28 will have the desired effects.

Clearly this schema will be unworkable in practice: given a value $\log f(x, y)$ we can't determine the values of its summands $\log i(x, y)$ and $\log r(x, y)$. A simpler way is to replace the dashed box in figure 7.28 with a high boost filter of the Fourier transform. This gives the simpler schema shown in figure 7.29.

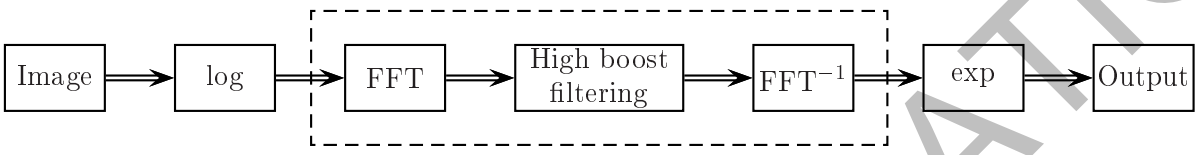

Figure 7.29: A simpler schema for homomorphic filtering

A simple function to apply homomorphic filtering to an image (using a Butterworth high boost filter) is given in figure 7.30.

```
function res=homfilt(im,cutoff,order,lowgain,highgain)

% HOMFILT(IMAGE,FILTER) applies homomorphic filtering to the image IMAGE
% with the given parameters

u=im2uint8(im);

u(find(u==0))=1;
l=log(double(u));
ft=fftshift(fft2(l));
f=hb_butter(im,cutoff,order,lowgain,highgain);
b=f.*ft;
ib=abs(ifft2(b));
res=exp(ib);
```

Figure 7.30: A function to apply homomorphic filtering

To see this in action, suppose we take an image of type `double` (so that its pixel values are between 0.0 and 1.0), and multiply it by a trigonometric function, scaled to between 0.1 and 1. If, for example we use $\sin x$, then the function

$$y = 0.5 + 0.4 \sin x$$

satisfies $0.1 \leq y \leq 1.0$. The result will be an image with varying illumination.

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Suppose we take an image `i` of size 256×256. Then we can superimpose a sine function with:

```
>> r=[1:256]'*ones(1,256);
>> x=i.*(0.5+0.4*sin((r-32)/16));
```

The transformation `(r-32)/16` was chosen by trial and error to produce bands of suitable width, and on such places as to obscure detail in our image. If the original image is in figure 7.31(a), then the result of this is shown in figure 7.31(b).

(a) The "newborn" image

(b) Altering the illumination with a sine function

Figure 7.31: Varying illumination across an image

If we apply homomorphic filtering with

```
>> xh=homfilt(x,10,2,0.5,2);
>> imshow(xh/16)
```

the result is shown in figure 7.32.

The result shows that detail originally unobservable in the original image—especially in regions of poor illumination—is now clear. Even though the dark bands have not been completely removed, they do not obscure underlying detail as they originally did.

Figure 7.33(a) shows a picture of a ruined archway; owing to the bright light through the archway, much of the details are too dark to be seen clearly. Given this image `x`, we can perform homomorphic filtering and display the result with:

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Figure 7.32: The result of homomorphic filtering

```
>> ah=homfilt(a,128,2,0.5,2);
>> imshow(xh/14)
```

and this is shown in figure 7.33(b).

Note that the arch details are now much clearer, and it is even possible to make out the figure of a person standing at its base.

Exercises

1. By hand, compute the DFT of each of the following sequences:

   (a) $[2, \quad 3, \quad 4, \quad 5]$ (b) $[2, \quad -3, \quad 4, \quad -5]$ (c) $[-9, \quad -8, \quad -7, \quad -6]$

   (d) $[-9, \quad 8, \quad -7, \quad 6]$

   Compare your answers with those given by MATLAB's `fft` function.

2. For each of the transforms you computed in the previous question, compute the inverse transform by hand.

3. By hand, verify the convolution theorem for each of the following pairs of sequences:

   (a) $[2, \quad 4, \quad 6, \quad 8]$ and $[-1, \quad 2 \quad -3, \quad 4]$ (b) $[4, \quad 5, \quad 6, \quad 7]$ and $[3, \quad 1 \quad 5, \quad -1]$

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(a) The arch

(b) After homomorphic filtering

Figure 7.33: Applying homomorphic filtering to an image

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4. Using MATLAB, verify the convolution theorem for the following pairs of sequences:

   (a) $[2, \quad -3, \quad 5, \quad 6, \quad -2, \quad -1, \quad 3, \quad 7]$ and $[-1, \quad 5, \quad 6, \quad 4, \quad -3, \quad -5, \quad 1, \quad 2]$

   (b) $[7, \quad 6, \quad 5, \quad 4, \quad -4, \quad -5, \quad -6, \quad -7]$ and $[2, \quad 2, \quad -5, \quad -5, \quad 6, \quad 6, \quad -7, \quad -7]$

5. Consider the following matrix:

$$\begin{bmatrix} 4 & 5 & -9 & -5 \\ 3 & -7 & 1 & 2 \\ 6 & -1 & -6 & 1 \\ 3 & -1 & 7 & -5 \end{bmatrix}$$

   Using MATLAB, calculate the DFT of each row. You can do this with the commands:

```
>> a=[4 5 -9 -5;3 -7 1 2;6 -1 -6 1;3 -1 7 -5];
>> a1=fft(a')'
```

   (The `fft` function, applied to a matrix, produces the individual DFTs of all the columns. Here we transpose first, so that the rows become columns, then transpose back afterwards.)

   Now use similar commands to calculate the DFT of each column of `a1`.

   Compare the result with the output of the command `fft2(a)`.

6. Perform similar calculations as in the previous question with the matrices produced by the commands `magic(4)` and `hilb(6)`.

7. How do you think filtering with an averaging filter will effect the output of a Fourier transform?

   Compare the DFTs of the cameraman image, and of the image after filtering with a 5×5 averaging filter.

   Can you account for the result?

   What happens if the averaging filter increases in size?

8. What is the result of two DFTs performed in succession? Apply a DFT to an image, and then another DFT to the result. Can you account for what you see?

9. Open up the image `engineer.tif`:

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```
>> en=imread('engineer.tif');
```

Experiment with applying the Fourier transform to this image and the following filters:

(a) ideal filters (both low and high pass),

(b) Butterworth filters,

(c) Gaussian filters.

What is the smallest radius of a low pass ideal filter for which the face is still recognizable?

10. If you have access to a digital camera, or a scanner, produce a digital image of the face of somebody you know, and perform the same calculations as in the previous question.

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Chapter 8

Image Restoration

8.1 Introduction

Image *restoration* concerns the removal or reduction of degradations which have occurred during the acquisition of the image. Such degradations may include *noise*, which are errors in the pixel values, or optical effects such as out of focus blurring, or blurring due to camera motion. We shall see that some restoration techniques can be performed very successfully using neighbourhood operations, while others require the use of frequency domain processes. Image restoration remains one of the most important areas of image processing, but in this chapter the emphasis will be on the techniques for dealing with restoration, rather than with the degradations themselves, or the properties of electronic equipment which give rise to image degradation.

A model of image degradation

In the spatial domain, we might have an image $f(x, y)$, and a spatial filter $h(x, y)$ for which convolution with the image results in some form of degradation. For example, if $h(x, y)$ consists of a single line of ones, the result of the convolution will be a motion blur in the direction of the line. Thus we may write

$$g(x, y) = f(x, y) * h(x, y)$$

for the degraded image, where the symbol $*$ represents spatial filtering. However, this is not all. We must consider noise, which can be modelled as an additive function to the convolution. Thus if

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$n(x, y)$ represents random errors which may occur, we have as our degraded image:

$$g(x, y) = f(x, y) * h(x, y) + n(x, y).$$

We ca perform the same operations in the frequency domain, where convolution is replaced by multiplication, and addition remains as addition, because of the linearity of the Fourier transform. Thus

$$G(i, j) = F(i, j)H(i, j) + N(i, j)$$

represents a general image degradation, where of course F, H and N are the Fourier transforms of f, h and n respectively.

If we knew the values of H and N we could recover F by writing the above equation as

$$F(i, j) = (G(i, j) - N(i, j))/H(i, j).$$

However, as we shall see, this approach may not be practical. Even though we may have some statistical information about the noise, we will not know the value of $n(x, y)$ or $N(i, j)$ for all, or even any, values. As well, dividing by $H(i, j)$ will cause difficulties if there are values which are close to, or equal to, zero.

8.2 Noise

We may define *noise* to be any degradation in the image signal, caused by external disturbance. If an image is being sent electronically from one place to another, via satellite or wireless transmission, or through networked cable, we may expect errors to occur in the image signal. These errors will appear on the image output in different ways depending on the type of disturbance in the signal. Usually we know what type of errors to expect, and hence the type of noise on the image; hence we can choose the most appropriate method for reducing the effects. Cleaning an image corrupted by noise is thus an important area of *image restoration*.

In this chapter we will investigate some of the standard noise forms, and the different methods of eliminating or reducing their effects on the image.

We will look at four different noise types, and how they appear on an image.

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Salt and pepper noise

Also called *impulse noise*, *shot noise*, or *binary noise*. This degradation can be caused by sharp, sudden disturbances in the image signal; its appearance is randomly scattered white or black (or both) pixels over the image.

To demonstrate its appearance, we will first generate a grey-scale image, starting with a colour image:

```
>> tw=imread('twins.tif');
>> t=rgb2gray(tw);
```

To add noise, we use the MATLAB function `imnoise`, which takes a number of different parameters. To add salt and pepper noise:

```
>> t_sp=imnoise(t,'salt & pepper');
```

The amount of noise added defaults to 10%; to add more or less noise we include an optional parameter, being a value between 0 and 1 indicating the fraction of pixels to be corrupted. Thus, for example

```
>> imnoise(t,'salt & pepper',0.2);
```

would produce an image with 20% of its pixels corrupted by salt and pepper noise.

The twins image is shown in figure 8.1(a) and the image with noise is shown in figure 8.1(b).

Gaussian noise

Gaussian noise is an idealized form of *white noise*, which is caused by random fluctuations in the signal. We can observe white noise by watching a television which is slightly mistuned to a particular channel. Gaussian noise is white noise which is normally distributed. If the image is represented as I, and the Gaussian noise by N, then we can model a noisy image by simply adding the two:

$$I + N.$$

Here we may assume that I is a matrix whose elements are the pixel values of our image, and N is a matrix whose elements are normally distributed. It can be shown that this is an appropriate model for noise. The effect can again be demonstrated by the `imnoise` function:

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(a) Original image (b) With added salt & pepper noise

Figure 8.1: Noise on an image

```
>> t_ga=inoise(t,'gaussian');
```

As with salt and pepper noise, the "`gaussian`" parameter also can take optional values, giving the mean and variance of the noise. The default values are 0 and 0.01, and the result is shown in figure 8.2(a).

Speckle noise

Whereas Gaussian noise can be modelled by random values *added* to an image; *speckle noise* (or more simply just *speckle*) can be modelled by random values *multiplied* by pixel values, hence it is also called *multiplicative noise*. Speckle noise is a major problem in some radar applications. As above, `imnoise` can do speckle:

```
>> t_spk=imnoise(t,'speckle');
```

and the result is shown in figure 8.2(b). In MATLAB, speckle noise is implemented as

$$I(1+N)$$

where I is the image matrix, and N consists of normally distributed values with mean 0. An optional parameter gives the variance of N; its default value is 0.04.

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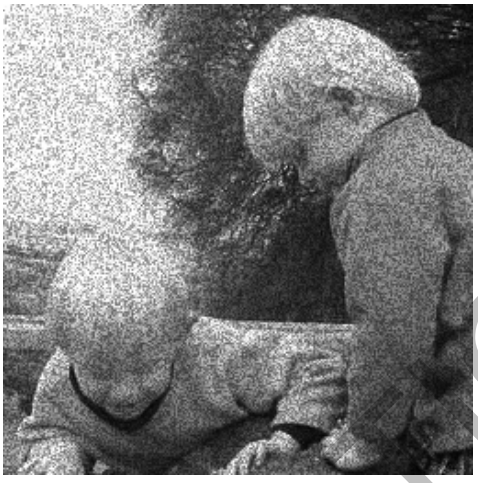
(a) Gaussian noise



(b) Speckle noise

Figure 8.2: The twins image corrupted by Gaussian and speckle noise

Although Gaussian noise and speckle noise appear superficially similar, they are formed by two totally different methods, and, as we shall see, require different approaches for their removal.

Periodic noise

If the image signal is subject to a periodic, rather than a random disturbance, we might obtain an image corrupted by *periodic noise*. The effect is of bars over the image. The function `imnoise` does not have a periodic option, but it is quite easy to create our own, by adding a periodic matrix (using a trigonometric function), to our image:

```
>> s=size(t);
>> [x,y]=meshgrid(1:s(1),1:s(2));
>> p=sin(x/3+y/5)+1;
>> t_pn=(im2double(t)+p/2)/2;
```

and the resulting image is shown in figure 8.3.

Salt and pepper noise, Gaussian noise and speckle noise can all be cleaned by using spatial filtering techniques. Periodic noise, however, requires the use of frequency domain filtering. This is because whereas the other forms of noise can be modelled as *local* degradations, periodic noise is a *global* effect.

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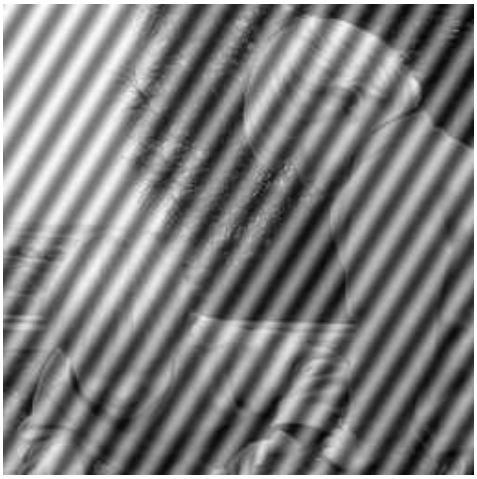

Figure 8.3: The twins image corrupted by periodic noise

8.3 Cleaning salt and pepper noise

Low pass filtering

Given that pixels corrupted by salt and pepper noise are high frequency components of an image, we should expect a low-pass filter should reduce them. So we might try filtering with an average filter:

```
>> a3=fspecial('average');
>> t_sp_a3=filter2(a3,t_sp);
```

and the result is shown in figure 8.4(a). Notice, however, that the noise is not so much removed as "smeared" over the image; the result is not noticeably "better" than the noisy image. The effect is even more pronounced if we use a larger averaging filter:

```
>> a7=fspecial('average',[7,7]);
>> t_sp_a7=filter2(a7,t_sp);
```

and the result is shown in figure 8.4(b).

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(a) 3×3 averaging

(b) 7×7 averaging

Figure 8.4: Attempting to clean salt & pepper noise with average filtering

Median filtering

Median filtering seems almost tailor-made for removal of salt and pepper noise. Recall that the *median* of a set is the middle value when they are sorted. If there are an even number of values, the median is the mean of the middle two. A median filter is an example of a non-linear spatial filter; using a 3×3 mask, the output value is the median of the values in the mask. For example:

$$\begin{array}{|c|c|c|}\hline 50 & 65 & 52 \\ \hline 63 & 255 & 58 \\ \hline 61 & 60 & 57 \\ \hline \end{array} \longrightarrow 50 \quad 52 \quad 57 \quad 58 \quad \boxed{60} \quad 61 \quad 63 \quad 65 \quad 255 \longrightarrow 60$$

The operation of obtaining the median means that very large or very small values—noisy values—will end up at the top or bottom of the sorted list. Thus the median will in general replace a noisy value with one closer to its surroundings.

In MATLAB, median filtering is implemented by the `medfilt2` function:

```
>> t_sp_m3=medfilt2(t_sp);
```

and the result is shown in figure 8.5. The result is a vast improvement on using averaging filters. As with most functions, `medfilt2` takes an optional parameter; in this case a 2 element vector giving the size of the mask to be used.

If we corrupt more pixels with noise:

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Figure 8.5: Cleaning salt and pepper noise with a median filter

```
>> t_sp2=imnoise(t,'salt & pepper',0.2);
```

then `medfilt2` still does a remarkably good job, as shown in figure 8.6. To remove noise completely,

(a) 20% salt & pepper noise

(b) After median fitering

Figure 8.6: Using a 3×3 median filter on more noise

we can either try a second application of the 3×3 median filter, the result of which is shown in figure 8.7(a) or try a 5×5 median filter on the original noisy image:

```
>> t_sp2_m5=medfilt2(t_sp2,[5,5]);
```

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the result of which is shown in figure 8.7(b).

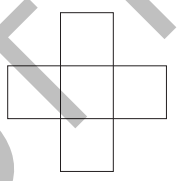
(a) Using `medfilt2` twice

(b) using a 5×5 median filter

Figure 8.7: Cleaning 20% salt & pepper noise with median filtering

Rank-order filtering

Median filtering is a special case of a more general process called *rank-order filtering*. Rather than take the median of a set, we order the set and take the n-th value, for some predetermined value of n. Thus median filtering using a 3×3 mask is equivalent to rank-order filtering with $n = 5$. Similarly, median filtering using a 5×5 mask is equivalent to rank-order filtering with $n = 13$. MATLAB implements rank-order filtering with the `ordfilt2` function; in fact the procedure for `medfilt2` is really just a wrapper for a procedure which calls `ordfilt2`. There is only one reason for using rank-order filtering instead of median filtering, and that is that it allows us to choose the median of non-rectangular masks. For example, if we decided to use as a mask a 3×3 cross shape:



then the median would be the *third* of these values when sorted. The command to do this is

```
>> ordfilt2(t_sp,3,[0 1 0;1 1 1;0 1 0]);
```

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In general, the second argument of `ordfilt2` gives the value of the ordered set to take, and the third element gives the *domain*; the non-zero values of which specify the mask. If we wish to use a cross with size and width 5 (so containing nine elements), we can use:

```
>> ordfilt2(t_sp,5,[0 0 1 0 0;0 0 1 0 0;1 1 1 1 1;0 0 1 0 0;0 0 1 0 0])
```

An outlier method

Applying the median filter can in general be a slow operation: each pixel requires the sorting of at least nine values[1]. To overcome this difficulty, Pratt [23] has proposed the use of cleaning salt and pepper noise by treating noisy pixels as *outliers*; that is, pixels whose grey values are significantly different from those of their neighbours. This leads to the following approach for noise cleaning:

1. Choose a threshold value D.
2. For a given pixel, compare its value p with the mean m of the values of its eight neighbours.
3. If $|p - m| > D$, then classify the pixel as noisy, otherwise not.
4. If the pixel is noisy, replace its value with m; otherwise leave its value unchanged.

There is no MATLAB function for doing this, but it is very easy to write one. First, we can calculate the average of a pixel's eight neighbours by convolving with the linear filter

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.125 & 0.125 & 0.125 \\ 0.125 & 0 & 0.125 \\ 0.125 & 0.125 & 0.125 \end{bmatrix}$$

We can then produce a matrix r consisting of 1's at only those places where the difference of the original and the filter are greater than D; that is, where pixels are noisy. Then $1 - r$ will consist of ones at only those places where pixels are not noisy. Multiplying r by the filter replaces noisy values with averages; multiplying $1 - r$ with original values gives the rest of the output.

A MATLAB function for implementing this is shown in figure 8.8. An immediate problem with the outlier method is that is it not completely automatic—the threshold D must be chosen. An

[1] In fact, this is not the case with MATLAB, which uses a highly optimized method. Nonetheless, we introduce a different method to show that there are other ways of cleaning salt and pepper noise.

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appropriate way to use the outlier method is to apply it with several different thresholds, and choose the value which provides the best results. Suppose we attempt to use the outlier method to clean

```
function res=outlier(im,d)
% OUTLIER(IMAGE,D) removes salt and pepper noise using an outlier method.
% This is done by using the following algorithm:
%
% For each pixel in the image, if the difference between its grey value
% and the average of its eight neighbours is greater than D, it is
% classified as noisy, and its grey value is changed to that of the
% average of its neighbours.
%
% IMAGE can be of type UINT8 or DOUBLE; the output is of type
% UINT8.  The threshold value D must be chosen to be between 0 and 1.

f=[0.125 0.125 0.125; 0.125 0 0.125; 0.125 0.125 0.125];
imd=im2double(im);
imf=filter2(f,imd);
r=abs(imd-imf)-d>0;
res=im2uint8(r.*imf+(1-r).*imd);
```

Figure 8.8: A MATLAB function for cleaning salt and pepper noise using an outlier method

the noise from figure 8.1(b); that is, the twins image with 10% salt and pepper noise. Choosing $D = 0.2$ gives the image in figure 8.9(a). This is not as good a result as using a median filter: the affect of the noise has been lessened, but there are still noise "artifacts" over the image. In this case we have chosen a threshold which is too small. If we choose $D = 0.4$, we obtain the image in figure 8.9(b), which still has some noise artifacts, although in different places. We can see that a lower values of D tends to remove noise from dark areas, and a higher value of D tends to remove noise from light areas. A mid-way value, round about $D = 0.3$ does in fact produce an acceptable result, although not quite as good as median filtering.

Clearly using an appropriate value of D is essential for cleaning salt and pepper noise by this method. If D is too small, then too many "non-noisy" pixels will be classified as noisy, and their values changed to the average of their neighbours. This will result in a blurring effect, similar to that obtained by using an averaging filter. If D is chosen to be too large, then not enough noisy pixels will be classified as noisy, and there will be little change in the output.

The outlier method is not particularly suitable for cleaning large amounts of noise; for such situations the median filter is to be preferred. The outlier method may thus be considered as a

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"quick and dirty" method for cleaning salt and pepper noise when the median filter proves too slow.

(a) $D = 0.2$

(b) $D = 0.4$

Figure 8.9: Applying the outlier method to 10% salt and pepper noise

A further method for cleaning salt & pepper noise will be discussed in chapter 10.

8.4 Cleaning Gaussian noise

Image averaging

It may sometimes happen that instead of just *one* image corrupted with Gaussian noise, we have many different copies of it. An example is satellite imaging; if a satellite passes over the same spot many times, we will obtain many different images of the same place. Another example is in microscopy: we might take many different images of the same object. In such a case a very simple approach to cleaning Gaussian noise is to simply take the average—the mean—of all the images. T

To see why this works, suppose we have 100 copies of our image, each with noise; then the i-th noisy image will be:

$$M + N_i$$

where M is the matrix of original values, and N_i is a matrix of normally distributed random values with mean 0. We can find the mean M' of these images by the usual add and divide method:

$$M' = \frac{1}{100}\sum_{i=1}^{100}(M + N_i)$$

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$$
\begin{aligned}
&= \frac{1}{100}\sum_{i=1}^{100} M + \frac{1}{100}\sum_{i=1}^{100} N_i \\
&= M + \frac{1}{100}\sum_{i=1}^{100} N_i
\end{aligned}
$$

Since N_i is normally distributed with mean 0, it can be readily shown that the mean of all the N_i's will be close to zero—the greater the number of N_i's; the closer to zero. Thus

$$M' \approx M$$

and the approximation is closer for larger number of images $M + N_i$.

We can demonstrate this with the twins image. We first need to create different versions with Gaussian noise, and then take the average of them. We shall create 10 versions. One way is to create an empty three-dimensional array of depth 10, and fill each "level" with a noisy image:

```
>> s=size(t);
>> t_ga10=zeros(s(1),s(2),10);
>> for i=1:10 t_ga10(:,:,i)=imnoise(t,'gaussian'); end
```

Note here that the "`gaussian`" option of `imnoise` calls the random number generator `randn`, which creates normally distributed random numbers. Each time `randn` is called, it creates a *different* sequence of numbers. So we may be sure that all levels in our three-dimensional array do indeed contain different images. Now we can take the average:

```
>> t_ga10_av=mean(t_ga10,3);
```

The optional parameter 3 here indicates that we are taking the mean along the *third* dimension of our array. The result is shown in figure 8.10(a). This is not quite clear, but is a vast improvement on the noisy image of figure 8.2(a). An even better result is obtained by taking the average of 100 images; this can be done by replacing 10 with 100 in the commands above, and the result is shown in figure 8.10(b). Note that this method only works if the Gaussian noise has mean 0.

Average filtering

If the Gaussian noise has mean 0, then we would expect that an average filter would average the noise to 0. The larger the size of the filter mask, the closer to zero. Unfortunately, averaging tends

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(a) 10 images

(b) 100 images

Figure 8.10: Image averaging to remove Gaussian noise

to blur an image, as we have seen in chapter 5. However, if we are prepared to trade off blurring for noise reduction, then we can reduce noise significantly by this method.

Suppose we take the 3×3 and 5×5 averaging filters, and apply them to the noisy image `t_ga`.

```
>> a3=fspecial('average');
>> a5=fspecial('average',[5,5]);
>> tg3=filter2(a3,t_ga);
>> tg5=filter2(a5,t_ga);
```

The results are shown in figure 8.11. The results are not really particularly pleasing; although there has been some noise reduction, the "smeary" nature of the resulting images is unattractive.

Adaptive filtering

Adaptive filters are a class of filters which change their characteristics according to the values of the greyscales under the mask; they may act more like median filters, or more like average filters, depending on their position within the image. Such a filter can be used to clean Gaussian noise by using local statistical properties of the values under the mask.

One such filter is the *minimum mean-square error filter*; this is a non-linear spatial filter; and as with all spatial filters, is implemented by applying a function to the grey values under the mask.

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(a) 4×3 averaging

(b) 5×5 averaging

Figure 8.11: Using averaging filtering to remove Gaussian noise

Since we are dealing with additive noise, our noisy image M' can be written as

$$M' = M + N$$

where M is the original correct image, and N is the noise; which we assume to be normally distributed with mean 0. However, within our mask, the mean may not be zero; suppose the mean is m_f, and the variance in the mask is σ_f^2. Suppose also that the variance of the noise over the entire image is known to be σ_g^2. Then the output value can be calculated as

$$m_f + \frac{\sigma_f^2}{\sigma_f^2 + \sigma_g^2}(g - m_f)$$

where g is the current value of the pixel in the noisy image. Note that if the local variance σ_f^2 is high, then the fraction will be close to 1, and the output close to the original image value g. This is appropriate, as high variance implies high detail such as edges, which should be preserved. Conversely, if the local variance is low, such as in a background area of the image, the fraction is close to zero, and the value returned is close to the mean value m_f. See Lim [20] for details.

Another version of this filter [34] has output defined by

$$g - \frac{\sigma_g^2}{\sigma_f^2}(g - m_f)$$

and again the filter returns a value close to either g or m_f depending on whether the local variance is high or low.

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In practice, m_f can be calculated by simply taking the mean of all grey values under the mask, and σ_f^2 by calculating the variance of all grey values under the mask. The value σ_g^2 may not necessarily be known, so a slight variant of the first filter may be used:

$$m_f + \frac{\max\{0, \sigma_f^2 - n\}}{\max\{\sigma_f^2, n\}}(g - m_f)$$

where n is the computed noise variance, and is calculated by taking the mean of all values of σ_f^2 over the entire image. This particular filter is implemented in MATLAB with the function `wiener2`. The name reflects the fact that this filter attempts to minimize the square of the difference between the input and output images; such filters are in general known as *Wiener filters*. However, Wiener filters are more usually applied in the frequency domain; see section 8.7 below.

Suppose we take the noisy image shown in figure 8.2(a), and attempt to clean this image with adaptive filtering. We will use the `wiener2` function, which can take an optional parameter indicating the size of the mask to be used. The default size is 3×3. We shall create four images:

```
>> t1=wiener2(t_ga);
>> t2=wiener2(t_ga,[5,5]);
>> t3=wiener2(t_ga,[7,7]);
>> t4=wiener2(t_ga,[9,9]);
```

and these are shown in figure 8.12. Being a low pass filter, adaptive filtering does tend to blur edges and high frequency components of the image. But it does a far better job than using a low pass blurring filter.

We can achieve very good results for noise where the variance is not as high as that in our current image.

```
>> t2=imnoise(t,'gaussian',0,0.005);
>> imshow(t2)
>> t2w=wiener2(t2,[7,7]);
>> figure,imshow(t2w)
```

The image and its appearance after adaptive filtering as shown in figure 8.13. The result is a great improvement over the original noisy image. Notice in each case that there may be some blurring of

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(a) 3×3 filtering

(b) 5×5 filtering

(a) 7×7 filtering

(b) 9×9 filtering

Figure 8.12: Examples of adaptive filtering to remove Gaussian noise

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Figure 8.13: Using adaptive filtering to remove Gaussian noise with low variance

the background, but the edges are preserved well, as predicted by our analysis of the adaptive filter formulas above.

8.5 Removal of periodic noise

Periodic noise may occur if the imaging equipment (the acquisition or networking hardware) is subject to electronic disturbance of a repeating nature, such as may be caused by an electric motor. We can easily create periodic noise by overlaying an image with a trigonometric function:

```
>> [x,y]=meshgrid(1:256,1:256);
>> p=1+sin(x+y/1.5);
>> tp=(double(t)/128+p)/4;
```

where `cm` is the cameraman image from previous sections. The second line simply creates a sine function, and adjusts its output to be in the range 0–2. The last line first adjusts the cameraman image to be in the same range; adds the sine function to it, and divides by 4 to produce a matrix of type `double` with all elements in the range 0.0–1.0. This can be viewed directly with `imshow`, and it is shown in figure 8.14(a). We can produce its shifted DFT and this is shown in figure 8.14(b). The extra two "spikes" away from the centre correspond to the noise just added. In general the tighter the period of the noise, the further from the centre the two spikes will be. This is because a

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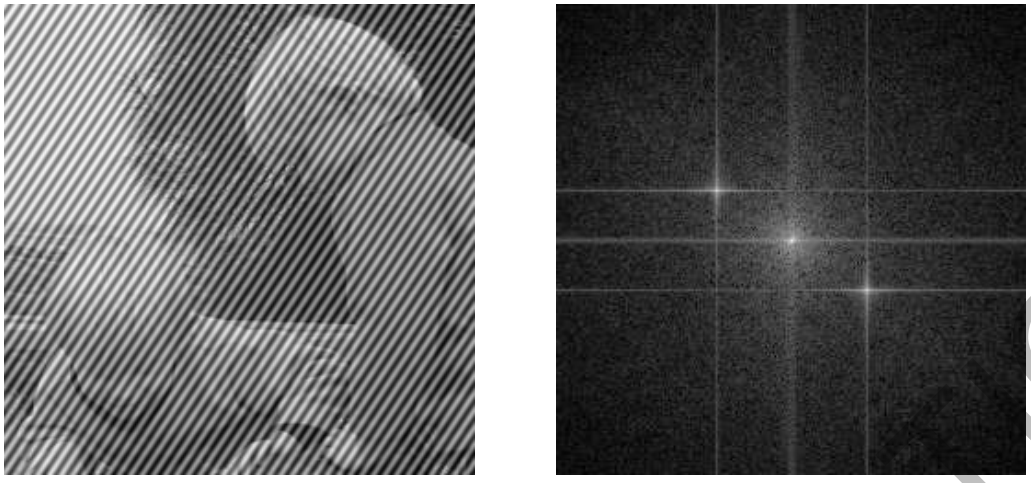

(a) (b)

Figure 8.14: The twins image (a) with periodic noise, and (b) its transform

small period corresponds to a high frequency (large change over a small distance), and is therefore further away from the centre of the shifted transform.

We will now remove these extra spikes, and invert the result. If we put `pixval on` and move around the image, we find that the spikes have row, column values of $(156, 170)$ and $(102, 88)$. These have the same distance from the centre: 49.0918. We can check this by

```
>> z=sqrt((x-129).^2+(y-129).^2);
>> z(156,170)
>> z(102,88)
```

There are two methods we can use to eliminate the spikes, we shall look at both of them.

Band reject filtering. We create a filter consisting of ones with a ring of zeroes; the zeroes lying at a radius of 49 from the centre:

```
>> br=(z < 47 | z > 51);
```

where `z` is the matrix consisting of distances from the origin. This particular ring will have a thickness large enough to cover the spikes. Then as before, we multiply this by the transform:

```
>> tbr=tf.*br;
```

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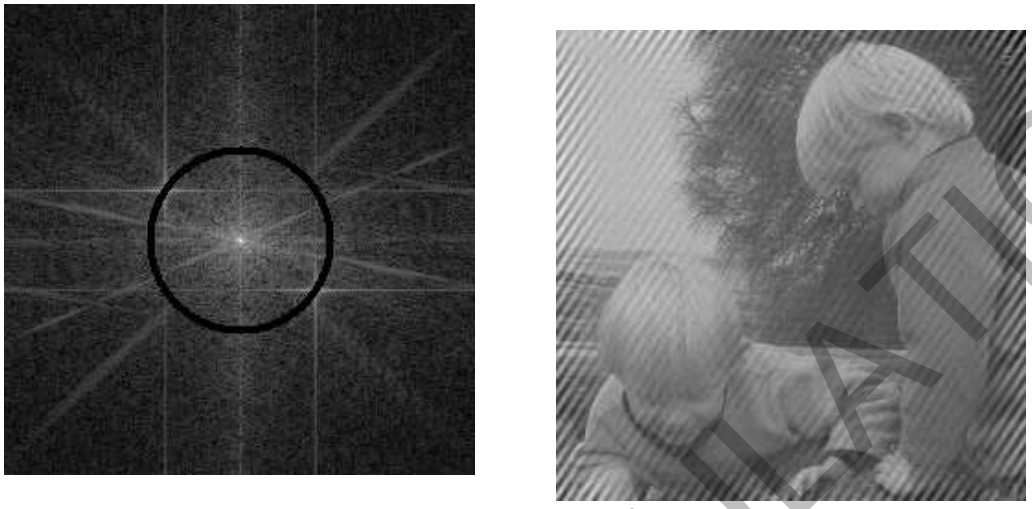
and this is shown in figure 8.15(a). The result is that the spikes have been blocked out by this filter. Taking the inverse transform produces the image shown in figure 8.15(b). Note that not all



(a) A band-reject filter (b) After inversion

Figure 8.15: Removing periodic noise with a band-reject filter

the noise has gone, but a significant amount has, especially in the centre of the image.

Notch filtering. With a notch filter, we simply make the rows and columns of the spikes zero:

```
>> tf(156,:)=0;
>> tf(102,:)=0;
>> tf(:,170)=0;
>> tf(:,88)=0;
```

and the result is shown in figure 8.16(a). The image after inversion is shown in figure 8.16(b). As before, much of the noise in the centre has been removed. Making more rows and columns of the transform zero would result in a larger reduction of noise.

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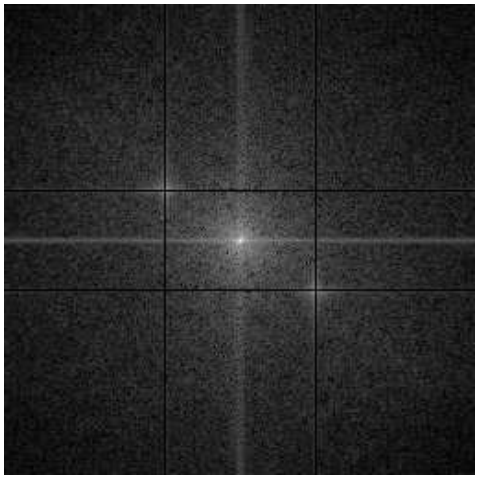

(a) A notch filter

(b) After inversion

Figure 8.16: Removing periodic noise with a notch filter

8.6 Inverse filtering

We have seen that we can perform filtering in the Fourier domain by multiplying the DFT of an image by the DFT of a filter: this is a direct use of the convolution theorem. We thus have

$$Y(i,j) = X(i,j)F(i,j)$$

where X is the DFT of the image; F is the DFT of the filter, and Y is the DFT of the result. If we are given Y and F, then we should be able to recover the (DFT of the) original image X simply by dividing by F:

$$X(i,j) = \frac{Y(i,j)}{F(i,j)}. \tag{8.1}$$

Suppose, for example we take the wombats image `wombats.tif`, and blur it using a low-pass Butterworth filter:

```
>> w=imread('wombats.tif');
>> wf=fftshift(fft2(w));
>> b=lbutter(w,15,2);
>> wb=wf.*b;
>> wba=abs(ifft2(wb));
>> wba=uint8(255*mat2gray(wba));
```

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```
>> imshow(wba)
```

The result is shown on the left in figure 8.17. We can attempt to recover the original image by dividing by the filter:

```
>> w1=fftshift(fft2(wba))./b;
>> w1a=abs(ifft2(w1));
>> imshow(mat2gray(w1a))
```

and the result is shown on the right in figure 8.17. This is no improvement! The trouble is that

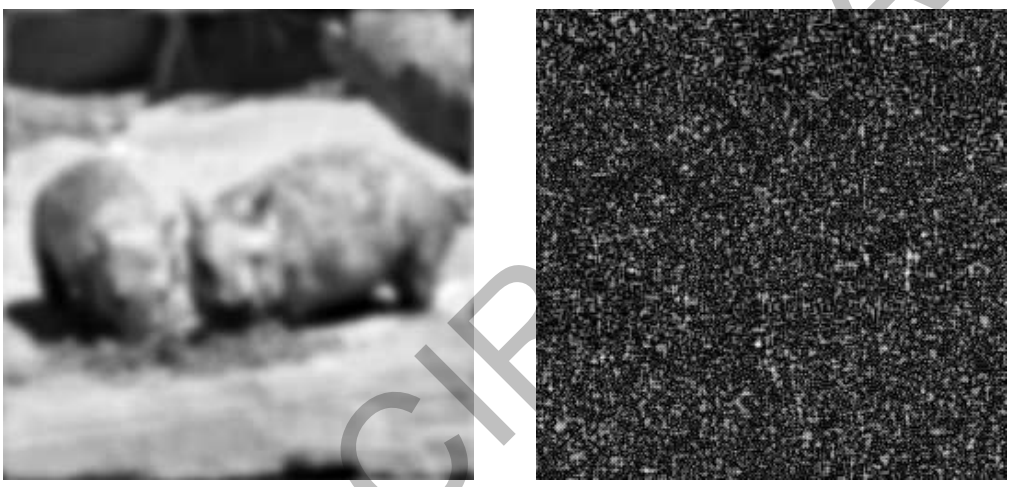

Figure 8.17: An attempt at inverse filtering

some elements of the Butterworth matrix are very small, so dividing produces very large values which dominate the output. We can deal with this problem in two ways:

1. Apply a low pass filter L to the division:

$$X(i,j) = \frac{Y(i,j)}{F(i,j)} L(i,j).$$

   This should eliminate very low (or zero) values.

2. "Constrained division": choose a threshold value d, and if $|F(i,j)| < d$, we don't perform a

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division, but just keep our original value. Thus:

$$X(i,j) = \begin{cases} \dfrac{Y(i,j)}{F(i,j)} & \text{if } |F(i,j)| \geq d, \\ \\ Y(i,j) & \text{if } |F(i,j)| < d. \end{cases}$$

We can apply the first method by multiplying a Butterworth low pass filter to the matrix `c1` above:

```
>> wbf=fftshift(fft2(wba));
>> w1=(wbf./b).*lbutter(w,40,10);
>> w1a=abs(ifft2(w1));
>> imshow(mat2gray(w1a))
```

Figure 8.18 shows the results obtained by using a different cutoff radius of the Butterworth filter each time: (a) uses 40 (as in the MATLAB commands just given); (b) uses 60; (c) uses 80, and (d) uses 100. It seems that using a low pass filter with a cutoff round about 60 will yield the best results. After we use larger cutoffs, the result degenerates.

We can try the second method; to implement it we simply make all values of the filter which are too small equal to 1:

```
>> d=0.01;
>> b=lbutter(w,15,2);b(find(b<d))=1;
>> w1=fftshift(fft2(wba))./b;
>> w1a=abs(ifft2(w1));
>> imshow(mat2gray(w1a))
```

Figure 8.19 shows the results obtained by using a different cutoff radius of the Butterworth filter each time: (a) uses $d = 0.01$ (as in the MATLAB commands just given); (b) uses $d = 0.005$; (c) uses $d = 0.002$, and (d) uses $d = 0.001$. It seems that using a threshold d in the range $0.002 \leq d \leq 0.005$ produces reasonable results.

Motion deblurring

We can consider the removal of blur caused by motion to be a special case of inverse filtering. Suppose we take an image and blur it by a small amount.

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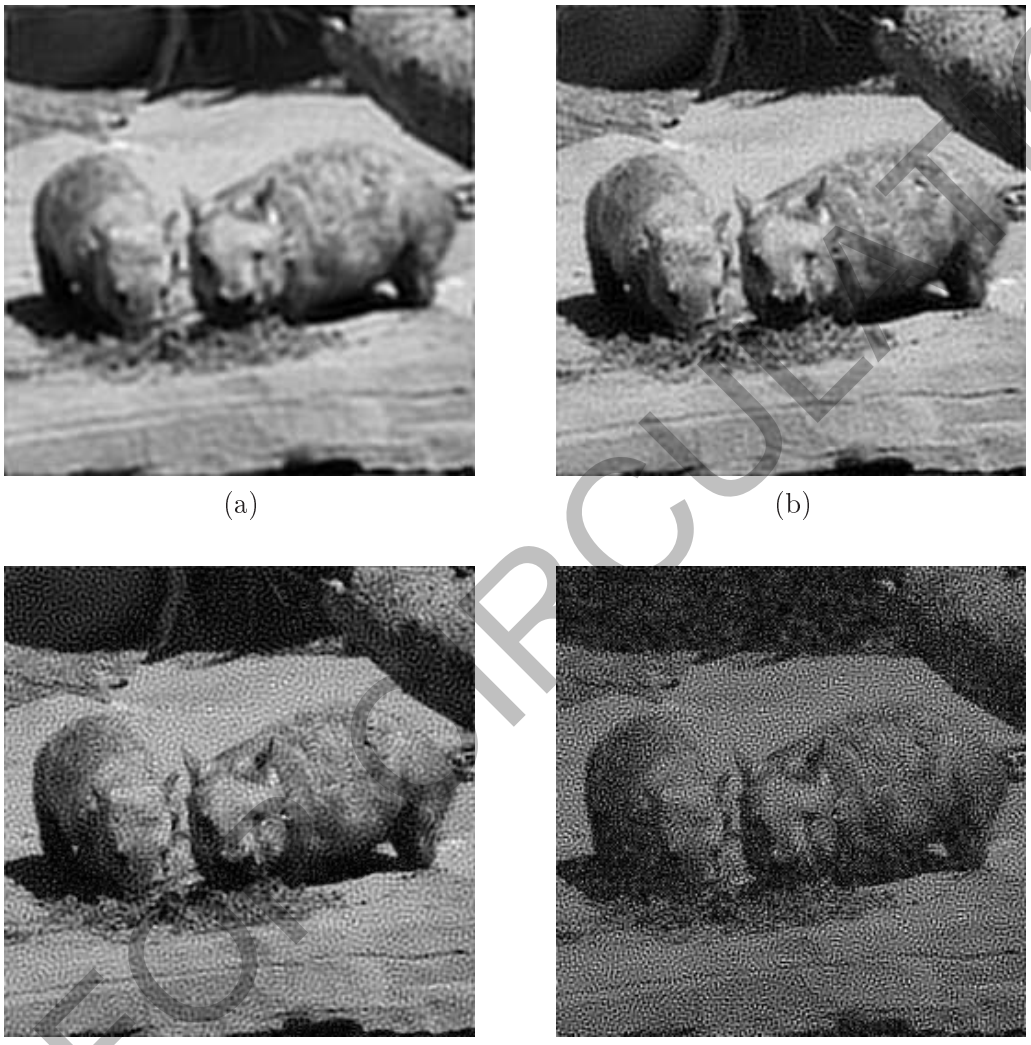

(a) (b) (c) (d)

Figure 8.18: Inverse filtering using low pass filtering to eliminate zeros

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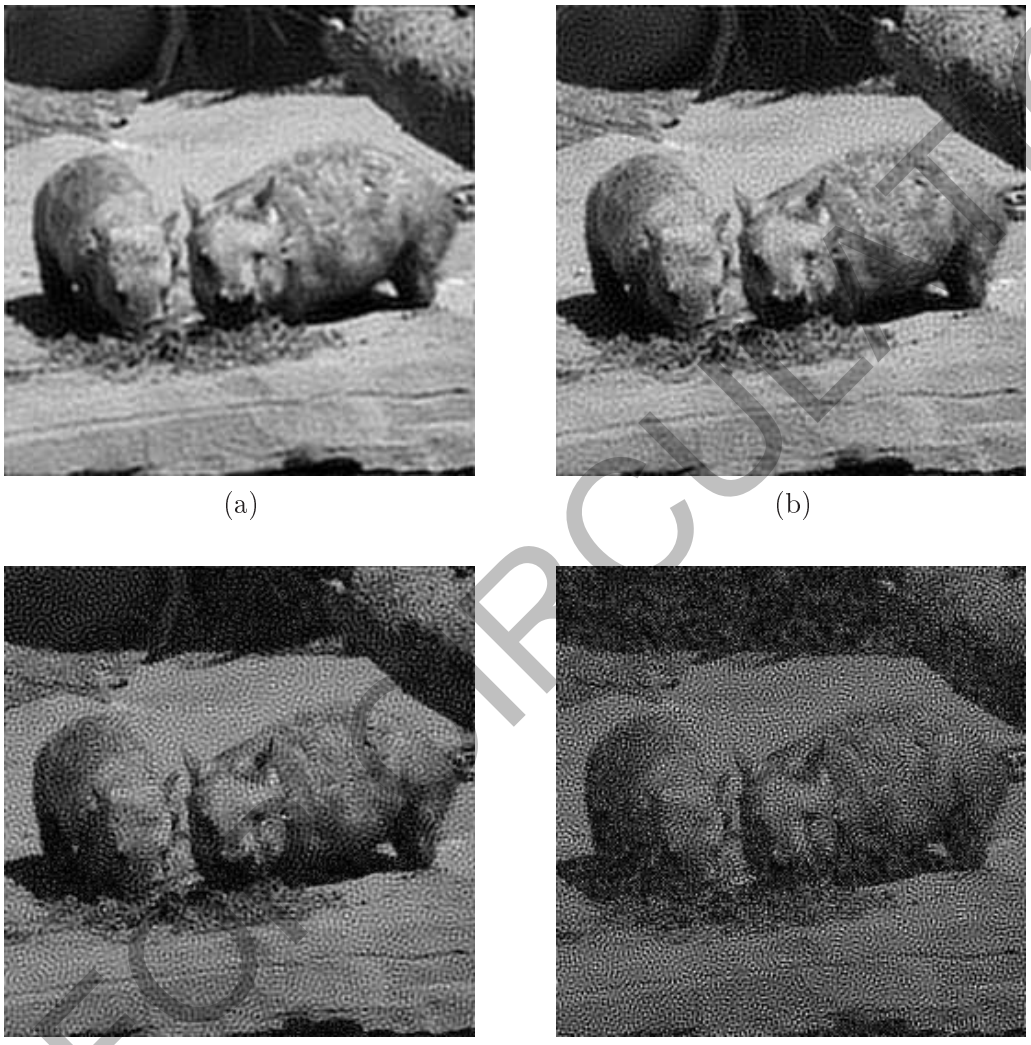

(a) (b)

(c) (d)

Figure 8.19: Inverse filtering using constrained division

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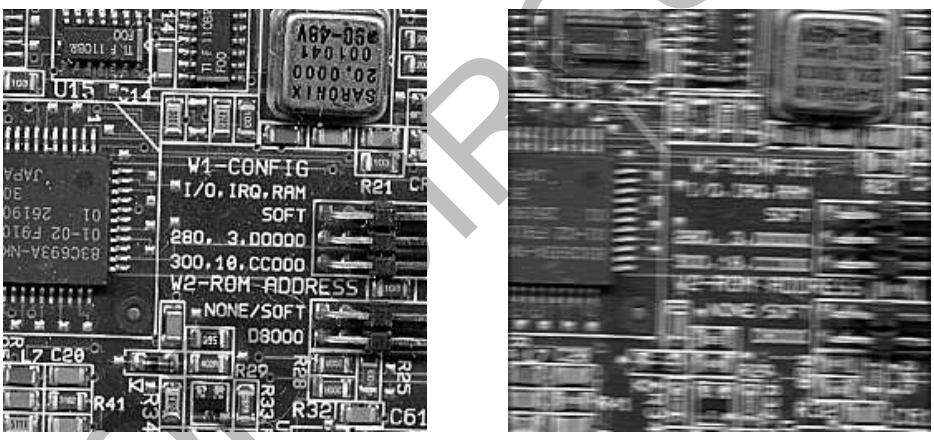
```
>> bc=imread('board.tif');
>> bg=im2uint8(rgb2gray(bc));
>> b=bg(100:355,50:305);
>> imshow(b)
```

These commands simply take the colour image of a circuit board (the image `board.tif`), makes a greyscale version of data type `uint8`, and picks out a square subimage. The result is shown as figure 8.20(a). To blur it, we can use the `blur` parameter of the `fspecial` function.

```
>> m=fspecial('motion',7,0);
>> bm=imfilter(b,m);
>> imshow(bm)
```

and the result is shown as figure 8.20(b). The result of the blur has effectively obliterated the text



(a) (b)

Figure 8.20: The result of motion blur

on the image.

To deblur the image, we need to divide its transform by the transform corresponding to the blur filter. This means that we first must create a matrix corresponding to the transform of the blur:

```
>> m2=zeros(256,256);
>> m2(1,1:7)=m;
>> mf=fft2(m2);
```

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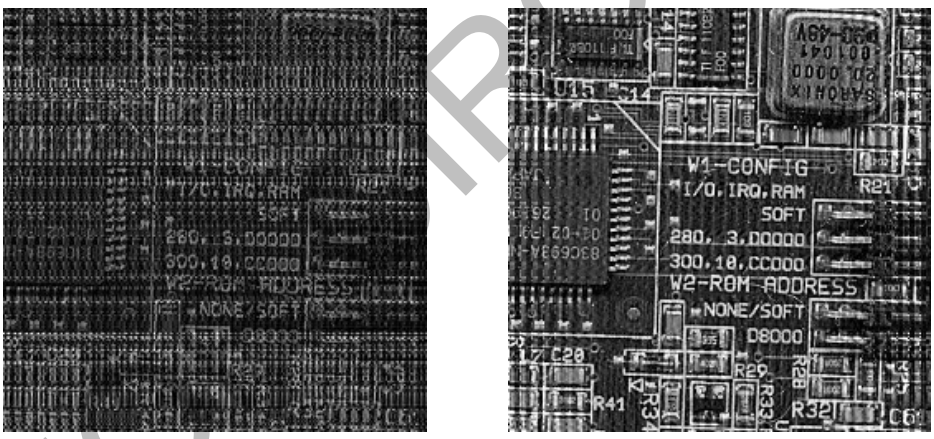
Now we can attempt to divide by this transform.

```
>> bmi=ifft2(fft2(bm)./mf);
>> fftshow(bmi,'abs')
```

and the result is shown in figure 8.21(a). As with inverse filtering, the result is not particularly good, because the values close to zero in the matrix `mf` have tended to dominate the result. As above, we can constrain the division by only dividing by values which are above a certain threshold.

```
>> d=0.02;
>> mf=fft2(m2);mf(find(abs(mf)<d))=1;
>> bmi=ifft2(fft2(bm)./mf);
>> imshow(mat2gray(abs(bmi))*2)
```

where the last multiplication by 2 just brightens the result, which is shown in figure 8.21(b). The



(a) Straight division (b) Constrained division

Figure 8.21: Attempts at removing motion blur

writing, especially in the centre of the image, is now quite legible.

8.7 Wiener filtering

As we have seen from the previous section, inverse filtering does not necessarily produce particularly pleasing results. The situation is even worse if the original image has been corrupted by noise. Here

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we would have an image X filtered with a filter F and corrupted by noise N. If the noise is additive (for example, Gaussian noise), then the linearity of the Fourier transform gives us

$$Y(i,j) = X(i,j)F(i,j) + N(i,j)$$

and so

$$X(i,j) = \frac{Y(i,j) - N(i,j)}{F(i,j)}$$

as we have seen in the introduction to this chapter. So not only do we have the problem of dividing by the filter, we have the problem of dealing with noise. In such a situation the presence of noise can have a catastrophic effect on the inverse filtering: the noise can completely dominate the output, making direct inverse filtering impossible.

To introduce Wiener filtering, we shall discuss a more general question: given a degraded image M' of some original image M and a restored version R, what measure can we use to say whether our restoration has done a good job? Clearly we would like R to be as close as possible to the "correct" image M. One way of measuring the closeness of R to M is by adding the squares of all differences:

$$\sum (m_{i,j} - r_{i,j})^2$$

where the sum is taken over all pixels of R and M (which we assume to be of the same size). This sum can be taken as a measure of the closeness of R to M. If we can minimize this value, we may be sure that our procedure has done as good a job as possible. Filters which operate on this principle of *least squares* are called *Wiener filters*. We can obtain X by

$$X(i,j) \approx \left[\frac{1}{F(i,j)} \frac{|F(i,j)|^2}{|F(i,j)|^2 + K} \right] Y(i,j) \tag{8.2}$$

where K is a constant [7]. This constant can be used to approximate the amount of noise: if the variance σ^2 of the noise is known, then $K = 2\sigma^2$ can be used. Otherwise, K can be chosen interactively (in other words, by trial and error) to yield the best result. Note that if $K = 0$, then equation 8.2 reduces to equation 8.1.

We can easily implement equation 8.2:

```
>> K=0.01;
>> wbf=fftshift(fft2(wba));
```

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```
>> w1=wbf.*(abs(b).^2./(abs(b).^2+K)./b);  % This is the equation
>> w1a=abs(ifft2(w1));
>> imshow(mat2gray(w1a))
```

The result is shown in figure 8.22(a). Images (b), (c) and (d) in this figure show the results with $K = 0.001$, $K = 0.0001$ and $K = 0.00001$ respectively. Thus as K becomes very small, noise starts

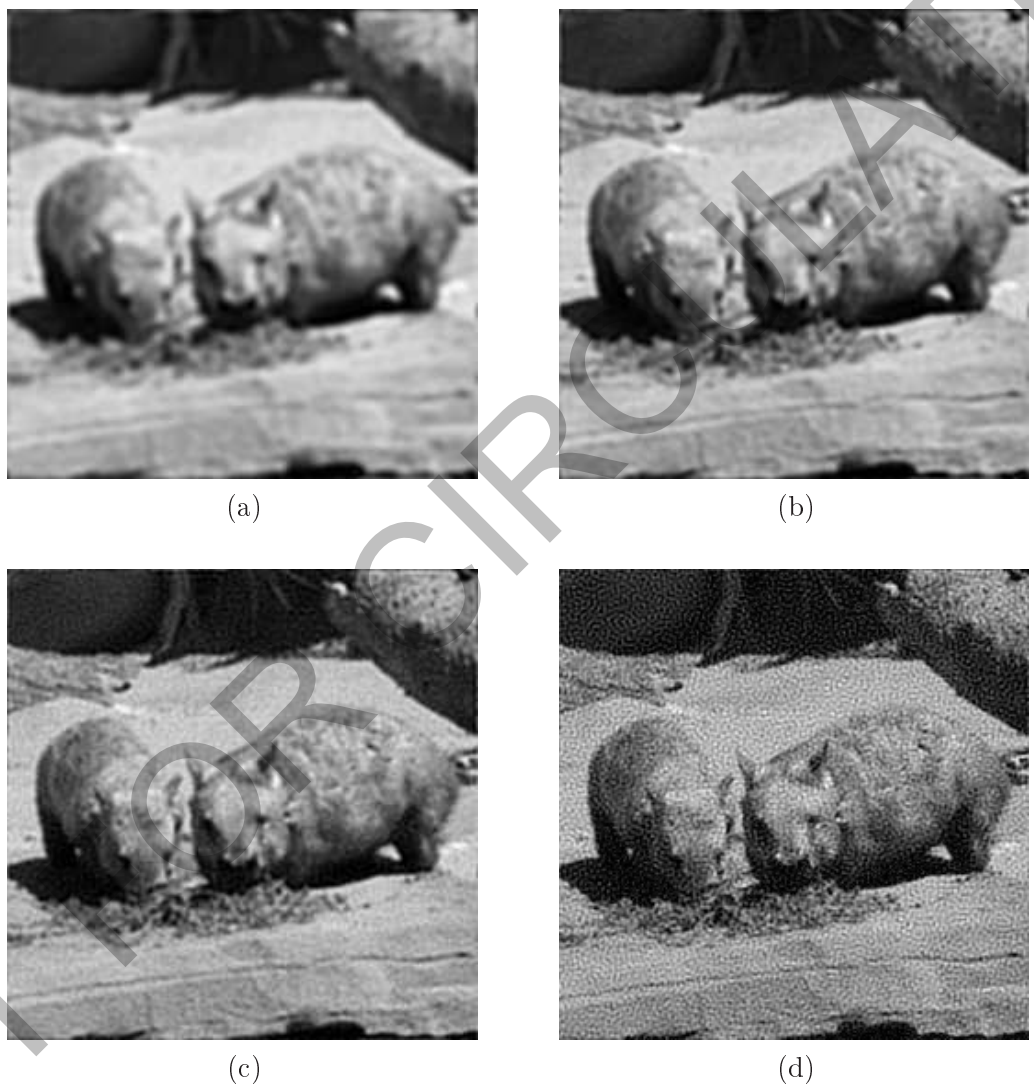

Figure 8.22: Wiener filtering

to dominate the image.

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Exercises

1. The arrays below represent small greyscale images. Compute the 4×4 image that would result in each case if the middle 16 pixels were transformed using a 3×3 median filter:

8	17	4	10	15	12
10	12	15	7	3	10
15	10	50	5	3	12
4	8	11	4	1	8
16	7	4	3	0	7
16	24	19	3	20	10

1	1	2	5	3	1
3	20	5	6	4	6
4	6	4	20	2	2
4	3	3	5	1	5
6	5	20	2	20	2
6	3	1	4	1	2

7	8	11	12	13	9
8	14	0	9	7	10
11	23	10	14	1	8
14	7	11	8	9	11
13	13	18	10	7	12
9	11	14	12	13	10

2. Using the same images as in question 1, transform them by using a 3×3 averaging filter.

3. Use the outlier method to find noisy pixels in each of the images given in question 1. What are the reasonable values to use for the difference between the grey value of a pixel and the average of its eight 8-neighbours?

4. Pratt [23] has proposed a "pseudo-median" filter, in order to overcome some of the speed disadvantages of the median filter. For example, given a five element sequence $\{a, b, c, d, e\}$, its pseudo-median is defined as

$$\begin{aligned} \text{psmed}(a,b,c,d,e) &= \tfrac{1}{2}\max\Big[\min(a,b,c)+\min(b,c,d)+\min(c,d,e)\Big] \\ &+ \tfrac{1}{2}\min\Big[\max(a,b,c)+\max(b,c,d)+\max(c,d,e)\Big] \end{aligned}$$

So for a sequence of length 5, we take the maxima and minima of all subsequences of length three. In general, for an odd-length sequence L of length $2n+1$, we take the maxima and minima of all subsequences of length $n+1$.

We can apply the pseudo-median to 3×3 neighbourhoods of an image, or cross-shaped neighbourhoods containing 5 pixels, or any other neighbourhood with an odd number of pixels.

Apply the pseudo-median to the images in question 1, using 3×3 neighbourhoods of each pixel.

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5. Write a MATLAB function to implement the pseudo-median, and apply it to the images above with the `nlfilter` function. Does it produce a good result?

6. Produce a grey subimage of the colour image `flowers.tif` by

```
>> f=imread('flowers.tif');
>> fg=rgb2gray(f);
>> f=im2uint8(f(30:285,60:315));
```

   Add 5% salt & pepper noise to the image. Attempt to remove the noise with

   (a) average filtering,

   (b) median filtering,

   (c) the outlier method,

   (d) pseudo-median filtering.

   Which method gives the best results?

7. Repeat the above question but with 10%, and then with 20% noise.

8. For 20% noise, compare the results with a 5×5 median filter, and two applications of a 3×3 median filter.

9. Add Gaussian noise to the greyscale flowers image with the following parameters:

   (a) mean 0, variance 0.01 (the default),

   (b) mean 0, variance 0.02,

   (c) mean 0, variance 0.05,

   (d) mean 0, variance 0.1.

   In each case, attempt to remove the noise with average filtering and with Wiener filtering.

   Can you produce satisfactory results with the last two noisy images?

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10. Gonzalez and Woods [7] mention the use of a *midpoint filter* for cleaning Gaussian noise. This is defined by

$$g(x,y) = \frac{1}{2}\Big(\max_{(x,y)\in B} f(x,y) + \min_{(x,y)\in B} f(x,y)\Big)$$

where the maximum and minimum are taken over all pixels in a neighbourhood B of (x,y). Use `ordfilt2` to find maxima and minima, and experiment with this approach to cleaning Gaussian noise, using different variances. Visually, how to the results compare with spatial Wiener filtering or using a blurring filter?

11. In chapter 5 we defined the alpha-trimmed mean filter, and the geometric mean filter. Using either `nlfilter` or `ordfilt2`, write MATLAB functions to implement these filters, and apply them to images corrupted with Gaussian noise.

    How well do they compare to average filtering, image averaging, or adaptive filtering?

12. Add the sine waves to the engineer face using the same commands as for the cameraman:

```
>> [x,y]=meshgrid(1:256,1:256);
>> s=1+sin(x+y/1.5);
>> ep=(double(en)/128+s)/4;
```

    Now attempt to remove the noise using band-reject filtering or notch filtering. Which one gives the best result?

13. For each of the following sine commands:

    (a) `s=1+sin(x/3+y/5);`

    (b) `s=1+sin(x/5+y/1.5);`

    (c) `s=1+sin(x/6+y/6);`

    add the sine wave to the image as shown in the previous question, and attempt to remove the resulting periodic noise using band-reject filtering or notch filtering.

    Which of the three is easiest to "clean up"?

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14. Apply a 5×5 blurring filter to the cameraman image with `imfilter`. Attempt to deblur the result using inverse filtering with constrained division. Which threshold gives the best results?

15. Repeat the previous question using a 7×7 blurring filter.

16. Work through the motion deblurring example, experimenting with different values of the threshold. What gives the best results?

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Chapter 9

Image Segmentation

9.1 Introduction

Segmentation refers to the operation of partitioning an image into component parts, or into separate objects. In this chapter, we shall investigate two very important topics: thresholding, and edge detection.

9.2 Thresholding

9.2.1 Single thresholding

A greyscale image is turned into a binary (black and white) image by first choosing a grey level T in the original image, and then turning every pixel black or white according to whether its grey value is greater than or less than T:

$$\text{A pixel becomes} \begin{cases} \text{white if its grey level is} > T, \\ \text{black if its grey level is} \leq T. \end{cases}$$

Thresholding is a vital part of image *segmentation*, where we wish to isolate objects from the background. It is also an important component of robot vision.

Thresholding can be done very simply in MATLAB. Suppose we have an 8 bit image, stored as the variable `X`. Then the command

```
X>T
```


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will perform the thresholding. We can view the result with `imshow`. For example, the commands

```
>> r=imread('rice.tif');
>> imshow(r),figure,imshow(r>110)
```

will produce the images shown in figure 9.1. The resulting image can then be further processed to

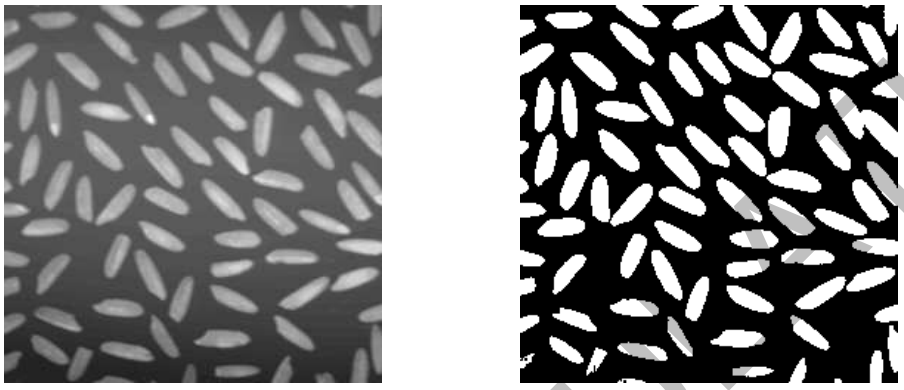

Figure 9.1: Thresholded image of rice grains

find the number, or average size of the grains.

To see how this works, recall that in MATLAB, an operation on a single number, when applied to a matrix, is interpreted as being applied simultaneously to all elements of the matrix; this is vectorization, which we have seen in chapter A. The command `X>T` will thus return 1 (for true) for all those pixels for which the grey values are greater than `T`, and 0 (for false) for all those pixels for which the grey values are less than or equal to `T`. We thus end up with a matrix of 0's and 1's, which can be viewed as a binary image.

The rice image shown above has light grains on a dark background; an image with dark objects over a light background may be treated the same::

```
>> b=imread('bacteria.tif');
>> imshow(b),figure,imshow(b>100)
```

will produce the images shown in figure 9.2.

As well as the above method, MATLAB has the `im2bw` function, which thresholds an image *of any data type*, using the general syntax

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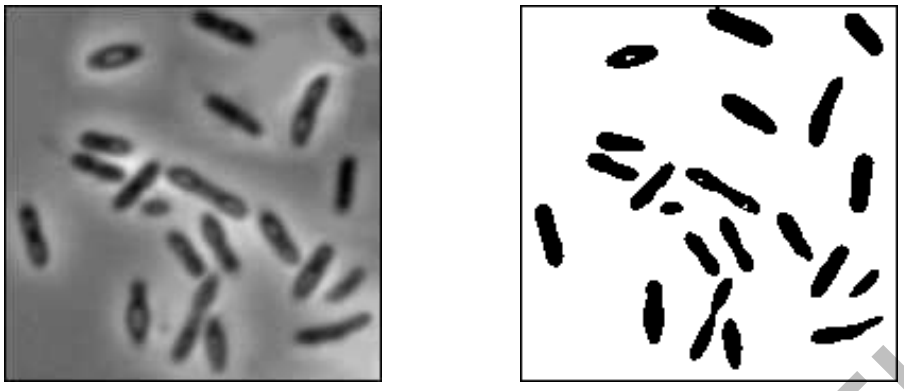

Figure 9.2: Thresholded image of bacteria

```
im2bw(image,level)
```

where `level` is a value between 0 and 1 (inclusive), indicating the fraction of grey values to be turned white. This command will work on greyscale, coloured and indexed images of data type `uint8`, uint16 or `double`. For example, the thresholded rice and bacteria images above could be obtained using

```
>> im2bw(r,0.43);
>> im2bw(b,0.39);
```

The `im2bw` function automatically scales the value `level` to a grey value appropriate to the image type, and then performs a thresholding by our first method.

As well as isolating objects from the background, thresholding provides a very simple way of showing hidden aspects of an image. For example, the image `paper.tif` appears all white, as nearly all the grey values are very high. However, thresholding at a high level produces an image of far greater interest. We can use the commands

```
>> p=imread('paper1.tif');
>> imshow(p),figure,imshow(p>241)
```

to provide the images shown in figure 9.3.

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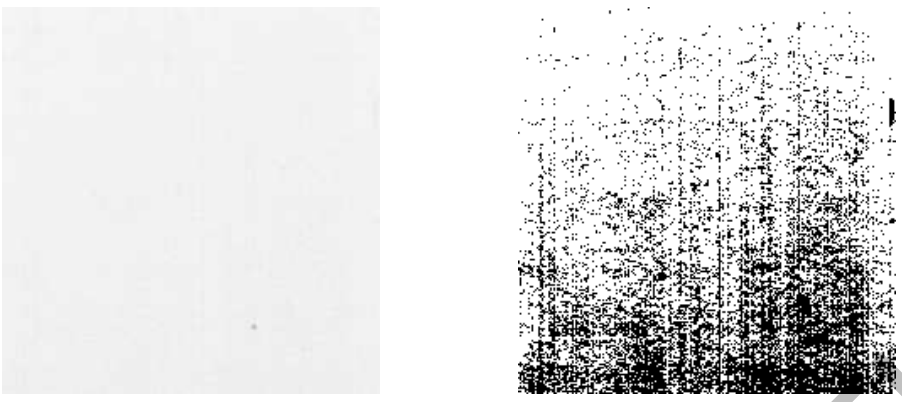

Figure 9.3: The paper image and result after thresholding

9.2.2 Double thresholding

Here we choose two values T_1 and T_2 and apply a thresholding operation as:

$$\text{A pixel becomes} \begin{cases} \text{white if its grey level is between } T_1 \text{ and } T_2, \\ \text{black if its grey level is otherwise.} \end{cases}$$

We can implement this by a simple variation on the above method:

```
X>T1 & X<T2
```

Since the ampersand acts as a logical "and", the result will only produce a one where both inequalities are satisfied. Consider the following sequence of commands, which start by producing an 8-bit grey version of the indexed image `spine.tif`:

```
>> [x,map]=imread('spine.tif');
>> s=uint8(256*ind2gray(x,map));
>> imshow(s),figure,imshow(s>115 & s<125)
```

The output is shown in figure 9.4. Note how double thresholding brings out subtle features of the spine which single thresholding would be unable to do. We can obtain similar results using `im2bw`:

```
imshow(im2bw(x,map,0.45)&~{}im2bw(x,map,0.5))}
```

but this is somewhat slower because of all the extra computation involved when dealing with an indexed image.

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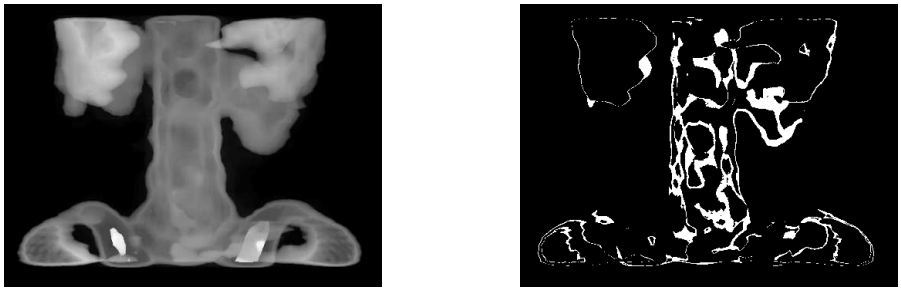

Figure 9.4: The image `spine.tif` an the result after double thresholding

9.3 Applications of thresholding

We have seen that thresholding can be useful in the following situations:

1. When we want to remove unnecessary detail from an image, to concentrate on essentials. Examples of this were given in the rice and bacteria images: by removing all grey level information, the rice and bacteria were reduced to binary blobs. But this information may be all we need to investigate sizes, shapes, or numbers of blobs.

2. To bring out hidden detail. This was illustrated with paper and spine images. In both, the detail was obscured because of the similarity of the grey levels involved.

But thresholding can be vital for other purposes. We list a few more:

3. When we want to remove a varying background from text or a drawing. We can simulate a varying background by taking the image `text.tif` and placing it on a random background. This can be easily implemented with some simple MATLAB commands:

```
>> r=rand(256)*128+127;
>> t=imread('text.tif');
>> tr=uint8(r.*double(not(t));
>> imshow(tr)
```

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The first command simply uses the `rand` function (which produces matrices of uniformly generated random numbers between 0.0 and 1.0), and scales the result so the random numbers are between 127 and 255. We then read in the text image, which shows white text on a dark background.

The third command does several things at once: `not(t)` reverses the text image so as to have black text on a white background; `double` changes the numeric type so that the matrix can be used with arithmetic operations; finally the result is multiplied into the random matrix, and the whole thing converted to `uint8` for display. The result is shown on the left in figure 9.5.

If we threshold this image and display the result with

```
>> imshow(tr>100)
```

the result is shown on the right in figure 9.5, and the background has been completely removed.

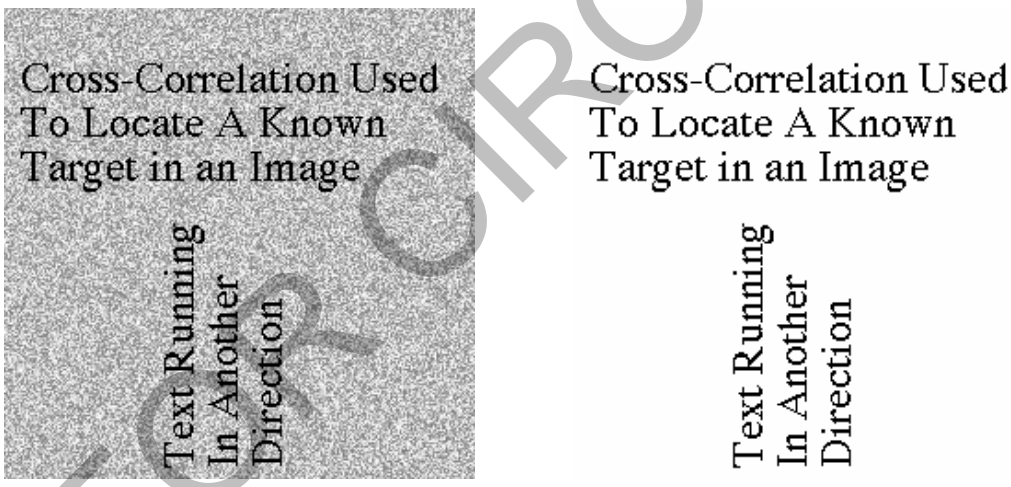

Figure 9.5: Text on a varying background, and thresholding

9.4 Choosing an appropriate threshold value

We have seen that one of the important uses of thresholding is to isolate objects from their background. We can then measure the sizes of the objects, or count them. Clearly the success of these operations depends very much on choosing an appropriate threshold level. If we choose a value too

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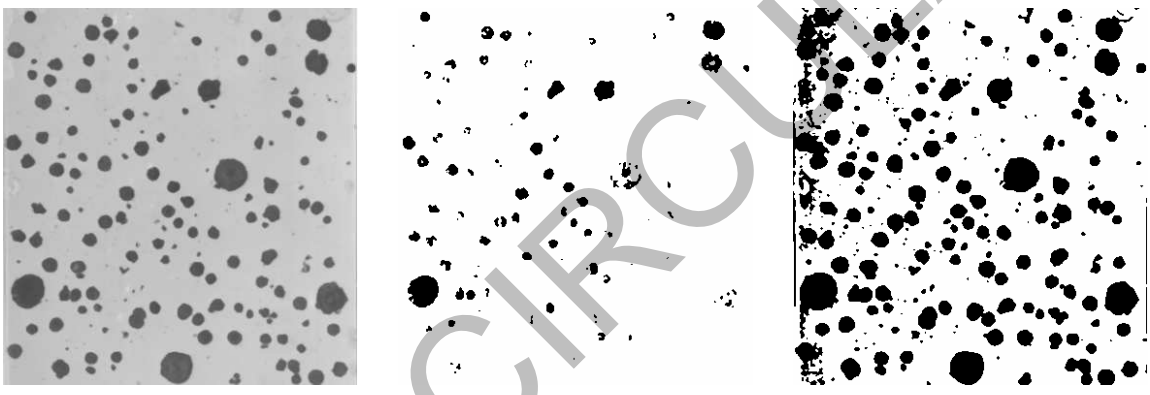
low, we may decrease the size of some of the objects, or reduce their number. Conversely, if we choose a value too high, we may begin to include extraneous background material.

Consider for example the image `nodules1.tif`, and suppose we try to threshold using the `im2bw` function and various threshold values t for $0 < t < 1$.

```
>> n=imread('nodules1.tif');
>> imshow(n);
>> n1=im2bw(n,0.35);
>> n2=im2bw(n,0.75);
>> figure,imshow(n1),figure,imshow(n2)
```

All the images are shown in figure 9.6. One approach is to investigate the histogram of the image,



`n`: Original image `n1`: Threshold too low `n2`: Threshold too high

Figure 9.6: Attempts at thresholding

and see if there is a clear spot to break it up. Sometimes this can work well, but not always.

Figure 9.7 shows various histograms. In each case the image consists of objects on a background. But only for some histograms is it easy to see where we can split it. In both the coins and nodules images, we could split it up about half way, but for the rice and bacteria images, it is not so clear.

The trouble is that in general the individual histograms of the objects and background will overlap, and without prior knowledge of the individual histograms it may be difficult to find a splitting point. Figure 9.8 illustrates this, assuming in each case that the histograms for the objects and backgrounds are those of a normal distribution. Then we choose the threshold values to be the place where the two histograms cross over.

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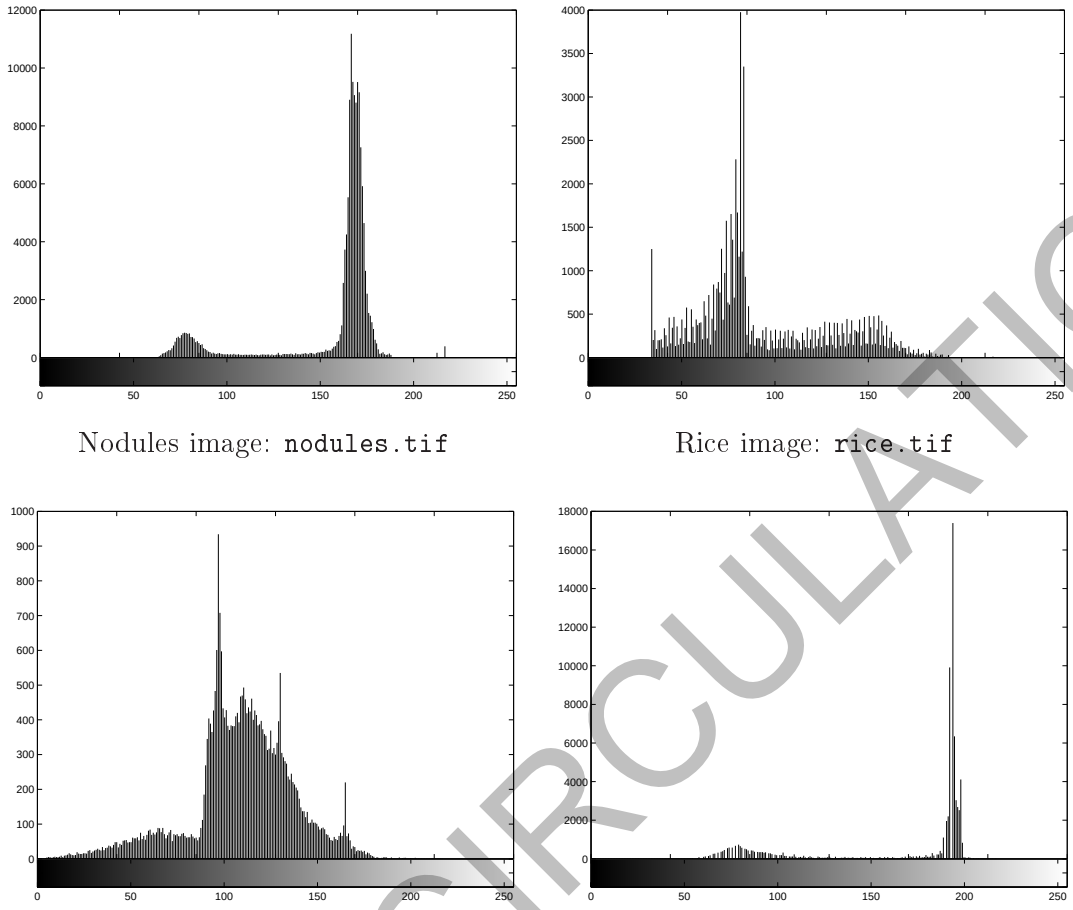

Nodules image: `nodules.tif`

Rice image: `rice.tif`

Bacteria image: `bacteria.tif`

Coins image: `eight.tif`

Figure 9.7: Histograms

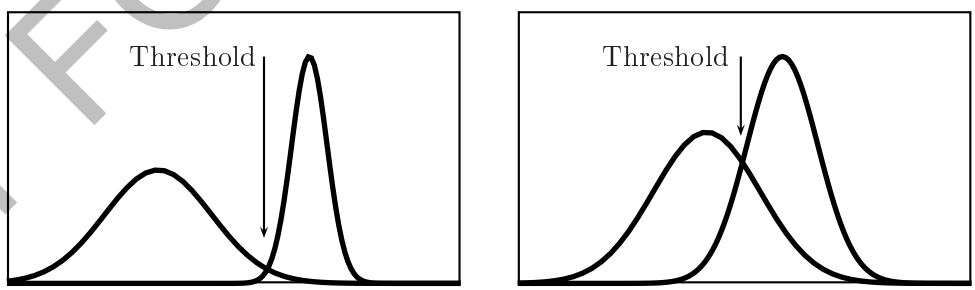

Figure 9.8: Splitting up a histogram for thresholding

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In practice though, the histograms won't be as clearly defined as those in figure 9.8, so we need some sort of automatic method for choosing a "best" threshold. One method is to describe the image histogram as a probability distribution, so that

$$p_i = n_i/N$$

where n_i is the number of pixels with grey level i, N is the total number of pixels, so that p_i is the probability of a pixel having grey level i. If we threshold at level k, we define

$$\begin{aligned}\omega(k) &= \sum_{i=0}^{k} p_i \\ \mu(k) &= \sum_{i=k+1}^{L-1} p_i\end{aligned}$$

where L is the number of greyscales, so that $L-1$ is the largest. By definition

$$\omega(k) + \mu(k) = \sum_{i=0}^{L-1} p_i = 1.$$

What we would like to do is to find k to maximize the difference between $\omega(k)$ and $\mu(k)$.

This can be done by first defining the image average as

$$\mu_T = \sum_{i=0}^{L-1} i p_i$$

and then finding k which maximizes

$$\frac{\Big(\mu_T\omega(k) - \mu(k)\Big)^2}{\omega(k)\mu(k)}.$$

This method of finding an optimal threshold is called *Otsu's method* after its developer, and is implemented by the MATLAB function `graythresh`. We can try this on our four images: nodules, rice, bacteria and coins.

```
>> tn=graythresh(n)


tn =


    0.5804
```

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```
>> r=imread('rice.tif');
>> tr=graythresh(r)

tr =

    0.4902

>> b=imread('bacteria.tif');
>> tb=graythresh(b)

tb =

    0.3765

>> e=imread('eight.tif');
>> te=graythresh(e)

te =

    0.6490
```

Now we can apply them to the images using `im2bw`.

```
>> imshow(im2bw(n,tn))
>> figure,imshow(im2bw(r,tr))
>> figure,imshow(im2bw(b,tb))
>> figure,imshow(im2bw(e,te))
```

The results are shown in figure 9.9. Note that for each image the result given is quite satisfactory.

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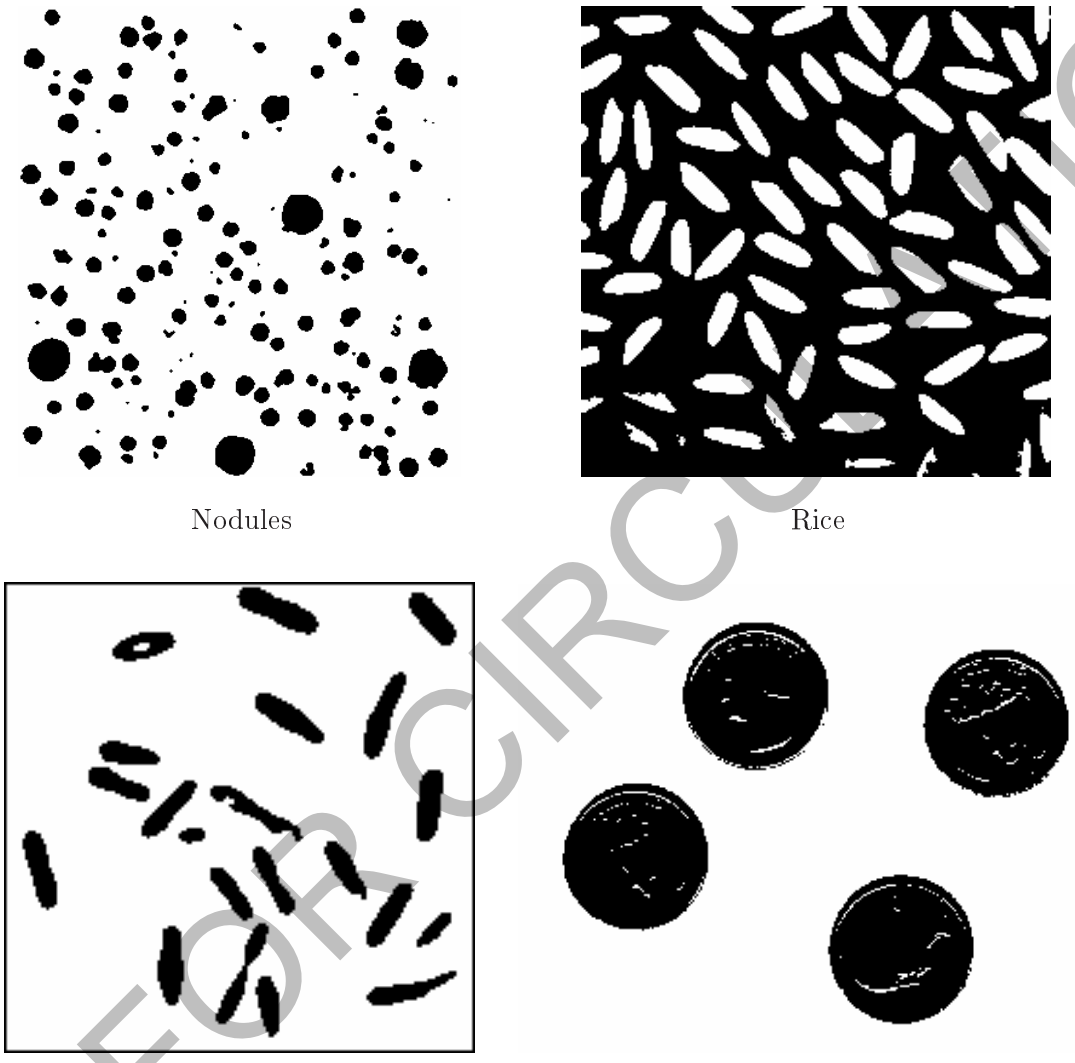

Figure 9.9: Thresholding with values from `graythresh`

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9.5 Adaptive thresholding

Sometimes it is not possible to obtain a single threshold value which will isolate an object completely. This may happen if both the object and its background vary. For example, suppose we take the circles image and adjust it so that both the circles and the background vary in brightness across the image.

```
>> c=imread('circles.tif');
>> x=ones(256,1)*[1:256];
>> c2=double(c).*(x/2+50)+(1-double(c)).*x/2;
>> c3=uint8(255*mat2gray(c2));
```

Figure 9.10 shows an attempt at thresholding, using `graythresh`.

```
>> t=graythresh(c3)


t =


    0.4196


>> ct=im2bw(c3,t);
```

As you see, the result is not particularly good; not all of the object has been isolated from its background. Even if different thresholds are used, the results are similar. Figure 9.11 illustrates the reason why a single threshold cannot work. In this figure the image is being shown as a function; the threshold is shown on the right as a horizontal plane. It can be seen that no position of the plane can cut off the circles from the background.

What can be done in a situation like this is to cut the image into small pieces, and apply thresholding to each piece individually. Since in this particular example the brightness changes from left to right, we shall cut up the image into four pieces:

```
>> p1=c3(:,1:64);
>> p2=c3(:,65:128);
```

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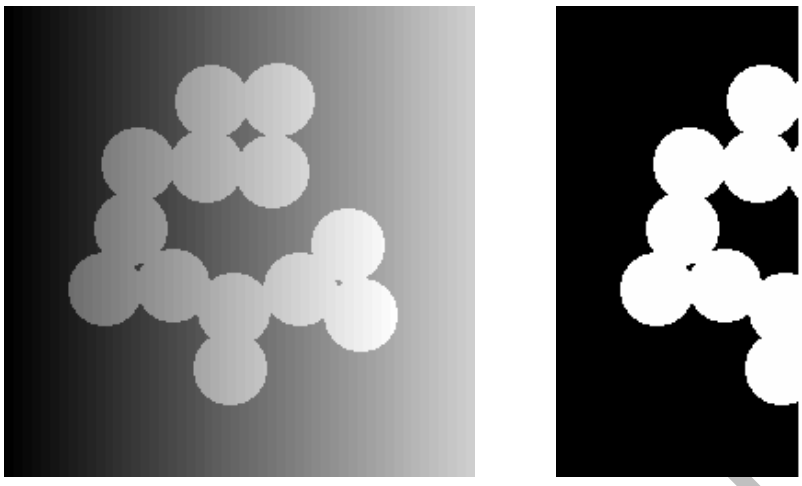
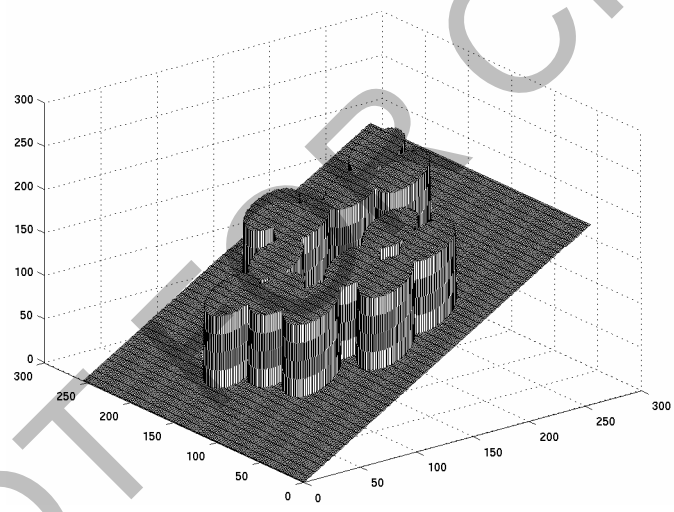
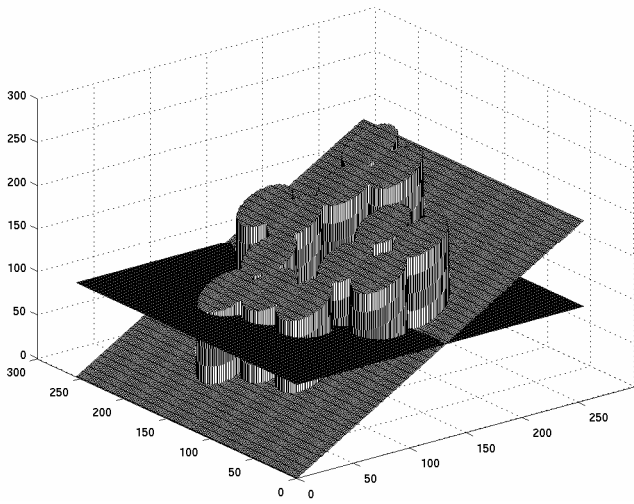


(a) Circles image: `c3` (b) Thresholding attempt: `ct`

Figure 9.10: An attempt at thresholding



(a) The image as a function



(b) Thresholding attempt

Figure 9.11: An attempt at thresholding—functional version

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```
>> p3=c3(:,129:192);
>> p4=c3(:,193:256);
```

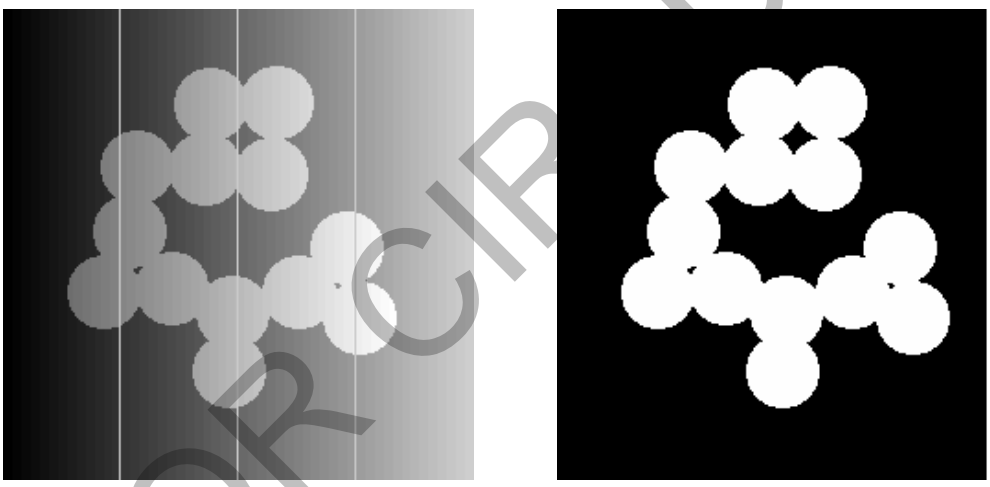
Figure 9.12(a) shows how the image is sliced up. Now we can threshold each piece:

```
>> g1=im2bw(p1,graythresh(p1));
>> g2=im2bw(p2,graythresh(p2));
>> g3=im2bw(p3,graythresh(p3));
>> g4=im2bw(p4,graythresh(p4));
```

and now display them as a single image:

```
>> imshow([g1 g2 g3 g4])
```

and the result is shown in figure 9.12(b). The above commands can be done much more simply by



(a) Cutting up the image (b) Thresholding each part separately

Figure 9.12: Adaptive thresholding

using the command `blkproc`, which applies a particular function to each block of the image. We can define our function with

```
>> fun=inline('im2bw(x,graythresh(x))');
```

Notice that this is the same as the commands used above to create `g1`, `g2`, `g3` and `g4` above, except that now `x` is used to represent a general input variable.

The function can then be applied it to the image `t3` with

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```
>> t4=blkproc(t3,[256,64],fun);
```

What this command means is that we apply our function `fun` to each distinct 256×64 block of our image.

9.6 Edge detection

Edges contain some of the most useful information in an image. We may use edges to measure the size of objects in an image; to isolate particular objects from their background; to recognize or classify objects. There are a large number of edge-finding algorithms in existence, and we shall look at some of the more straightforward of them. The general MATLAB command for finding edges is

`edge(image,'method',`*parameters*...`)`

where the parameters available depend on the method used. In this chapter, we shall show how to create edge images using basic filtering methods, and discuss the MATLAB edge function.

An *edge* may be loosely defined as a local discontinuity in the pixel values which exceeds a given threshold. More informally, an edge is an observable difference in pixel values. For example, consider the two 4 blocks of pixels shown in figure 9.13.

51	52	53	59
54	52	53	62
50	52	53	68
55	52	53	55

50	53	155	160
51	53	160	170
52	53	167	190
51	53	162	155

Figure 9.13: Blocks of pixels

In the right hand block, there is a clear difference between the grey values in the second and third columns, and for these values the differences exceed 100. This would be easily discernible in an image—the human eye can pick out grey differences of this magnitude with relative ease. Our aim is to develop methods which will enable us to pick out the edges of an image.

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9.7 Derivatives and edges

9.7.1 Fundamental definitions

Consider the image in figure 9.14, and suppose we plot the gray values as we traverse the image from left to right. Two types of edges are illustrated here: a *ramp edge*, where the grey values

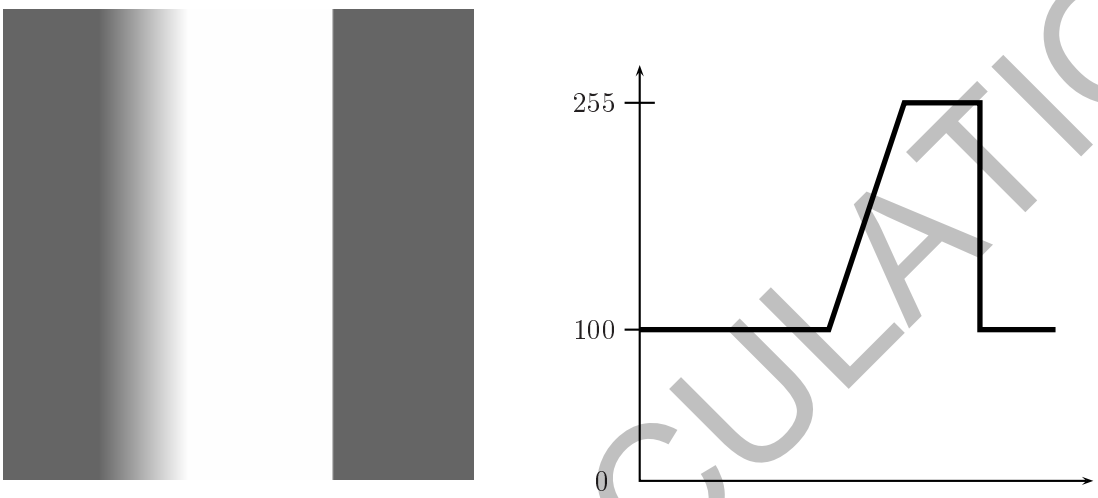

Figure 9.14: Edges and their profiles

change slowly, and a *step edge*, or an *ideal edge*, where the grey values change suddenly.

Suppose the function which provides the profile in figure 9.14 is $f(x)$; then its derivative $f'(x)$ can be plotted; this is shown in figure 9.15. The derivative, as expected, returns zero for all constant

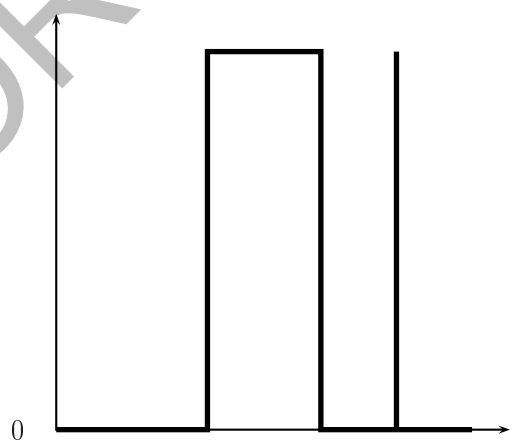

Figure 9.15: The derivative of the edge profile

sections of the profile, and is non zero (in this example) only in those parts of the image in which

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difference occur.

Many edge finding operators are based on differentiation; to apply the continuous derivative to a discrete image, first recall the definition of the derivative:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Since in an image, the smallest possible value of h is 1, being the difference between the index values of two adjacent pixels, a discrete version of the derivative expression is

$$f(x+1) - f(x).$$

Other expressions for the derivative are

$$\lim_{h \to 0} \frac{f(x) - f(x-h)}{h}, \qquad \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

with discrete counterparts

$$f(x) - f(x-1), \qquad (f(x+1) - f(x-1))/2.$$

For an image, with two dimensions, we use partial derivatives; an important expression is the *gradient*, which is the vector defined by

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

which for a function $f(x, y)$ points in the direction of its greatest increase. The direction of that increase is given by

$$\tan^{-1}\left(\frac{\partial f / \partial y}{\partial f / \partial x}\right)$$

and its magnitude by

$$\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

Most edge detection methods are concerned with finding the magnitude of the gradient, and then applying a threshold to the result.

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9.7.2 Some edge detection filters

Using the expression $f(x+1) - f(x-1)$ for the derivative, leaving the scaling factor out, produces horizontal and vertical filters:

$$\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

These filters will find vertical and horizontal edges in an image and produce a reasonably bright result. However, the edges in the result can be a bit "jerky"; this can be overcome by smoothing the result in the opposite direction; by using the filters

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Both filters can be applied at once, using the combined filter:

$$P_x = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

This filter, and its companion for finding horizontal edges:

$$P_y = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

are the *Prewitt* filters for edge detection.

If p_x and p_y are the grey values produced by applying P_x and P_y to an image, then the magnitude of the gradient is obtained with

$$\sqrt{p_x^2 + p_y^2}.$$

In practice, however, its is more convenient to use either of

$$\max\{|p_x|, |p_y|\}$$

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or

$$|p_x| + |p_y|.$$

For example, let us take the image of the integrated circuit shown in figure 9.16, which can be read into MATLAB with

```
>> ic=imread('ic.tif');
```

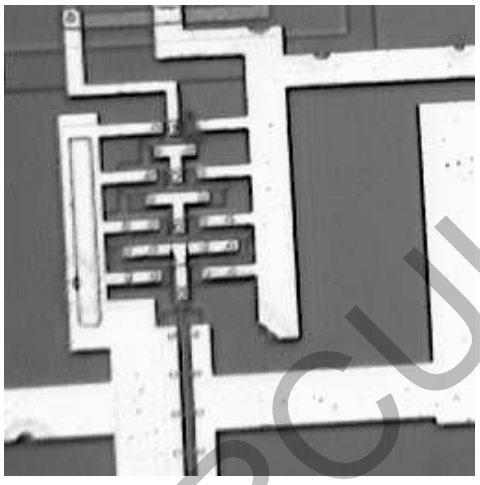

Figure 9.16: An integrated circuit

Applying each of P_x and P_y individually provides the results shown in figure 9.17 Figure 9.17(a) was produced with the following MATLAB commands:

```
>> px=[-1 0 1;-1 0 1;-1 0 1];
>> icx=filter2(px,ic);
>> figure,imshow(icx/255)
```

and figure 9.17(b) with

```
>> py=px';
>> icy=filter2(py,ic);
>> figure,imshow(icy/255)
```

Note that the filter P_x highlights vertical edges, and P_y horizontal edges. We can create a figure containing all the edges with:

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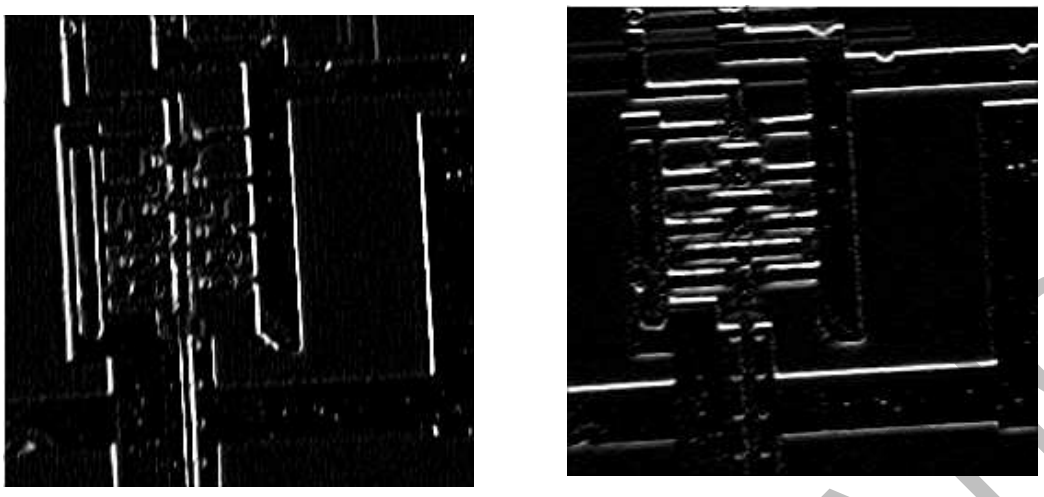

(a) (b)

Figure 9.17: The circuit after filtering with the Prewitt filters

```
>> edge_p=sqrt(icx.^2+icy.^2);
>> figure,imshow(edge_p/255)
```

and the result is shown in figure 9.18(a). This is a grey-scale image; a binary image containing edges only can be produced by thresholding. Figure 9.18(b) shows the result after the command

```
>> edge_t=im2bw(edge_p/255,0.3);
```

We can obtain edges by the Prewitt filters directly by using the command

```
>> edge_p=edge(ic,'prewitt');
```

and the `edge` function takes care of all the filtering, and of choosing a suitable threshold level; see its help text for more information. The result is shown in figure 9.19. Note that figures 9.18(b) and 9.19 seem different to each other. This is because the `edge` function does some extra processing over and above taking the square root of the sum of the squares of the filters.

Slightly different edge finding filters are the *Roberts cross-gradient filters*:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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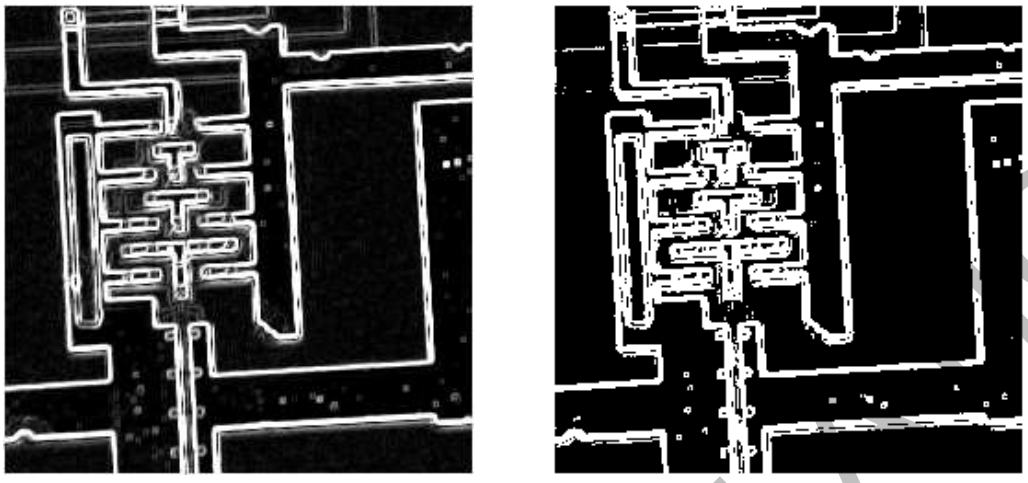

(a) (b)

Figure 9.18: All the edges of the circuit

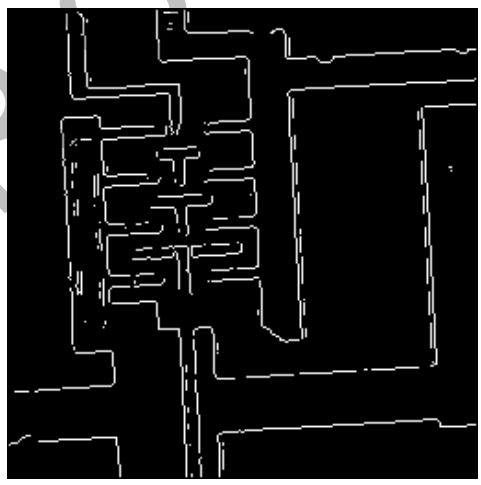

Figure 9.19: The `prewitt` option of `edge`

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and the *Sobel filters*:

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

The *Sobel filters* are similar to the Prewitt filters, in that they apply a smoothing filter in the opposite direction to the central difference filter. In the Sobel filters, the smoothing takes the form

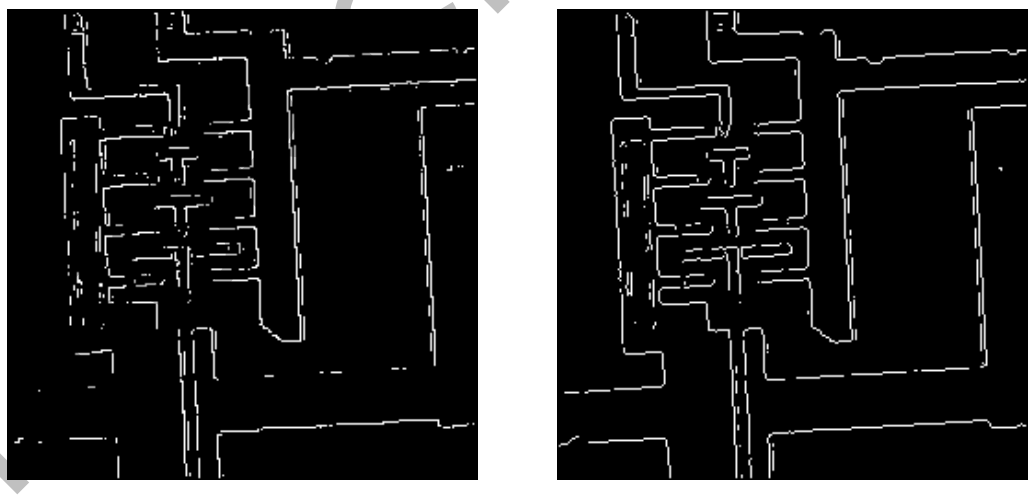
$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

which gives slightly more prominence to the central pixel. Figure 9.20 shows the respective results of the MATLAB commands

```
>> edge_r=edge(ic,'roberts');
>> figure,imshow(edge_r)
```

and

```
>> edge_s=edge(ic,'sobel');
>> figure,imshow(edge_s)
```



(a) Roberts edge detection (b) Sobel edge detection

Figure 9.20: Results of the Roberts and Sobel filters

The appearance of each of these can be changed by specifying a threshold level.

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Of the three filters, the Sobel filters are probably the best; they provide good edges, and they perform reasonably well in the presence of noise.

9.8 Second derivatives

9.8.1 The Laplacian

Another class of edge-detection method is obtained by considering the second derivatives.

The sum of second derivatives in both directions is called the *laplacian*; it is written as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

and it can be implemented by the filter

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is known as a *discrete Laplacian*. The laplacian has the advantage over first derivative methods in that it is an *isotropic filter* [27]; this means it is invariant under rotation. That is, if the laplacian is applied to an image, and the image then rotated, the same result would be obtained if the image were rotated first,. and the laplacian applied second. This would appear to make this class of filters ideal for edge detection. However, a major problem with all second derivative filters is that they are very sensitive to noise.

To see how the second derivative affects an edge, take the derivative of the pixel values as plotted in figure 9.14; the results are shown schematically in figure 9.21.

The Laplacian (after taking an absolute value, or squaring) gives double edges. To see an example, suppose we enter the MATLAB commands:

```
>> l=fspecial('laplacian',0);
>> ic_l=filter2(l,ic);
>> figure,imshow(mat2gray(ic_l))
```

the result of which is shown in figure 9.22.

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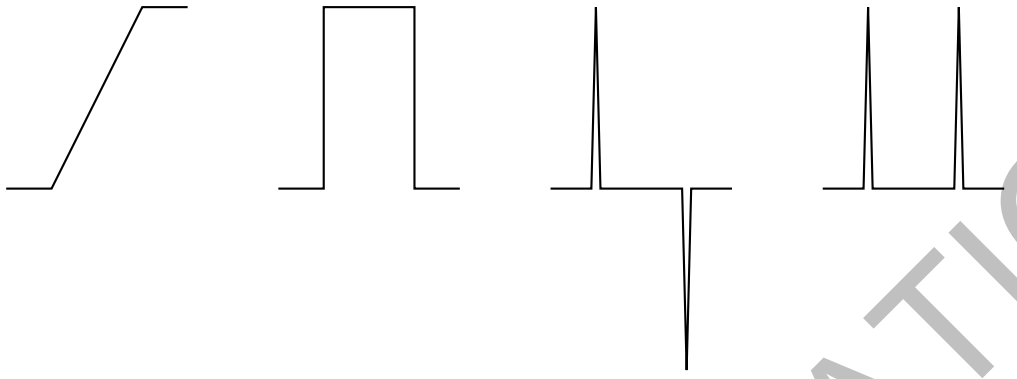

The edge First derivative Second derivative Absolute values

Figure 9.21: Second derivatives of an edge function

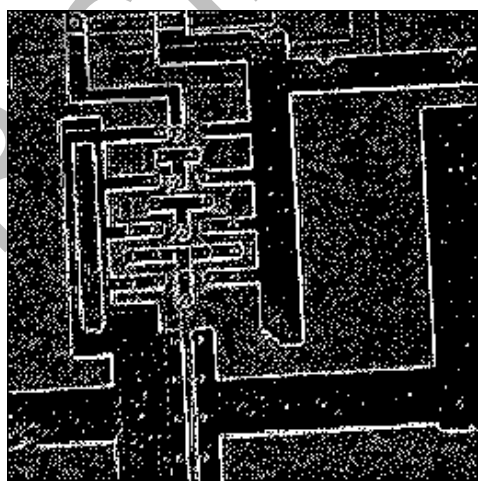

Figure 9.22: Result after filtering with a discrete laplacian

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Although the result is adequate, it is very messy when compared to the results of the Prewitt and Sobel methods discussed earlier. Other Laplacian masks can be used; some are:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 & 1 & -2 \\ 1 & 4 & 1 \\ -2 & 1 & -2 \end{bmatrix}.$$

In MATLAB, Laplacians of all sorts can be generated using the `fspecial` function, in the form

```
fspecial('laplacian',ALPHA)
```

which produces the Laplacian

$$\frac{1}{\alpha+1}\begin{bmatrix} \alpha & 1-\alpha & \alpha \\ 1-\alpha & -4 & 1-\alpha \\ \alpha & 1-\alpha & \alpha \end{bmatrix}.$$

If the parameter `ALPHA` (which is optional) is omitted, it is assumed to be 0.2. The value 0 gives the Laplacian developed earlier.

9.8.2 Zero crossings

A more appropriate use for the Laplacian is to find the *position* of edges by locating *zero crossings*. From figure 9.21, the position of the edge is given by the place where the value of the filter takes on a zero value. In general, these are places where the result of the filter changes sign. For example, consider the the simple image given in figure 9.23(a), and the result after filtering with a Laplacian mask in figure 9.23(b).

We define the *zero crossings* in such a filtered image to be pixels which satisfy either of the following:

1. they have a negative grey value and are next to (by four-adjacency) a pixel whose grey value is positive,

2. they have a value of zero, and are between negative and positive valued pixels.

To give an indication of the way zero-crossings work, look at the edge plots and their second differences in figure 9.24.

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50	50	50	50	50	50	50	50	50	50
50	50	50	50	50	50	50	50	50	50
50	50	200	200	200	200	200	200	50	50
50	50	200	200	200	200	200	200	50	50
50	50	200	200	200	200	200	200	50	50
50	50	200	200	200	200	200	200	50	50
50	50	50	50	200	200	200	200	50	50
50	50	50	50	200	200	200	200	50	50
50	50	50	50	50	50	50	50	50	50
50	50	50	50	50	50	50	50	50	50

(a) A simple image

−100	−50	−50	−50	−50	−50	−50	−50	−50	−100
−50	0	150	150	150	150	150	150	0	−50
−50	150	−300	−150	−150	−150	−150	−300	150	−50
−50	150	−150	0	0	0	0	−150	150	−50
−50	150	−150	0	0	0	0	−150	150	−50
−50	150	−300	−150	0	0	0	−150	150	−50
−50	0	150	300	−150	0	0	−150	150	−50
−50	0	0	150	−300	−150	−150	−300	150	−50
−50	0	0	0	150	150	150	150	0	−50
−100	−50	−50	−50	−50	−50	−50	−50	−50	−100

(b) After laplace filtering

Figure 9.23: Locating zero crossings in an image

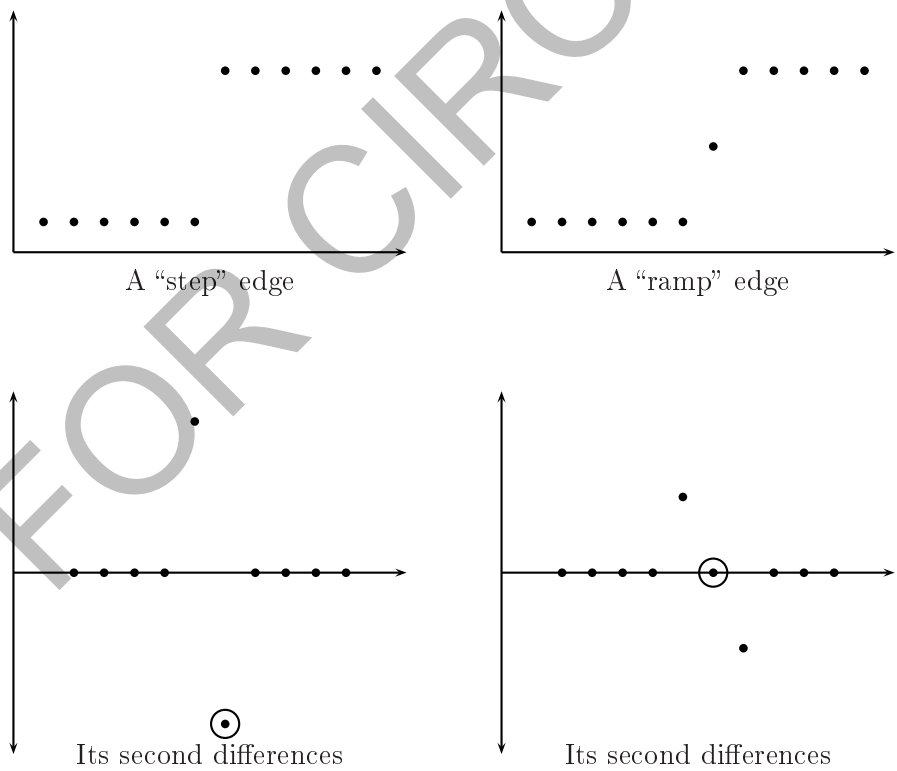

Figure 9.24: Edges and second differences

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In each case the zero-crossing is circled. The important point is to note that across any edge there can be only *one* zero-crossing. Thus an image formed from zero-crossings has the potential to be very neat.

In figure 9.23(b) the zero crossings are shaded. We now have a further method of edge detection: take the zero-crossings after a laplace filtering. This is implemented in MATLAB with the `zerocross` option of `edge`, which takes the zero crossings after filtering with a given filter:

```
>> l=fspecial('laplace',0);
>> icz=edge(ic,'zerocross',l);
>> imshow(icz)
```

The result is shown in figure 9.25(a). This is not in fact a very good result—far too many grey level changes have been interpreted as edges by this method. To eliminate them, we may first smooth the image with a Gaussian filter. This leads to the following sequence of steps for edge detection; the *Marr-Hildreth* method:

1. smooth the image with a Gaussian filter,
2. convolve the result with a laplacian,
3. find the zero crossings.

This method was designed to provide a edge detection method to be as close as possible to biological vision. The first two steps can be combined into one, to produce a "Laplacian of Gaussian" or "LoG" filter. These filters can be created with the `fspecial` function. If no extra parameters are provided to the `zerocross` edge option, then the filter is chosen to be the LoG filter found by

```
>> fspecial('log',13,2)
```

This means that the following command:

```
>> edge(ic,'log');
```

produces exactly the same result as the commands:

```
>> log=fspecial('log',13,2);
```

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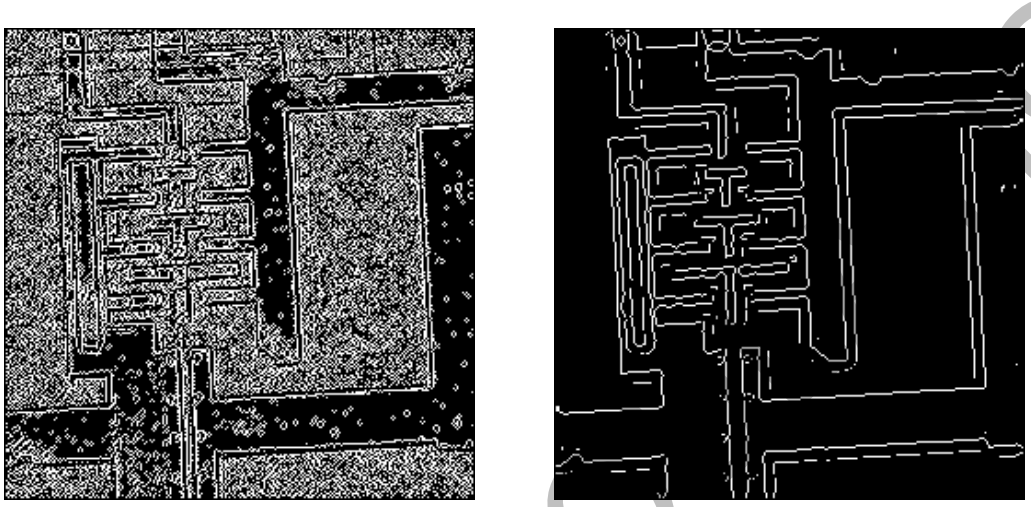
```
>> edge(ic,'zerocross',log);
```

In fact the `LoG` and `zerocross` options implement the same edge finding method; the difference being that the `zerocross` option allows you to specify your own filter. The result after applying an LoG filter and finding its zero crossings is given in figure 9.25(b).



(a) Zeros crossings (b) Using an LoG filter first

Figure 9.25: Edge detection using zero crossings

9.9 The Canny edge detector

The `edge` function has another option: `canny`, which implements the *Canny edge detector*, named after John Canny, who described this method in 1986 [3]. It was designed to meet three criteria for edge detection:

1. **Low error rate of detection.** It should find *all* edges, and nothing but edges.

2. **Localization of edges.** The distance between actual edges in the image and edges found by this algorithm should be minimized.

3. **Single response.** The algorithm should not return multiple edge pixels when only a single edge exists.

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Canny showed that the best filter to use for beginning his algorithm was a Gaussian (for smoothing), followed by the derivative of the Gaussian, which in one dimension is

$$\left(-\frac{x}{\sigma^2}\right) e^{-\frac{x^2}{2\sigma^2}}. \tag{9.1}$$

These have the effect of both smoothing noise and finding possible candidate pixels for edges. Since this filter is separable, it can be applied first as a column filter to the columns, next as a row filter to the rows. We can then put the two results together to form an edge image. Recall from chapter 5 that this is more efficient computationally than applying a two dimensional filter.

Thus at this stage we have the following sequence of steps:

1. Take our image `x`.
2. Create a one dimensional Gaussian filter `g`.
3. Create a one dimensional filter `dg` corresponding to the expression given in equation 9.1.
4. Convolve `g` with `dg` to obtain `gdg`.
5. Apply `gdg` to `x` producing `x1`.
6. Apply `gdg'` to `x` producing `x2`.

We can now form an edge image with

$$\texttt{xe} = \sqrt{\texttt{x1}^2 + \texttt{x2}^2}.$$

So far we have not achieved much more than we would achieve with a standard edge detection by using a filter.

The next step is that of *non-maximum suppression*. What we want to do is to threshold the edge image `xe` from above to keep only edge pixels and remove the others. However, thresholding alone will not produce acceptable results. The idea is that every pixel p has a direction ϕ_p (an edge "gradient") associated with it, and to be considered as an edge pixel, a p must have a greater magnitude than its neighbours in direction ϕ_p.

Just as we computed the magnitude `xe` above we can compute the edge gradient using the inverse tangent function:

$$\texttt{xg} = \tan^{-1}\left(\frac{\texttt{x2}}{\texttt{x1}}\right).$$

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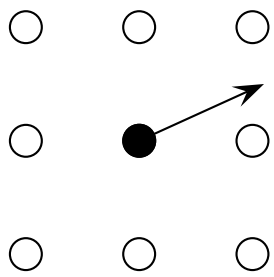
In general that direction will point between pixels in its 3×3 neighbourhood, as shown in figure 9.26. There are two approaches here: we can compare the gradient of the current (centre)



The edge direction at a pixel

Figure 9.26: Non-maximum suppression in the Canny edge detector

pixel with the value obtained from linear interpolation (see chapter 6); basically we just take the weighted average of the gradients of the two pixels. So in figure 9.26 we take the weighted average of the upper two pixels on the right.

The second approach is to quantize the gradient to one of the values 0°, 45°, 90° or 135°, and compare the original gradient to the gradient of the pixel to which the quantized gradient points. That is, suppose the gradient at position (x, y) was $\phi(x, y)$. We quantize this to one of the four angles given to obtain $\phi'(x, y)$. Consider the two pixels in direction $\phi'(x, y)$ and $\phi'(x, y) + 180°$ from (x, y). If the edge magnitude of either of those is greater than the magnitude of the current pixel, we mark the current pixel for deletion. After we have passed over the entire image, we delete all marked pixels.

We can in fact compute the quantized gradients without using the inverse tangent: we simply compare the values in the two filter results `x1` and `x2`. Depending on the relative values of `x1`(x, y) and `x2`(x, y) we can place the gradient at (x, y) into one of the four gradient classes. Figure 9.27 shows how this is done. We divide the image plane into eight regions separated by lines at 45°, as shown, with the x axis positive to the right, and the y axis positive down. Then we can assign regions, and degrees, to pixel values according to this table:

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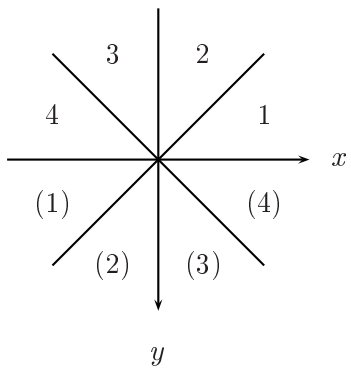

Figure 9.27: Using pixel locations to quantize the gradient

Region	Degree	Pixel location
1	0°	$y \leq 0$ and $x > -y$
(1)	0°	$y \geq 0$ and $x < -y$
2	45°	$x > 0$ and $x \leq -y$
(2)	45°	$x < 0$ and $x \geq -y$
3	90°	$x \leq 0$ and $x > y$
(3)	90°	$x \geq 0$ and $x < y$
4	135°	$y < 0$ and $x \leq y$
(4)	135°	$y > 0$ and $x \geq y$

Suppose we had a neighbourhood for a pixel whose gradient had been quantized to 45°, as shown in figure 9.28. The dashed arrow indicates the opposite direction to the current gradient. In this figure both magnitudes at the ends of the arrows are smaller than the centre magnitude, so we keep the centre pixel as an edge pixel. If, however, one of those two values was *greater* than 150, we would mark the central pixel for deletion.

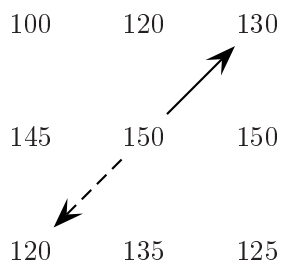

Figure 9.28: Quantizing in non-maximum suppression

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After performing non-maximum suppression, we can threshold to obtain a binary edge image. Canny proposed, rather than using a single threshold, a technique called *hysteresis thresholding*, which uses two threshold values: a low value t_L, and a high value t_H. Any pixel with a value greater than t_H is assumed to be an edge pixel. Also, any pixel with a value p where $t_L \leq p \leq t_H$ and which is adjacent to an edge pixel, is also considered to be an edge pixel.

The Canny edge detector is implemented by the `canny` option of the `edge` function. We can enter the threshold values, or else they will be chosen automatically:

```
>> [icc,t]=edge(ic,'canny');
>> t


t =


    0.0500    0.1250


>> imshow(icc)
```

and the result is shown in figure 9.29. Two other results with different thresholds are given in

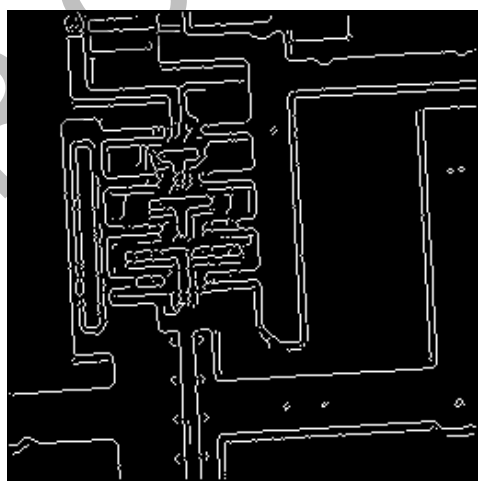

Figure 9.29: Canny edge detection

figure 9.30. The higher we make the upper threshold, the less edges will be shown. We can also

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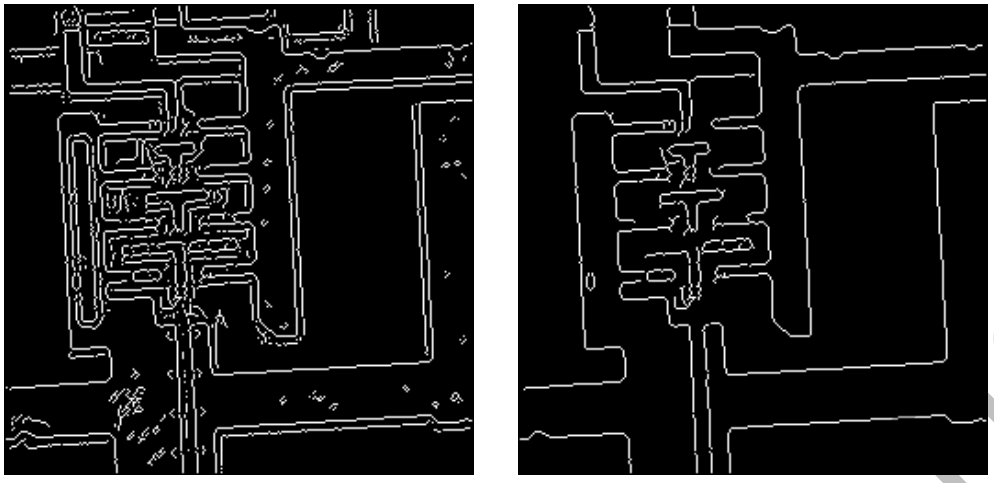

`edge(ic,'canny',[0,0.05])` `edge(ic,'canny',[0.01,0.5])`

Figure 9.30: Canny edge detection with different thresholds

vary the standard deviation of the original Gaussian filter.

The Canny edge detector is the most complex of the edge detectors we have discussed; however it is not the last word in edge detectors. A good account of edge detectors is given by Parker [21], and an interesting account of some advanced edge detection techniques can be found in Heath et. al [11].

9.10 The Hough transform

If the edge points found by the above edge detection methods are sparse, the resulting edge image may consist of individual points, rather than straight lines or curves. Thus in order to establish a boundary between the regions, it might be necessary to fit a line to those points. This can be a time consuming and computationally inefficient process, especially if there are many such edge points. One way of finding such boundary lines is by use of the "Hough transform".

The *Hough transform*[1] is designed to find lines in images, but it can be easily varied to find other shapes. The idea is simple. Suppose (x, y) is a point in the image (which we shall assume to be binary). We can write $y = ax + b$, and consider all pairs (a, b) which satisfy this equation, and plot them into an "accumulator array". The (a, b) array is the "transform array".

[1] "Hough" is pronounced "Huff".

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For example, take $(x, y) = (1, 1)$. Since the equation relating a and b is

$$1 = a.1 + b$$

we can write

$$b = -a + 1.$$

Thus the line $b = -a + 1$ consists of all pairs of points relating to the single point $(1, 1)$. This is shown in figure 9.31.

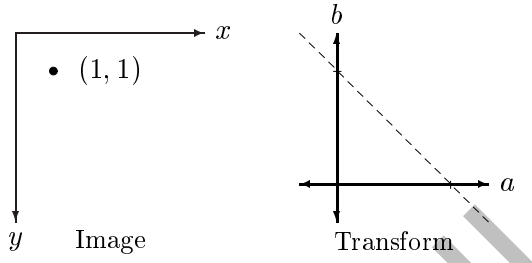

Figure 9.31: A point in an image and its corresponding line in the transform

Each point in the image is mapped onto a line in the transform. The points in the transform corresponding to the greatest number of intersections correspond to the strongest line in the image.

For example, suppose we consider an image with five points: $(1, 0)$, $(1, 1)$, $(2, 1)$, $(4, 1)$ and $(3, 2)$. Each of these points corresponds to a line as follows:

$$\begin{aligned}
(1,0) &\rightarrow b = -a \\
(1,1) &\rightarrow b = -a + 1 \\
(2,1) &\rightarrow b = -2a + 1 \\
(4,1) &\rightarrow b = -4a + 1 \\
(3,2) &\rightarrow b = -3a + 2.
\end{aligned}$$

Each of these lines appears in the transform as shown in figure 9.32.

The dots in the transform indicate places where there are maximum intersections of lines: at each dot three lines intersect. The coordinates of these dots are $(a, b) = (1, 0)$ and $(a, b) = (1, -1)$. These values correspond to the lines

$$y = 1.x + 0$$

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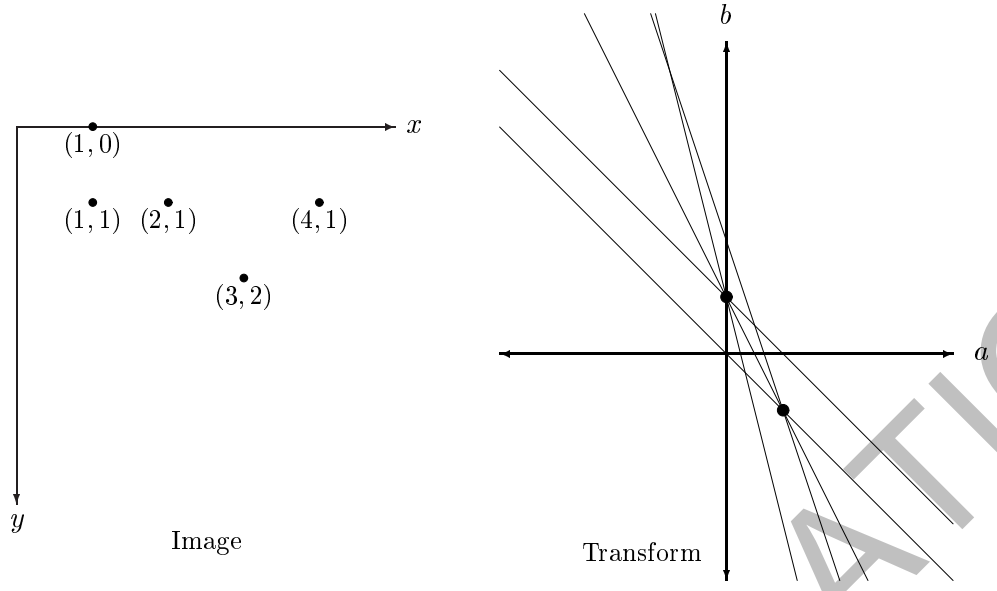

Figure 9.32: An image and its corresponding lines in the transform

and

$$y = 1.x + (-1)$$

or $y = x$ and $y = x - 1$. These lines are shown on the image in figure 9.33.

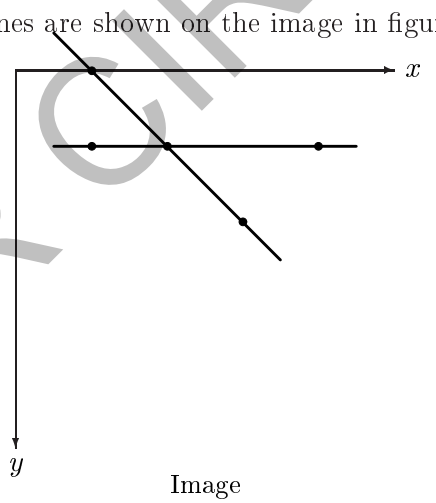

Figure 9.33: Lines found by the Hough transform

These are indeed the "strongest" lines in the image in that they contain the greatest number of points.

There is a problem with this implementation of the Hough transform, and that is that it can't

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find vertical lines: we can't express a vertical line in the form $y = mx + c$, as m represents the gradient, and a vertical line has infinite gradient. We need another parameterization of lines.

Consider a general line, as shown in figure 9.34. Clearly any line can be described in terms of

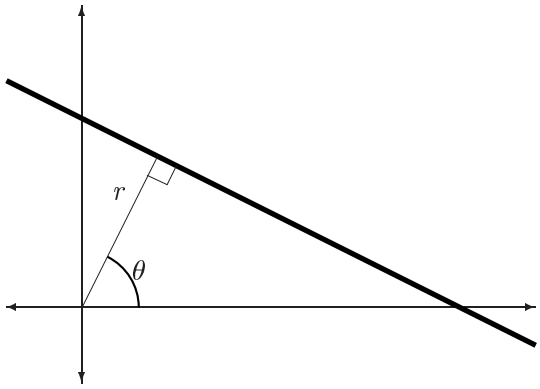

Figure 9.34: A line and its parameters

the two parameters r and θ: r is the perpendicular distance from the line to the origin, and θ is the angle of the line's perpendicular to the x-axis. In this parameterization, vertical lines are simply those which have $\theta = 0$. If we allow r to have negative values, we can restrict θ to the range

$$-90 < \theta \leq 90.$$

Given this parameterization, we need to be able to find the equation of the line. First note that the point (p, q) where the perpendicular to the line meets the line is $(p, q) = (r\cos\theta, r\sin\theta)$. Also note that the gradient of the perpendicular is $\tan\theta = \sin\theta/\cos\theta$. Now let (x, y) be any point on then line. The gradient of the line is

$$\begin{aligned}\frac{\text{rise}}{\text{run}} &= \frac{y-q}{x-p} \\ &= \frac{y - r\sin\theta}{x - r\cos\theta}.\end{aligned}$$

But since the gradient of the line's perpendicular is $\tan\theta$, the gradient of the line itself must be

$$-\frac{1}{\tan\theta} = -\frac{\cos\theta}{\sin\theta}.$$

Putting these two expressions for the gradient together produces:

$$\frac{y - r\sin\theta}{x - r\cos\theta} = -\frac{\cos\theta}{\sin\theta}.$$

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If we now multiply out these fractions we obtain:

$$y \sin\theta - r\sin^2\theta = -x\cos\theta + r\cos^2\theta$$

and this equation can be rewritten as

$$\begin{aligned} y\sin\theta + x\cos\theta &= r\sin^2\theta + r\cos^2\theta \\ &= r(\sin^2\theta + \cos^2\theta) \\ &= r. \end{aligned}$$

We finally have the required equation for the line as:

$$x\cos\theta + y\sin\theta = r.$$

The Hough transform can then be implemented as follows: we start by choosing a discrete set of values of r and θ to use. For each pixel (x, y) in the image, we compute

$$x\cos\theta + y\sin\theta$$

for each value of θ, and place the result in the appropriate position in the (r, θ) array. At the end, the values of (r, θ) with the highest values in the array will correspond to strongest lines in the image.

An example will clarify this: consider the image shown in figure 9.35.

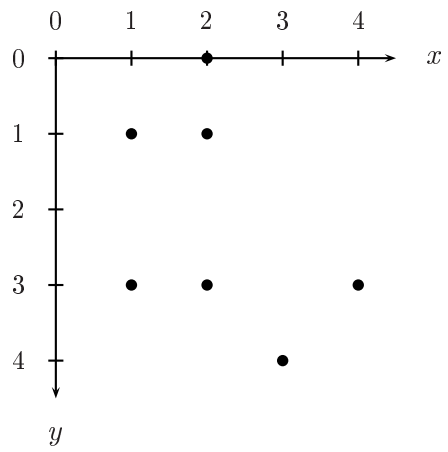

Figure 9.35: A small image

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We shall discretize θ to use only the values

$$-45^\circ, \quad 0^\circ, \quad 45^\circ, \quad 90^\circ.$$

We can start by making a table containing all values $x\cos\theta + y\sin\theta$ for each point, and for each value of θ:

(x, y)	-45°	0°	45°	90°
$(2, 0)$	1.4	2	1.4	0
$(1, 1)$	0	1	1.4	1
$(2, 1)$	0.7	2	2.1	1
$(1, 3)$	-1.4	1	2.8	3
$(2, 3)$	-0.7	2	3.5	3
$(4, 3)$	0.7	4	4.9	3
$(3, 4)$	-0.7	3	4.9	4

The accumulator array contains the number of times each value of (r, θ) appears in the above table:

	-1.4	-0.7	0	0.7	1	1.4	2	2.1	2.8	3	3.5	4	4.9
-45°	1	2	1	2		1							
0°					2		3			1		1	
45°						2		1	1		1		2
90°			1		2					3		2	

In practice this array will be very large, and can be displayed as an image. In this example the two equal largest values occur at $(r, \theta) = (2, 0^\circ)$ and $(r, \theta) = (3, 90^\circ)$. The lines then are

$$x\cos 0 + y\sin 0 = 2$$

or $x = 2$, and

$$x\cos 90 + y\sin 90 = 3$$

or $y = 3$. These lines are shown in figure 9.36

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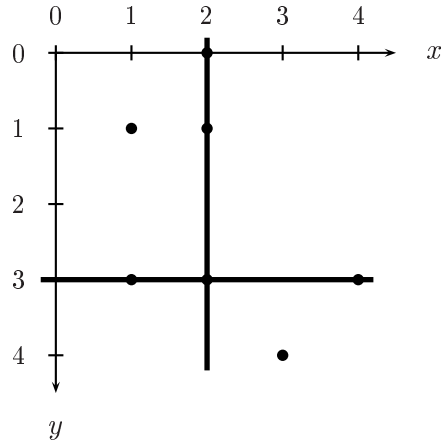

Figure 9.36: Lines found by the Hough transform

9.11 Implementing the Hough transform in MATLAB

The Image Processing Toolbox does not contain a `hough` routine, and although the Hough transform can be obtained by other means, it is a pleasant programming exercise to write our own. Following the procedure outlined above, we will:

1. decide on a discrete set of values of θ and r to use,
2. for each foreground pixel (x, y) in the image calculate the values of $r = x\cos\theta + y\sin\theta$ for all of our chosen values of θ,
3. create an accumulator array whose sizes are the numbers of angles θ and values r in our chosen discretizations from step 1,
4. step through all of our r values updating the accumulator array as we go.

We shall consider each step in turn:

Discretizing r and θ. We observe that even though a particular pair (r, θ) corresponds to only one line, a given line can be parameterized in different ways. For example, the line $x + y = -3$ can be parameterized using $\theta = 5\pi/4$ and $r = \sqrt{3/2}$ as shown in figure 9.37(a).

However, the same line can be parameterized using $\theta = \pi/4$ and $r = -\sqrt{3/2}$. So we have a choice: we may restrict the values of θ, say to the range $-\pi/2 < \theta \leq \pi/2$, and let r have both

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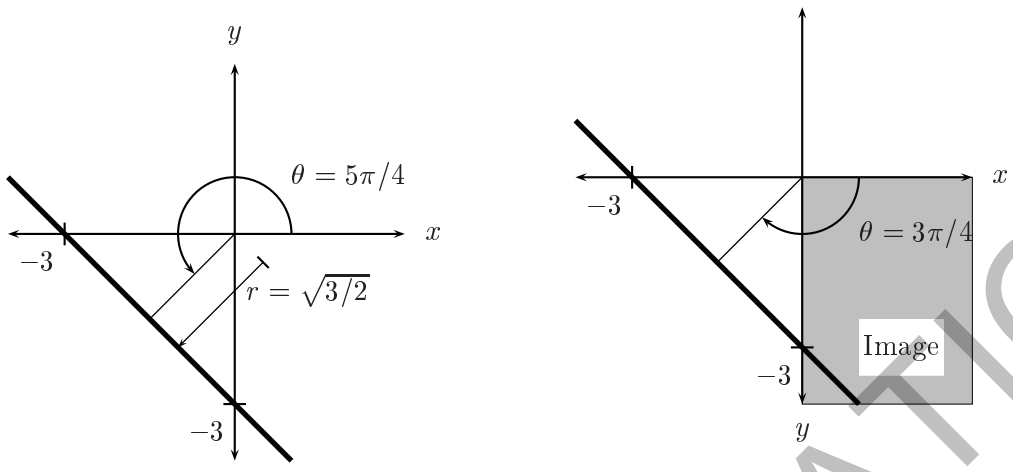

(a) Using ordinary Cartesian axes (b) Using matrix axes

Figure 9.37: A line parameterized with r and θ

positive and negative values, or we may let θ take any value chosen from the range $0 \leq \theta < 2\pi$ and restrict r to non-negative values. Since we are going to use the values of θ and r as indices into the accumulator array, it is simplest to take the second option. However, consider figure 9.37(a) slightly redrawn to take into account the directions of x and y as column and row indices of a matrix, as shown in (b). Since our image, for which both x and y will be positive, will sit in the bottom right quadrant, we will only have positive r values for $-90 \leq \theta \leq 180$. For if θ is outside that range, the perpendicular will point into the second (upper left) quadrant, which would require a negative value of r if the line was to intersect the image.

We can choose any discrete set of values we like, but let us take all the integer degree values in the given range, but convert them to radians for the purposes of calculation:

```
>> angles=[-90:180]*pi/180;
```

Calculating the r values. Suppose that the image is binary, then we can find the positions of all the foreground pixels with a simple application of the `find` function:

```
>> [x,y]=find(im);
```

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If the image is not binary, we can create a binary edge image by use of the `edge` function. Now we can calculate all the r values with one simple command:

```
>> r=floor(x*cos(angles)+y*sin(angles));
```

where we use `floor` so as to obtain only integer values.

Forming the accumulator array. If the image is of size $m \times n$ pixels, then supposing the origin to be at the upper left corner, the maximum value of r will be $\sqrt{m^2+n^2}$. So the size of the accumulator array can be set at $\sqrt{m^2+n^2} \times 270$. However, we are only interested in positive values of r: given our choice of range of θ, negative values can be discarded. So we can first find the largest positive value of r, and use that as one dimension of the array:

```
>> rmax=max(r(find(r>0)));
>> acc=zeros(rmax+1,270);
```

The `rmax+1` allows us to have values of $r = 0$.

Updating the accumulator array. We now have to step through the array of r values, and for each value (r, θ), increase the corresponding accumulator value by one. First notice that the array `r`, as define above, has size $N \times 360$, where N is the number of foreground pixels. Hence the second index of both `r` and `acc` corresponds to the angle.

This can be done with nested loops, at the heart of which will be the command

```
  if r(i,j)>=0, acc(r(i,j)+1,i)=acc(r(i,j)+1,i)+1;
```

Note that this is not a very efficient method of creating the accumulator array, but it does have the advantage of following the theory very closely. The entire program is shown in figure 9.38.

An example

Let us take the cameraman image, and apply our Hough transform procedure to it:

```
>> c=imread('cameraman.tif');
>> hc=hough(c);
```

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```
function res=hough(image)

%
% HOUGH(IMAGE) creates the Hough transform corresponding to the image IMAGE
%

if ~isbw(image)
  edges=edge(image,'canny');
else
  edges=image;
end;
[x,y]=find(edges);
angles=[-90:180]*pi/180;
r=floor(x*cos(angles)+y*sin(angles));
rmax=max(r(find(r>0)));
acc=zeros(rmax+1,270);
for i=1:length(x),
  for j=1:270,
    if r(i,j)>=0
      acc(r(i,j)+1,j)=acc(r(i,j)+1,j)+1;
    end;
  end;
end;
res=acc;
```

Figure 9.38: A simple MATLAB function for implementing the Hough transform

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This last command may take some time, because of the inefficient nested loops. The first thing we may do is to view the result:

```
>> imshow(mat2gray(hc)*1.5)
```

where the extra 1.5 at the end is just to brighten up the image. The result is shown in figure 9.39.

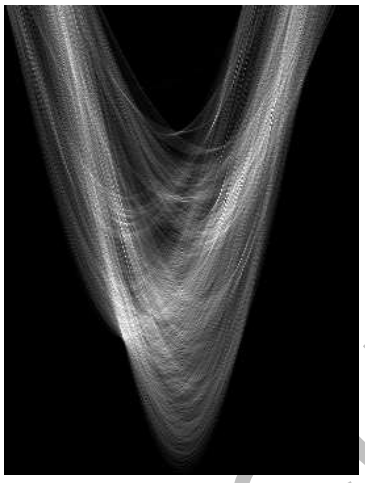

Figure 9.39: The result of a Hough transform

What we are viewing here is the accumulator array. And it is what we should expect: a series of curves, with some bright points indicating places of maximum intersection. Because of the use of sines and cosines in the construction of the transform, we can expect the curves to be of a sinusoidal nature.

What now? Let us find the maximum value of the transform:

```
>> max(hc(:))


ans =


    91
```

We can now find the r and θ values corresponding to the maximum:

```
>> [r,theta]=find(hc==91)
```

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```
r =

   138



theta =

   181
```

However, since we are using the pixel coordinates as cartesian coordinates, our x and y axes have been rotated counter-clockwise by 90°. Thus we are measuring θ in a clockwise direction from the left vertical edge of our image. Figure 9.40 shows how this works:

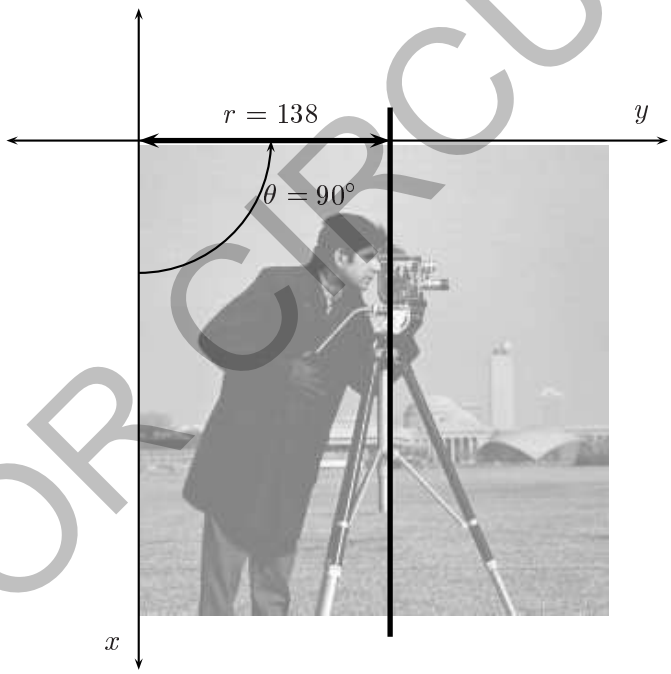

Figure 9.40: A line from the Hough transform

We have also got the problem that We can easily create a small function to draw lines for us, given their perpendicular distance from the origin, and the angle of the perpendicular from the x axis. The tool for this will be the `line` function, which in the form

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```
>> line([x1,x2],[y1,y2])
```

draws a line between coordinates (`x1`, `y1`) and (`x2`, `y2`) over the current image. The only difficulty here is that `line` takes the x axis to be on the top, and the y axis going down on the left. This is easily dealt with by replacing θ with $\pi/2 - \theta$, and a simple function for drawing lines is shown in figure 9.41.

```
function houghline(image,r,theta)
%
% Draws a line at perpendicular distance R from the upper left corner of the
% current figure, with perpendicular angle THETA to the left vertical axis.
% THETA is assumed to be in degrees.
%
[x,y]=size(image);
angle=pi*(181-theta)/180;
X=[1:x];
if sin(angle)==0
  line([r r],[0,y],'Color','black')
else
  line([0,y],[r/sin(angle),(r-y*cos(angle))/sin(angle)],'Color','black')
end;
```

Figure 9.41: A simple MATLAB function for drawing lines on an image

The only thing to notice is the expression `181-theta` in the line

```
angle=pi*(181-theta)/180;
```

This can be interpreted as $180+1-\theta$; the initial 180 swaps between cartesian and matrix coordinates, and the extra 1 counters the accumulation of `r+1` in the function `hough.m`.

We can use this as follows:

```
>> c2=imadd(imdivide(cm,4),192);
>> imshow(c2)
>> houghline(c2,r,theta)
```

The idea of using `c2` is simply to lighten the image so as to show the line more clearly. The result is shown on the left in figure 9.42. To draw other lines, we can extract some more points from the transform:

```
>> [r,theta]=find(hc>80)
```

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Figure 9.42: The `houghline` function in action

```
r =

   148
   138
   131
    85
    90

theta =

   169
   181
   182
   204
   204
```

Then the command

```
>> houghline(c,r(1),theta(1))
```

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will place the line as shown on the right in figure 9.42. Clearly we can use these functions to find any lines we like.

The Hough transform can be generalized to finding other shapes in images; for example circles or ellipses. For an account of these generalizations (as well as very good discussions of the basic Hough transform), see Leavers [19] or Sonka et. al. [32].

Exercises

Thresholding

1. Suppose you thresholded an image at value t_1, and thresholded the result at value t_2. Describe the result if

   (a) $t_1 > t_2$,

   (b) $t_1 < t_2$.

2. Create a simple image with

```
>> [x,y]=meshgrid(1:256,1:256);
>> z=sqrt((x-128).^2+(y-128).^2);
>> z2=1-mat2gray(z);
```

   Using `im2bw`, threshold `z2` at different values, and comment on the results. What happens to the amount of white as the threshold value increases? Can you state and prove a general result?

3. Repeat the above question, but with the image `cameraman.tif`.

4. Can you can create a small image which produces an "X" shape when thresholded at one level, and a cross shape "+" when thresholded at another level?

   If not, why not?

5. Superimpose the image `text.tif` onto the image `cameraman.tif`. You can do this with:

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```
>> t=imread('text.tif');}
>> c=imread('cameraman.tif');}
>> m=uint8(double(c)+255*double(t));}
```

Can you threshold this new image `m` to isolate the text?

6. Try the same problem as above, but define `m` as:

```
>> m=uint8(double(c).*double(~t));
```

7. Create a version of the circles image with

```
>> t=imread('circles.tif');
>> [x,y]=meshgrid(1:256,1:256);
>> t2=double(t).*((x+y)/2+64)+x+y;
>> t3=uint8(255*mat2gray(t2));
```

Attempt to threshold the image `t3` to obtain the circles alone, using adaptive thresholding and the `blkproc` function. What sized blocks produce the best result?

Edge detection

8. Enter the following matrix into MATLAB:

201	195	203	203	199	200	204	190	198	203
201	204	209	197	210	202	205	195	202	199
205	198	46	60	53	37	50	51	194	205
208	203	54	50	51	50	55	48	193	194
200	193	50	56	42	53	55	49	196	211
200	198	203	49	51	60	51	205	207	198
205	196	202	53	52	34	46	202	199	193
199	202	194	47	51	55	48	191	190	197
194	206	198	212	195	196	204	204	199	200

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```
201   189   203   200   191   196   207   203   193   204
```

and use `imfilter` to apply each of the Roberts, Prewitt, Sobel, Laplacian, and zero-crossing edge-finding methods to the image. In the case of applying two filters (such as with Roberts, Prewitt, or Sobel) apply each filter separately, and join the results.

Apply thresholding if necessary to obtain a binary image showing only the edges.

Which method seems to produce the best results?

9. Now with the same matrix as above, use the `edge` function with all possible parameters.

   Which method seems to produce the best results?

10. Open up the image `cameraman.tif` in MATLAB, and apply each of the following edge finding techniques in turn:

    (a) Roberts

    (b) Prewitt

    (c) Sobel

    (d) Laplacian

    (e) Zero-crossings of a laplacian

    (f) the Marr-Hildreth method

    (g) Canny

    Which seems to you to provide the best looking result?

11. Repeat the above exercise, but use the image `tire.tif`.

12. Obtain a grey-scale flower image with:

```
fl=imread('flowers.tif');
f=im2uint8(rgb2gray(fl));
```

    Now repeat question 10.

13. Pick a grey-scale image, and add some noise to it; say with

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```
c=imread('cameraman.tif');
c1=imnoise(c,'salt & pepper',0.1);
c2=imnoise(c,'gaussian',0,0.02);
```

Now apply the edge finding techniques to each of the "noisy" images `c1` and `c2`.

Which technique seems to give

(a) the best results in the presence of noise?

(b) the worst results in the presence of noise?

The Hough transform

14. Write the lines $y = x - 2$, $y = 1 - x/2$ in (r, θ) form.

15. Use the Hough transform to detect the strongest line in the binary image shown below. Use the form $x \cos\theta + y \sin\theta = r$ with θ in steps of $45°$ from $-45°$ to $90°$ and place the results in an accumulator array.

		x						
		-3	-2	-1	0	1	2	3
	-3	0	0	0	0	0	1	0
	-2	0	0	0	0	0	0	0
	-1	0	1	0	1	0	1	0
y	0	0	0	1	0	0	0	0
	1	0	0	0	0	0	0	0
	2	1	0	0	0	0	1	0
	3	0	0	0	0	0	0	0

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16. Repeat the above question with the images:

$y \backslash x$	-3	-2	-1	0	1	2	3
-3	0	0	0	0	1	0	0
-2	0	0	0	0	0	0	0
-1	0	0	1	0	0	0	1
0	0	1	0	0	1	0	0
1	1	0	0	0	0	0	1
2	0	0	0	0	1	0	0
3	0	0	0	1	1	0	0

$y \backslash x$	-3	-2	-1	0	1	2	3
-3	0	0	0	1	0	0	0
-2	1	0	0	0	1	0	0
-1	0	0	0	0	0	0	0
0	0	0	1	0	0	1	0
1	0	1	0	1	0	0	0
2	1	0	0	0	0	0	1
3	0	0	1	0	0	1	0

17. Find some more lines on the cameraman image, and plot them with `houghline`.

18. Read and display the image `alumgrns.tif`.

    (a) Where does it appear that the "strongest" lines will be?

    (b) Using `hough` and `houghline`, plot the five strongest lines.

19. Experiment with the two routines by changing the initial edge detection of `hough`. Can you affect the lines found by the Hough transform?

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Chapter 10

Mathematical morphology

10.1 Introduction

Morphology, or *morphology* for short, is a branch of image processing which is particularly useful for analyzing shapes in images. We shall develop basic morphological tools for investigation of binary images, and then show how to extend these tools to greyscale images. MATLAB has many tools for binary morphology in the image processing toolbox; most of which can be used for greyscale morphology as well.

10.2 Basic ideas

The theory of mathematical morphology can be developed in many different ways. We shall adopt one standard method which uses operations on sets of points. A very solid and detailed account can be found in Haralick and Shapiro [9].

Translation

Suppose that A is a set of pixels in a binary image, and $w = (x, y)$ is a particular coordinate point. Then A_w is the set A "translated" in direction (x, y). That is

$$A_x = \{(a, b) + (x, y) : (a, b) \in A\}.$$

For example, in figure 10.1, A is the cross shaped set, and $w = (2, 2)$. The set A has been shifted

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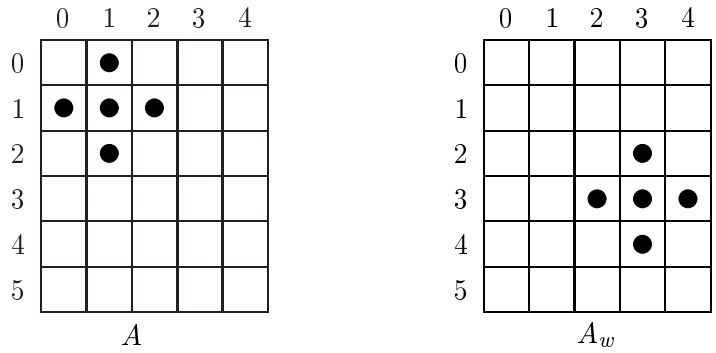

Figure 10.1: Translation

in the x and y directions by the values given in w. Note that here we are using matrix coordinates, rather than Cartesian coordinates, so that the origin is at the top left, x goes down and y goes across.

Reflection

If A is set of pixels, then its *reflection*, denoted $\hat{A}$, is obtained by reflecting A in the origin:

$$\hat{A} = \{(-x, -y) : (x, y) \in A\}.$$

For examples, in figure 10.2, the open and closed circles form sets which are reflections of each other.

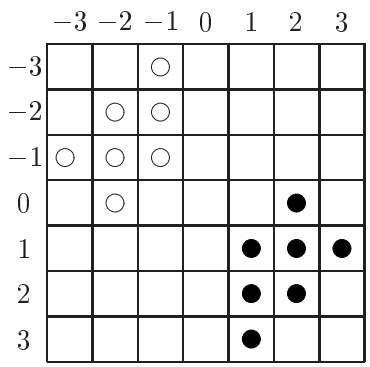

Figure 10.2: Reflection

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10.3 Dilation and erosion

These are the basic operations of morphology, in the sense that all other operations are built from a combination of these two.

10.3.1 Dilation

Suppose A and B are sets of pixels. Then the *dilation of A by B*, denoted $A \oplus B$, is defined as

$$A \oplus B = \bigcup_{x \in B} A_x.$$

What this means is that for every point $x \in B$, we translate A by those coordinates. Then we take the union of all these translations.

An equivalent definition is that

$$A \oplus B = \{(x, y) + (u, v) : (x, y) \in A, (u, v) \in B\}.$$

From this last definition, dilation is shown to be commutative; that

$$A \oplus B = B \oplus A.$$

An example of a dilation is given in figure 10.3. In the translation diagrams, the grey squares show the original position of the object. Note that $A_{(0,0)}$ is of course just A itself. In this example, we have

$$B = \{(0, 0), (1, 1), (-1, 1), (1, -1), (-1, -1)\}$$

and those these are the coordinates by which we translate A.

In general, $A \oplus B$ can be obtained by replacing every point (x, y) in A with a copy of B, placing the $(0, 0)$ point of B at (x, y). Equivalently, we can replace every point (u, v) of B with a copy of A.

Dilation is also known as *Minkowski addition*; see Haralick and Shapiro [9] for more information.

As you see in figure 10.3, dilation has the effect of increasing the size of an object. However, it is not necessarily true that the original object A will lie within its dilation $A \oplus B$. Depending on the coordinates of B, $A \oplus B$ may end up quite a long way from A. Figure 10.4 gives an example of

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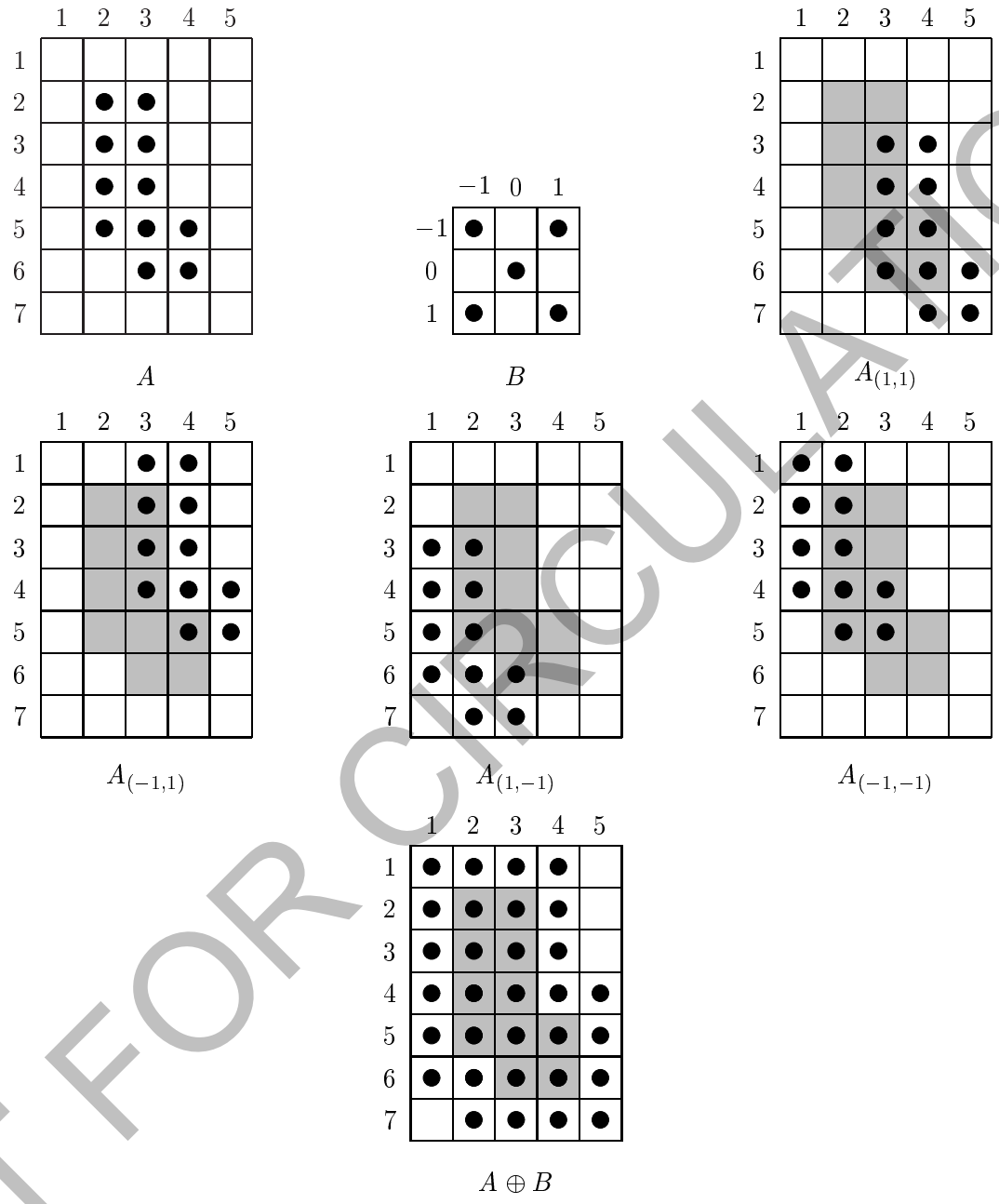

Figure 10.3: Dilation

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this: A is the same as in figure 10.3; B has the same shape but a different position. In this figure, we have

$$B = \{(7,3),(6,2),(6,4),(8,2),(8,4)\}$$

so that

$$A \oplus B = A_{(7,3)} \cup A_{(6,2)} \cup A_{(6,4)} \cup A_{(8,2)} \cup A_{(8,4)}.$$

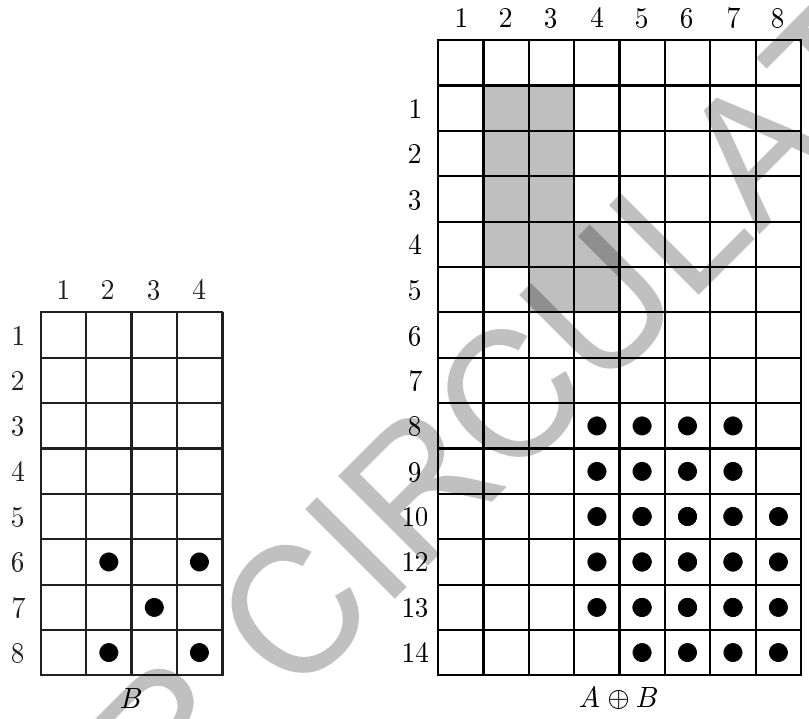

Figure 10.4: A dilation for which $A \not\subseteq A \oplus B$

For dilation, we generally assume that A is the image being processed, and B is a small set of pixels. In this case B is referred to as a *structuring element* or as a *kernel*.

Dilation in MATLAB is performed with the command

```
>> imdilate(image,kernel)
```

To see an example of dilation, consider the commands:

```
>> t=imread('text.tif');
>> sq=ones(3,3);
```

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```
>> td=imdilate(t,sq);
>> subplot(1,2,1),imshow(t)
>> subplot(1,2,2),imshow(td)
```

The result is shown in figure 10.5. Notice how the image has been "thickened". This is really what dilation does; hence its name.

Figure 10.5: Dilation of a binary image

10.3.2 Erosion

Given sets A and B, the *erosion of A by B*, written $A \ominus B$, is defined as:

$$A \ominus B = \{w : B_w \subseteq A\}.$$

In other words the erosion of A by B consists of all points $w = (x, y)$ for which B_w is in A. To perform an erosion, we can move B over A, and find all the places it will fit, and for each such place mark down the corresponding $(0, 0)$ point of B. The set of all such points will form the erosion.

An example of erosion is given in figures 10.6.

Note that in the example, the erosion $A \ominus B$ was a subset of A. This is not necessarily the case; it depends on the position of the origin in B. If B contains the origin (as it did in figure 10.6), then the erosion will be a subset of the original object.

Figure 10.7 shows an example where B does not contain the origin. In this figure, the open circles in the right hand figure form the erosion.

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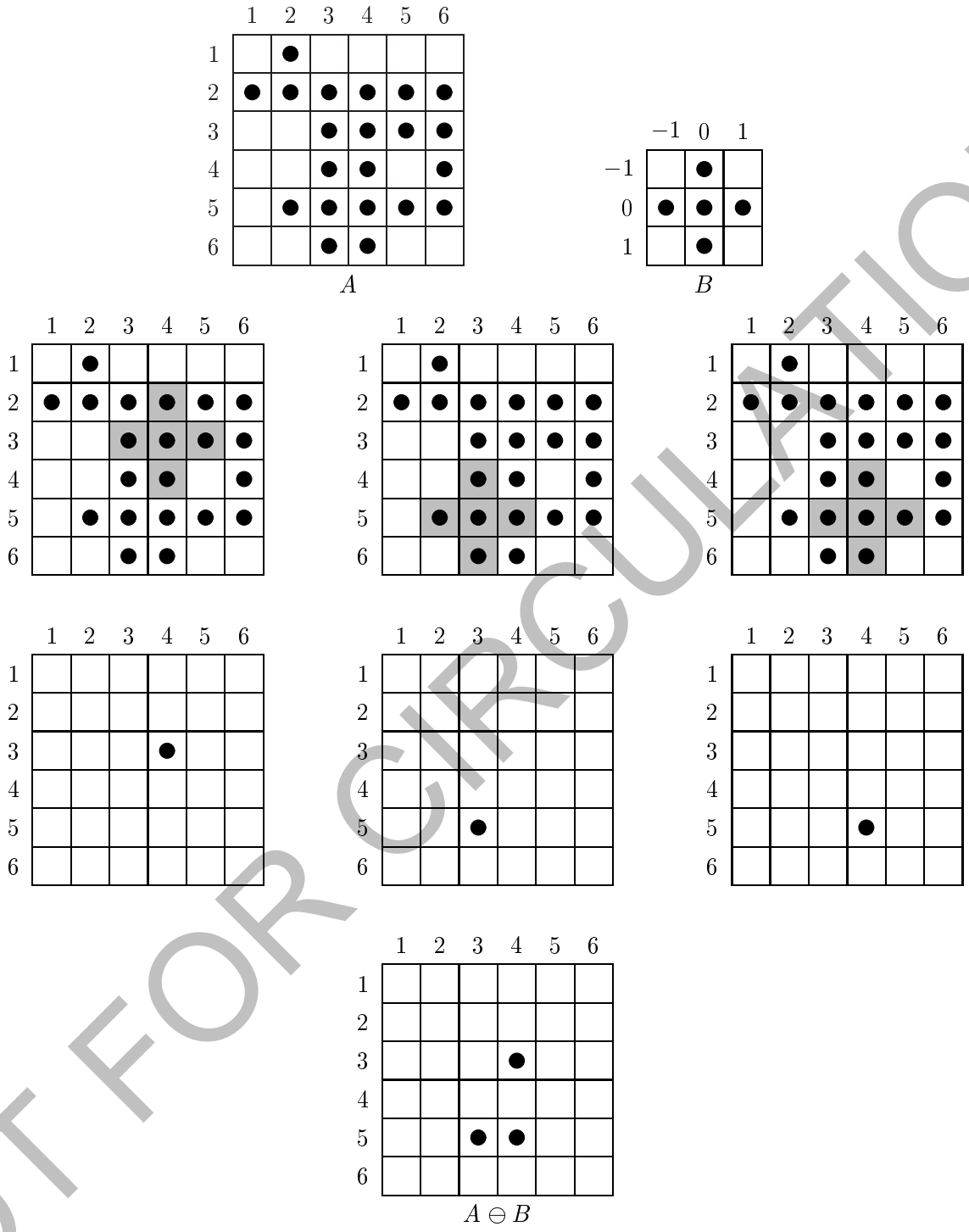

Figure 10.6: Erosion with a cross-shaped structuring element

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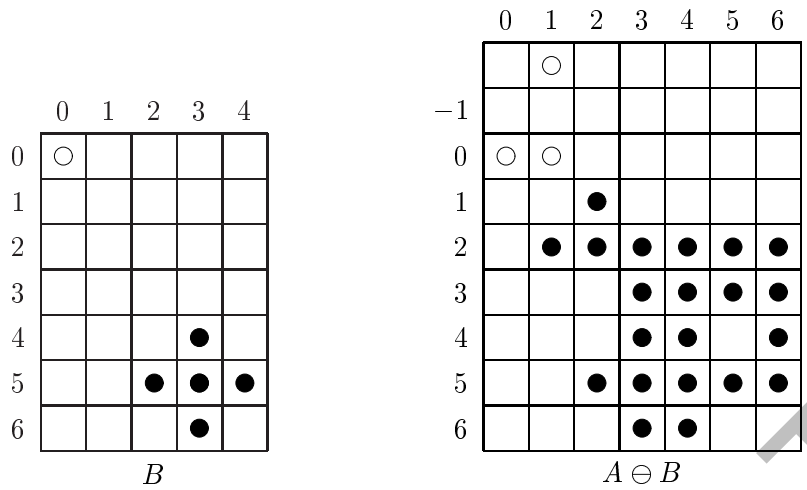

Figure 10.7: Erosion with a structuring element not containing the origin

Note that in figure 10.7, the *shape* of the erosion is the same as that in figure 10.6; however its *position* is different. Since the origin of B in figure 10.7 is translated by $(-4, -3)$ from its position in figure 10.6, we can assume that the erosion will be translated by the same amount. And if we compare figures 10.6 and 10.7, we can see that the second erosion has indeed been shifted by $(-4, -3)$ from the first.

For erosion, as for dilation, we generally assume that A is the image being processed, and B is a small set of pixels: the structuring element or kernel.

Erosion is related to *Minkowski subtraction*: the Minkowski subtraction of B from A is defined as

$$A - B = \bigcap_{b \in B} A_b.$$

Erosion in MATLAB is performed with the command

```
>> imerode(image,kernel)
```

We shall give an example; using a different binary image:

```
>> c=imread('circbw.tif');
>> ce=imerode(c,sq);
>> subplot(1,2,1),imshow(c)
>> subplot(1,2,2),imshow(ce)
```

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The result is shown in figure 10.8. Notice how the image has been "thinned". This is the expected result of an erosion; hence its name. If we kept on eroding the image, we would end up with a completely black result.

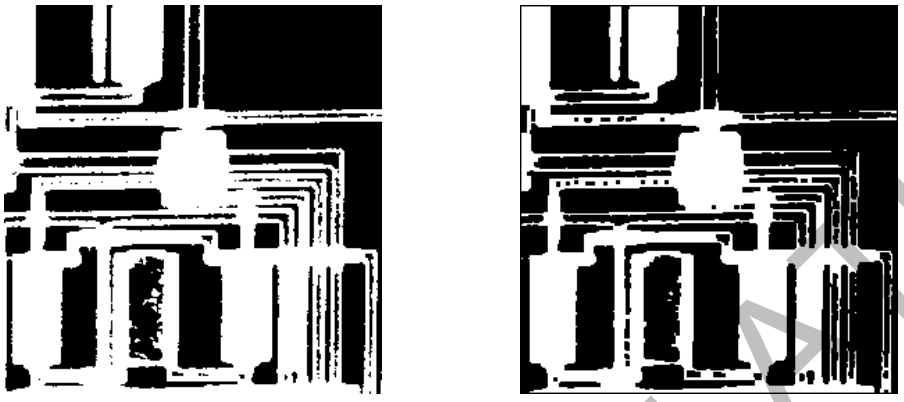

Figure 10.8: Erosion of a binary image

Relationship between erosion and dilation

It can be shown that erosion and dilation are "inverses" of each other; more precisely, the complement of an erosion is equal to the dilation of the complement. Thus:

$$\overline{A \ominus B} = \overline{A} \oplus \hat{B}.$$

A proof of this can be found in Haralick and Shapiro [9].

It can be similarly shown that the same relationship holds if erosion and dilation are interchanged; that

$$\overline{A \oplus B} = \overline{A} \ominus \hat{B}.$$

We can demonstrate the truth of these using MATLAB commands; all we need to know is that the complement of a binary image

```
b
```

is obtained using

```
>> ~b
```

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and that given two images `a` and `b`; their equality is determined with

```
>> all(a(:)==b(:))
```

To demonstrate the equality

$$\overline{A \ominus B} = \overline{A} \oplus \hat{B}.$$

pick a binary image, say the text image, and a structuring element. Then the left hand side of this equation is produced with

```
>> lhs=~imerode(t,sq);
```

and the right hand side with

```
>> rhs=imdilate(~t,sq);
```

Finally, the command

```
>> all(lhs(:)==rhs(:))
```

should return 1, for true.

10.3.3 An application: boundary detection

If A is an image, and B a small structuring element consisting of point symmetrically places about the origin, then we can define the boundary of A by any of the following methods:

(i) $A - (A \ominus B)$ " internal boundary"

(ii) $(A \oplus B) - A$ "external boundary"

(iii) $(A \oplus B) - (A \ominus B)$ "morphological gradient"

In each definition the minus refers to set difference. For some examples, see figure 10.9. Note that the internal boundary consists of those pixels in A which are at its edge; the external boundary consists of pixels outside A which are just next to it, and that the morphological gradient is a combination of both the internal and external boundaries.

To see some examples, choose the image `rice.tif`, and threshold it to obtain a binary image:

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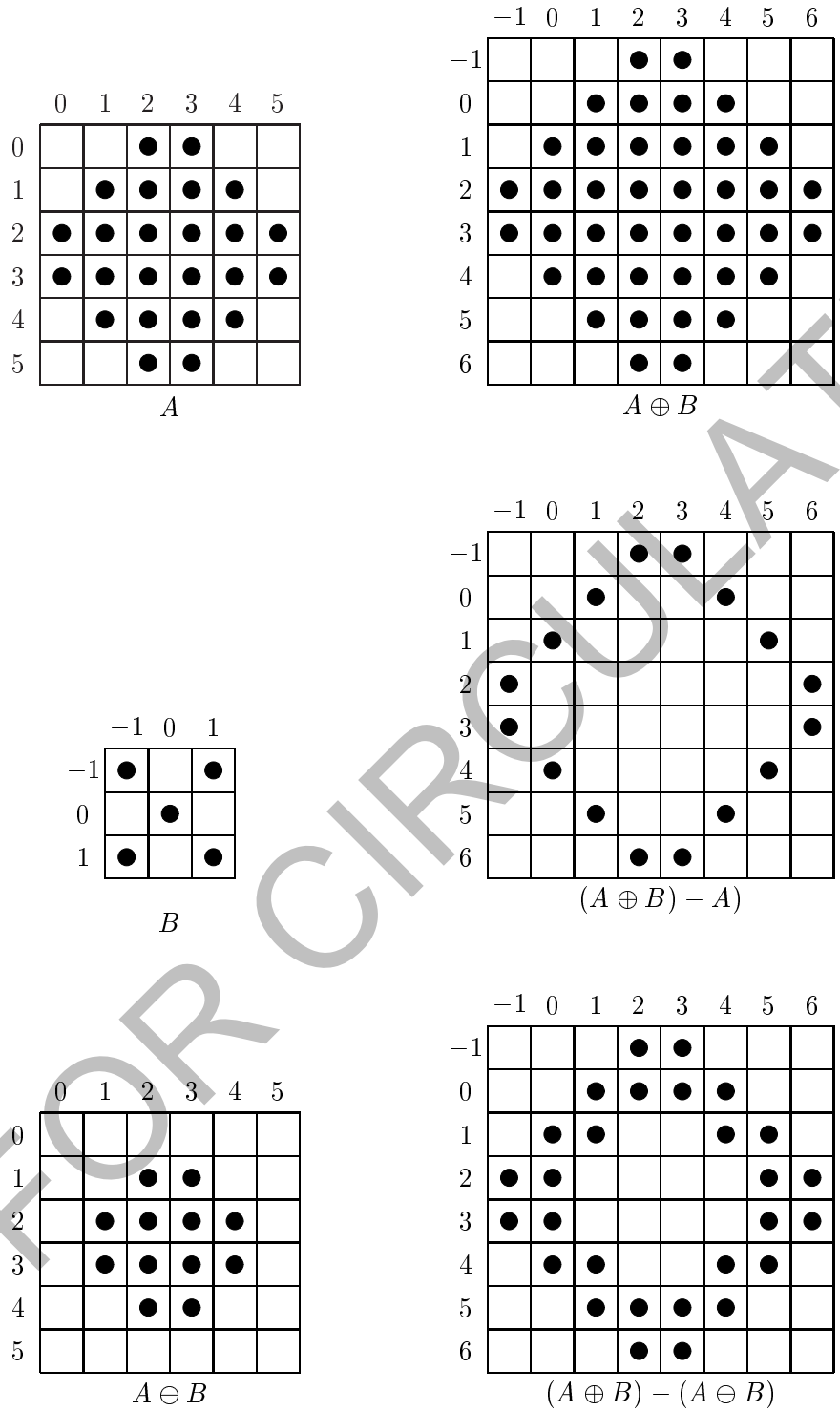

Figure 10.9: Boundaries

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```
>> rice=imread('rice.tif');
>> r=rice>110;
```

Then the internal boundary is obtained with:

```
>> re=imerode(r,sq);
>> r_int=r&~re;
>> subplot(1,2,1),imshow(r)
>> subplot(1,2,2),imshow(r_int)
```

The result is shown in figure 10.10.

Figure 10.10: "Internal boundary" of a binary image

The external boundary and morphological gradients can be obtained similarly:

```
>> rd=imdilate(r,sq);
>> r_ext=rd&~r;
>> r_grad=rd&~re;
>> subplot(1,2,1),imshow(r_ext)
>> subplot(1,2,2),imshow(r_grad)
```

The results are shown in figure 10.11.

Note that the external boundaries are larger than the internal boundaries. This is because the internal boundaries show the outer edge of the image components; whereas the external boundaries

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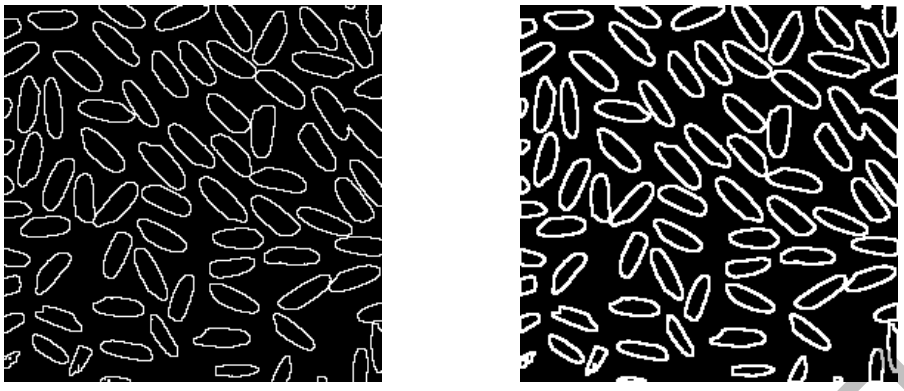

Figure 10.11: "External boundary" and the morphological gradient of a binary image

show the pixels just outside the components. The morphological gradient is thicker than either, and is in fact the union of both.

10.4 Opening and closing

These operations may be considered as "second level" operations; in that they build on the basic operations of dilation and erosion. They are also, as we shall see, better behaved mathematically.

10.4.1 Opening

Given A and a structuring element B, the *opening of A by B*, denoted $A \circ B$, is defined as:

$$A \circ B = (A \ominus B) \oplus B.$$

So an opening consists of an erosion followed by a dilation. An equivalent definition is

$$A \circ B = \cup\{B_w : B_w \subseteq A\}.$$

That is, $A \circ B$ is the union of all translations of B which fit inside A. Note the difference with erosion: the erosion consists only of the $(0,0)$ point of B for those translations which fit inside A; the opening consists of all of B. An example of opening is given in figure 10.12.

The opening operation satisfies the following properties:

1. $(A \circ B) \subseteq A$. Note that this is not the case with erosion; as we have seen, an erosion may not necessarily be a subset.

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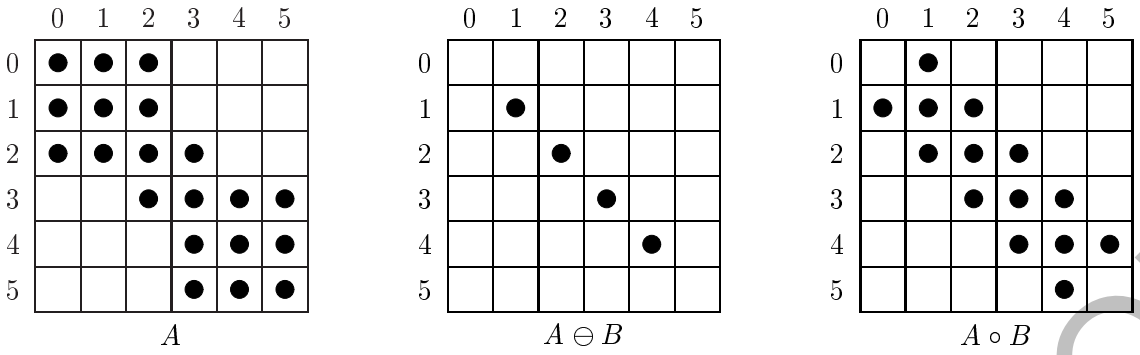

Figure 10.12: Opening

2. $(A \circ B) \circ B = A \circ B$. That is, an opening can never be done more than once. This property is called *idempotence*. Again, this is not the case with erosion; you can keep on applying a sequence of erosions to an image until nothing is left.

3. If $A \subseteq C$, then $(A \circ B) \subseteq (C \circ B)$.

4. Opening tends to "smooth" an image, to break narrow joins, and to remove thin protrusions.

10.4.2 Closing

Analogous to opening we can define *closing*, which may be considered as a dilation followed by an erosion, and is denoted $A \bullet B$:

$$A \bullet B = (A \oplus B) \ominus B.$$

Another definition of closing is that $x \in A \bullet B$ if *all* translations B_w which contain x have non-empty intersections with A. An example of closing is given in figure 10.13. The closing operation satisfies the following properties:

1. $A \subseteq (A \bullet B)$.

2. $(A \bullet B) \bullet B = A \bullet B$; that is, closing, like opening, is idempotent.

3. If $A \subseteq C$, then $(A \bullet B) \subseteq (C \bullet B)$.

4. Closing tends also to smooth an image, but it fuses narrow breaks and thin gulfs, and eliminates small holes.

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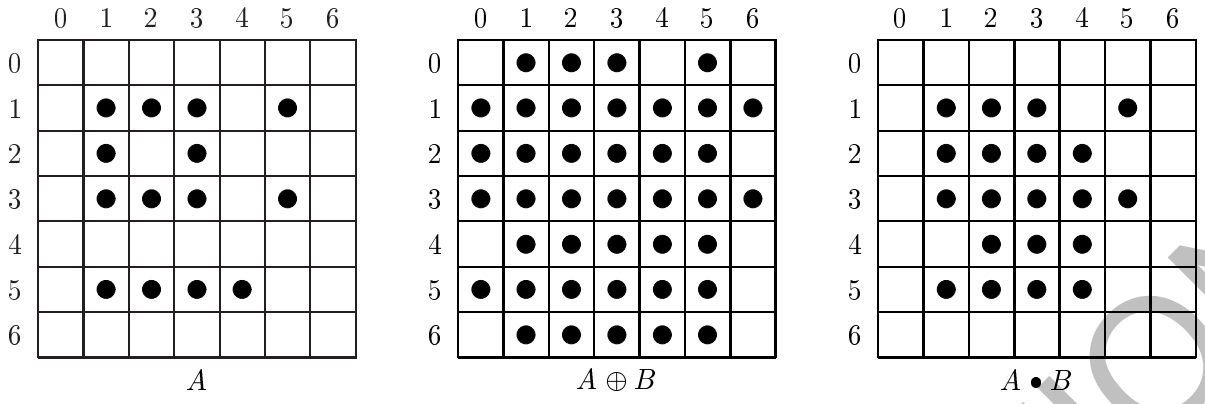

Figure 10.13: Closing

Opening and closing are implemented by the `imopen` and `imclose` functions respectively. We can see the effects on a simple image using the square and cross structuring elements.

```
>> cr=[0 1 0;1 1 1;0 1 0];
>> >> test=zeros(10,10);test(2:6,2:4)=1;test(3:5,6:9)=1;test(8:9,4:8)=1;test(4,5)=1


test =


     0     0     0     0     0     0     0     0     0     0
     0     1     1     1     0     0     0     0     0     0
     0     1     1     1     0     1     1     1     1     0
     0     1     1     1     1     1     1     1     1     0
     0     1     1     1     0     1     1     1     1     0
     0     1     1     1     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     1     1     1     1     1     0     0
     0     0     0     1     1     1     1     1     0     0
     0     0     0     0     0     0     0     0     0     0


>> imopen(test,sq)
```

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```
ans =

     0     0     0     0     0     0     0     0     0     0
     0     1     1     1     0     0     0     0     0     0
     0     1     1     1     0     1     1     1     1     0
     0     1     1     1     0     1     1     1     1     0
     0     1     1     1     0     1     1     1     1     0
     0     1     1     1     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0

>> imopen(test,cr)

ans =

     0     0     0     0     0     0     0     0     0     0
     0     0     1     0     0     0     0     0     0     0
     0     1     1     1     0     1     1     1     0     0
     0     1     1     1     1     1     1     1     1     0
     0     1     1     1     0     1     1     1     0     0
     0     0     1     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
```

Note that in each case the image has been separated into distinct components, and the lower part has been removed completely.

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```
>> imclose(test,sq)


ans =


     1     1     1     1     0     0     0     0     0     0
     1     1     1     1     0     0     0     0     0     0
     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     1     1     1     1     0     0
     0     0     0     1     1     1     1     1     0     0
     0     0     0     1     1     1     1     1     0     0
     0     0     0     1     1     1     1     1     0     0
     0     0     0     1     1     1     1     1     0     0


>> imclose(test,cr)


ans =


     0     0     1     0     0     0     0     0     0     0
     0     1     1     1     0     0     0     0     0     0
     1     1     1     1     1     1     1     1     1     0
     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     1     1     1     1     1     0
     0     1     1     1     1     1     1     1     0     0
     0     0     1     1     1     1     1     0     0     0
     0     0     0     1     1     1     1     1     0     0
     0     0     0     1     1     1     1     1     0     0
```

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```
     0     0     0     0     1     1     1     0     0     0
```

With closing, the image is now fully "joined up". We can obtain a joining-up effect with the text image, using a diagonal structuring element.

```
>> diag=[0 0 1;0 1 0;1 0 0]

diag =

     0     0     1
     0     1     0
     1     0     0

>> tc=imclose(t,diag);
>> imshow(tc)
```

The result is shown in figure 10.14.

Figure 10.14: An example of closing

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An application: noise removal

Suppose A is a binary image corrupted by impulse noise—some of the black pixels are white, and some of the white pixels are back. An example is given in figure 10.15. Then $A \ominus B$ will remove the single black pixels, but will enlarge the holes. We can fill the holes by dilating twice:

$$((A \ominus B) \oplus B) \oplus B.$$

The first dilation returns the holes to their original size; the second dilation removes them. But this will enlarge the objects in the image. To reduce them to their correct size, perform a final erosion:

$$(((A \ominus B) \oplus B) \oplus B) \ominus B.$$

The inner two operations constitute an opening; the outer two operations a closing. Thus this noise removal method is in fact an opening followed by a closing:

$$(A \circ B) \bullet B).$$

This is called *morphological filtering*.

Suppose we take an image and apply 10% shot noise to it:

```
>> c=imread('circles.tif');
>> x=rand(size(c));
>> d1=find(x<=0.05);
>> d2=find(x>=0.95);
>> c(d1)=0;
>> c(d2)=1;
>> imshow(c)
```

The result is shown as figure 10.15(a). The filtering process can be implemented with

```
>> cf1=imclose(imopen(c,sq),sq);
>> figure,imshow(cf1)
>> cf2=imclose(imopen(c,cr),cr);
>> figure,imshow(cf2)
```

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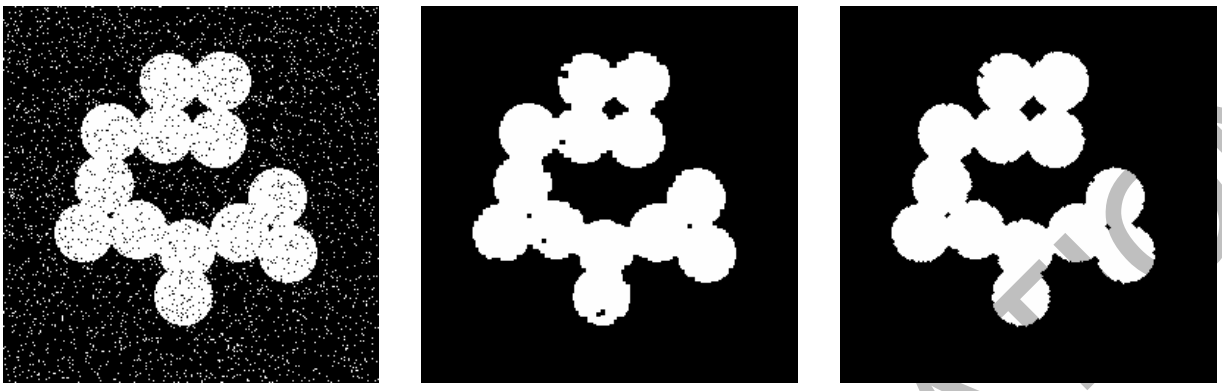
and the results are shown as figures 10.15(b) and (c). The results are rather "blocky"; although less so with the cross structuring element.



(a) (b) (c)

Figure 10.15: A noisy binary image and results after morphological filtering with different structuring elements.

Relationship between opening and closing

Opening and closing share a relationship very similar to that of erosion and dilation: the complement of an opening is equal to the closing of a complement, and complement of an closing is equal to the opening of a complement. Specifically:

$$\overline{A \bullet B} = \overline{A} \circ \hat{B}$$

and

$$\overline{A \circ B} = \overline{A} \bullet \hat{B}.$$

Again see Haralick and Shapiro [9] for a formal proof.

10.5 The hit-or-miss transform

This is a powerful method for finding shapes in images. As with all other morphological algorithms, it can be defined entirely in terms of dilation and erosion; in this case, erosion only.

Suppose we wish to locate 3×3 square shapes, such as is in the centre of the image A in figure 10.16.

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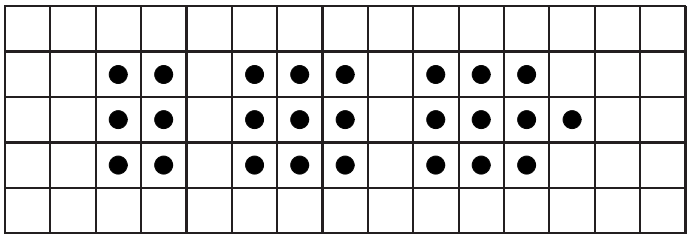


Figure 10.16: An image A containing a shape to be found

If we performed an erosion $A \ominus B$ with B being the square structuring element, we would obtain the result given in figure 10.17.

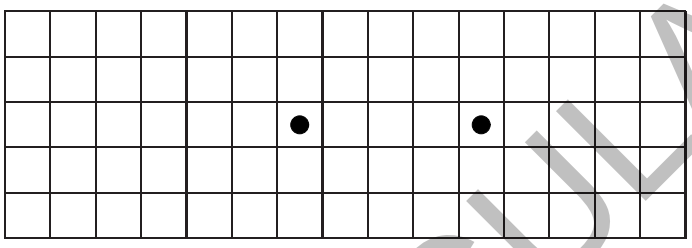
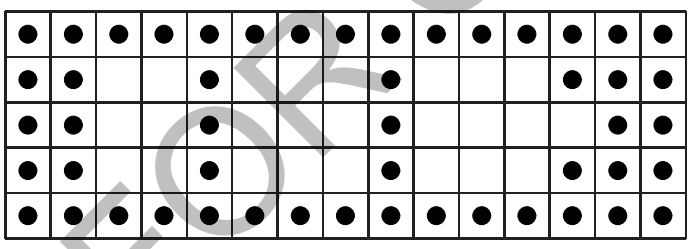


Figure 10.17: The erosion $A \ominus B$

The result contains two pixels, as there are exactly two places in A where B will fit. Now suppose we also erode the complement of A with a structuring element C which fits exactly around the 3×3 square; $\overline{A}$ and C are shown in figure 10.18. (We assume that $(0, 0)$ is at the centre of C.)

$\overline{A}$:



C:

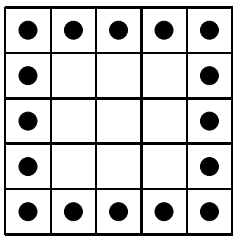


Figure 10.18: The complement and the second structuring element

If we now perform the erosion $\overline{A} \ominus C$ we would obtain the result shown in figure 10.19.

The intersection of the two erosion operations would produce just one pixel at the position of the centre of the 3×3 square in A, which is just what we want. If A had contained more than one square, the final result would have been single pixels at the positions of the centres of each. This combination of erosions forms the hit-or-miss transform.

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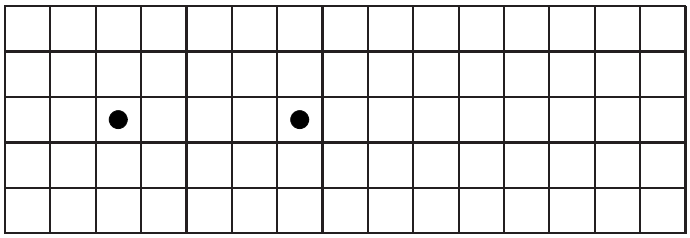

Figure 10.19: The erosion $\overline{A} \ominus C$

In general, if we are looking for a particular shape in an image, we design two structuring elements: B_1 which is the same shape, and B_2 which fits around the shape. We then write $B = (B_1, B_2)$ and

$$A \circledast B = (A \ominus B_1) \cap (\overline{A} \ominus B_2)$$

for the hit-or-miss transform.

As an example, we shall attempt to find the hyphen in "Cross-Correlation" in the text image shown in figure 10.5. This is in fact a line of pixels of length six. We thus can create our two structuring elements as:

```
>> b1=ones(1,6);
>> b2=[1 1 1 1 1 1 1 1;1 0 0 0 0 0 0 1; 1 1 1 1 1 1 1 1];
>> tb1=erode(t,b1);
>> tb2=erode(~t,b2);
>> hit_or_miss=tb1&tb2;
>> [x,y]=find(hit_or_miss==1)
```

and this returns a coordinate of $(41, 76)$, which is right in the middle of the hyphen. Note that the command

```
>> tb1=erode(t,b1);
```

is not sufficient, as there are quite a few lines of length six in this image. We can see this by viewing the image `tb1`, which is given in figure 10.20.

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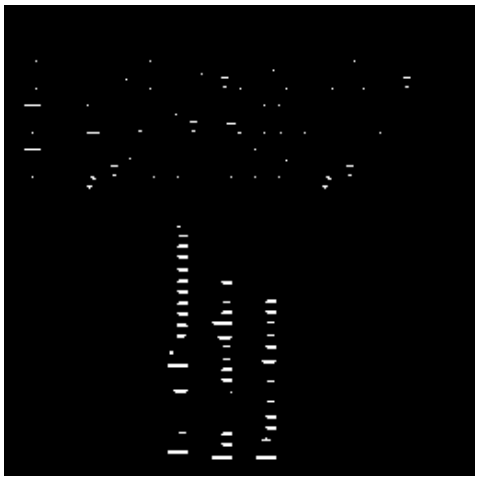

Figure 10.20: Text eroded by a hyphen-shaped structuring element

10.6 Some morphological algorithms

In this section we shall investigate some simple algorithms which use some of the morphological techniques we have discussed in previous sections.

10.6.1 Region filling

Suppose in an image we have a region bounded by an 8-connected boundary, as shown in figure 10.21.

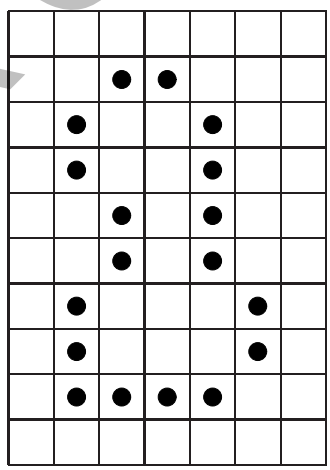

Figure 10.21: An 8-connected boundary of a region to be filled

Given a pixel p within the region, we wish to fill up the entire region. To do this, we start with

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p, and dilate as many times as necessary with the cross-shaped structuring element B (as used in figure 10.6), each time taking an intersection with $\overline{A}$ before continuing. We thus create a sequence of sets:

$$\{p\} = X_0, X_1, X_2, \ldots, X_k = X_{k+1}$$

for which

$$X_n = (X_{n-1} \oplus B) \cap \overline{A}.$$

Finally $X_k \cup A$ is the filled region. Figure 10.22 shows how this is done.

$\overline{A}$:

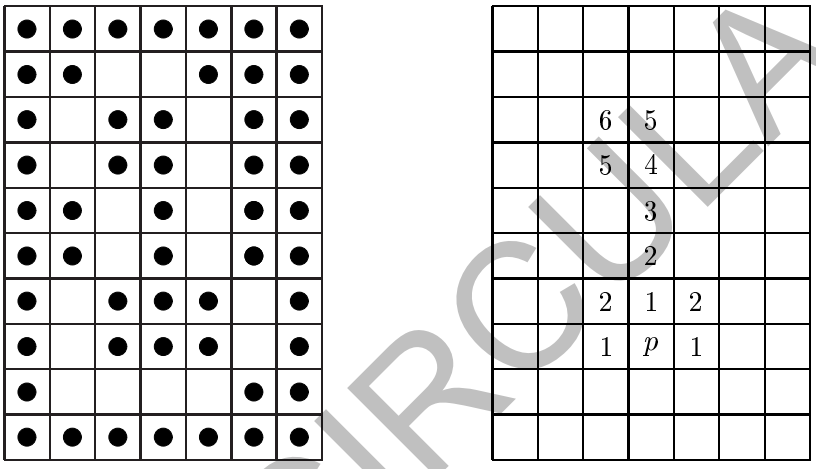

Figure 10.22: The process of filling a region

In the right hand grid, we have

$$X_0 = \{p\}, X_1 = \{p, 1\}, X_2 = \{p, 1, 2\}, \ldots$$

Note that the use of the cross-shaped structuring element means that we never cross the boundary.

10.6.2 Connected components

We use a very similar algorithm to fill a connected component; we use the cross-shaped structuring element for 4-connected components, and the square structuring element for 8-connected components. Starting with a pixel p, we fill up the rest of the component by creating a sequence of sets

$$X_0 = \{p\}, X_1, X_2, \ldots$$

NOT FOR CIRCULATION

such that

$$X_n = (X_{n-1} \oplus B) \cap A$$

until $X_k = X_{k-1}$. Figure 10.23 shows an example.

●	●		●	●	
●	●	●		●	
			●	●	●
●	●	●			
●	●	●			
●	●	●			

2	1	2			
1	p	1			
2	1	2			

Using the cross

5	4		4	4	
5	4	3		3	
			2	3	4
1	1	1			
1	p	1			
1	1	1			

Using the square

Figure 10.23: Filling connected components

In each case we are starting in the centre of the square in the lower left. As this square is itself a 4-connected component, the cross structuring element cannot go beyond it.

Both of these algorithms can be very easily implemented by MATLAB functions. To implement region filling, we keep track of two images: `current` and `previous`, and stop when there is no difference between them. We start with `previous` being the single point p in the region, and `current` the dilation $(p \oplus B) \cap \overline{A}$. At the next step we set

$$\begin{aligned} \texttt{previous} &\leftarrow \texttt{current}, \\ \texttt{current} &\leftarrow (\texttt{current} \oplus B) \cap \overline{A}. \end{aligned}$$

Given B, we can implement the last step in MATLAB by

```
imdilate(current,B)&~A.
```

The function is shown in figure 10.24 We can use this to fill a particular region delineated by a boundary.

```
>> n=imread('nicework.tif');
>> imshow(n),pixval on
>> nb=n&~imerode(n,sq);
```

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```
function out=regfill(im,pos,kernel)
% REGFILL(IM,POS,KERNEL) performs region filling of binary image IMAGE,
% with kernel KERNEL, starting at point with coordinates given by POS.
%
% Example:
%           n=imread('nicework.tif');
%           nb=n&~imerode(n,ones(3,3));
%           nr=regfill(nb,[74,52],ones(3,3));
%
current=zeros(size(im));
last=zeros(size(im));
last(pos(1),pos(2))=1;
current=imdilate(last,kernel)&~im;
while any(current(:)~=last(:)),
  last=current;
  current=imdilate(last,kernel)&~im;
end;
out=current;
```

Figure 10.24: A simple program for filling regions

```
>> figure,imshow(nb)

>> nf=regfill(nb,[74,52],sq);

>> figure,imshow(nf)
```

The results are shown in figure 10.25. Image (a) is the original; (b) the boundary, and (c) the result of a region fill. Figure (d) shows a variation on the region filling, we just include all boundaries. This was obtained with

```
>> figure,imshow(nf|nb)
```

(a) (b) (c) (d)

Figure 10.25: Region filling

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The function for connected components is almost exactly the same as that for region filling, except that whereas for region filling we took an intersection with the *complement* of our image, for connected components we take the intersection with the image itself. Thus we need only change one line, and the resulting function is shown in 10.26 We can experiment with this function with

```
function out=components(im,pos,kernel)
% COMPONENTS(IM,POS,KERNEL) produces the connected component of binary image
% IMAGE which nicludes the point with coordinates given by POS, using
% kernel KERNEL.
%
% Example:
%           n=imread('nicework.tif');
%           nc=components(nb,[74,52],ones(3,3));
%
current=zeros(size(im));
last=zeros(size(im));
last(pos(1),pos(2))=1;
current=imdilate(last,kernel)&im;
while any(current(:)~=last(:)),
  last=current;
  current=imdilate(last,kernel)&im;
end;
out=current;
```

Figure 10.26: A simple program for connected components

the "nice work" image. We shall use the square structuring element, and also a larger structuring element of size 11×11.

```
>> sq2=ones(11,11);
>> nc=components(n,[57,97],sq);
>> imshow(nc)
>> nc2=components(n,[57,97],sq2);
>> figure,imshow(nc2)
```

and the results are shown in figure 10.27. Image (a) uses the 3×3 square; image (b) uses the 11×11 square.

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(a) (b)

Figure 10.27: Connected components

10.6.3 Skeletonization

Recall that the *skeleton* of an object can be defined by the "medial axis transform"; we may imagine fires burning in along all edges of the object. The places where the lines of fire meet form the skeleton. The skeleton may be produced by morphological methods.

Consider the table of operations as shown in table 10.1.

Erosions	**Openings**	**Set differences**
A	$A \circ B$	$A - (A \circ B)$
$A \ominus B$	$(A \ominus B) \circ B$	$(A \ominus B) - ((A \ominus B) \circ B)$
$A \ominus 2B$	$(A \ominus 2B) \circ B$	$(A \ominus 2B) - ((A \ominus 2B) \circ B)$
$A \ominus 3B$	$(A \ominus 3B) \circ B$	$(A \ominus 3B) - ((A \ominus 3B) \circ B)$
$\vdots$	$\vdots$	$\vdots$
$A \ominus kB$	$(A \ominus kB) \circ B$	$(A \ominus kB) - ((A \ominus kB) \circ B)$

Table 10.1: Operations used to construct the skeleton

Here we use the convention that a sequence of k erosions using the same structuring element B is denoted $A \ominus kB$. We continue the table until $(A \ominus kB) \circ B$ is empty. The skeleton is then obtained by taking the unions of all the set differences. An example is given in figure 10.28, using the cross structuring element.

Since $(A \ominus 2B) \circ B$ is empty, we stop here. The skeleton is the union of all the sets in the third

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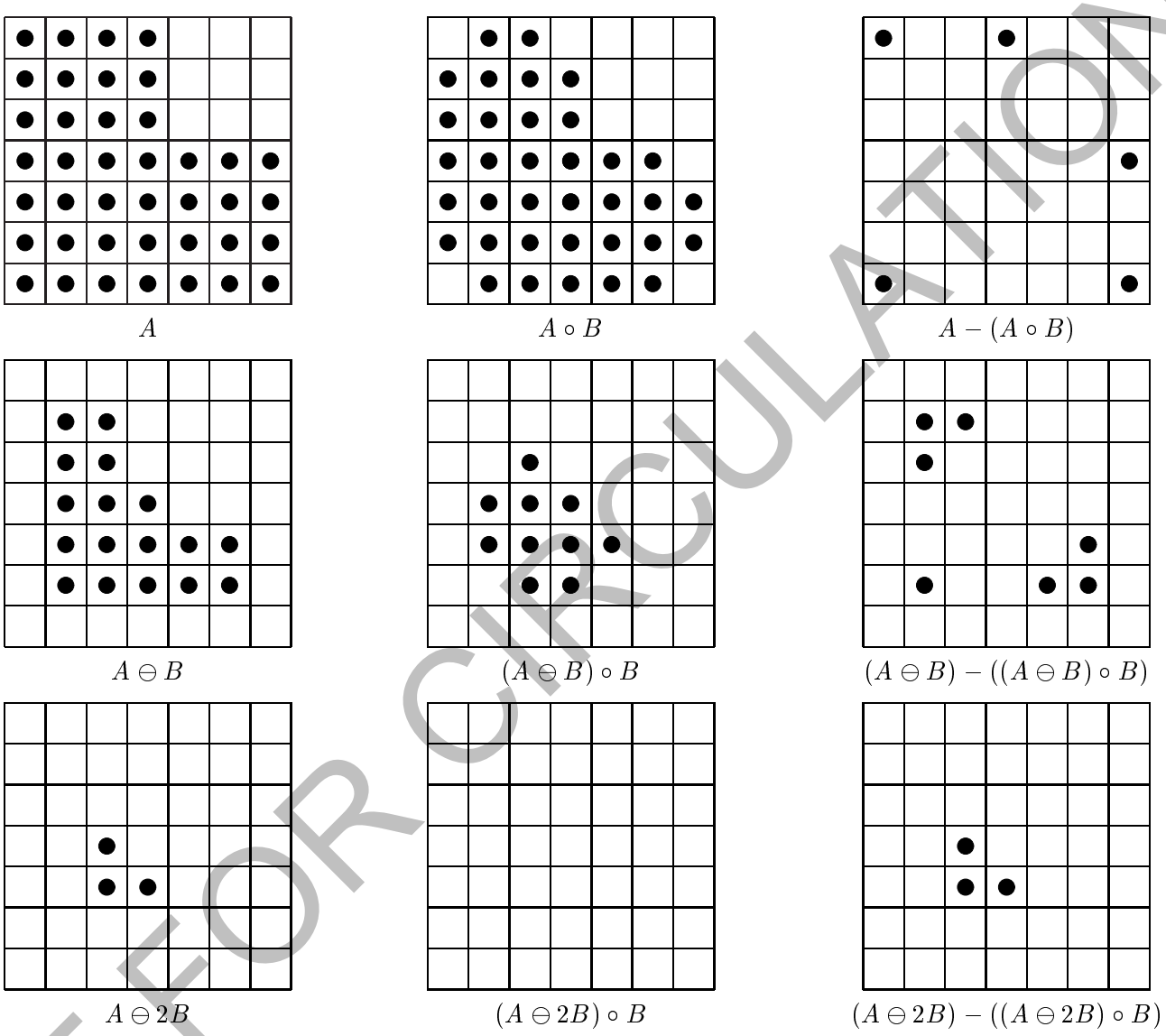

Figure 10.28: Skeletonization

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column; it is shown in figure 10.29. This method of skeletonization is called *Lantuéjoul's method*; for details see Serra [30].

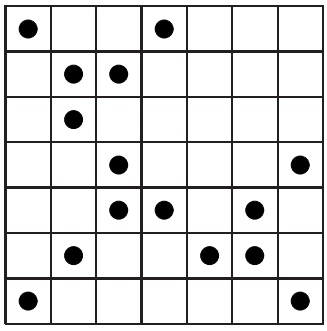

Figure 10.29: The final skeleton

This algorithm again can be implemented very easily; a function to do so is shown in figure 10.30. We shall experiment with the nice work image.

```
function skel = imskel(image,str)
% IMSKEL(IMAGE,STR) - Calculates the skeleton of binary image IMAGE using
% structuring element STR.  This function uses Lantejoul's algorithm.
%
skel=zeros(size(image));
e=image;
while (any(e(:))),
    o=imopen(e,str);
    skel=skel | (e&~o);
    e=imerode(e,str);
end
```

Figure 10.30: A simple program for computing skeletons

```
>> nk=imskel(n,sq);
>> imshow(nk)
>> nk2=imskel(n,cr);
>> figure,imshow(nk2)
```

The result is shown in figure 10.31. Image (a) is the result using the square structuring element; Image (b) is the result using the cross structuring element.

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(a) (b)

Figure 10.31: Skeletonization of a binary image

10.7 A note on MATLAB's bwmorph function

The theory of morphology developed so far uses versions of erosion and dilation sometimes called *generalized* erosion and dilation; so called because the definitions allow for general structuring elements. This is the method used in the `imerode` and `imdilate` functions. However, the `bwmorph` function actually uses a different approach to morphology; that based on *lookup tables*. We have seen the use of lookup tables for binary operations in chapter 11; they can be just as easily applied to implement morphological algorithms.

The idea is simple. Consider the 3×3 neighbourhood of a pixel. Since each pixel in the neighbourhood can only have two values, there are $2^9 = 512$ different possible neighbourhoods. We define a morphological operation to be a function which maps these neighbourhoods to the values 0 and 1. Each possible neighbourhood state can be associated with a numeric value from 0 (all pixels have value 0), to 511 (all pixels have value 1). The lookup table is then a binary vector of length 512; its k-th element is the value of the function for state k.

With this approach, we can define dilation as follows: a 0-valued pixel is changed to 1 if at least one of its eight neighbours is black has value 1. Conversely, we may define erosion as changing a 1-valued pixel to 0 is at least one of its eight neighbours has value 0.

Many other operations can be defined by this method (see the help file for `bwmorph`). The advantage is that any operation can be implemented extremely easily, simply by listing the lookup table. Moreover, the use of a lookup table allows us to satisfy certain requirements; the skeleton, for

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example, can be connected and have exactly one pixel thickness. This is not necessarily true of the algorithm presented in section 10.6.3. The disadvantage of lookup tables is that we are restricted to using 3×3 neighbourhoods. For more details, see "Digital Image Processing" by W. K. Pratt.[1]

10.8 Greyscale morphology

The operations of erosion and dilation can be generalized to be applied to greyscale images. But before we do, we shall reconsider binary erosion and dilation. We have defined binary erosion, $A \ominus B$, to be the union of the $(0,0)$ positions of all translations B_x for which $B_x \subseteq A$.

Suppose we take B to be a 3×3 square consisting entirely of zeros. Let A be the image as shown in figure 10.32. Now suppose we move over the image A, and for each point p we perform

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

A $\qquad$ B

Figure 10.32: An example for erosion

the following steps:

1. find the 3×3 neighbourhood N_p of p,
2. compute the matrix $N_p - B$,
3. find the minimum of that result.

We note that since B consists of all zeros, the second and third items could be reduced to finding the minimum of N_p. However, we shall see that for generalization it will be more convenient to have this expanded form. An immediate consequence of these steps is that if a neighbourhood contains at least one zero, the output will be zero. The output is one only if the neighbourhood contains all ones. For example:

[1] William K. Pratt, "Digital Image Processing", John Wiley and Sons, second edition, 1991

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$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \longrightarrow 0$$

If we perform this operation, we will obtain:

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

which you can verify is exactly the erosion $A \ominus B$.

For dilation, we perform a sequence of step very similar to those for erosion:

1. find the 3×3 neighbourhood N_p of p,

2. compute the matrix $N_p + B$,

3. find the maximum of that result.

We note again that since B consists of all zeros, the second and third items could be reduced to finding the maximum of N_p. If the neighbourhood contains at least one `1`, then the output will be `1`. The output will be `0` only if the neighbourhood contains all zeroes.

Suppose A to be surrounded by zeros, so that neighbourhoods are defined for all points in A above. Applying these steps produces:

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```
1 1 1 1 1 1 0 0
1 1 1 1 1 1 1 0
1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1
0 1 1 1 1 1 1 1
0 0 1 1 1 1 1 1
```

which again can be verified to be the dilation $A \oplus B$.

If A is a greyscale image, and B is a structuring element, which will be an array of integers, we define greyscale erosion by using the steps above; for each pixel p in the image:

1. position B so that $(0, 0)$ lies over p,

2. find the neighbourhood N_p of p corresponding to the shape of B,

3. find the value $\min(N_p - B)$.

We note that there is nothing in this definition which requires B to be any particular shape or size; or that the elements of B be positive. And as for binary dilation, B does not have to contain the origin $(0, 0)$.

We can define this more formally; let B be a set of points with associated values. For example,

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for our square of zeros we would have:

Point	Value
$(-1,-1)$	0
$(-1,0)$	0
$(-1,1)$	0
$(0,-1)$	0
$(0,0)$	0
$(0,1)$	0
$(1,-1)$	0
$(1,0)$	0
$(1,1)$	0

The set of points forming B is called the *domain* of B and is denoted D_B. Now we can define:

$$\begin{aligned}(A \ominus B)(x,y) &= \min\{A(x+s,y+t) - B(s,t), (s,t) \in D_B\},\\ (A \oplus B)(x,y) &= \max\{A(x+s,y+t) + B(s,t), (s,t) \in D_B\}.\end{aligned}$$

We note that some published definitions use $s-x$ and $t-y$ instead of $x+s$ and $y+t$. This just requires that the structuring element is rotated 180°.

An example. Suppose we take A and B as given in figure 10.33.

		y				
		1	2	3	4	5
	1	10	20	20	20	30
	2	20	30	30	40	50
x	3	20	30	30	50	60
	4	20	40	50	50	60
	5	30	50	60	60	70

A

		t		
		-1	0	1
	-1	1	2	3
s	0	4	5	6
	1	7	8	9

B

Figure 10.33: An example for greyscale erosion and dilation

In this example $(0,0) \in D_B$. Consider $(A \ominus B)(1,1)$. By the definition, we have:

$$(A \ominus B)(1,1) = \min\{A(1+s,1+t) - B(s,t), (s,t) \in D_B\}.$$

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Since $D_B = \{(s,t) : -1 \le s \le 1; -1 \le t \le 1\}$, we have

$$(A \ominus B)(1,1) = \min\{A(1+s, 1+t) - B(s,t) : -1 \le s \le 1; -1 \le t \le 1\}.$$

In order to ensure we don't require matrix indices which move outside A, we simply "cut off" the structuring element so that we restrict it to elements in A.

The values of $A(1+s, 1+t) - B(s,t)$ can then be obtained by matrix arithmetic:

$$\begin{bmatrix} 10 & 20 \\ 20 & 30 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 14 \\ 12 & 21 \end{bmatrix}$$

and the minimum of the result is 5. Note that to create the matrices we ensure that the $(0,0)$ point of B sits over the current point of A, in this case the point $(1,1)$. Another example:

$$(A \ominus B)(3,4) = \min\left\{\begin{bmatrix} 30 & 40 & 50 \\ 30 & 50 & 60 \\ 50 & 50 & 60 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right\} = \min\left\{\begin{bmatrix} 29 & 38 & 47 \\ 26 & 45 & 54 \\ 43 & 42 & 51 \end{bmatrix}\right\} = 26.$$

Finally, we have

$$A \ominus B = \begin{matrix} 5 & 6 & 14 & 15 & 16 \\ 8 & 9 & 17 & 18 & 19 \\ 12 & 13 & 25 & 26 & 39 \\ 15 & 16 & 28 & 29 & 46 \\ 18 & 19 & 39 & 48 & 49 \end{matrix}$$

Dilation is very similar to erosion, except that we add B and take the maximum of the result. As for erosion we restrict the structuring element so that it doesn't goes outside of A. For example:

$$(A \oplus B)(1,1) = \max\left\{\begin{bmatrix} 10 & 20 \\ 20 & 30 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right\} = \max\left\{\begin{bmatrix} 15 & 26 \\ 28 & 39 \end{bmatrix}\right\} = 39,$$

$$(A \oplus B)(3,4) = \max\left\{\begin{bmatrix} 30 & 40 & 50 \\ 30 & 50 & 60 \\ 50 & 50 & 60 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right\} = \max\left\{\begin{bmatrix} 31 & 42 & 53 \\ 34 & 55 & 66 \\ 57 & 58 & 69 \end{bmatrix}\right\} = 69.$$

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After all the calculations, we have

$$A \oplus B = \begin{matrix} 39 & 39 & 49 & 59 & 58 \\ 39 & 39 & 59 & 69 & 68 \\ 49 & 59 & 59 & 69 & 68 \\ 59 & 69 & 69 & 79 & 78 \\ 56 & 66 & 66 & 76 & 75 \end{matrix}$$

Note that an erosion may contain negative values, or a dilation values greater than 255. In order to render the result suitable for display, we have the same choices as for spatial filtering: we may apply a linear transformation; or we may clip values.

In general, and this can be seen from the examples, erosion will tend to decrease and darken objects in an image, and dilation will tend to enlarge and lighten objects.

Relationship between greyscale erosion and dilation

By definition of maximum and minimum we have, if X and Y are two matrices:

$$\max\{X + Y\} = -\min\{-X - Y\}.$$

Since $\max\{X + Y\}$ corresponds to $A \oplus B$ and $\min\{X - Y\}$ to $A \ominus B$, we have

$$\begin{aligned} A \oplus B &= -(-A \ominus B), \\ A \ominus B &= -(-A \oplus B), \end{aligned}$$

or

$$\begin{aligned} -(A \oplus B) &= -A \ominus B, \\ -(A \ominus B) &= -A \oplus B. \end{aligned}$$

We can use the `imerode` and `imdilate` functions for greyscale erosion and dilation, but we have to me more careful about the structuring element. To create a structuring element for use with greyscale morphology, we have to provide both the neighbourhood D_B, and the values. To do this we need to use the `strel` function.

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For example, we shall use MATLAB with the previous examples. First we need to create the structuring element.

```
>> str=strel('arbitrary',ones(3,3),[1 2 3;4 5 6;7 8 9])


str =


Nonflat STREL object containing 9 neighbors.


Neighborhood:
     1     1     1
     1     1     1
     1     1     1



Height:
     1     2     3
     4     5     6
     7     8     9
```

Here we use the `arbitrary` parameter of `strel`; this allows us to create a structuring element containing any values we like. The first matrix: `ones(3,3)` provides the neighbourhood; the second matrix provides the values. Now we can test it.

```
>> A=[10 20 20 20 30;20 30 30 40 50;20 30 30 50 60;20 40 50 50 60;30 50 60 60 70];
>> imerode(A,str)


ans =


     5     6    14    15    16
     8     9    17    18    19
```

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```
    12    13    25    26    39

    15    16    28    29    46

    18    19    39    48    49
```

For dilation MATLAB implements the convention of the structuring element being rotated by 180°. So to obtain the result we produced above, we need to rotate `str` to obtain `str2`.

```
>> str2=strel('arbitrary',ones(3,3),[9 8 7;6 5 4;3 2 1])


str2 =


Nonflat STREL object containing 9 neighbors.


Neighborhood:

     1     1     1

     1     1     1

     1     1     1



Height:

     9     8     7

     6     5     4

     3     2     1



>> imdilate(A,str2)


ans =


    39    39    49    59    58
```

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```
    39    39    59    69    68

    49    59    59    69    68

    59    69    69    79    78

    56    66    66    76    75
```

Now we can experiment with an image. We would expect that dilation would increase light areas in an image, and erosion would decrease them.

```
>> c=imread('caribou.tif');
>> str=strel('square',5)


str =


Flat STREL object containing 25 neighbors.
Decomposition: 2 STREL objects containing a total of 10 neighbors


Neighborhood:
     1     1     1     1     1

     1     1     1     1     1

     1     1     1     1     1

     1     1     1     1     1

     1     1     1     1     1



>> cd=imdilate(c,str);
>> ce=imerode(c,str);
>> imshow(cd),figure,imshow(ce)
```

The results are shown in figure 10.34. Image (a) is the dilation and image (b) the erosion.

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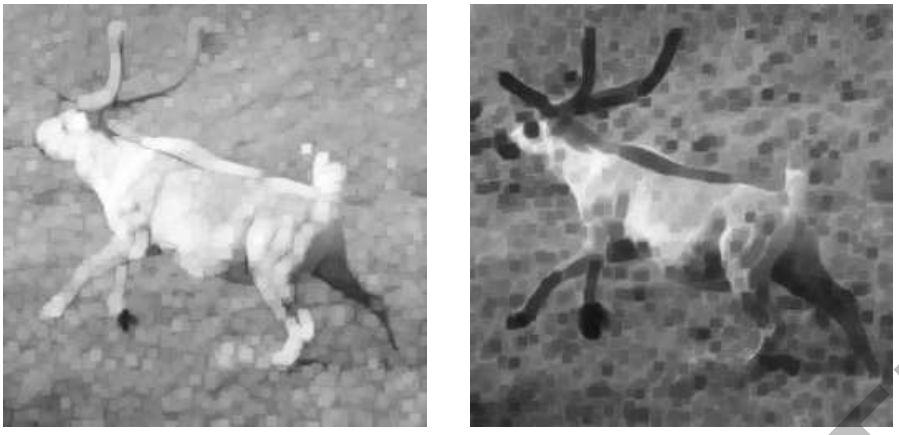

(a) (b)

Figure 10.34: Greyscale dilation and erosion

Opening and closing

Opening and closing are defined exactly as for binary morphology: opening is an erosion followed by a dilation, and closing is a dilation followed by an erosion. The `imopen` and `imclose` functions can be applied to greyscale images, if a structuring element has been created with `strel`.

Using the caribou image and the same 5×5 square structuring element as above:

```
>> co=imopen(c,str);
>> cc=imclose(c,str);
>> imshow(co),figure,imshow(cc)
```

and the results are shown in figure 10.35; image (a) being the opening and (b) the closing.

10.9 Applications of greyscale morphology

Almost all the applications we saw for binary images can be carried over directly to greyscale images.

Edge detection

We can use the morphological gradient

$$(A \oplus B) - (A \ominus B)$$

for greyscale edge detection. We can try this with two different structuring elements.

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(a) (b)

Figure 10.35: Greyscale opening and closing

```
>> str1=strel('square',3);
>> str2=strel('square',5);
>> ce1=imerode(c,str1);
>> ce2=imerode(c,str2);
>> cd1=imdilate(c,str1);
>> cd2=imdilate(c,str2);
>> cg1=imsubtract(cd1,ce1);
>> cg2=imsubtract(cd2,ce2);
>> imshow(cg1),figure,imshow(cg2)
```

The results are shown in figure 10.36. Image (a) uses the 3×3 square, and image (b) the 5×5 square.

Noise removal

As before we can attempt to remove noise with morphological filtering: an opening followed by a closing.

```
>> cn=imnoise(c,'salt & pepper');
>> cf=imclose(imopen(cn,str),str);
```

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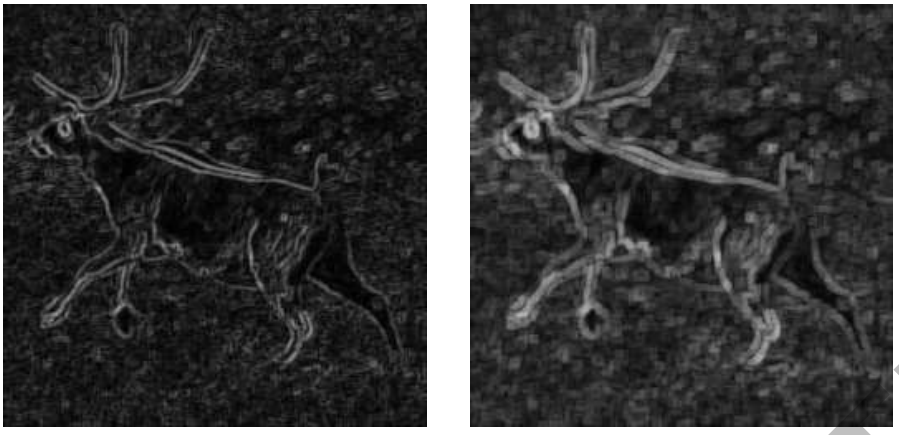

(a) (b)

Figure 10.36: Use of the morphological gradient

```
>> imshow(cn),figure,imshow(cf)
```

These are shown in figure 10.37. The result is reasonable, if slightly blurry. Morphological filtering

(a) (b)

Figure 10.37: Use of morphological filtering to remove noise

does not perform particularly well on Gaussian noise.

Exercises

1. For each of the following images A and structuring elements B:

   $A =$

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```
0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0
0 0 0 1 1 1 1 0    0 1 1 1 1 1 1 0    0 0 0 0 0 1 1 0
0 0 0 1 1 1 1 0    0 1 1 1 1 1 1 0    0 1 1 1 0 1 1 0
0 1 1 1 1 1 1 0    0 1 1 0 0 1 1 0    0 1 1 1 0 1 1 0
0 1 1 1 1 1 1 0    0 1 1 0 0 1 1 0    0 1 1 1 0 1 1 0
0 1 1 1 1 0 0 0    0 1 1 1 1 1 1 0    0 1 1 1 0 0 0 0
0 1 1 1 1 0 0 0    0 1 1 1 1 1 1 0    0 1 1 1 0 0 0 0
0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0
```

$B =$

```
0 1 0    1 1 1    1 0 0
1 1 1    1 1 1    0 0 0
0 1 0    1 1 1    0 0 1
```

calculate the erosion $A \ominus B$, the dilation $A \oplus B$, the opening $A \circ B$ and the closing $A \bullet B$.

Check your answers with MATLAB.

2. Suppose a square object was eroded by a circle whose radius was about one quarter the side of the square. Draw the result.

3. Repeat the previous question with dilation.

4. Using the binary images `circbw.tif`, `circles.tif`, `circlesm.tif`, `logo.tif` and `testpat2.tif`, view the erosion and dilation with both the square and the cross structuring elements.

   Can you see any differences?

5. Read in the image `circlesm.tif`.

   (a) Erode with squares of increasing size until the image starts to split into disconnected components.

   (b) Using `pixval on`, find the coordinates of a pixel in one of the components.

   (c) Use the `components` function to isolate that particular component.

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6. (a) With your disconnected image from the previous question, compute its boundary.

   (b) Again with `pixval on`, find a pixel inside one of the boundaries.

   (c) Use the `regfill` function to fill that region.

   (d) Display the image as a boundary with one of the regions filled in.

7. Using the 3×3 square structuring element, compute the skeletons of

   (a) a 7 square,

   (b) a 5×9 rectangle,

   (c) an L shaped figure formed from an 8×8 square with a 3×3 square taken from a corner,

   (d) an H shaped figure formed from a 15×15 square with 5×5 squares taken from the centres of the top and bottom,

   (e) a cross formed from an 11×11 square with 3×3 squares taken from each corner.

   In each case check your answer with MATLAB

8. Repeat the above question but use the cross structuring element.

9. For the images listed in question 4, obtain their skeletons by both the `bwmorph` function, and by using the function given in figure 10.30. Which seems to provide the best result?

10. Use the hit-or-miss transform with appropriate structuring elements to find the dot on the "i" in the word "in" in the image `text.tif`.

11. Let A be the matrix obtained with the MATLAB command

```
>> A=magic(6)
```

   By hand, compute the greyscale erosions and dilations with the structuring elements

   $B =$

```
10  10  10      5  20   5
10  10  10     20   5  20
10  10  10      5  20   5
```

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In each case, check your answers with MATLAB.

12. Perform greyscale erosions, dilations, openings and closings on the cameraman image using the above structuring elements.

13. (a) Obtain the greyscale version of the twins image as was used in chapter 8.
    (b) Apply salt & pepper noise to the image.
    (c) Attempt to remove the noise using morphological techniques.
    (d) Compare the results to median filtering.

14. Repeat the above question but use Gaussian noise.

15. Use morphological methods to find the edges of the image `ic.tif`. Do this in two ways:
    (a) By thresholding first and using binary morphology.
    (b) By using greyscale morphology and thresholding second.

    Which seems to give the best results?

16. Compare the results of the previous question with standard edge detection techniques.

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Chapter 11

Image Topology

11.1 Introduction

We are often interested in only the very basic aspects of an image: the number of occurrences of a particular object; whether or not there are holes, and so on. The investigation of these fundamental properties of an image is called *digital topology* or *image topology* and in this chapter we shall investigate some of the more elementary aspects of this subject.

For example, consider an image thresholded and cleaned up with a morphological opening to show a collection of blobs.

```
>> n=imread('nodules1.tif');
>> nt=~im2bw(e,0.5);
>> n2=imopen(nt,strel('disk',5));
```

The image `n2` is shown in figure 11.1. The number of blobs can be determined by morphological methods. However, the study of image topology provides alternative, and very powerful, methods for such tasks as object counting. Topology provides very rigorous definitions for concepts such as adjacency and distance. As we shall see, skeletonization can be performed very efficiently using topological methods.

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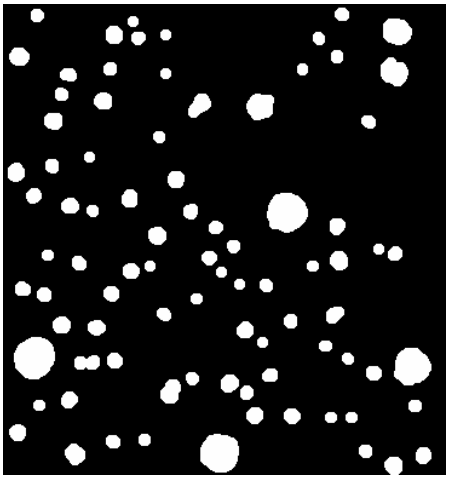

Figure 11.1: How many blobs?

11.2 Neighbours and adjacency

A first task is to define the concepts of adjacency: under what conditions a pixel may be regarded as being "next to" another pixel. For this chapter, the concern will be with binary images only, and so we will be dealing only with positions of pixels.

A pixel P has four 4-*neighbours*:

	•	
•	P	•
	•	

and eight 8-*neighbours*:

•	•	•
•	P	•
•	•	•

Two pixels P and Q are 4-*adjacent* if they are 4-neighbours of one another, and 8-*adjacent* if they are 8-neighbours of one another.

11.3 Paths and components

Suppose that P and Q are any two (not necessarily adjacent) pixels, and suppose that P and Q can be joined by a sequence of pixels as shown:

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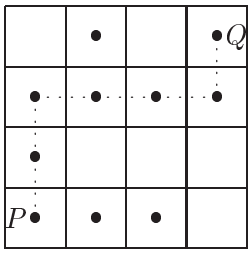
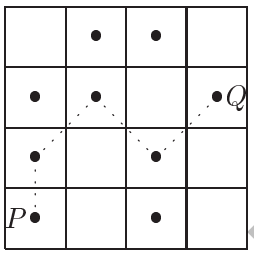
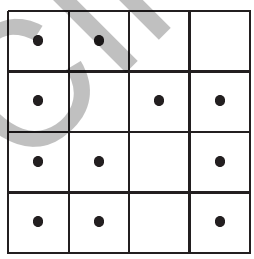


If the path contains only 4-adjacent pixels, as the path does in the above diagram, then P and Q are 4-*connected*. If the path contains 8-adjacent pixels, then P and Q are 8-*connected*. The following picture shows an example of 8-connected pixels.



A set of pixels all of which are 4-connected to each other is called a 4-*component*; if all the pixels are 8-connected the set is an 8-*component*.

For example, the following image:



has two 4-components (one component containing all the pixels in the left two columns, the other component containing all the pixels in the right two columns), but only one 8-component.

We can define *path* more formally as follows:

A 4-*path* from P to Q is a sequence of pixels

$$P = p_0, p_1, p_2, \ldots, p_n = Q$$

such that for each $i = 0, 1, \ldots, n-1$, pixel p_i is 4-adjacent to pixel p_{i+1}.

An 8-*path* is where the pixels in the sequence connecting P and Q are 8-adjacent.

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11.4 Equivalence relations

A relation $x \sim y$ between two objects x and y is an *equivalence relation* if the relation is

1. *reflexive*; $x \sim x$ for all x,
2. *symmetric*; $x \sim y \iff y \sim x$ for all x and y,
3. *transitive*; if $x \sim y$ and $y \sim z$ then $x \sim z$ for all x, y and z.

Some examples:

1. Numeric equality: $x \sim y$ if x and y are two numbers for which $x = y$.
2. Divisors: $x \sim y$ if x and y are two numbers which have the same remainder when divided by 7.
3. Set cardinality: $S \sim T$ if S and T are two sets with the same number of elements.
4. Connectedness: $P \sim Q$ if P and Q are two connected pixels.

And here are some relations which are not equivalence relations:

1. Personal relations. Define $x \sim y$ if x and y are two people who are related to each other. This is not an equivalence relation. It is reflexive (a person is certainly related to himself or herself); symmetric; but not transitive (can you give an example?).
2. Pixel adjacency. This is not transitive.
3. Subset relation. Define $S \sim T$ if $S \subseteq T$. This is reflexive (a set is a subset of itself); transitive; but not reflexive: if $S \subseteq T$ then it is not necessarily true that $T \subseteq S$.

The importance of the equivalence relation concept is that it allows us a very neat way of dealing with issues of connectedness. We need another definition: an *equivalence class* is the set of all objects equivalent to each other.

We can now define the components of a binary image as being the equivalence classes of the connectedness equivalence relation.

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11.5 Component labelling

In this section we give an algorithm for labelling all the 4-components of a binary image, starting a the top left and working across and down. If p is the current pixel, let u be its upper 4-neighbour, and l its left-hand 4-neighbour:

	u	
l	p	

We will "scan" the image row by row moving across from left to right. We will assign labels to pixels in the image; these labels will be the numbers of the components of the image.

For descriptive purposes, a pixel in the image will be called a *foreground pixel*; a pixel not in the image will be called a *background pixel*. And now for the algorithm:

1. Check the state of p. If it is a background pixel, move on to the next scanning position. If it is a foreground pixel, check the state of u and l. If the are both background pixels , assign a new label to p. (This is the case when a new component has been encountered.)

   - If just *one* of u or l is a foreground pixel, then assign its label to p.
   - If both of u and l are foreground pixels and have the same label, assign that label to p.
   - If both of u and l are foreground pixels but have different labels, assign either of those labels to p, and make a note that those two labels are equivalent (since u and l belong to the same 4-component connected through p).

2. At the end of the scan, all foreground pixels have been labelled, but some labels may be equivalent. We now sort the labels into equivalence classes, and assign a different label to each class.

3. Do a second pass through the image, replacing the label on each foreground pixel with the label assigned to its equivalence class in the previous step.

We now give an example of this algorithm in practice, on the binary image:

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	•		
•	•		•
		•	•
	•	•	

which has two 4-components: one the three pixels in the top left; the other the five pixels in the bottom right.

Step 1. We start moving along the top row. The first foreground pixel is the bullet in the second place, and since its upper and left neighbours are either background pixels or non-existent, we assign it the label 1:

	•1		
•	•		•
		•	•
	•	•	

In the second row, the first (foreground) pixel again has its upper or left neighbours either background or non-existent, so we assign it a new label—2:

	•1		
•2	•		•
		•	•
	•	•	

The second (foreground) pixel in the second row now has both its upper and left neighbours being foreground pixels. However, they have different labels. We thus assign either of these labels to this second pixel, say label 1, and make a note that labels 1 and 2 are equivalent:

	•1		
•2	•1		•
		•	•
	•	•	

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The third foreground pixel in the second row has both its upper and left neighbours being background pixels, so we assign it a new label—3:

	•1		
•2	•1		•3
		•	•
	•	•	

In the third row, the first foreground pixel has both its upper and left neighbours being background pixels, so we assign it a new label—4. The second (foreground) pixel in the third row now has both its upper and left neighbours being foreground pixels. However, they have different labels. We thus assign either of these labels to this second pixel, say label 3, and make a note that labels 3 and 4 are equivalent:

	•1		
•2	•1		•3
		•4	•3
	•	•	

In the fourth row, the first foreground pixel has both its upper and left neighbours being background pixels, so we assign it a new label—5. The second (foreground) pixel in the fourth row now has both its upper and left neighbours being foreground pixels. However, they have different labels. We thus assign either of these labels to this second pixel, say label 4, and make a note that labels 4 and 5 are equivalent:

	•1		
•2	•1		•3
		•4	•3
	•5	•4	

This completes step 1.

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Step 2. We have the following equivalence classes of labels:

$$\{1, 2\} \qquad \text{and} \qquad \{3, 4, 5\}.$$

Assign label 1 to the first class, and label 2 to the second class.

Step 3. Each pixel with labels 1 or 2 from the first step will be assigned label 1, and each pixel with labels 3, 4 or 5 from the first step will be assigned label 2:

	•1		
•1	•1		•2
		•2	•2
	•2	•2	

This completes the algorithm.

The algorithm can be modified to label 8-components of an image, but in step 1 we need to consider diagonal elements of p:

d	u	e
l	p	

The algorithm is similar to the previous algorithm; step 1 is changed as follows:

> If p is a background pixel, move on to the next scanning position. If it is a foreground pixel, check d, u, e and l. If they are all background pixels, assign a new label to p. If just one is a foreground pixel, assign its label to p. If two or more are foreground pixels, assign any of their labels to p, and make a note that all their labels are equivalent.

Steps 2 and 3 are as before.

This algorithm is implemented by the `bwlabel` function. We shall give an example on a small image:

```
>> i=zeros(8,8);
>> i(2:4,3:6)=1;
```

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```
>> i(5:7,2)=1;
>> i(6:7,5:8)=1;
>> i(8,4:5)=1;
>> i
```

and the result is

```
     0     0     0     0     0     0     0     0
     0     0     1     1     1     1     0     0
     0     0     1     1     1     1     0     0
     0     0     1     1     1     1     0     0
     0     1     0     0     0     0     0     0
     0     1     0     0     1     1     1     1
     0     1     0     0     1     1     1     1
     0     0     0     1     1     0     0     0
```

This image can be seen to have two 8-components and three 4-components. To see these, we can enter

```
>> bwabel(i,4)
```

which produces

```
     0     0     0     0     0     0     0     0
     0     0     2     2     2     2     0     0
     0     0     2     2     2     2     0     0
     0     0     2     2     2     2     0     0
     0     1     0     0     0     0     0     0
     0     1     0     0     3     3     3     3
     0     1     0     0     3     3     3     3
     0     0     0     3     3     0     0     0
```

and

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```
>> bwlabel(i,8)
```

which produces

```
     0     0     0     0     0     0     0     0
     0     0     1     1     1     1     0     0
     0     0     1     1     1     1     0     0
     0     0     1     1     1     1     0     0
     0     1     0     0     0     0     0     0
     0     1     0     0     2     2     2     2
     0     1     0     0     2     2     2     2
     0     0     0     2     2     0     0     0
```

To experiment with a real image, let's try to count the number of bacteria in the image `bacteria.tif`. The image must first be thresholded to obtain a binary image showing only the bacteria, and then `bwlabel` applied to the result. We can find the number of objects simply by finding the largest label produced.

```
>> b=imread('bacteria.tif');
>> bt=b<100;
>> bl=bwlabel(bt);
>> max(bl(:))
```

and the result is given as

```
>> 21
```

which is the required number.

11.6 Lookup tables

Lookup tables provide a very neat and efficient method of binary image processing. Consider the 3×3 neighbourhood of a pixel. Since there are 9 pixels in this neighbourhood, each with two

NOT FOR CIRCULATION

possible states, the total number of different neighbourhoods is $2^9 = 512$. Since the output of any binary operation will be either zero or one, a *lookup table* is a binary vector of length 512, each element representing the output from the corresponding neighbourhoods. Note that this is slightly different from the lookup tables discussed in section 4.4. In that section lookup tables were applied to single pixel grey values; here they are applied to neighbourhoods.

The trick is then to be able to order all the neighbourhoods, so that we have a one-one correspondence between neighbourhoods and output values. This is done by giving each pixel in the neighbourhood a weighting:

1	8	64
2	16	128
4	32	256

The "value" of the neighbourhood is obtained by adding up the weights of one-valued pixels. That value is then the index to the lookup table. For example, the following show neighbourhoods and their values:

0	1	0
1	1	0
0	0	1

Value $= 2 + 8 + 16 + 256 = 282$

1	0	0
0	1	1
1	1	1

Value $= 1 + 4 + 16 + 32 + 128 + 256 = 437$

To apply a lookup table, we have to make it first. It would be tedious to create a lookup table element by element, so we use the `makelut` function, which defines a lookup table according to a rule. Its syntax is

```
makelut(function, n, P1, P2, ...)
```

where `function` is a string defining a MATLAB matrix function, `n` is either 2 or 3, and `P1`, `P2` and so on are optional parameters to be passed to the function. We note that `makelut` allows for lookup tables on 2×2 neighbourhoods; in this case the lookup table has only $2^4 = 16$ elements.

NOT FOR CIRCULATION

Suppose we wish to find the 4- boundary of an image. We define a pixel to be a boundary pixel if it is a foreground pixel which is 4-adjacent to a background pixel. So the function to be used in `makelut` is a function which returns one if and only if the central pixel of the 3×3 neighbourhood is a boundary pixel:

```
>> f=inline('x(5)&~(x(2)*x(4)*x(6)*x(8))');
```

Note that for this function we are using the single value matrix indexing scheme, so that `x(5)` is the central pixel of a 3×3 matrix `x`, and `x(2)`, `x(4)`, `x(6)`, `x(8)` are the pixels 4-adjacent to the centre.

Now we can make the lookup table:

```
>> lut=makelut(f,3);
```

and apply it to an image:

```
>> c=imread('circles.tif');
>> cw=applylut(c,lut);
>> imshow(c),figure,imshow(cw)
```

and the result is shown in figure 11.2.

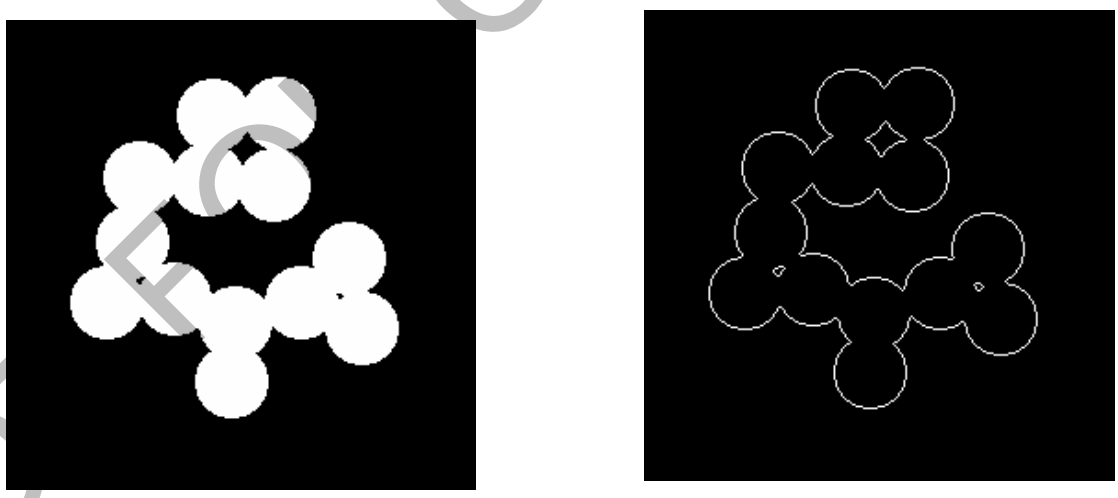

Figure 11.2: The circles image and its boundary

We can easily adjust the function to find the 8-boundary pixels; these being foreground pixels

NOT FOR CIRCULATION

which are 8-adjacent to background pixels:

```
>> f8=inline('x(5)&~(x(1)*x(2)*x(3)*x(4)*x(6)*x(7)*x(8)*x(9))');
>> lut=makelut(f8,3);
>> cw=applylut(c,lut);
>> imshow(cw)
```

and the result is shown in figure 11.3. Note that this is a "stronger" boundary, since more pixels are classified as boundary pixels.

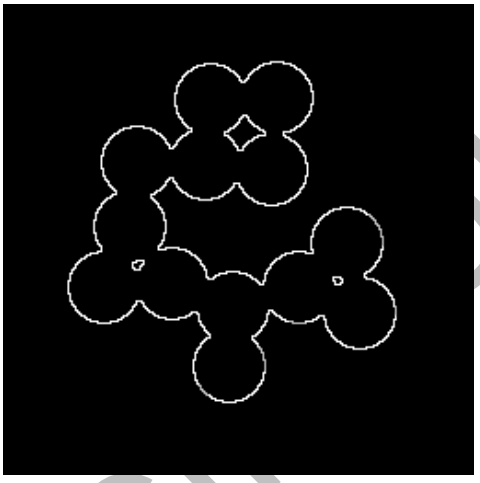

Figure 11.3: The 8-boundary of the circles image

As we shall see in section 11.8, lookup tables can be used to great effect in performing skeletonization.

11.7 Distances and metrics

It is necessary to define a function which provides a measure of *distance* between two points x and y on a grid. A distance function $d(x, y)$ is called a *metric* if it satisfies all of:

1. $d(x, y) = d(y, x)$ (symmetry)
2. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$ (positivity)
3. $d(x, y) + d(y, z) \leq d(x, z)$ (the triangle inequality)

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A standard distance metric is provided by *Euclidean distance*, where if $x = (x_1, x_2)$ and $y = (y_1, y_1)$ then

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This is just the length of the straight line between the points x and y. It is easy to see that the first two properties are satisfied by this metric: it is always positive, and is zero only when $x_1 = y_1$ and $x_2 = y_2$; that is, when $x = y$. The third property may be proved very easily; it simply says that given three points x, y and z then it is shorter to go from x to z directly than via y.

However, if we are constrained to points on a grid, then the Euclidean metric may not be applicable. Figure 11.4 show the shortest paths for the Euclidean metric, and for 4-connected and 8-connected paths. In this figure, the Euclidean distance is $\sqrt{5^2 + 2^2} \approx 5.39$, and the 4-path (the

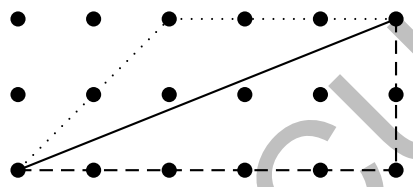

Figure 11.4: Comparison of three metrics

dashed line) and 8-path (dotted line) have length, measured by the number of line segments in each, of 7 and 5 respectively.

Metrics for measuring distance by 4-paths and 8-paths are given by the following two functions:

$$\begin{aligned} d_4(x, y) &= |x_1 - y_1| + |x_2 - y_2| \\ d_8(x, y) &= \max\{|x_1 - y_1|, |x_2 - y_2|\} \end{aligned}$$

As with the Euclidean distance, the first two properties are immediate; to prove the triangle inequality takes a little effort. The metric d_4 is sometimes known as the *taxicab metric*, as it gives the length of the shortest taxicab route through a rectangular grid of streets.

11.7.1 The distance transform

In many applications, it is necessary to find the distance of every pixel from a region R. This can be done using the standard Euclidean distance defined above. However, this means that to calculate the distance of (x, y) to R, we need to determine all possible distances from (x, y) to pixels in R,

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and take the smallest value as our distance. This is very computationally inefficient: if R is large, then we have to compute many square roots. A saving can be obtained by noting that since the square root function is increasing, we can write the minimum distance as

$$md(x, y) = \sqrt{\min_{(p,q)\in R} \left((x-p)^2 + (y-q)^2\right)}$$

which involves only one square root. But even this definition is slow to compute. There is a great deal of arithmetic involved, and the finding of a smallest value.

The *distance transform* is a computationally efficient method of finding such distance. We shall describe it in a sequence of steps:

Step 1. Attach to each pixel (x, y) in the image a label $d(x, y)$ giving its distance from R. Start with labelling each pixel in R with 0, and each pixel not in R with ∞.

Step 2. We now travel through the image pixel by pixel. For each pixel (x, y) replace its label with

$$\min\{d(x, y), d(x+1, y)+1, d(x-1, y)+1, d(x, y-1)+1, d(x, y+1)+1\}$$

using $\infty + 1 = \infty$.

Step 3. Repeat step 2 until all labels have been converted to finite values.

To give some examples of step 2, suppose we have this neighbourhood:

$$\begin{array}{ccc} \infty & \infty & \infty \\ 2 & \infty & \infty \\ \infty & 3 & \infty \end{array}$$

We are only interested in the centre pixel (whose label we are about to change), and the four pixels above, below, and to the left and right. To these four pixels we add 1:

$$\begin{array}{ccc} & \infty & \\ 3 & \infty & \infty \\ & 4 & \end{array}$$

The minimum of these five values is 3, and so that is the new label for the centre pixel.

NOT FOR CIRCULATION

Suppose we have this neighbourhood:

$$\begin{matrix} 2 & \infty & \infty \\ \infty & \infty & \infty \\ 3 & \infty & 5 \end{matrix}$$

Again, we add 1 to each of the four pixels above, below, to the left and right, and keep the centre value:

$$\begin{matrix} & \infty & \\ \infty & \infty & \infty \\ & \infty & \end{matrix}$$

The minimum of these values is ∞, and so at this stage the pixel's label is not changed.

Suppose we take an image whose labels after step 1 are given below

$$\begin{matrix} \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & 0 & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty \end{matrix}$$

Step 1

$$\begin{matrix} \infty & \infty & 1 & 1 & \infty & \infty \\ \infty & 1 & 0 & 0 & 1 & \infty \\ \infty & \infty & 1 & 0 & 1 & \infty \\ \infty & \infty & 1 & 0 & 1 & \infty \\ \infty & \infty & 1 & 0 & 0 & 1 \\ \infty & \infty & \infty & 1 & 1 & \infty \end{matrix}$$

Step 2 (first pass)

$$\begin{matrix} \infty & 2 & 1 & 1 & 2 & \infty \\ 2 & 1 & 0 & 0 & 1 & 2 \\ \infty & 2 & 1 & 0 & 1 & 2 \\ \infty & 2 & 1 & 0 & 1 & 2 \\ \infty & 2 & 1 & 0 & 0 & 1 \\ \infty & \infty & 2 & 1 & 1 & 2 \end{matrix}$$

Step 2 (second pass)

$$\begin{matrix} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ \infty & 3 & 2 & 1 & 1 & 2 \end{matrix}$$

Step 2 (third pass)

$$\begin{matrix} 3 & 2 & 1 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 4 & 3 & 2 & 1 & 1 & 2 \end{matrix}$$

Step 2 (final pass)

At this stage we stop, as all label values are finite.

An immediate observation is that the distance values given are not in fact a very good approximation to "real" distances. To provide better accuracy, we need to generalize the above transform.

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One way to do this is to use the concept of a "mask"; the mask used above was

$$\begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{array}$$

Step two in the transform then consists of adding the corresponding mask elements to labels of the neighbouring pixels, and taking the minimum. To obtain good accuracy with simple arithmetic, the mask will generally consist of integer values, but the final result may require scaling.

Suppose we apply the mask

$$\begin{array}{ccc} 4 & 3 & 4 \\ 3 & 0 & 3 \\ 4 & 3 & 4 \end{array}$$

to the above image. Step 1 is as above; steps 2 are:

$$\begin{array}{cccccc} \infty & 4 & 3 & 3 & 4 & \infty \\ \infty & 3 & 0 & 0 & 3 & \infty \\ \infty & 4 & 3 & 0 & 3 & \infty \\ \infty & \infty & 3 & 0 & 3 & \infty \\ \infty & \infty & 3 & 0 & 0 & 3 \\ \infty & \infty & 4 & 3 & 3 & 4 \end{array} \qquad \begin{array}{cccccc} 7 & 4 & 3 & 3 & 4 & 7 \\ 6 & 3 & 0 & 0 & 3 & 6 \\ 7 & 4 & 3 & 0 & 3 & 6 \\ 8 & 6 & 3 & 0 & 3 & 6 \\ \infty & 6 & 3 & 0 & 0 & 3 \\ \infty & 7 & 4 & 3 & 3 & 4 \end{array} \qquad \begin{array}{cccccc} 7 & 4 & 3 & 3 & 4 & 7 \\ 6 & 3 & 0 & 0 & 3 & 6 \\ 7 & 4 & 3 & 0 & 3 & 6 \\ 8 & 6 & 3 & 0 & 3 & 6 \\ 9 & 6 & 3 & 0 & 0 & 3 \\ 10 & 7 & 4 & 3 & 3 & 4 \end{array}$$

Step 2 (first pass) Step 2 (second) Step 2 (third)

at which point we stop, and divide all values by three:

$$\begin{array}{cccccc} 2.3 & 1.3 & 1 & 1 & 1.3 & 2.3 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2.3 & 1.3 & 1 & 0 & 1 & 2 \\ 2.7 & 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 3.3 & 2.3 & 1.3 & 1 & 1 & 1.3 \end{array}$$

These values are much closer to the Euclidean distances than those provided by the first transform.

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Even better accuracy can be obtained by using the mask

	11		11	
11	7	5	7	11
	5	0	5	
11	7	5	7	11
	11		11	

and dividing the final result by 5:

11	7	5	5	7	11	2.2	1.4	1	1	1.4	2.2
10	5	0	0	5	10	2	1	0	0	1	2
11	7	5	0	5	10	2.2	1.4	1	0	1	2
14	10	5	0	5	7	2.8	2	1	0	1	1.4
16	10	5	0	0	5	3.2	2	1	0	0	1
16	11	7	5	5	7	3.2	2.2	1.4	1	1	1.4
Result of transform						After division by 5					

The method we have described can in fact be very slow; for a large image we may require many passes before all distance labels become finite. A quicker method requires *two* passes only: the first pass starts at the top left of the image, and moves left to right, and top to bottom. The second pass starts at the bottom right of the image, and moves right to left, and from bottom to top. For this method we break the mask up into two halves; one half inspects only values to the left and above (this is used for the first pass), and the other half inspects only values to the right and below (this is used for the second pass).

Such pairs of masks are shown in figure 11.5, and the solid lines show how the original mask is broken up into its two halves.

We apply these masks as follows: first surround the image with zeros (such as for spatial filtering), so that at the edges of the image the masks have values to work with. For the forward pass, for each pixel at position (i, j), add the values given in the forward mask to its neighbours and take the minimum, and replace the current label with this minimum. This is similar to step 2 in the original algorithm, except that we are using less pixels. We do the same for the backward pass, except that we use the backwards mask, and we start at the bottom right.

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Mask 1

	1	
1	0	

Forward mask

	0	1
	1	

Backward mask

Mask 2

4	3	4
3	0	

Forward mask

	0	3
4	3	4

Backward mask

Mask 3

	11		11	
11	7	5	7	11
	5	0		

Forward mask

		0	5	
11	7	5	7	11
	11		11	

Backward mask

Figure 11.5: Pairs of masks for the two pass distance transform

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If we apply forward masks 1 and 2 to our image, the results of the forward passes are:

∞	∞	∞	∞	∞	∞
∞	∞	0	0	1	2
∞	∞	1	0	1	2
∞	∞	2	0	1	2
∞	∞	3	0	0	1
∞	∞	4	1	1	2

Use of mask 1

∞	∞	∞	∞	∞	∞
∞	∞	0	0	3	6
∞	4	3	0	3	6
8	7	5	0	3	6
11	8	4	0	0	3
12	8	4	3	3	4

Use of mask 2

After applying the backward masks, we will obtain the distance transforms as above.

Implementation in MATLAB

We can easily write a function to perform the distance transform using the second method as above. Our function will implement the transform as follows:

1. Using the size of the mask, pad the image with an appropriate number of zeros.
2. Change each zero to infinity, and each one to zero.
3. Create forward and backwards masks.
4. Perform a forward pass: replace each label with the minimum of its neighbourhood plus the forward mask.
5. Perform a backwards pass: replace each label with the minimum of its neighbourhood plus the backwards mask.

Let's consider each of these steps separately:

1. Suppose the image is of size $r \times c$, and the mask is of size $r_m \times c_m$, where both mask dimensions are odd numbers. We need to add, on each side of the image, a number of columns equal to $(c_m - 1)/2$, and we need to add, on both top and bottom of the image, a number of rows equal to $(r_m - 1)/2$. In other words we can embed the image in a larger array of size $(r + r_m - 1) \times (c + c_m - 1)$:

NOT FOR CIRCULATION

```
>> [r,c]=size(image);
>> [mr,mc]=size(mask);
>> nr=(mr-1)/2;
>> nc=(mc-1)/2;
>> image2=zeros(r+mr-1,c+mc-1);
>> image2(nr+1:r+nr,nc+1:c+nc)=image;
```

2. This can be easily done using the `find` function: first change all zeros to infinity, and next change all ones to zeros:

```
>> image2(find(image2)==0)=Inf;
>> image2(find(image2)==1)=0;
```

3. Suppose we are given a forward mask. We can do this most simply by making all blank entries in the masks shown in figure 11.5 infinity; this means that pixels in these positions will have no effect on the final minimum. The backwards mask can be obtained by two 90° rotations. For example:

```
>> mask1=[Inf 1 Inf;1 0 Inf;Inf Inf Inf];
>> backmask=rot90(rot90(mask1));
```

4. We can implement the forward mask with nested loops:

```
>> for i=nr+1:r+nr,
     for j=nc+1:c+nc,
       image2(i,j)=min(min(image2(i-nr:i+nr,j-nc:j+nc)+mask));
     end;
   end;
```

5. The backward pass is done similarly:

```
>> for i=r+nr:-1:nr+1,
```

NOT FOR CIRCULATION

```
    for j=c+nc:-1:nc+1,
      image2(i,j)=min(min(image2(i-nr:i+nr,j-nc:j+nc)+backmask));
    end;
  end;
```

The full function is shown in figure 11.6.

Let's try it out, first creating our image, and the three masks:

```
>> im=[0 0 0 0 0 0;...
0 0 1 1 0 0;...
0 0 0 1 0 0;...
0 0 0 1 0 0;...
0 0 0 1 1 0;...
0 0 0 0 0 0]

im =

     0     0     0     0     0     0
     0     0     1     1     0     0
     0     0     0     1     0     0
     0     0     0     1     0     0
     0     0     0     1     1     0
     0     0     0     0     0     0

>> mask1=[Inf 1 Inf;1 0 Inf;Inf Inf Inf]

mask1 =

   Inf     1   Inf
     1     0   Inf
```

NOT FOR CIRCULATION

```
function res=disttrans(image,mask)
%
% This function implements the distance transform by applying MASK to
% IMAGE, using the two step algorithm with "forward" and "backwards" masks.
%
backmask=rot90(rot90(mask));
[mr,mc]=size(mask);
if ((floor(mr/2)==ceil(mr/2)) | (floor(mc/2)==ceil(mc/2))) then
     error('The mask must have odd dimensions.')
     end;
[r,c]=size(image);
nr=(mr-1)/2;
nc=(mc-1)/2;
image2=zeros(r+mr-1,c+mc-1);
image2(nr+1:r+nr,nc+1:c+nc)=image;
%
% This is the first step; replacing R values with 0 and other values
% with infinity
%
image2(find(image2==0))=Inf;
image2(find(image2==1))=0;
%
% Forward pass
%
for i=nr+1:r+nr,
  for j=nc+1:c+nc,
    image2(i,j)=min(min(image2(i-nr:i+nr,j-nc:j+nc)+mask));
  end;
end;
%
% Backward pass
%
for i=r+nr:-1:nr+1,
  for j=c+nc:-1:nc+1,
    image2(i,j)=min(min(image2(i-nr:i+nr,j-nc:j+nc)+backmask));
  end;
end;

res=image2(nr+1:r+nr,nc+1:c+nc);
```

Figure 11.6: A function for computing the distance transform

NOT FOR CIRCULATION

```
   Inf   Inf   Inf


>> mask2=[4 3 4;3 0 Inf;Inf Inf Inf]


mask2 =


     4     3     4
     3     0   Inf
   Inf   Inf   Inf


>> mask3=[Inf 11 Inf 11 Inf;...
11 7 5 7 11;...
Inf 5 0 Inf Inf;...
Inf Inf Inf Inf Inf;...
Inf Inf Inf Inf Inf]


mask3 =


   Inf    11   Inf    11   Inf
    11     7     5     7    11
   Inf     5     0   Inf   Inf
   Inf   Inf   Inf   Inf   Inf
   Inf   Inf   Inf   Inf   Inf
```

Now we can apply the transform:

```
>> disttrans(im,mask1)


ans =
```

NOT FOR CIRCULATION

```
     3     2     1     1     2     3

     2     1     0     0     1     2

     3     2     1     0     1     2

     3     2     1     0     1     2

     3     2     1     0     0     1

     4     3     2     1     1     2


>> disttrans(im,mask2)


ans =


     7     4     3     3     4     7

     6     3     0     0     3     6

     7     4     3     0     3     6

     8     6     3     0     3     4

     9     6     3     0     0     3

    10     7     4     3     3     4


>> disttrans(im,mask3)


ans =


    11     7     5     5     7    11

    10     5     0     0     5    10

    11     7     5     0     5    10

    14    10     5     0     5     7

    15    10     5     0     0     5

    16    11     7     5     5     7
```

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Now the results of the transforms using masks 2 and 3 should be divided by appropriate values to get approximations to the true distances.

We can of course apply the distance transform to a large image; say `circles.tif`:

```
>> c=~imread('circles.tif');
>> imshow(c)
>> cd=disttrans(c,mask1);
>> figure,imshow(mat2gray(cd))
```

Note that we invert the circles image to produce black circles on a white background. This means that our transform will find distances towards the inside of the original circles. The only reason for doing this is that it is easier to make sense of the resulting image. The circles image is shown on the left in figure 11.7 and the distance transform on the right.

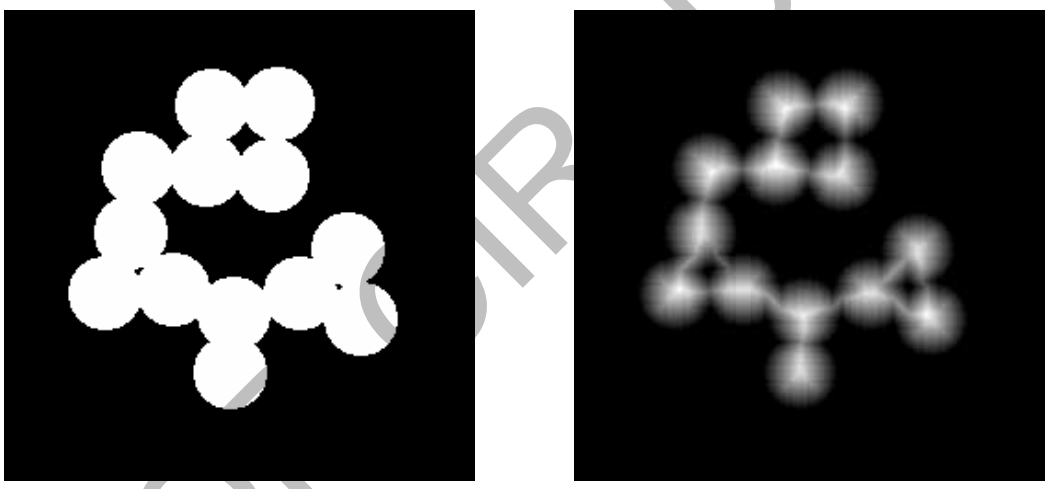

Figure 11.7: An example of a distance transform

11.8 Skeletonization

The *skeleton* of a binary object is a collection of lines and curves which encapsulate the size and shape of the object. There are in fact many different methods of defining a skeleton, and for a given object, many different possible skeletons. But to give a general idea, we show a few examples in figure 11.8.

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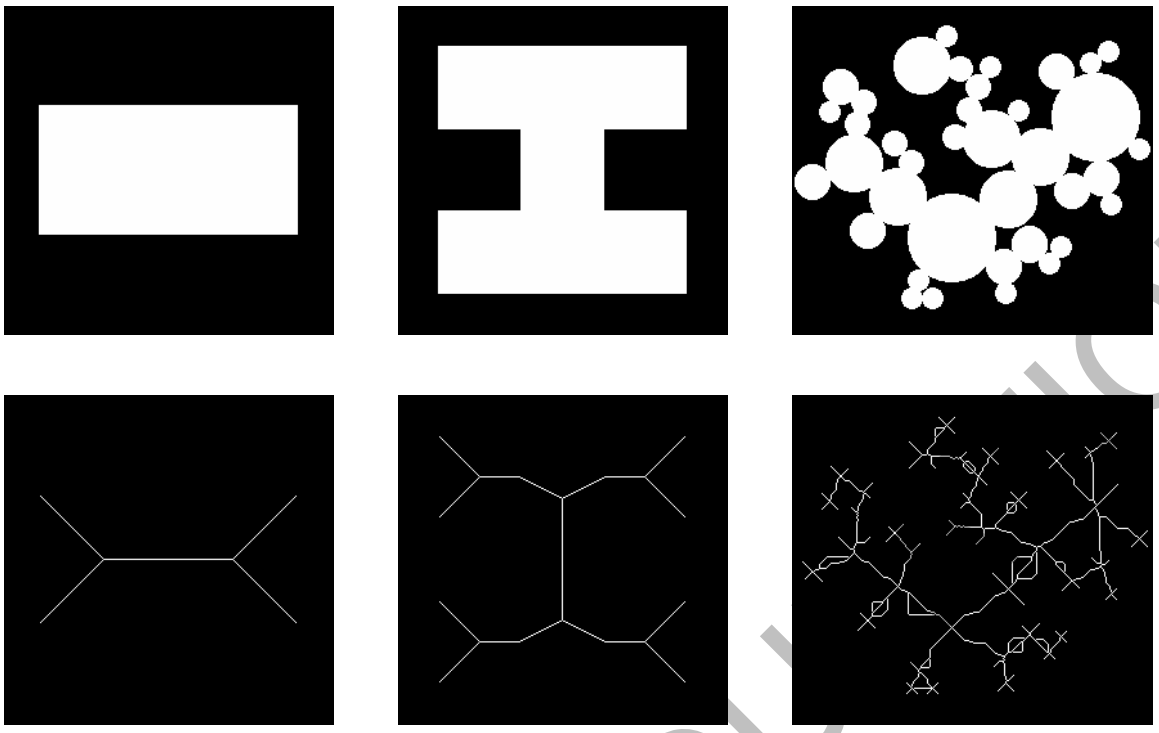

Figure 11.8: Examples of skeletonization

A problem with skeletonization is that very small changes to an object can result in large changes to the skeleton. Figure 11.9 shows what happens if we take away or add to the central image of figure 11.8.

One very popular way of defining a skeleton is is by the *medial axis* of an object: a pixel is on the medial axis if is equidistant from at least two pixels on the boundary of the object. To implement this definition directly requires a quick and efficient method of obtaining approximate distances; one way is to use the distance transform Other ways of approaching the medial axis are:

- imagine the object to be burning up by a fire which advances at a constant rate from the boundary. The places where two "lines" of fire meet form the medial axis.

- consider the set of all circles lying within the object which touch at least two points on the boundary. The centres of all such circles form the medial axis. Figure 11.10 shows this in action.

A very good account of the medial axis (and of skeletonization in general), is given by Parker [21].

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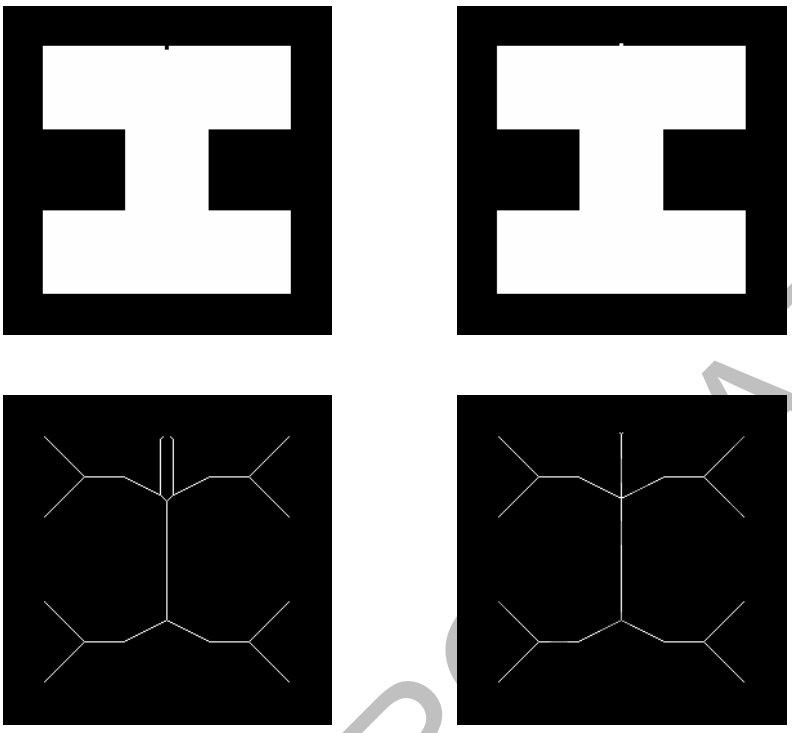

Figure 11.9: Skeletons after small changes to an object

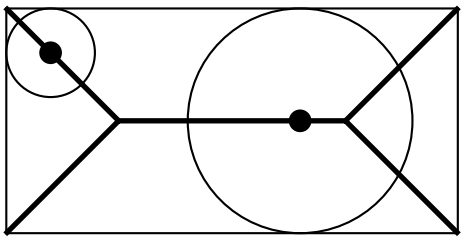

Figure 11.10: The medial axis of an object

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Topological methods provide some powerful methods of skeletonization; in that we can directly define those pixels which are to be deleted to obtain the final skeleton. In general, we want to delete pixels which can be deleted without changing the *connectivity* of an object: which do not change the number of components, change the number of holes, or the relationship of objects and holes. For example, figure 11.11 shows a non-deletable pixel; deleting the centre (boxed) pixel introduces a hole into the object. In figure 11.12 there is another example of a non-deletable pixel; in this case

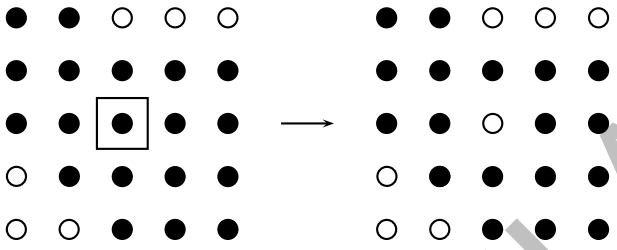

Figure 11.11: A non-deletable pixel: creates a hole

deletion *removes* a hole, in that the hole and the exterior become joined. In figure 11.13 there is

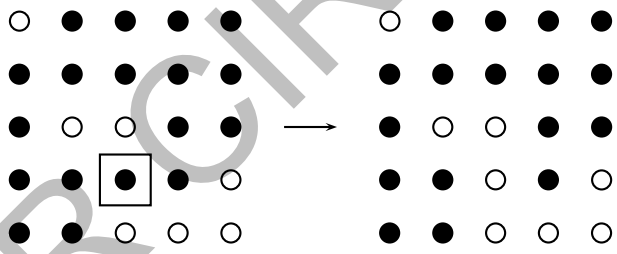

Figure 11.12: A non-deletable pixel: removes a hole

another example of a non-deletable pixel; in this case deletion breaks the object into two separate components. Sometimes we need to consider whether the object is 4-connected or 8-connected. In the previous examples this wasn't a problem. However, look at the examples in figure 11.14. In 11.14(a), the central point can't be deleted without changing both the 4-connectivity and the 8-connectivity. In 11.14(b) deleting the central pixel will change the 4-connectivity, but not the 8-connectivity. A pixel which can be deleted without changing the 4-connectivity of the object is called *4-simple*; similarly a pixel which can be deleted without changing the 8-connectivity of the

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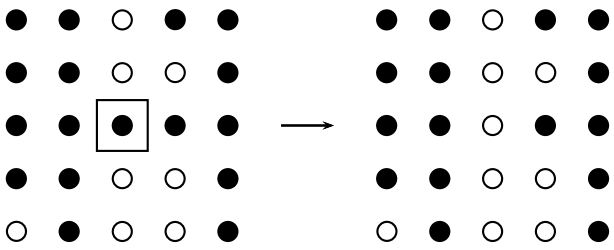

Figure 11.13: A non-deletable pixel: disconnects an object

Figure 11.14: Simple points

object is called *8-simple*. Thus the central pixel in 11.14(a) is neither 4-simple nor 8-simple. but the central pixel in 11.14(b) is 8-simple but not 4-simple.

A pixel can be tested for deletability by checking its 3×3 neighbourhood. Look again at figure 11.14(a). Suppose the central pixel is deleted. There are two options:

1. The top two pixels and the bottom two pixels become separated; in effect breaking up the object.

2. The top two pixels and the bottom two pixels are joined by a chain of pixels outside the shown neighbourhood. In this case all pixels will encircle a hole, and removing the central pixel will remove the hole, as in figure 11.12.

To check whether a pixel is 4-simple or 8-simple, we introduce some numbers associated with the neighbourhood of a foreground pixel p. First define N_p to be the 3×3 neighbourhood of p, and N_p^* to be the 3×3 neighbourhood *excluding* p. Then:

$$\begin{aligned} A(p) &= \text{the number of 4-components in } N_p^* \\ C(p) &= \text{the number of 8-components in } N_p^* \\ B(p) &= \text{the number of foreground pixels in } N_p^* \end{aligned}$$

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For example, in figure 11.14(a), we have

$$\begin{aligned} A(p) &= 2 \\ C(p) &= 2 \\ B(p) &= 4 \end{aligned}$$

and in figure 11.14(b) we have

$$\begin{aligned} A(p) &= 2 \\ C(p) &= 1 \\ B(p) &= 5 \end{aligned}$$

We can see the last example by deleting the central pixel and enumerating the components of the remaining foreground pixels. This is shown in figure 11.15. The importance of simple points for

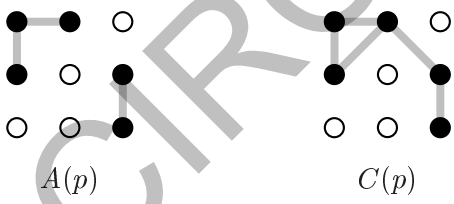

Figure 11.15: Components of N_p^*

deletion can now be stated:

> A foreground pixel p is 4-simple if and only if $A(p) = 1$, and is 8-simple if and only if $C(p) = 1$.

Returning again to figure 11.14(b), since $C(p) = 1$ the central pixel p is 8-simple and so can be deleted without affecting the 8-connectivity of the object. But since $A(p) \neq 1$, the central pixel p is *not* 4-simple and so cannot be deleted without affecting the 4-connectivity of the object. This is exemplified in figure 11.15.

NOT FOR CIRCULATION

Calculating $A(p)$ and $C(p)$

To implement any algorithm involving the values $A(p)$ and $C(p)$, we need efficient method of calculating them. For $A(p)$ we are only interested in the case where $A(p) = 1$, and this can be determined by calculating the *crossing number* $X(p)$ of a foreground pixel.

> The crossing number $X(p)$ of a foreground pixel p is defined to be the number of times a 0 is followed by a 1 as we traverse the 8-neighbours of p in a clockwise direction.

Suppose we label the neighbours of p as follows:

$$\begin{matrix} p_1 & p_2 & p_3 \\ p_8 & p & p_4 \\ p_7 & p_6 & p_5 \end{matrix}$$

Consider the sequence

$$p_1, \quad p_2, \quad p_3, \quad p_4, \quad p_5, \quad p_6, \quad p_7, \quad p_8, \quad p_1.$$

The number of times the subsequence

$$0, \quad 1$$

occurs is the crossing number. The importance of the crossing number is that

1. it is easy to compute,
2. If $X(p) = 1$, then $A(p) = 1$ and so p is 4-simple.

Consider the neighbourhoods in figure 11.14, but this time using 0's and 1s for background and foreground pixels:

$$\begin{matrix} 1 & 1 & 0 \\ 0 & p & 0 \\ 0 & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 \\ 1 & p & 1 \\ 0 & 0 & 1 \end{matrix}$$

Reading the neighbours of p from the left hand neighbourhood produces:

$$1, \quad 1, \quad 0, \quad 0, \quad 1, \quad 1, \quad 0, \quad 0, \quad 1.$$

The subsequence "0, 1" occurs twice, so $X(p) = 2$.

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For the right hand neighbourhood, we have:

$$1, \quad 1, \quad 0, \quad 1, \quad 1, \quad 0, \quad 0, \quad 1, \quad 1.$$

Again the subsequence "0, 1" occurs twice, so $X(p) = 2$.

Here are two more:

$$\begin{matrix} 1 & 1 & 1 \\ 1 & p & 1 \\ 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 \\ 0 & p & 0 \\ 1 & 0 & 1 \end{matrix}$$

For the left neighbourhood, the sequence is:

$$1, \quad 1, \quad 1, \quad 1, \quad 0, \quad 0, \quad 1, \quad 1, \quad 1.$$

The subsequence "0, 1" occurs only once, so $X(p) = 1$.

For the right neighbourhood, the sequence is:

$$1, \quad 1, \quad 0, \quad 0, \quad 1, \quad 0, \quad 1, \quad 0, \quad 1.$$

The subsequence "0, 1" occurs three times, so $X(p) = 3$.

To implement the crossing number in MATLAB, we first note that a 3×3 matrix `a` in MATLAB can be indexed using single indexing as:

```
a(1)  a(4)  a(7)
a(2)  a(5)  a(8)
a(3)  a(6)  a(9)
```

We can now create two sequences: the neighbouring pixels in clockwise order:

```
>> p=[a(1) a(4) a(7) a(8) a(9) a(6) a(3) a(2)];
```

and the neighbouring pixels starting at `a(4)`:

```
>> pp=[p(2:8) p(1)];
```

So `pp(i)=p(i+1)` with `pp(8)=p(1)`. A "0, 1" is counted if for any i with $1 \leq i \leq 8$ we have `p(i)=0` and `pp(i)=1`. This will occur if and only if

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```
(1-p(i))*pp(i)
```

is equal to 1.

Now we can calculate the crossing number as the sum:

```
>> crossnum=sum((1-p).*pp)
```

Calculating $C(p)$ can be done with this simple function:

$$C(p) = [\overline{p_2} \wedge (p_3 \vee p_4)] + [\overline{p_4} \wedge (p_5 \vee p_6)] + [\overline{p_7} \wedge (p_8 \vee p_9)] + [\overline{p_8} \wedge (p_1 \vee p_2)]$$

where

$$\begin{aligned} \overline{p_i} &= \text{complement of } p_i \ (= 1 - p_i) \\ \wedge, \vee &= \text{usual boolean and, or} \\ + &= \text{arithmetic addition.} \end{aligned}$$

Again look at the neighbourhoods from figure 11.14 given above as 0's and 1's. For the left hand neighbourhood we have

$$\begin{aligned} C(p) &= [\overline{1} \wedge (0 \vee 0)] + [\overline{0} \wedge (1 \vee 1)] + [\overline{1} \wedge (0 \vee 0)] + [\overline{0} \wedge (1 \vee 1)] \\ &= [0 \wedge 0] + [1 \wedge 1] + [0 \wedge 0] + [1 \wedge 1] \\ &= 0 + 1 + 0 + 1 \\ &= 2 \end{aligned}$$

and this is indeed the number of 8-components in N_p^*. For the right hand neighbourhood (which is shown in figure 11.15), we have:

$$\begin{aligned} C(p) &= [\overline{1} \wedge (0 \vee 1)] + [\overline{1} \wedge (1 \vee 0)] + [\overline{0} \wedge (0 \vee 1)] + [\overline{1} \wedge (1 \vee 1)] \\ &= [0 \wedge 1] + [0 \wedge 1] + [1 \wedge 1] + [0 \wedge 1] \\ &= 0 + 0 + 0 + 1 \\ &= 1 \end{aligned}$$

which is the number of eight components in N_p^*.

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How not to do skeletonization

So now we know how to check if a pixel can be deleted without effecting the connectivity of the object. In general, a skeletonization algorithm works by an iteration process: at each step identifying deletable pixels, and deleting them. The algorithm will continue until no further deletions are possible.

One way to remove pixels is as follows:

> At each step, find all foreground pixels which are 4-simple, and delete them all.

Sounds good? Let's try it on a small rectangle of size 2×4:

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

If we check the pixels in this object carefully, we will find that they are *all* 4-simple. Deleting them all will thus remove the object completely: a very undesirable result. Clearly we need to be a bit more careful about which pixels we can delete, and when. We need to add an extra test for deletability, so that we don't delete too many pixels.

We have two options here:

1. we can provide a step-wise algorithm, and change the test for deletability at each step,
2. we can apply a different test for deletability according to where the pixel lies on the image grid.

Algorithms which work according to the first option are called *subiteration algorithms*; algorithms which work according to the second option are called *subfield algorithms*.

The Zhang-Suen skeletonization algorithm

This algorithm has attained the status of a modern classic; it has some faults (as we shall see), but it works fairly fast, and most of the time produces acceptable results. It is an example of a

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subiteration algorithm, in that we apply a slightly different deletablity test for different steps of the algorithm.

Step N

Flag a foreground pixel $p = 1$ to be deletable if

1. $2 \leq B(p) \leq 6$,
2. $X(p) = 1$,
3. If N is odd, then

$$\begin{aligned} p_2 \cdot p_4 \cdot p_6 &= 0 \\ p_4 \cdot p_6 \cdot p_8 &= 0. \end{aligned}$$

   If N is even, then

$$\begin{aligned} p_2 \cdot p_4 \cdot p_8 &= 0 \\ p_2 \cdot p_6 \cdot p_8 &= 0. \end{aligned}$$

Delete all flagged pixels.

Continue Until there are no more deletable pixels in two successive iterations.

Item 3 can be written alternatively as:

3. If N is odd, then

   $p_4 = 0$, or $p_6 = 0$, or $p_2 = p_8 = 0$

   If N is even, then

   $p_2 = 0$, or $p_8 = 0$, or $p_4 = p_6 = 0$.

If we check the diagram for the neighbours of a foreground pixel p, we see that we can rephrase this item as:

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For odd iterations, delete only pixels which are on the right hand side, or bottom of an object, or on a north-west corner.

For even iterations, delete only pixels which are on the left hand side, or top of an object, or on a south-east corner.

Figure 11.16 shows pixels which may be considered for deletion at different iterations. Item 1 in

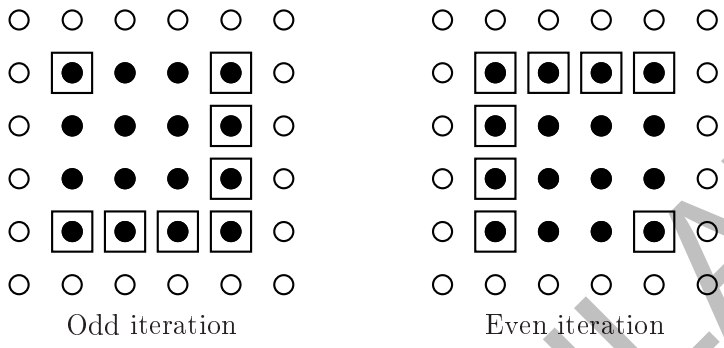

Figure 11.16: Deletion in the Zhang-Suen algorithm

the algorithm ensures that we don't delete pixels which have only one neighbour, or have seven or more. If a pixel has only one neighbour, it would be at the end of a skeletal line, and we would not want it to be deleted. If a pixel has seven neighbours, then deleting it would start an unacceptable erosion into the object's shape. This item thus ensures that the basic shape of the object is kept by the skeleton. Item 2 is our standard connectivity condition.

A major fault with this algorithm is that there are objects which will be deleted completely. Consider a 2×2 square:

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

We can check carefully that every item is satisfied by every pixel, and hence every pixel will be deleted.

An example. Consider the L shape shown in figure 11.17, where for ease of viewing we have replaced all the zeros (background pixels) with dots. The boxed pixels show those which will be

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·	·	·	·	·	·	·	·	·	·
·	1	1	1	1	1	·	·	·	·
·	1	1	1	1	1	·	·	·	·
·	1	1	1	1	1	·	·	·	·
·	1	1	1	1	1	·	·	·	·
·	1	1	1	1	1	·	·	·	·
·	1	1	1	1	1	1	1	1	·
·	1	1	1	1	1	1	1	1	·
·	1	1	1	1	1	1	1	1	·
·	1	1	1	1	1	1	1	1	·
·	1	1	1	1	1	1	1	1	·
·	·	·	·	·	·	·	·	·	·

Step 1.

·	·	·	·	·	·	·	·	·	·
·	·	1	1	1	·	·	·	·	·
·	1	1	1	1	·	·	·	·	·
·	1	1	1	1	·	·	·	·	·
·	1	1	1	1	·	·	·	·	·
·	1	1	1	1	·	·	·	·	·
·	1	1	1	1	1	1	1	·	·
·	1	1	1	1	1	1	1	·	·
·	1	1	1	1	1	1	1	·	·
·	1	1	1	1	1	1	1	·	·
·	·	·	·	·	·	·	·	·	·

Step 2.

Figure 11.17: Steps 1 and 2 of a Zhang-Suen skeletonization

deleted by steps 1 and 2 of the algorithm. Figure 11.18 shows steps 3 and 4 of the skeletonization. After step 4 no more deletions are possible, and so the skeleton consists of the unboxed foreground pixels in the right hand diagram of figure 11.18. Note that the skeleton does not include the corners of the original object.

We can implement this algorithm easily in MATLAB; we use lookup tables: one for the odd iterations, and one for the even. We then apply these lookup tables alternately until there is no change in the image for two successive iterations. We manage this by keeping three images at any given time: the *current* image, the *previous* image, and the *last* (that is, before the previous) image. If the current and last images are equal, we stop. Otherwise, "push" the images back: the previous image becomes last, and the current image become previous. We then apply whichever lookup table is appropriate to the current image to create the new current image. That is:

$$\begin{aligned} \textit{last} &\leftarrow \textit{previous} \\ \textit{previous} &\leftarrow \textit{current} \\ \textit{current} &\leftarrow \texttt{applylut}(\textit{current}, \texttt{lut}) \end{aligned}$$

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·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·
·	·	$\boxed{1}$	1	$\boxed{1}$	·	·	·	·	·
·	·	1	1	$\boxed{1}$	·	·	·	·	·
·	·	1	1	$\boxed{1}$	·	·	·	·	·
·	·	1	1	$\boxed{1}$	·	·	·	·	·
·	·	1	1	$\boxed{1}$	·	·	·	·	·
·	·	1	1	1	1	1	$\boxed{1}$	·	·
·	·	1	1	1	1	1	$\boxed{1}$	·	·
·	·	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	·	·	·
·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·

Step 3.

·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·
·	·	·	1	·	·	·	·	·	·
·	·	$\boxed{1}$	1	·	·	·	·	·	·
·	·	$\boxed{1}$	1	·	·	·	·	·	·
·	·	$\boxed{1}$	1	·	·	·	·	·	·
·	·	$\boxed{1}$	1	·	·	·	·	·	·
·	·	$\boxed{1}$	1	1	1	1	·	·	·
·	·	$\boxed{1}$	1	1	1	$\boxed{1}$	·	·	·
·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·

Step 4.

Figure 11.18: Steps 3 and 4 of a Zhang-Suen skeletonization

The functions for creating the lookup tables are given in figures 11.19 and 11.20

Now we create a function which applies these as described above; it is shown in 11.21 We can test this first on our L shape of above.

```
>> L=zeros(12,10);L(2:11,2:6)=1;L(7:11,7:9)=1


L =


     0     0     0     0     0     0     0     0     0     0

     0     1     1     1     1     1     0     0     0     0

     0     1     1     1     1     1     0     0     0     0

     0     1     1     1     1     1     0     0     0     0

     0     1     1     1     1     1     0     0     0     0

     0     1     1     1     1     1     0     0     0     0

     0     1     1     1     1     1     1     1     1     0

     0     1     1     1     1     1     1     1     1     0

     0     1     1     1     1     1     1     1     1     0
```

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```
function out=zsodd(nbhd);
s=sum(nbhd(:))-nbhd(5);
temp1=(2<=s)&(s<=6);
p=[nbhd(1) nbhd(4) nbhd(7) nbhd(8) nbhd(9) nbhd(6) nbhd(3) nbhd(2)];
pp=[p(2:8) p(1)];
xp=sum((1-p).*pp);
temp2=(xp==1);
prod1=nbhd(4)*nbhd(8)*nbhd(6);
prod2=nbhd(8)*nbhd(6)*nbhd(2);
temp3=(prod1==0)&(prod2==0);
if temp1&temp2&temp3&nbhd(5)==1
  out=0;
else
  out=nbhd(5);
end;
```

Figure 11.19: A MATLAB function for Zhang-Suen odd iterations

```
function out=zseven(nbhd);
s=sum(nbhd(:))-nbhd(5);
temp1=(2<=s)&(s<=6);
p=[nbhd(1) nbhd(4) nbhd(7) nbhd(8) nbhd(9) nbhd(6) nbhd(3) nbhd(2)];
pp=[p(2:8) p(1)];
xp=sum((1-p).*pp);
temp2=(xp==1);
prod1=nbhd(4)*nbhd(8)*nbhd(2);
prod2=nbhd(4)*nbhd(6)*nbhd(2);
temp3=(prod1==0)&(prod2==0);
if temp1&temp2&temp3&nbhd(5)==1
  out=0;
else
  out=nbhd(5);
end;
```

Figure 11.20: A MATLAB function for Zhang-Suen even iterations

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```
function out=zs(im)
%
% ZS(IM) applises the Zhang-Suen skeletonization algorithm to image IM.  IM
% must be binary.
%
luteven=makelut('zseven',3);
lutodd=makelut('zsodd',3);
done=0;
N=2;
last=im;
previous=applylut(last,lutodd);
current=applylut(previous,luteven);
while done==0,
  if all(current(:)==last(:)),
    done=1;
  end;
  N=N+1;
  last=previous;
  previous=current;
  if mod(N,2)==0,
    current=applylut(current,luteven);
  else
    current=applylut(current,lutodd);
  end;
end;

out=current;
```

Figure 11.21: A MATLAB function implementing Zhang-Suen skeletonization

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```
     0     1     1     1     1     1     1     1     1     0
     0     1     1     1     1     1     1     1     1     0
     0     0     0     0     0     0     0     0     0     0

>> Ls=zs(L)

Ls =

     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     1     0     0     0     0     0     0
     0     0     0     1     0     0     0     0     0     0
     0     0     0     1     0     0     0     0     0     0
     0     0     0     1     0     0     0     0     0     0
     0     0     0     1     0     0     0     0     0     0
     0     0     0     1     1     1     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0
```

This result is exactly the same as that obtained previously.

Two more examples; the circle image, and the "nice work" image. Both the images and their skeletonizations are shown in figure 11.22.

The Guo-Hall skeletonization algorithm

There are in fact a number of Guo-Hall algorithms; we will investigate one which has the advantages of being simple to describe, easy to implement, fast to run, and gives good results. What could be better?

The Guo-Hall algorithm is an example of a subfield algorithm. We imagine the image grid being

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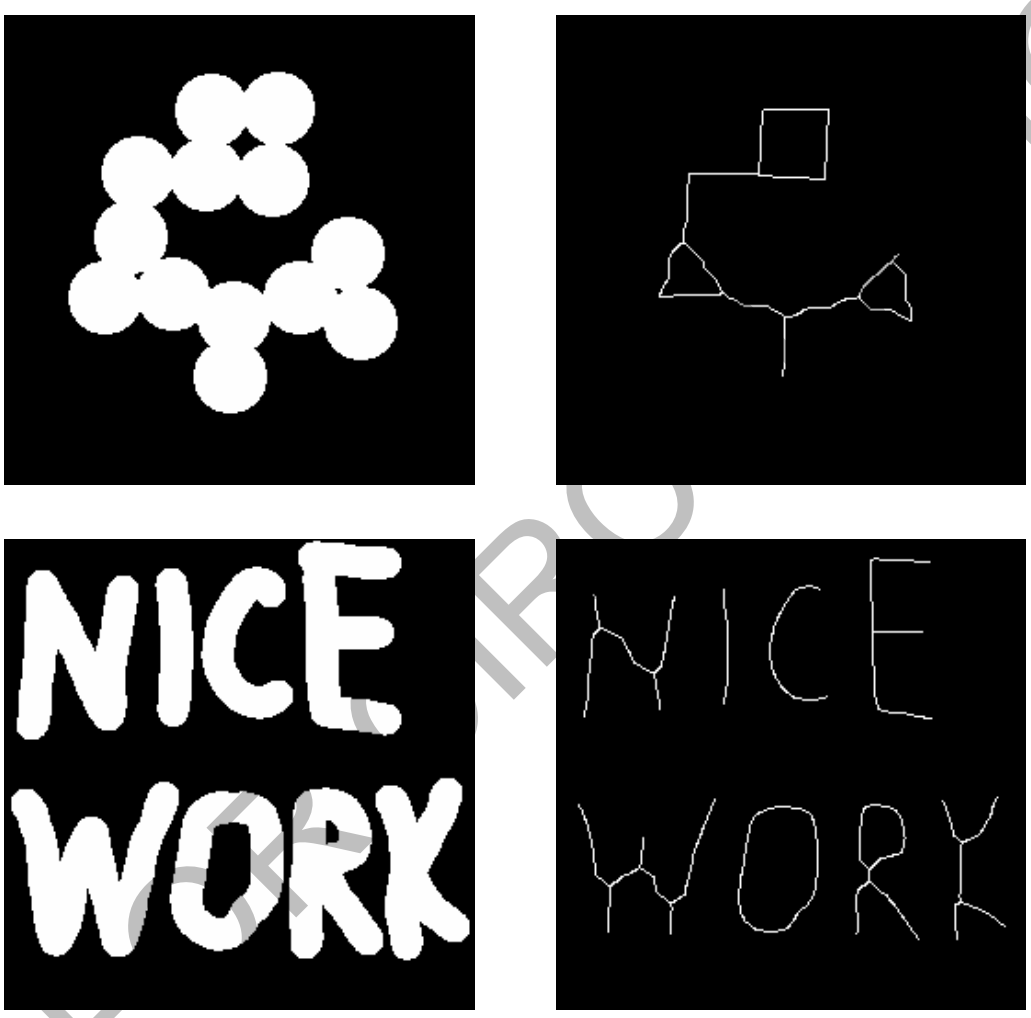

Figure 11.22: Examples of the Zhang-Suen skeletonization

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labelled with 1's and 2's in a chessboard configuration:

$$\begin{matrix} 1 & 2 & 1 & 2 & \cdots \\ 2 & 1 & 2 & 1 & \cdots \\ 1 & 2 & 1 & 2 & \cdots \\ 2 & 1 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

At step one, we only consider foreground pixels whose labels are 1. At step two, we only consider foreground pixels whose labels are 2. We continue alternating between pixels labelled 1 and pixels labelled 2 from step to step until no more deletions are possible. Here is the algorithm:

> Flag a foreground pixel p as deletable if $C(p) = 1$ and $B(p) > 1$, and then delete all flagged pixels. This is done, in parallel, at alternate iterations for each of the two subfields.
>
> Continue until no deletions are possible in two successive iterations.

Consider our "L" example from above. We first superimpose a chessboard of 1's and 2's. In step 1 we just consider 1's only. Step 1 shown in figure 11.23 illustrates the first step: we delete only those 1's satisfying the Guo-Hall deletablity conditions. These pixels are shown in squares. Having deleted them, we now consider 2's only; the deletable 2's are shown in step .2

Steps three and four continue the work; by step 4 there are no more deletions to be done, and we stop. We notice two aspects of the Guo-Hall algorithm as compared with Zhang-Suen:

1. More pixels may be deleted at each step, so we would expect the algorithm to work faster.
2. The final result includes more corner information than the Zhang-Suen algorithm.

We can implement this in `Matlab` using very similar means to our implementation of Zhang-Suen. We have a base program, almost identical to `zs.m`, which applies odd and even iterations until no more deletions occur after two iterations.

We then require three supplementary programs: the deletions at odd iterations, the deletions at even iterations, and the program to compute $C(p)$. To perform our deletions, we first place a chessboard pattern over the image. For odd deletions, we move over the image pixel by pixel. If the pixel has value `1`, we consider if for deletion, and if it deletable, we replace its value with `3`. This

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.	.	.	.	.	.	.	.	.	.
.	$\boxed{1}$	2	$\boxed{1}$	2	$\boxed{1}$	.	.	.	.
.	2	1	2	1	2	.	.	.	.
.	$\boxed{1}$	2	1	2	$\boxed{1}$	.	.	.	.
.	2	1	2	1	2	.	.	.	.
.	$\boxed{1}$	2	1	2	$\boxed{1}$	.	.	.	.
.	2	1	2	1	2	$\boxed{1}$	2	$\boxed{1}$	.
.	$\boxed{1}$	2	1	2	1	2	1	2	.
.	2	1	2	1	2	1	2	$\boxed{1}$	.
.	$\boxed{1}$	2	1	2	1	2	1	2	.
.	2	$\boxed{1}$	2	$\boxed{1}$	2	$\boxed{1}$	2	$\boxed{1}$	.
.	.	.	.	.	.	.	.	.	.

Step 1

.	.	.	.	.	.	.	.	.	.
.	.	$\boxed{2}$	.	$\boxed{2}$	.	.	.	.	.
.	$\boxed{2}$	1	$\boxed{2}$	1	$\boxed{2}$	.	.	.	.
.	.	$\boxed{2}$	1	$\boxed{2}$	.	.	.	.	.
.	$\boxed{2}$	1	2	1	$\boxed{2}$	.	.	.	.
.	.	$\boxed{2}$	1	$\boxed{2}$	.	.	.	.	.
.	$\boxed{2}$	1	2	1	$\boxed{2}$	.	$\boxed{2}$	.	.
.	.	$\boxed{2}$	1	2	1	$\boxed{2}$	1	$\boxed{2}$	.
.	$\boxed{2}$	1	2	1	2	1	$\boxed{2}$	.	.
.	.	2	1	$\boxed{2}$	1	$\boxed{2}$	1	$\boxed{2}$	.
.	2	.	$\boxed{2}$	.	$\boxed{2}$	.	$\boxed{2}$	.	.
.	.	.	.	.	.	.	.	.	.

Step 2

Figure 11.23: Steps 1 and 2 of a Guo-Hall skeletonization

.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	1	.	$\boxed{1}$	.	.	.	.	.
.	.	.	1	.	.	.	.	.	.
.	.	$\boxed{1}$	2	$\boxed{1}$	.	.	.	.	.
.	.	.	1	.	.	.	.	.	.
.	.	$\boxed{1}$	2	$\boxed{1}$	.	.	.	.	.
.	.	.	$\boxed{1}$	2	$\boxed{1}$	.	1	.	.
.	.	$\boxed{1}$	2	1	2	1	.	.	.
.	.	2	$\boxed{1}$	.	$\boxed{1}$	.	1	.	.
.	2	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

Step 3

.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	1	.	1	.	.	.	.	.
.	.	.	1	.	.	.	.	.	.
.	.	.	2	.	.	.	.	.	.
.	.	.	1	.	.	.	.	.	.
.	.	.	2	.	.	.	.	.	.
.	.	.	.	2	.	.	1	.	.
.	.	.	2	.	2	1	.	.	.
.	.	2	.	.	.	.	1	.	.
.	2	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.

Step 4

Figure 11.24: Steps 3 and 4 of a Guo-Hall skeletonization

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does not change the deletablity of neighbouring pixels. After we have tested all pixels, we change all the 3's to 0's. Even iterations are similar, except that here we only check pixels whose value is 2 for deletability.

Figure 11.25 shows the base program, and figures 11.26 and 11.27 the programs for dealing with odd and even iterations respectively. Figure 11.28 is the program for testing if $C(p) = 1$.

```
function out=gh(a)

a1=ceil(size(a,1)/2);
a2=ceil(size(a,2)/2);
aa=repmat([1 2;2 1],a1,a2);
aa=aa(1:size(a,1),1:size(a,2)).*a;

done=0;
N=2;
last=aa;
previous=gh_odd(last);
current=gh_even(previous);
while done==0,
  if all(current(:)==last(:)),
    done=1;
  end;
  N=N+1;
  last=previous;
  previous=current;
  if mod(N,2)==0,
    current=gh_even(current);
  else
    current=gh_odd(current);
  end;
end;

current(find(current>0))=1;
out=current;
```

Figure 11.25: A MATLAB function for Guo-Hall skeletonization

Having done all this hard work, we can now test the algorithm: first on the L shape:

```
>> gh(L)


ans =


     0     0     0     0     0     0     0     0     0     0
```

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```
function out=gh_odd(a)

height=size(a,1);
width=size(a,2);
out=a;

for i=2:height-1,
  for j=2:width-1,
    if a(i,j)==1,
      nbhd=a(i-1:i+1,j-1:j+1);
      nbhd(find(nbhd>0))=1;
      b=sum(nbhd(:))-1;
      if eight_comps(nbhd)==1 & b>1,
        out(i,j)=3;
      end;
    end;
  end;
end;

out(find(out==3))=0;
```

Figure 11.26: A MATLAB function for Guo-Hall odd iterations

```
a

function out=gh_even(a)

height=size(a,1);
width=size(a,2);
out=a;

for i=2:height-1,
  for j=2:width-1,
    if a(i,j)==2,
      nbhd=a(i-1:i+1,j-1:j+1);
      nbhd(find(nbhd>0))=1;
      b=sum(nbhd(:))-1;
      if eight_comps(nbhd)==1 & b>1,
        out(i,j)=3;
      end;
    end;
  end;
end;

out(find(out==3))=0;
```

Figure 11.27: A MATLAB function for Guo-Hall even iterations

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```
function out=eight_comps(a)

  out=min(1-a(4),max(a(7),a(8)))+...
      min(1-a(8),max(a(9),a(6)))+...
      min(1-a(6),max(a(3),a(2)))+...
      min(1-a(2),max(a(1),a(4)));
```

Figure 11.28: A MATLAB function for testing if $C(p) = 1$

0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	1	0	0	1	0	0
0	0	0	1	0	1	1	0	0	0
0	0	1	0	0	0	0	1	0	0
0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

and on the circles and "nice work" images, for which the results are shown in figure 11.29. Note the

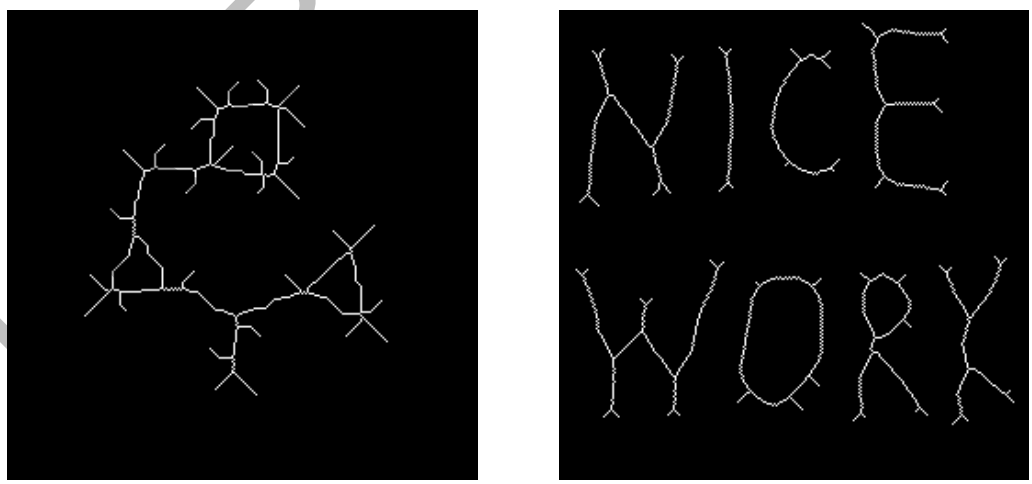

Figure 11.29: Examples of Guo-Hall skeletonization

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differences between these skeletons, and those produce by the Zhang-Suen algorithm as shown in figure 11.22.

Although the Guo-Hall algorithm is faster than that of Zhang-Suen, our implementation is in fact slower. This is because we were able to use lookup tables in our program `zs.m`, but the nature of subfields does not allow the use of lookup tables for `gh.m`.

Skeletonization using the distance transform

The distance transform can be used to provide the skeleton of a region R. We apply the distance transform, using mask 1, to the image negative, just as we did for the circle previously. Then the skeleton consists of those pixels (i, j) for which

$$d(i, j) \geq \max\{d(i-1, j), d(i+1, j), d(i, j-1), d(i, j+1)\}.$$

For example, suppose we take a small region consisting of a single rectangle, and find the distance transform of its negative:

```
>> c=zeros(7,9);c(2:6,2:8)=1


c =


     0     0     0     0     0     0     0     0     0
     0     1     1     1     1     1     1     1     0
     0     1     1     1     1     1     1     1     0
     0     1     1     1     1     1     1     1     0
     0     1     1     1     1     1     1     1     0
     0     1     1     1     1     1     1     1     0
     0     0     0     0     0     0     0     0     0


>> cd=disttrans(~c,mask1)


cd =
```

NOT FOR CIRCULATION

```
     0     0     0     0     0     0     0     0     0
     0     1     1     1     1     1     1     1     0
     0     1     2     2     2     2     2     1     0
     0     1     2     3     3     3     2     1     0
     0     1     2     2     2     2     2     1     0
     0     1     1     1     1     1     1     1     0
     0     0     0     0     0     0     0     0     0
```

We can obtain the skeleton using a double loop:

```
>> skel=zeros(size(c));
>> for i=2:6,
     for j=2:8,
       if cd(i,j)>=max([cd(i-1,j),cd(i+1,j),cd(i,j-1),cd(i,j+1)])
         skel(i,j)=1;
       end;
     end;
   end;

>> skel

skel =

     0     0     0     0     0     0     0     0     0
     0     1     0     0     0     0     0     1     0
     0     0     1     0     0     0     1     0     0
     0     0     0     1     1     1     0     0     0
     0     0     1     0     0     0     1     0     0
     0     1     0     0     0     0     0     1     0
```

NOT FOR CIRCULATION

```
     0     0     0     0     0     0     0     0     0
```

In fact we can produce the skeleton more efficiently by using MATLAB's `ordfilt2` function, which was introduced in chapter 5. This can be used to find the largest value in a neighbourhood, and the neighbourhood can be very precisely defined:

```
>> cd2=ordfilt2(cd,5,[0 1 0;1 1 1;0 1 0])


cd2 =


     0     1     1     1     1     1     1     1     0
     1     1     2     2     2     2     2     1     1
     1     2     2     3     3     3     2     2     1
     1     2     3     3     3     3     3     2     1
     1     2     2     3     3     3     2     2     1
     1     1     2     2     2     2     2     1     1
     0     1     1     1     1     1     1     1     0


>> cd2<=cd


ans =


     1     0     0     0     0     0     0     0     1
     0     1     0     0     0     0     0     1     0
     0     0     1     0     0     0     1     0     0
     0     0     0     1     1     1     0     0     0
     0     0     1     0     0     0     1     0     0
     0     1     0     0     0     0     0     1     0
     1     0     0     0     0     0     0     0     1
```

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We can easily restrict the image so as not to obtain the extra 1's in the corners of the result. Now let's do the same thing with our circles image:

```
>> c=~imread('circles.tif');
>> cd=disttrans(c,mask1);
>> cd2=ordfilt2(cd,5,[0 1 0;1 1 1;0 1 0]);
>> imshow((cd2<=cd)&~c)
```

and the result is shown in figure 11.30. The use of the command `(cd2<=cd)&~c` blocks out the

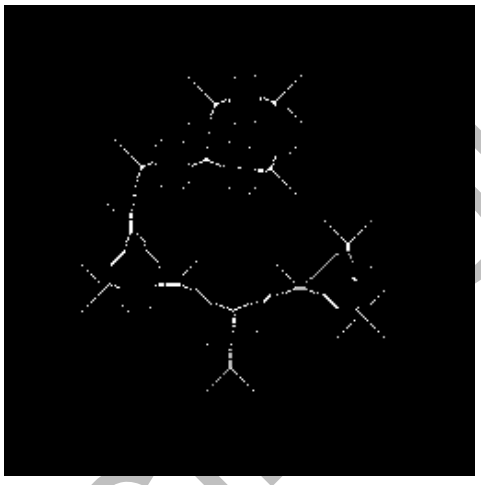

Figure 11.30: Skeletonization using the distance transform

outside of the circles, so that just the skeleton is left. We will see in chapter 10 how to thicken this skeleton, and also other ways of obtaining the skeleton.

Details about both the Guo-Hall and Zhang-Suen algorithms, and others, with discussions and

Exercises

1. What are the coordinates of the 4-neighbours of the pixel (i, j)? What are the coordinates of its 8-neighbours?

2. Find the length of the shortest 4-path from

   (a) pixel $(1, 1)$ to pixel $(5, 4)$

NOT FOR CIRCULATION

(b) pixel $(3, 1)$ to pixel $(1, 6)$

(c) pixel (i, j) to pixel (l, m)

For this question define the length of a path be the number of pixels in it.

3. Find the shortest 8-paths between each pair of pixels in the preceding question.

4. Consider the two images

$$\begin{array}{cccccccccc}
\bullet & \bullet & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \cdot \\
\bullet & \bullet & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \cdot \\
\cdot & \bullet & \bullet & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \cdot \\
\bullet & \bullet & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \cdot \\
\bullet & \bullet & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \\
\bullet & \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet
\end{array}, \qquad
\begin{array}{cccccccccc}
\bullet & \bullet & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot \\
\bullet & \bullet & \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \bullet & \cdot \\
\cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot \\
\bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot \\
\bullet & \bullet & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \\
\bullet & \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

A $\qquad\qquad$ B

Find the 4-components and 8-components of each image.

5. The above matrices were obtained with the MATLAB commands

```
>> A=magic(10)>50
>> B=magic(10)>60
```

Check your answers of the previous question with the `bwlabel` function.

6. We can define the 6-*neighbours* of a pixel p with coordinates (x, y) to be the pixels with coordinates $(x+1, y)$, $(x-1, y)$, $(x, y+1)$, $(x, y-1)$, $(x+1, y+1)$ and $(x-1, y-1)$. Draw a diagram showing the six 6-neighbours of a pixel.

7. Prove the triangle inequality for the Euclidean distance metric, and for the 4-path and 8-path metrics d_4 and d_8.

NOT FOR CIRCULATION

8. Define the relation 6-*connectedness* as:

   p is 6-connected to q if there is a path $p = p_1, p_2, \ldots, p_n = q$ such that for each $i = 1, 2, \ldots, n-1$, p_i is 6-adjacent to p_{i+1}.

   Show that this is an equivalence relation.

9. Find the lengths of the shortest 6-paths between pixels with the following coordinates:

   (a) $(0,0)$ and $(2,2)$, (b) $(1,2)$ and $(5,4)$, (c) $(2,1)$ and $(6,8)$,

   (d) $(3,1)$ and $(7,4)$

   Can you develop an expression for a 6-path metric?

10. Show how to refine the algorithms for component labelling to label the 6-components of a binary image.

11. For the following image:

$$
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot \\
\cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot \\
\cdot & \cdot & \bullet & \bullet & \cdot & \bullet & \bullet & \cdot \\
\cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot \\
\cdot & \bullet & \bullet & \bullet & \bullet & \cdot & \bullet & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

   use your algorithm developed in the previous question to label the 6-components.

12. Use `bwlabel` to determine the number of blobs in the example given at the beginning of this chapter. Is there any difference between the results using 4 and 8 adjacency? Can you account for your answer?

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13. Obtain the 6-components of the images A and B given in question 4.

14. Let C be the image:

$$
\begin{array}{ccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot \\
\cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot \\
\cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot \\
\cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot \\
\cdot & \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Obtain the 4-boundary and 8-boundary of this image.

15. The image in the previous question was obtained with the MATLAB commands:

```
>> [x,y]=meshgrid(-5:5,-5:5);
>> C=(x.^2+y.^2)<20
```

Check your answers to question 14 with the `bwperim` function.

16. Determine if each of the following foreground pixels is (a) 4-simple, (b) 8-simple.

$$
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & p & \cdot \\
\bullet & \bullet & \cdot
\end{array}
\qquad
\begin{array}{ccc}
\bullet & \cdot & \cdot \\
\bullet & p & \bullet \\
\cdot & \bullet & \cdot
\end{array}
\qquad
\begin{array}{ccc}
\cdot & \bullet & \cdot \\
\bullet & p & \bullet \\
\cdot & \cdot & \bullet
\end{array}
\qquad
\begin{array}{ccc}
\cdot & \bullet & \bullet \\
\cdot & p & \bullet \\
\bullet & \bullet & \cdot
\end{array}
\qquad
\begin{array}{ccc}
\cdot & \bullet & \bullet \\
\cdot & p & \cdot \\
\bullet & \bullet & \cdot
\end{array}
\qquad
\begin{array}{ccc}
\cdot & \cdot & \bullet \\
\bullet & p & \cdot \\
\bullet & \cdot & \bullet
\end{array}
$$

17. Calculate $C_S(p)$ for each configuration in the previous question, and show that the result is equal to $C(p)$ in each case.

18. Find configurations of pixels surrounding p for which

NOT FOR CIRCULATION

(a) $C(p) = A(p) - 2$

(b) $C(p) = A(p) - 3$

(c) $C(p) = A(p) - 4$

19. Show that $C(p) \leq A(p)$ for every configuration of pixels.

20. Skeletonize each of the following images using (a) the Zhang-Suen algorithm, (b) the Guo-Hall algorithm:

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \cdot & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \qquad \begin{array}{ccccccc} \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot \\ \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot \\ \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bullet & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccccccc} \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot \\ \bullet & \bullet & \bullet & \cdot & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \cdot & \bullet & \bullet & \bullet \\ \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet \\ \bullet & \bullet & \bullet & \cdot & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \cdot & \bullet & \bullet & \bullet \\ \cdot & \bullet & \bullet & \bullet & \bullet & \bullet & \cdot \end{array}$$

21. Consider the cross shape formed by starting with an 11×11 square and removing a 3×3 square from each corner. Skeletonize it by each of the algorithms.

22. Check your answers to the previous two questions with MATLAB.

23. Which of the algorithms seems to be the fastest, in requiring the least number of iterations, or removing the greatest number of pixels at each step?

24. Sketch the skeletons of: (a) a 2×1 rectangle, (b) a triangle.

25. Apply both algorithms to the image `circbw.tif`.

    (a) Which one works faster?

    (b) Which gives the best looking results?

26. Repeat the previous question on some other binary images.

NOT FOR CIRCULATION

27. For each of the following images:

0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0
0	1	1	1	1	0	0	0
0	0	1	1	1	1	0	0
0	0	0	1	1	1	1	1
0	1	1	1	1	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	1	1	1	1	1	1	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	1	0
0	1	1	1	1	1	1	0
0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0
0	0	0	1	1	0	0	0
0	0	0	1	1	0	0	0
0	0	1	1	1	1	0	0
0	1	1	1	1	1	1	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	1	0
0	0	0	0	0	0	0	0

apply the distance transform to approximate distances from the region containing 1's to all other pixels in the image, using the masks:

(i)
$$\begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{array}$$

(ii)
$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{array}$$

(iii)
$$\begin{array}{ccc} 4 & 3 & 4 \\ 3 & 0 & 3 \\ 4 & 3 & 4 \end{array}$$

(iv)
$$\begin{array}{ccccc} & 11 & & 11 & \\ 11 & 7 & 5 & 7 & 11 \\ & 5 & 0 & 5 & \\ 11 & 7 & 5 & 7 & 11 \\ & 11 & & 11 & \end{array}$$

and applying any necessary scaling at the end.

28. Apply the distance transform and use it to find the skeleton in the images `circlesm.tif` and `nicework.tif`.

29. Compare the result of the previous question with the results given by the Zhang-Suen and Guo-Hall methods.

    Which method produces the most visually appealing results?

    Which method seems fastest?

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Chapter 12

Shapes and boundaries

12.1 Introduction

In this chapter we shall investigate tools for examining image shapes. Some tools have already been discussed in chapter 10; now we look at specific methods for examining shapes of objects. Questions we might ask about shapes include:

- how do we tell if two objects have the same shape?
- how can we classify shape?
- how can we describe the shape of an object?

Formal means of describing shapes are called *shape descriptors*. Shape descriptors may include size, symmetry, length of perimeter. A precise definition of the exact shape in some efficient manner is a *shape representation*.

In this chapter we shall be concerned with the *boundary* of objects. A boundary differs from an edge in that whereas an edge is an example of a local property of an image, a boundary is a global property.

12.2 Chain codes and shape numbers

The idea of a chain code is quite straightforward; we walk around the boundary of an object, taking note of the directions we take. The resulting list of direction is the chain code.

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We need to consider two types of boundaries: 4-connected and 8-connected (see chapter 11. If the boundary is 4-connected, there are four possible directions in which to walk; If the boundary is 8-connected, there are eight possible directions. These are shown in figure 12.1.

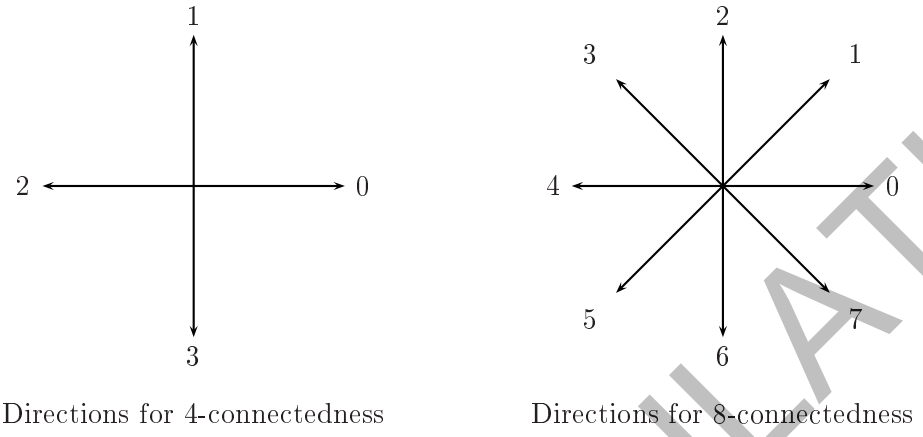

Figure 12.1: Directions for chain codes

To see how they work, consider the object and its boundary shown in figure 12.2.

0	1	1	1	0
0	1	1	1	1
0	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	0

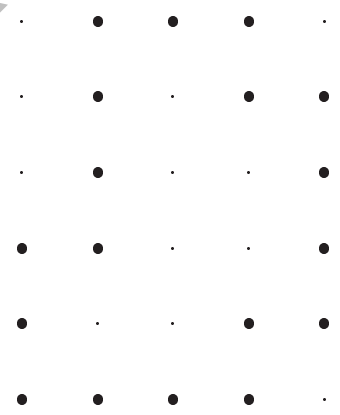

Figure 12.2: A 4-connected object and its boundary

Suppose we walk along the boundary in a clockwise direction starting at the left-most point in the bottom row, and list the directions as we go. This is shown in figure 12.3.

We can thus read off the chain code as:

3 3 3 2 3 3 0 0 0 1 0 1 1 1 2 1 2 2

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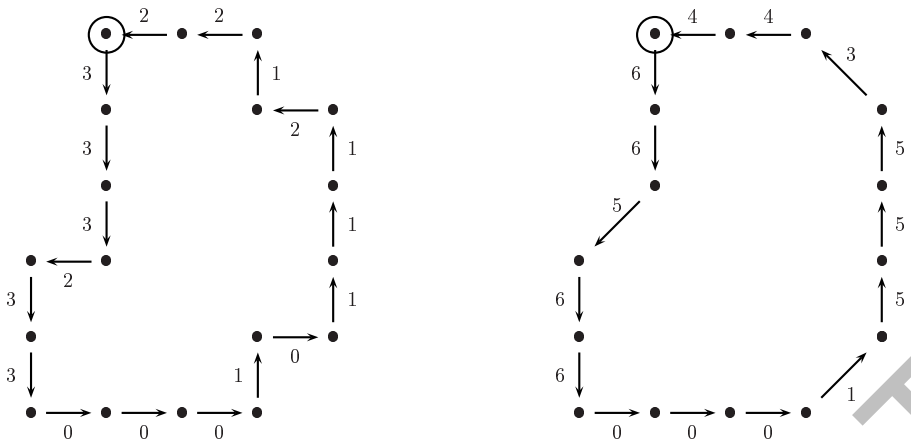

Figure 12.3: Obtaining the chain code from the object in figure 12.2

If we treat the object in figure 12.2 as being 8-connected, then its boundary and chain code are generated as in the right hand diagram in figure 12.3. In this case, the resulting chain code is:

6 6 5 6 6 0 0 0 1 5 5 5 3 4 4

To obtain the chain code in MATLAB, we have to write a small function to do the job for us. We have to first be able to trace out the boundary of our object, and once we can do that, we can write down our directions to form the chain code. A simple boundary following algorithm has been given by Sonka et. al. [32]. For simplicity, we shall just give the version for 4-connected boundaries:

1. Start by finding the pixel in the object which has the left most value in the topmost row; call this pixel P_0. Define a variable `dir` (for direction), and set it equal to 3. (Since P_0 is the top left pixel in the object, the direction to the next pixel must be 3.)

2. Traverse the 3×3 neighbourhood of the current pixel in an anti-clockwise direction, beginning the search at the pixel in direction

   $$\texttt{dir} + 3 \pmod 4$$

   This simply sets the current direction to the first direction anticlockwise from `dir`:

`dir`	0	1	2	3
$\texttt{dir} + 3 \pmod 4$	3	0	1	2

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The first foreground pixel will be the new boundary element. Update `dir`.

3. Stop when the current boundary element P_n is equal to the second element P_1 and the previous boundary pixel P_{n-1} is equal to the first boundary element P_0.

Suppose we have a binary image `im` consisting of a single object. We can find the top left pixel with the following MATLAB commands:

```
>> [x,y]=find(im==1);
>> x=min(x)
>> imx=im(x,:);
>> y=min(imx)
```

The first command simply finds the coordinates of all the foreground pixels. The second command finds the minimum. Thus `x` is the top row of the object. The third command isolates this top row, and the final command finds the left-most column in it.

Given the indices `x` and `y` of the current pixel, and the value `dir`, how do we implement step 2? We give a simple example, shown in figure 12.4. Suppose that in this particular example the current value of `dir` is 0, so that

$$\texttt{dir} + 3 \pmod 4 = 3.$$

The dotted arrows indicate the direction of traversing (starting from the pixel in direction 3 from P_k) until we reach a new foreground pixel. This pixel is then P_{k+1}.

We start by noting the row and column increments to the four-neighbours of a pixel:

$$\begin{array}{ccc} & -1 & 0 \\ 0 & -1 & \\ & 0 & 0 \\ & 1 & 0 \end{array} \qquad \begin{array}{cc} 0 & 1 \end{array}$$

Since we are looking at four-connected boundaries, these are the only neighbours we need. We can put these into a matrix, with the first row corresponding to the increments in direction 0:

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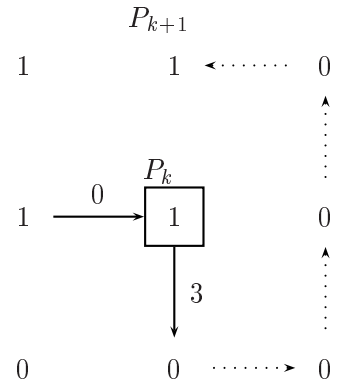

Figure 12.4: Traversing a neighbourhood

```
>> n=[0 1;-1 0;0 -1;1 0]
```

This means that the indices in `n(j,:)` correspond to the increments in indices in direction `j` − 1. Thus given a direction `dir`, we can enter:

```
>> newdir=mod(dir+3,4);
>> for i=0:3,j=mod(newdir+i,4)+1;im(x+n(j,1),y+n(j,2)),end
```

and this will traverse the neighbourhood of our image `im` at position (`x`, `y`) starting from the correct direction. Notice that we set

```
>> j=mod(newdir+i,4)+1;
```

The extra "`+1`" takes account of the fact that the modulus function returns values $0, 1, 2, 3$, but the rows of `n` are $1, 2, 3, 4$. As the neighbourhood is traversed, all the values will be placed into a small vector `tt`, so that

```
>> tt(i+1)=image(x+n(j,1),y+n(j,2));
```

We can then easily find the first non-zero value with

```
>> d=min(find(tt==1));
```

and now we can update `dir`, and the position of the current pixel:

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```
>> dir=mod(newdir+d-1,4);

>> x=x+n(dir+1,1);y=y+n(dir+1,2);
```

As we do this, we will place the most recent value of `dir` into a vector which will be our final chain code.

The function shown in figure 12.5 is the complete code for the above algorithm with one change: we stop when we reach the original pixel.

```
function out=chaincode4(image)

n=[0 1;-1 0;0 -1;1 0];

flag=1;
cc=[];
[x y]=find(image==1);
x=min(x);
imx=image(x,:);
y=min(find(imx==1));
first=[x y];
dir=3;

while flag==1,
  tt=zeros(1,4);
  newdir=mod(dir+3,4);
  for i=0:3,
    j=mod(newdir+i,4)+1;
    tt(i+1)=image(x+n(j,1),y+n(j,2));
  end
  d=min(find(tt==1));
  dir=mod(newdir+d-1,4);
  cc=[cc,dir];
  x=x+n(dir+1,1);y=y+n(dir+1,2);
  if x==first(1) & y==first(2)
    flag=0;
  end;
end;

out=cc;
```

Figure 12.5: A MATLAB function for obtaining the chain code of a 4-connected object

We can test this out with the shape in figure 12.2 (we surround our image with zeros first):

```
>> test=[0 0 0 0 0 0 0;...
```

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```
        0 0 1 1 1 0 0;...
        0 0 1 1 1 1 0;...
        0 0 1 1 1 1 0;...
        0 1 1 1 1 1 0;...
        0 1 1 1 1 1 0;...
        0 1 1 1 1 0 0;...
        0 0 0 0 0 0 0];

>> chaincode4(test)

  ans =

    Columns 1 through 12

       3     3     3     2     3     3     0     0     0     1     0     1

    Columns 13 through 18

       1     1     2     1     2     2
```

and comparing this with the code given earlier, the function has indeed returned the correct chain code.

We can easily modify the program to perform chain codes for 8-connected boundaries. The above algorithm must be slightly changed; again, see [32]

1. Start by finding the pixel in the object which has the left most value in the topmost row; call this pixel P_0. Define a variable `dir` (for direction), and set it equal to 7 (Since P_0 is the top left pixel in the object, the direction to the next pixel must be 7)

2. Traverse the 3×3 neighbourhood of the current pixel in an anti-clockwise direction, beginning

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the search at the pixel in direction

$$\texttt{dir} + 7 \pmod 8 \quad \text{if } \texttt{dir} \text{ is even}$$
$$\texttt{dir} + 6 \pmod 8 \quad \text{if } \texttt{dir} \text{ is odd}$$

This simply sets the current direction to the first direction anticlockwise from `dir`:

`dir`		0	1	2	3	4	5	6	7
`dir` + 7	(mod 8)	7	0	1	2	3	4	5	6
`dir` + 6	(mod 8)	6	7	0	1	2	3	4	5

The first foreground pixel will be the new boundary element. Update `dir`.

3. Stop when the current boundary element P_n is equal to the second element P_1 and the previous boundary pixel P_{n-1} is equal to the first boundary element P_0.

Note that the choosing of the starting direction in step 2 can be implemented by

```
>> newdir=mod(dir+7-mod(dir,2),8);
```

which produces `dir` + 7 if `dir` is even, and `dir` + 6 if `dir` is odd. As well as this we need to take into account *all* possible directions from a pixel; the index increments are:

$$\begin{array}{cc} -1 & -1 \\ \\ 0 & -1 \\ \\ 1 & -1 \end{array} \qquad \begin{array}{cc} -1 & 0 \\ \\ 0 & 0 \\ \\ 1 & 0 \end{array} \qquad \begin{array}{cc} -1 & 1 \\ \\ 0 & 1 \\ \\ 1 & 1 \end{array}$$

These will all be placed in the array `n`:

```
>> n=[0 1;-1 1;-1 0;-1 -1;0 -1;1 -1;1 0;1 1];
```

where as before, direction 0 corresponds to row 1 of `n`.

The program is given in figure 12.6.

This can be tested on out test image `test`:

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```
function out=chaincode8(image)

n=[0 1;-1 1;-1 0;-1 -1;0 -1;1 -1;1 0;1 1];

flag=1;
cc=[];
[x y]=find(image==1);
x=min(x);
imx=image(x,:);
y=min(find(imx==1));
first=[x y];
dir=7;

while flag==1,
  tt=zeros(1,8);
  newdir=mod(dir+7-mod(dir,2),8);
  for i=0:7,
    j=mod(newdir+i,8)+1;
    tt(i+1)=image(x+n(j,1),y+n(j,2));
  end
  d=min(find(tt==1));
  dir=mod(newdir+d-1,8);
  cc=[cc,dir];
  x=x+n(dir+1,1);y=y+n(dir+1,2);
  if x==first(1) & y==first(2)
    flag=0;
  end;
end;

out=cc;
```

Figure 12.6: A MATLAB function for obtaining the chain code of an 8-connected object

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```
>> chaincode8(test)

  ans =

    Columns 1 through 12

       6     6     5     6     6     0     0     0     1     2     2     2

    Columns 13 through 15

       3     4     4
```

and the answer is indeed the code we obtained earlier by following the arrows around the object.

12.2.1 Normalization of chain codes

There are two problems with the definition of the chain code as given in previous sections:

1. the chain code is dependent on the starting pixel,
2. the chain code is dependent on the orientation of the object.

Here we will look at (1). The idea is to "normalize" the chain code as follows: imaging the code to be written around the edge of a circle. We choose as our starting place the position for which the code, when read off, will be the lowest possible integer. The result is the *normalized chain code* for our object.

For example, suppose we have an object consisting of a 3×3 square:

```
>> a=zeros(5,5);a(2:4,2:4)=1

     0     0     0     0     0
     0     1     1     1     0
```

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```
     0     1     1     1     0

     0     1     1     1     0

     0     0     0     0     0


>> c=chaincode4(a)


c =


     3     3     0     0     1     1     2     2
```

Now let's put these codes around a circle as shown in figure 12.7. The arrow indicates where we

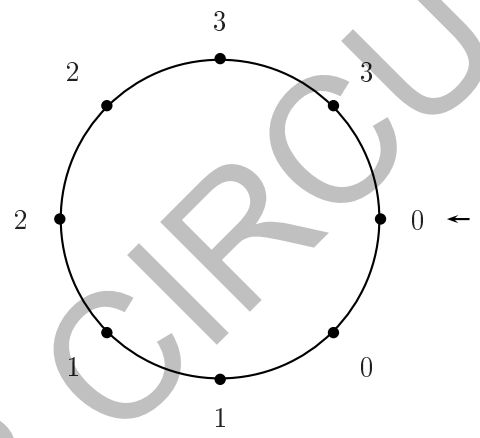

Figure 12.7: A chain code written cyclically

should start reading off the code to obtain the lowest integer; in this case it is

```
0 1 1 2 2 3 3
```

But we can do this easily in MATLAB. What we do is to create a "cyclic" matrix: the top row is the chain code, and every row consists of a possible starting place around our circle. This can be done by advancing from one row to the next and each time moving the last element to the beginning. For our chain code above we can do this:

```
>> m=c;
```

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```
>> for i=2:8,m=[m;[m(i-1,8),m(i-1,1:7)]];end
>> m


m =


     3     3     0     0     1     1     2     2
     2     3     3     0     0     1     1     2
     2     2     3     3     0     0     1     1
     1     2     2     3     3     0     0     1
     1     1     2     2     3     3     0     0
     0     1     1     2     2     3     3     0
     0     0     1     1     2     2     3     3
     3     0     0     1     1     2     2     3
```

To find the row which contains the least integer we use the handy `sortrows` function which sorts the rows "lexicographically": first on the first element, then on the second element, and so on:

```
>> ms=sortrows(m)


ms =


     0     0     1     1     2     2     3     3
     0     1     1     2     2     3     3     0
     1     1     2     2     3     3     0     0
     1     2     2     3     3     0     0     1
     2     2     3     3     0     0     1     1
     2     3     3     0     0     1     1     2
     3     0     0     1     1     2     2     3
     3     3     0     0     1     1     2     2
```

Now the normalized chain code is just the first row:

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```
>> ms(1,:)


ans =


     0     0     1     1     2     2     3     3
```

Let's try this with a slightly larger example: our `test` object from figure 12.2:

```
>> c=chaincode4(test);
>> lc=length(c);
>> m=c;
>> for i=2:lc,m=[m;[m(i-1,lc),m(i-1,1:lc-1)]];end
>> ms=sortrows(m);
>> ms(1,:)


ans =


  Columns 1 through 11


     0     0     0     1     0     1     1     1     2     1     2


  Columns 12 through 18


     2     3     3     3     2     3     3
```

The only difference here with the previous list of commands is that we replaces the numbers 8 and 7 with `lc` and `lc-1`. These commands can easily be put into a function; as shown in figure 12.8.

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```
function out=normalize(c)
%
% NORMALIZE returns the vector which is the least integer of all cyclic
% shifts of V.
%
m=c;
lc=length(c);
for i=2:lc,m=[m;[m(i-1,lc),m(i-1,1:lc-1)]];end
ms=sortrows(m);
out=ms(1,:);
```

Figure 12.8: A MATLAB function for normalizing a chain code

12.2.2 Shape numbers

We now consider the problem of defining a chain code which is independent of orientation of the object. For example, let's create a simple "L" shape:

```
>> L=zeros(7,6);L(2:6,2:3)=1;L(5:6,4:5)=1


L =


     0     0     0     0     0     0

     0     1     1     0     0     0

     0     1     1     0     0     0

     0     1     1     0     0     0

     0     1     1     1     1     0

     0     1     1     1     1     0

     0     0     0     0     0     0
```

Then we can take its chain code:

```
>> c=chaincode4(L)


c =


  Columns 1 through 11
```

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```
     3     3     3     3     0     0     0     1     2     2     1

  Columns 12 through 14

     1     1     2
```

and normalize it:

```
>> normalize(c)

  Columns 1 through 11

     0     0     0     1     2     2     1     1     1     2     3

  Columns 12 through 14

     3     3     3
```

Now suppose we rotate the shape so that it has a different orientation (we use the `rot90` function which rotates a matrix by 90 degrees):

```
>> L2=rot90(L)

L2 =

     0     0     0     0     0     0     0
     0     0     0     0     1     1     0
     0     0     0     0     1     1     0
     0     1     1     1     1     1     0
     0     1     1     1     1     1     0
     0     0     0     0     0     0     0
```

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Now we can find the normalized chain code for this new orientation

```
>> c2=chaincode4(L2)

c2 =

  Columns 1 through 11

     3     3     2     2     2     3     0     0     0     0     1

  Columns 12 through 14

     1     1     2
>> normalize(c2)

ans =

  Columns 1 through 11

     0     0     0     0     1     1     1     2     3     3     2

  Columns 12 through 14

     2     2     3
```

Even when normalized, the chain codes are different.

To overcome this, we take *differences* of the chain code: for each two consecutive elements c_i and c_{i+1} their difference is defined as

$$c_{i+1} - c_i \pmod 4.$$

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(If we were dealing with 8-connected boundaries, then we would take differences mod 8). For an example, take the simple L shape shown in figure 12.9. The chain code can be read off easily starting

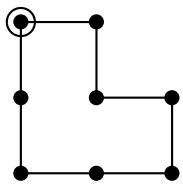

Figure 12.9: A simple L-shape

from the point shown; it is

$$3 \quad 3 \quad 0 \quad 0 \quad 1 \quad 2 \quad 1 \quad 2.$$

To apply differences, repeat the first chain code number at the end, then take differences along the entire length of the code:

$$\begin{array}{ccccccccc}
3 & 3 & 0 & 0 & 1 & 2 & 1 & 2 & 3 \\
 & 0 & -3 & 0 & 1 & 1 & -1 & 1 & 1 \\
 & 0 & 1 & 0 & 1 & 1 & 3 & 1 & 1
\end{array}$$

The first row is just the chain code, with the first element repeated at the end; the second row are the differences, and the third row are these differences mod 4. The normalized version of these differences (which here is just the last row itself), is the *shape number* for this L shape.

Suppose this shape is rotated by 90° to obtain the shape shown in figure 12.10. The chain code

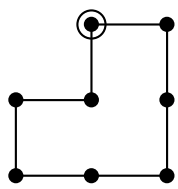

Figure 12.10: The shape from figure 12.9 rotated

for this shape is

$$3 \quad 2 \quad 3 \quad 0 \quad 0 \quad 1 \quad 1 \quad 2.$$

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Even when normalized the result is not the same as the chain code for this shape in its original orientation. But if the differences are obtained:

$$\begin{array}{ccccccccc} 3 & 2 & 3 & 0 & 0 & 1 & 1 & 2 & 3 \\ -1 & 1 & -3 & 0 & 1 & 0 & 1 & 1 & \\ 3 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & \end{array}$$

then the normalized third row is

$$0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 3 \quad 1 \quad 1$$

which is exactly the same code obtained above.

This can be done easily in MATLAB: given our chain code `c`, we create a cyclically shifted code by moving the first element to the end:

```
>> c=chaincode4(a)

c =

     3     3     0     0     1     1     2     2

>> c1=[c(2:8) c(1)]

c1 =

     3     0     0     1     1     2     2     3
```

Now we just subtract the two, mod 4:

```
>> mod(c1-c,4)

ans =

     0     1     0     1     0     1     0     1
```

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The normalized version of this code is the shape number of our object. Let's try this with our L shape and its rotation.

```
>> c=chaincode4(L);
>> lc=length(c);
>> c1=[c(2:lc) c(1)];
>> mod(c1-c,4)

ans =

  Columns 1 through 11

     0     0     0     1     0     0     1     1     0     3     0

  Columns 12 through 14

     0     1     1
```

This is already normalized. Now for L2:

```
>> c=chaincode4(L2);
>> lc=length(c);
>> c1=[c(2:lc) c(1)];
>> mod(c1-c,4)

ans =

  Columns 1 through 11

     0     3     0     0     1     1     0     0     0     1     0
```

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```
  Columns 12 through 14


     0     1     1
```

This needs to be normalized:

```
>> normalize(ans)


  Columns 1 through 11


     0     0     0     1     0     0     1     1     0     3     0


  Columns 12 through 14


     0     1     1
```

and this is exactly the result we obtained for `L`.

12.3 Fourier descriptors

The idea is this: suppose we walk around an object, but instead of writing down the directions, we write down the boundary coordinates. The final list of (x, y) coordinates can be turned into a list of complex numbers $z = x + yi$. The Fourier transform of this list of numbers is a *Fourier descriptor* of the object.

The beauty of a Fourier descriptor is that often only a few terms at the beginning of the transform are enough to distinguish objects, or to classify them.

We can easily modify our function `chaincode4.m` to `boundary4.m` by replacing the lines

```
cc=[cc,dir];
x=x+n(dir+1,1);y=y+n(dir+1,2);
```

with

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```
x=x+n(dir+1,1);y=y+n(dir+1,2);
cc=[cc;x y];
```

Thus the variable `cc` contains a list of the boundary pixels. Suppose we have created our function. Then for example:

```
>> a=zeros(5,5);a(2:4,2:4)=1


a =


     0     0     0     0     0
     0     1     1     1     0
     0     1     1     1     0
     0     1     1     1     0
     0     0     0     0     0


>> b=boundary4(a)


b =


     3     2
     4     2
     4     3
     4     4
     3     4
     2     4
     2     3
     2     2
```

Turning these into complex numbers is easy:

```
>> c=complex(b(:,1),b(:,2))
```

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```
c =

   3.0000 + 2.0000i
   4.0000 + 2.0000i
   4.0000 + 3.0000i
   4.0000 + 4.0000i
   3.0000 + 4.0000i
   2.0000 + 4.0000i
   2.0000 + 3.0000i
   2.0000 + 2.0000i
```

These can be plotted:

```
>> plot(c,'o'),axis([1,5,1,5]),axis equal
```

and the result is shown in figure 12.11. Supposing we take the Fourier transform, and from it extract

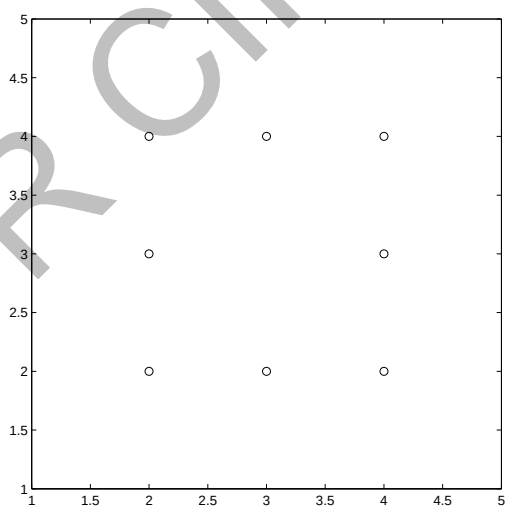

Figure 12.11: Boundary pixels

only a few terms:

```
>> f=fft(c)
```

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```
f =

  24.0000 +24.0000i
        0 - 9.6569i
        0
        0
        0
        0 + 1.6569i
        0
        0

>> f1=zeros(size(f));
>> f1(1:2)=f(1:2);
>> plot(ifft(f1),'o'),axis([1.0,5.0,1.0,5.0]),axis square
```

and the result is shown in figure 12.12. We note several aspects:

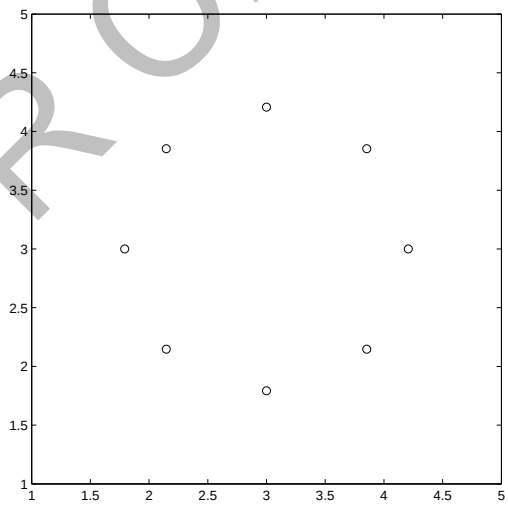

Figure 12.12: The use of Fourier descriptors for approximating a boundary

1. The Fourier transform of `c` contains only three non-zero terms.

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2. Only two terms of the transform is enough to begin to get some idea as the shape, size and symmetry of the object.

3. Even though the shape itself has been greatly changed—a square has become a circle—many shape descriptors are still little changed (such as size and symmetry).

To give a slightly more substantial example, we shall create a cross shape, and experiment with it.

```
>> a=zeros(14,14);a(6:9,2:13)=1;a(2:13,6:9)=1;
>> b=boundary4(a);
>> c=complex(b(:,1),b(:,2));
>> plot(c,'o'),axis([1,14,1,14]),axis square
```

and this is shown in figure 12.13. Now we can obtain the Fourier transform, and experiment with

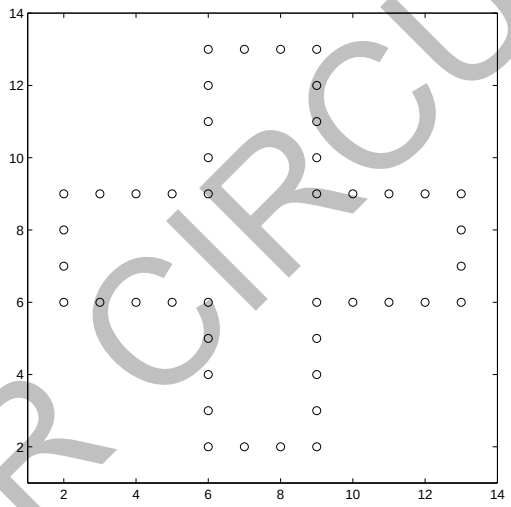

Figure 12.13: The boundary of a cross shape

extracting different elements from it.

```
>> f=fft(c);
>> f1=zeros(size(f));
>> f1(1:2)=f(1:2);
>> plot(ifft(f1),'o'),axis([1,14,1,14]),axis square
```

This is shown in figure 12.14(a). Taking eight terms:

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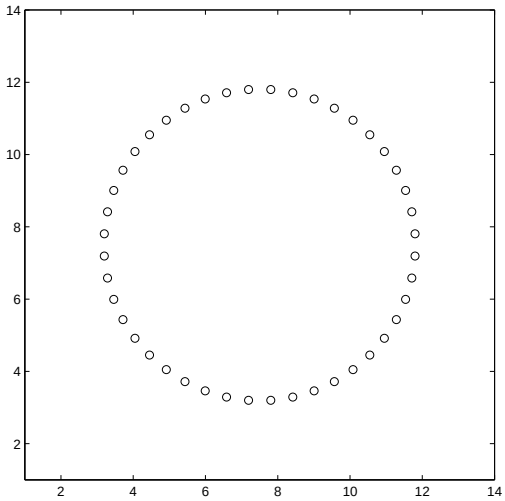
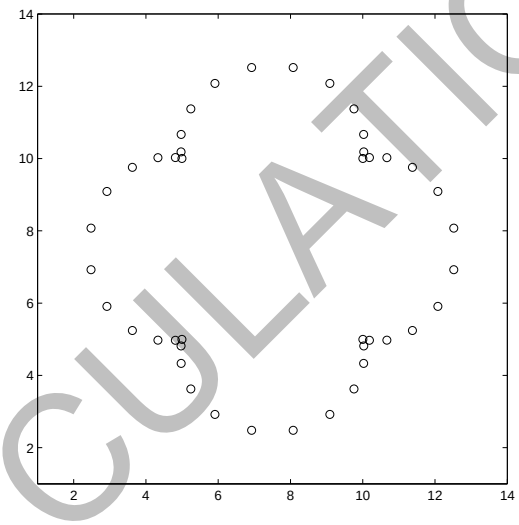
```
>> f1(1:8)=f(1:8);
>> plot(ifft(f1),'o'),axis([1,14,1,14]),axis square
```

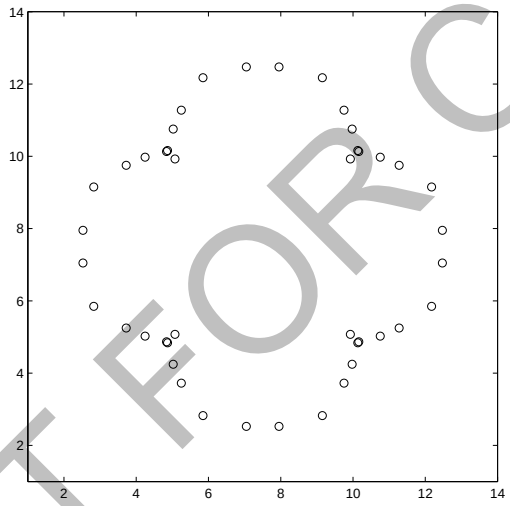
is shown in figure 12.14(b). The results after using 24 and 36 elements from the transform are shown in figures 12.14(c) and (d).



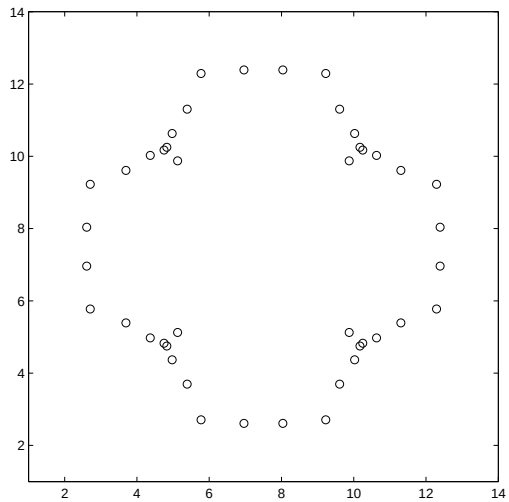
(a) After using 2 terms from the transform



(b) After using 8 terms from the transform



(c) After using 24 terms from the transform



(d) After using 36 terms from the transform

Figure 12.14: Results after extracting different numbers of elements from the Fourier transform of the boundary

We will not, of course, obtain the exact boundary until we take all the terms of the transform.

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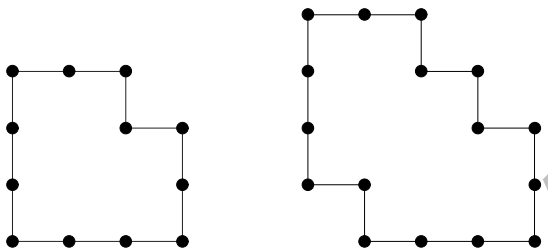
But even 8 terms from 44 give a rough idea of the shape and size.

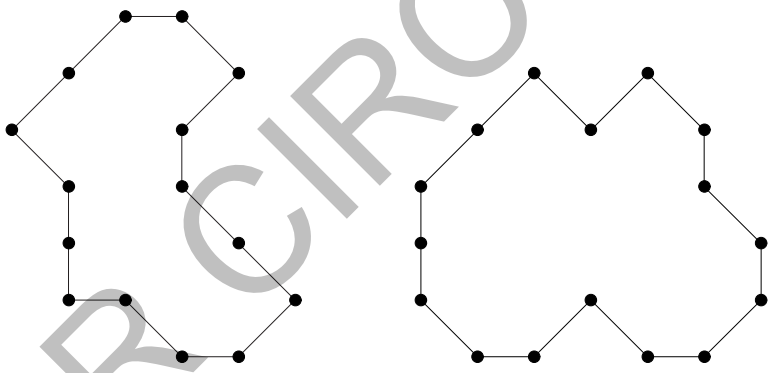
Exercises

1. Find the chain codes, normalized chain codes, and shape numbers for each of the following 4-connected shapes:



2. Now repeat the above exercise for all possible reflections and rotations of those shapes.

3. Repeat the above exercises for the 8-connected shapes:



4. Check your answers to the previous questions with MATLAB.

5. Generate the shapes with the following 4-connected chain codes:

   (a) `3 3 3 0 0 0 0 1 1 2 2 1 2 2`

   (b) `3 3 3 0 3 0 0 1 0 1 1 2 2 1 2 2`

6. Generate the shapes with the following 8-connected chain codes:

   (a) `5 6 7 6 0 0 1 2 2 4 3 4`

   (b) `5 6 7 7 1 7 1 2 2 3 5 3 4`

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7. Obtain the normalized chaincodes and shape numbers of all the shapes in the previous questions.

   Check your answers with MATLAB.

8. Use MATLAB to generate a T shape, and obtain its Fourier descriptors. How many terms are required to identify the

   (a) symmetry of the object?

   (b) size of the object?

   (c) shape of the object?

9. Repeat the above question but use an X shape.

10. Compare the results of the previous two questions with the results of the cross shape discussed in section 12.3.

    How many terms are required to distinguish the

    (a) symmetries

    (b) sizes

    (c) shapes

    of the three objects?

11. How does the size of an object affect its Fourier descriptors? Experiment with a 6×4 rectangle, and then with a 12×8 rectangle. Generalize your findings.

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Chapter 13

Colour processing

For human beings, colour provides one of the most important descriptors of the world around us. The human visual system is particularly attuned to two things: edges, and colour. We have mentioned that the human visual system is not particularly good at recognizing subtle changes in grey values. In this section we shall investigate colour briefly, and then some methods of processing colour images

13.1 What is colour?

Colour study consists of

1. the physical properties of light which give rise to colour,
2. the nature of the human eye and the ways in which it detects colour,
3. the nature of the human vision centre in the brain, and the ways in which messages from the eye are perceived as colour.

Physical aspects of colour

As we have seen in chapter 1, visible light is part of the electromagnetic spectrum. The values for the wavelengths of blue, green and red were set in 1931 by the CIE (Commission Internationale d'Eclairage), an organization responsible for colour standards.

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Perceptual aspects of colour

The human visual system tends to perceive colour as being made up of varying amounts of red, green and blue. That is, human vision is particularly sensitive to these colours; this is a function of the cone cells in the retina of the eye. These values are called the *primary colours*. If we add together any two primary colours we obtain the *secondary colours*:

$$\begin{aligned} \text{magenta (purple)} &= \text{red} + \text{blue}, \\ \text{cyan} &= \text{green} + \text{blue}, \\ \text{yellow} &= \text{red} + \text{green}. \end{aligned}$$

The amounts of red, green, and blue which make up a given colour can be determined by a *colour matching experiment*. In such an experiment, people are asked to match a given colour (a *colour source*) with different amounts of the additive primaries red, green and blue. Such an experiment was performed in 1931 by the CIE, and the results are shown in figure 13.1. Note that for some

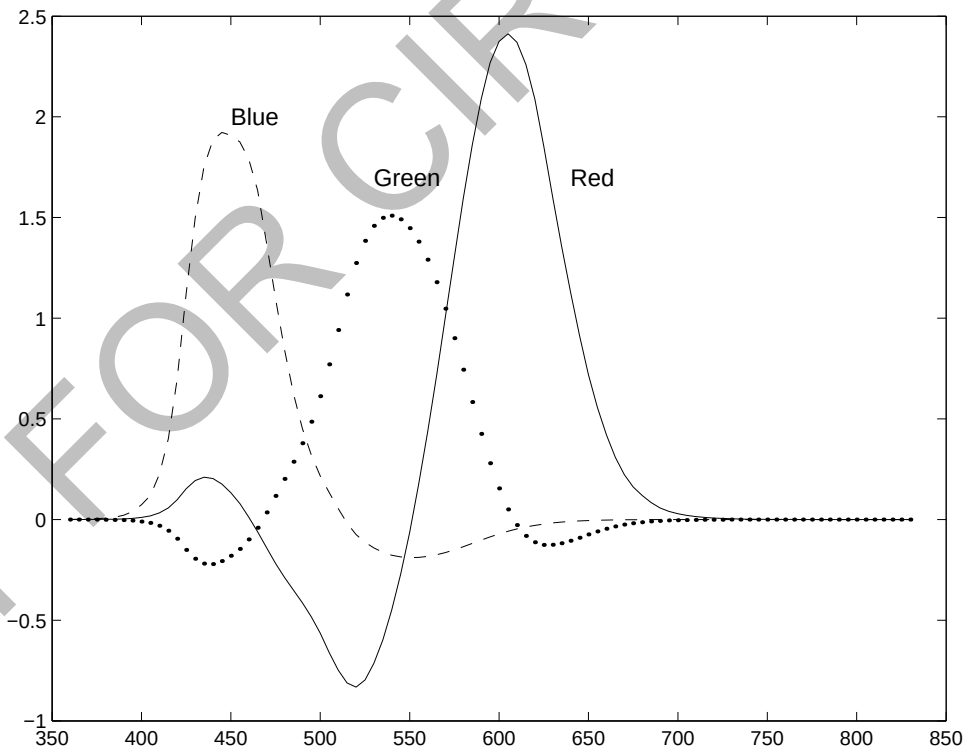

Figure 13.1: RGB colour matching functions (CIE, 1931)

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wavelengths, various of the red, green or blue values are *negative*. This is a physical impossibility, but it can be interpreted by adding the primary beam to the colour source, to maintain a colour match.

To remove negative values from colour information, the CIE introduced the XYZ colour model. The values of X, Y and Z can be obtained from the corresponding R, G and B values by a linear transformation:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0.431 & 0.342 & 0.178 \\ 0.222 & 0.707 & 0.071 \\ 0.020 & 0.130 & 0.939 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}.$$

The inverse transformation is easily obtained by inverting the matrix:

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} 3.063 & -1.393 & -0.476 \\ -0.969 & 1.876 & 0.042 \\ 0.068 & -0.229 & 1.069 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}.$$

The XYZ colour matching functions corresponding to the R, G, B curves of figure 13.1 are shown in figure 13.2. The matrices given are not fixed; other matrices can be defined according to the definition of the colour white. Different definitions of white will lead to different transformation matrices.

The CIE required that the Y component corresponded with *luminance*, or perceived brightness of the colour. That is why the row corresponding to Y in the first matrix (that is, the second row) sums to 1, and also why the Y curve in figure 13.2 is symmetric about the middle of the visible spectrum.

In general, the values of X, Y and Z needed to form any particular colour are called the *tristimulus values*. Values corresponding to particular colours can be obtained from published tables. In order to discuss colour independent of brightness, the tristimulus values can be normalized by dividing by $X + Y + Z$:

$$\begin{aligned} x &= \frac{X}{X+Y+Z} \\ y &= \frac{Y}{X+Y+Z} \\ z &= \frac{Z}{X+Y+Z} \end{aligned}$$

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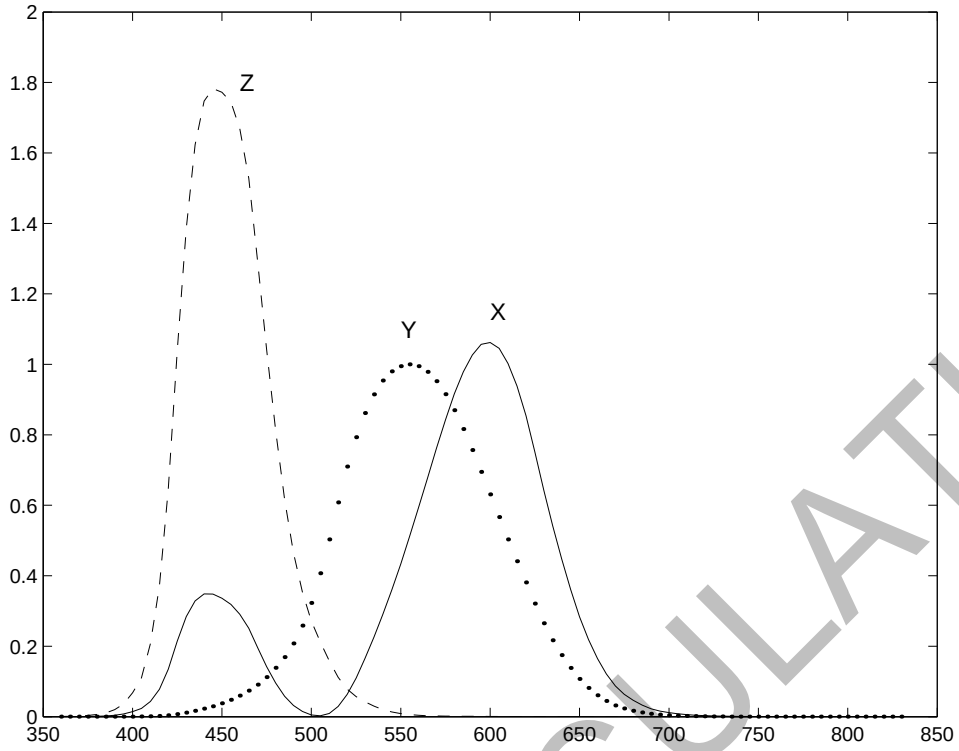

Figure 13.2: XYZ colour matching functions (CIE, 1931)

and so $x + y + z = 1$. Thus a colour can be specified by x and y alone, called the *chromaticity coordinates*. Given x, y, and Y, we can obtain the tristimulus values X and Z by working through the above equations backwards:

$$\begin{aligned} X &= \frac{x}{y}Y \\ Z &= \frac{1-x-y}{y}Y. \end{aligned}$$

We can plot a chromaticity diagram, using the `ciexyz31.txt`[1] file of XYZ values:

```
>> wxyz=load('ciexyz31.txt');
>> xyz=wxyz(:,2:4)';
>> xy=xyz'./(sum(xyz)'*[1 1 1]);
```

[1]This file can be obtained from the *Colour & Vision Research Laboratories* web page `http://www.cvrl.org`.

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```
>> x=xy(:,1)';
>> y=xy(:,2)';
>> figure,plot([x x(1)],[y y(1)]),xlabel('x'),ylabel('y'),axis square
```

Here the matrix `xyz` consists of the second, third and fourth columns of the data, and `plot` is a function which draws a polygon with vertices taken from the `x` and `y` vectors. The extra `x(1)` and `y(1)` ensures that the polygon joins up. The result is shown in figure 13.3. The values of x and y

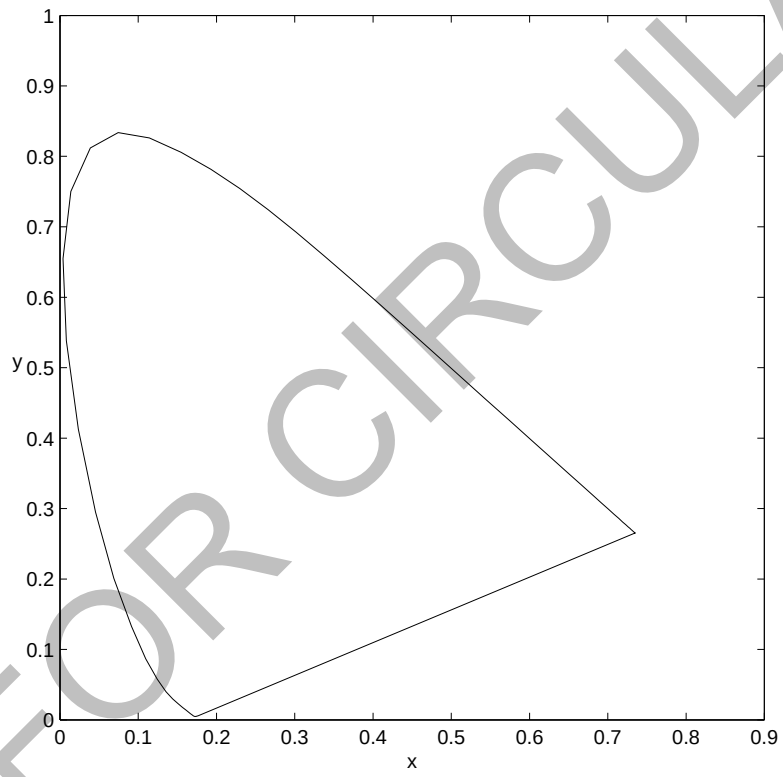

Figure 13.3: A chromaticity diagram

which lie within the horseshoe shape in figure 13.3 represent values which correspond to physically realizable colours. A good account of the XYZ model and associated colour theory can be found in Foley et. al [6].

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13.2 Colour models

A *colour model* is a method for specifying colours in some standard way. It generally consists of a three dimensional coordinate system and a subspace of that system in which each colour is represented by a single point. We shall investigate three systems.

13.2.1 RGB

In this model, each colour is represented as three values R, G and B, indicating the amounts of red, green and blue which make up the colour. This model is used for displays on computer screens; a monitor has three independent electron "guns" for the red, green and blue component of each colour. We have met this model in chapter 2.

Note also from figure 13.1 that some colours require negative values of R, G or B. These colours are not realizable on a computer monitor or TV set, on which only positive values are possible. The colours corresponding to positive values form the *RGB gamut*; in general a colour "gamut" consists of all the colours realizable with a particular colour model. We can plot the RGB gamut on a chromaticity diagram, using the `xy` coordinates obtained above. To define the gamut, we shall create a $100 \times 100 \times 3$ array, and to each point (i, j) in the array, associate an XYZ triple defined by $(i/100, j/100, 1 - i/100 - j/100)$. We can then compute the corresponding RGB triple, and if any of the RGB values are negative, make the output value white. This is easily done with the simple function shown in figure 13.4.

```
function res=gamut()

global cg;
x2r=[3.063 -1.393 -0.476;-0.969 1.876 0.042;0.068 -0.229 1.069];
cg=zeros(100,100,3);
for i=1:100,
 for j=1:100,
   cg(i,j,:)=x2r*[j/100 i/100 1-i/100-j/100]';
   if min(cg(i,j,:))<0,
     cg(i,j,:)=[1 1 1];
   end;
 end;
end;
res=cg;
```

Figure 13.4: Computing the RGB gamut

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We can then display the gamut inside the chromaticity figure by

```
>> imshow(cG),line([x' x(1)],[y' y(1)]),axis square,axis xy,axis on
```

and the result is shown in figure 13.5.

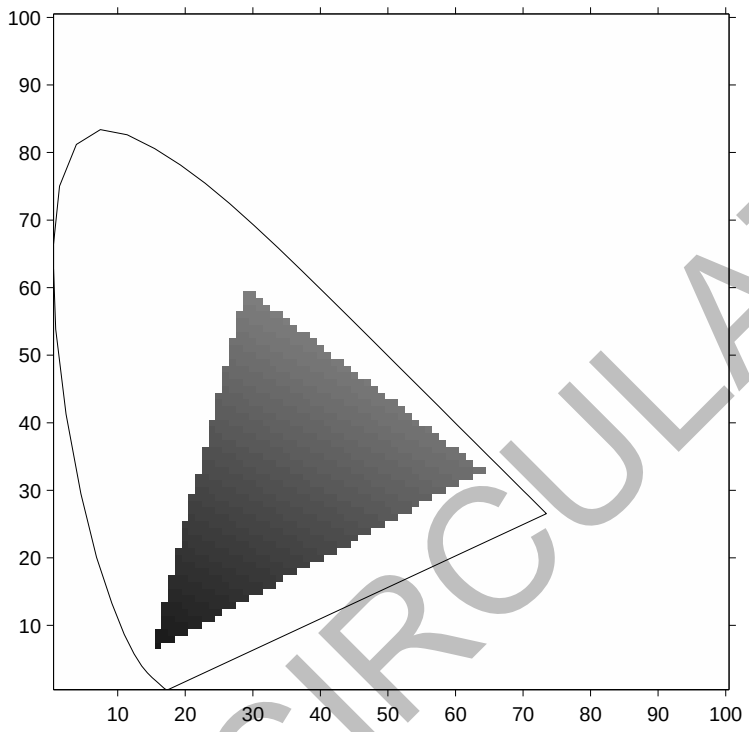

Figure 13.5: The RGB gamut

13.2.2 HSV

HSV stands for Hue, Saturation, Value. These terms have the following meanings:

Hue: The "true colour" attribute (red, green, blue, orange, yellow, and so on).

Saturation: The amount by which the colour as been diluted with white. The more white in the colour, the lower the saturation. So a deep red has high saturation, and a light red (a pinkish colour) has low saturation.

Value: The degree of brightness: a well lit colour has high intensity; a dark colour has low intensity.

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This is a more intuitive method of describing colours, and as the intensity is independent of the colour information, this is a very useful model for image processing. We can visualize this model as a cone, as shown in figure 13.6.

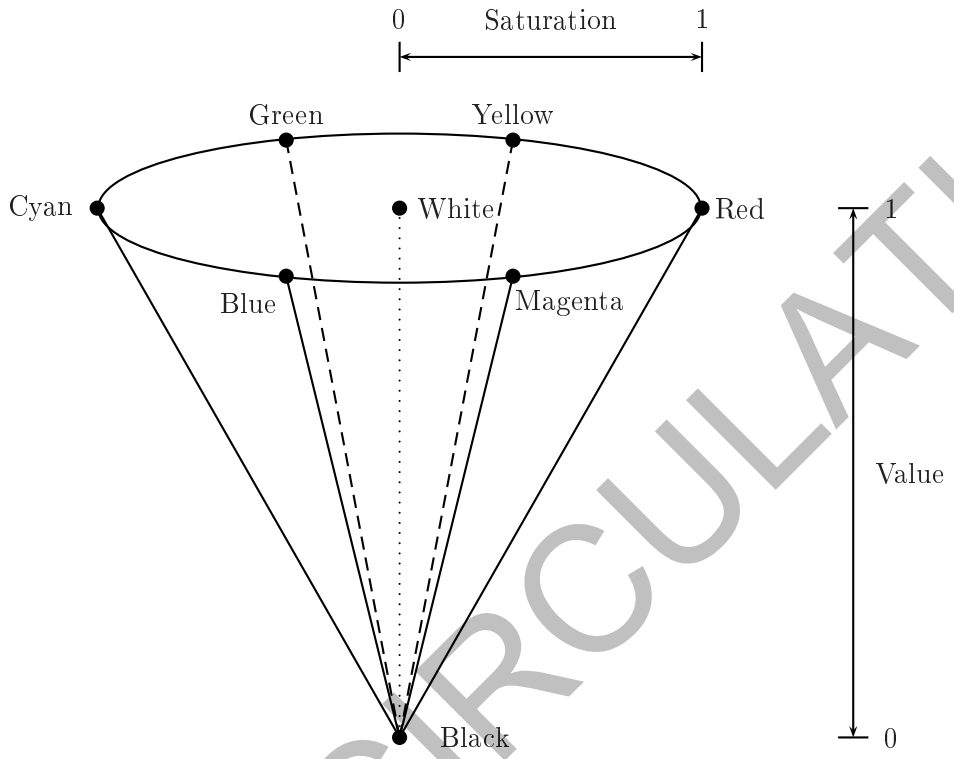

Figure 13.6: The colour space HSV as a cone

Any point on the surface represents a purely saturated colour. The saturation is thus given as the relative distance to the surface from the central axis of the structure. Hue is defined to be the angle measurement from a pre-determined axis, say red.

13.2.3 Conversion between RGB and HSV

Suppose a colour is specified by its RGB values. If all the three values are equal, then the colour will be a grey scale; that is, an intensity of white. Such a colour, containing just white, will thus have a saturation of zero. Conversely, if the RGB values are very different, we would expect the resulting colour to have a high saturation. In particular, if one or two of the RGB values are zero, the saturation will be one, the highest possible value.

Hue is defined as the fraction around the circle starting from red, which thus has a hue of zero.

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Reading around the circle in figure 13.6 produces the following hues:

Colour	Hue
Red	0
Yellow	0.1667
Green	0.3333
Cyan	0.5
Blue	0.6667
Magenta	0.8333

Suppose we are given three R, G, B values, which we suppose to be between 0 and 1. So if they are between 0 and 255, we first divide each value by 255. We then define:

$$\begin{aligned} V &= \max\{R, G, B\} \\ \delta &= V - \min\{R, G, B\} \\ S &= \frac{\delta}{V} \end{aligned}$$

To obtain a value for Hue, we consider several cases:

1. if $R = V$ then $H = \frac{1}{6}\frac{G-B}{\delta}$,
2. if $G = V$ then $H = \frac{1}{6}\left(2 + \frac{B-R}{\delta}\right)$,
3. if $B = V$ then $H = \frac{1}{6}\left(4 + \frac{R-G}{\delta}\right)$.

If H ends up with a negative value, we add 1. In the particular case $(R, G, B) = (0, 0, 0)$, for which both $V = \delta = 0$, we define $(H, S, V) = (0, 0, 0)$.

For example, suppose $(R, G, B) = (0.2, 0.4, 0.6)$ We have

$$\begin{aligned} V &= \max\{0.2, 0.4, 0.6\} = 0.6 \\ \delta &= V - \min\{0.2, 0.4, 0.6\} = 0.6 - 0.2 = 0.4 \\ S &= \frac{0.4}{0.6} = 0.6667 \end{aligned}$$

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Since $B = G$ we have

$$H = \frac{1}{6}\left(4 + \frac{0.2 - 0.4}{0.4}\right) = 0.5833.$$

Conversion in this direction is implemented by the `rgb2hsv` function. This is of course designed to be used on $m \times n \times 3$ arrays, but let's just experiment with our previous example:

```
>> rgb2hsv([0.2 0.4 0.6])


ans =


    0.5833    0.6667    0.6000
```

and these are indeed the H, S and V values we have just calculated.

To go the other way, we start by defining:

$$\begin{aligned} H' &= \lfloor 6H \rfloor \\ F &= 6H - H' \\ P &= V(1 - S) \\ Q &= V(1 - SF) \\ T &= V(1 - S(1 - F)) \end{aligned}$$

Since H' is a integer between 0 and 5, we have six cases to consider:

H'	R	G	B
0	V	T	P
1	Q	V	P
2	P	V	T
3	P	Q	V
4	T	P	V
5	V	P	Q

Let's take the HSV values we computed above. We have:

$$H' = \lfloor 6(0.5833) \rfloor = 3$$

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$$
\begin{aligned}
F &= 6(0.5833) - 3 = 0.5 \\
P &= 0.6(1 - 0.6667) = 0.2 \\
Q &= 0.6(1 - (0.6667)(0.5)) = 0.4 \\
T &= 0.6(1 - 0.6667(1 - 0.5)) = 0.4
\end{aligned}
$$

Since $H' = 3$ we have

$$(R, G, B) = (P, Q, V) = (0.2, 0.4, 0.6).$$

Conversion from HSV to RGB is implemented by the `hsv2rgb` function.

13.2.4 YIQ

This colour space is used for TV/video in America and other countries where NTSC is the video standard (Australia uses PAL). In this scheme Y is the "luminance" (this corresponds roughly with intensity), and I and Q carry the colour information. The conversion between RGB is straightforward:

$$
\begin{bmatrix} Y \\ I \\ Q \end{bmatrix} = \begin{bmatrix} 0.299 & 0.587 & 0.114 \\ 0.596 & -0.274 & -0.322 \\ 0.211 & -0.523 & 0.312 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}
$$

and

$$
\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} 1.000 & 0.956 & 0.621 \\ 1.000 & -0.272 & -0.647 \\ 1.000 & -1.106 & 1.703 \end{bmatrix} \begin{bmatrix} Y \\ I \\ Q \end{bmatrix}
$$

The two conversion matrices are of course inverses of each other. Note the difference between Y and V:

$$
\begin{aligned}
Y &= 0.299R + 0.587G + 0.114B \\
V &= \max\{R, G, B\}.
\end{aligned}
$$

This reflects the fact that the human visual system assigns more intensity to the green component of an image than to the red and blue components. We note here that other transformations [7]

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$RGB \leftrightarrow HSV$ have

$$V = 0.333R + 0.333G + 0.333B$$

where the intensity is a simple average of the primary values. Note also that the Y of YIQ is different to the Y of XYZ, with the similarity that both represent luminance.

Since YIQ is a linear transformation of RGB, we can picture YIQ to be a parallelepiped (a rectangular box which has been skewed in each direction) for which the Y axis lies along the central $(0, 0, 0)$ to $(1, 1, 1)$ line of RGB. Figure 13.7 shows this.

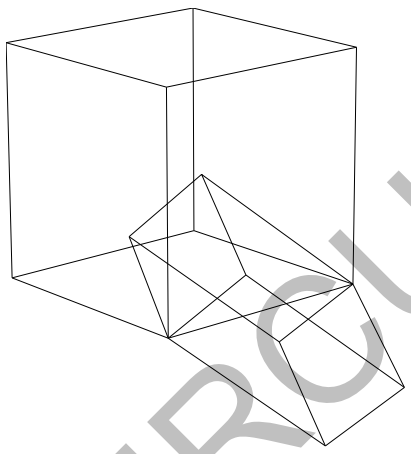

Figure 13.7: The RGB cube and its YIQ transformation

That the conversions are linear, and hence easy to do, makes this a good choice for colour image processing. Conversion between RGB and YIQ are implemented with the MATLAB functions `rgb2ntsc` and `ntsc2rgb`.

13.3 Colour images in MATLAB

Since a colour image requires three separate items of information for each pixel, a (true) colour image of size $m \times n$ is represented in MATLAB by an array of size $m \times n \times 3$: a three dimensional array. We can think of such an array as a single entity consisting of three separate matrices aligned vertically. Figure 13.8 shows a diagram illustrating this idea. Suppose we read in an RGB image:

```
>> x=imread('lily.tif');
>> size(x)
```

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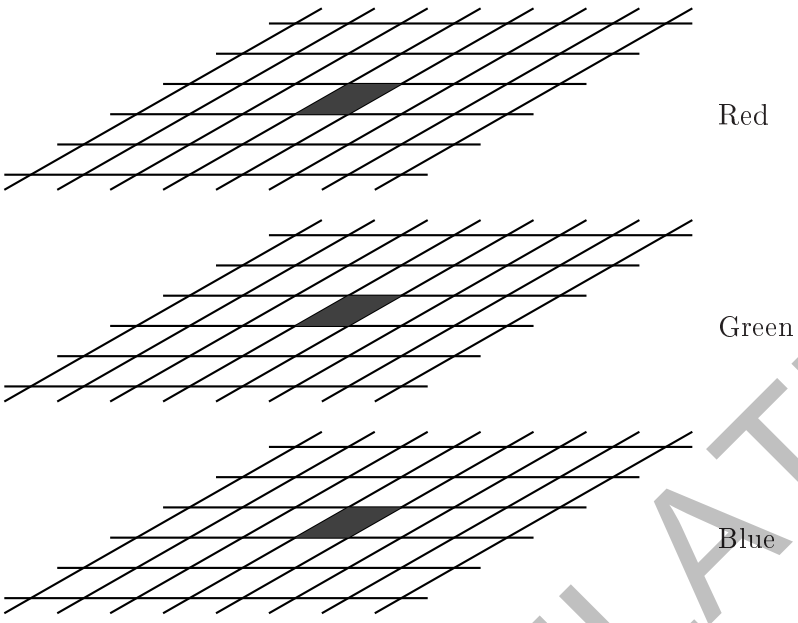

Figure 13.8: A three dimensional array for an RGB image

```
ans =

   186   230     3
```

We can isolate each colour component by the colon operator:

`x(:,:,1)`	The first, or red component
`x(:,:,2)`	The second, or green component
`x(:,:,3)`	The third, or blue component

These can all be viewed with `imshow`:

```
>> imshow(x)
>> figure,imshow(x(:,:,1))
>> figure,imshow(x(:,:,1))
>> figure,imshow(x(:,:,2))
```

These are all shown in figure 13.9. Notice how the colours with particular hues show up with high

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A colour image | Red component | Green component | Blue component

Figure 13.9: An RGB colour image and its components

intensities in their respective components. For the rose in the top right, and the flower in the bottom left, both of which are predominantly red, the red component shows a very high intensity for these two flowers. The green and blue components show much lower intensities. Similarly the green leaves—at the top left and bottom right—show up with higher intensity in the green component than the other two.

We can convert to YIQ or HSV and view the components again:

```
>> xh=rgb2hsv(x);
>> imshow(xh(:,:,1))
>> figure,imshow(xh(:,:,2))
>> figure,imshow(xh(:,:,3))
```

and these are shown in figure 13.10. We can do precisely the same thing for the YIQ colour space:

Hue | Saturation | Value

Figure 13.10: The HSV components

```
>> xn=rgb2ntsc(x);
>> imshow(xn(:,:,1))
```

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```
>> figure,imshow(xn(:,:,2))
>> figure,imshow(xn(:,:,3))
```

and these are shown in figure 13.11. Notice that the Y component of YIQ gives a better greyscale

Y I Q

Figure 13.11: The YIQ components

version of the image than the value of HSV. The top right rose, in particular, is quite washed out in figure 13.10 (Value), but shows better contrast in figure 13.11 (Y).

We shall see below how to put three matrices, obtained by operations on the separate components, back into a single three dimensional array for display.

13.4 Pseudocolouring

This means assigning colours to a grey-scale image in order to make certain aspects of the image more amenable for visual interpretation—for example, for medical images. There are different methods of pseudocolouring.

13.4.1 Intensity slicing

In this method, we break up the image into various grey level ranges. We simply assign a different colour to each range. For example:

grey level:	0–63	64–127	128–191	192–255
colour:	blue	magenta	green	red

We can consider this as a mapping, as shown in figure 13.12.

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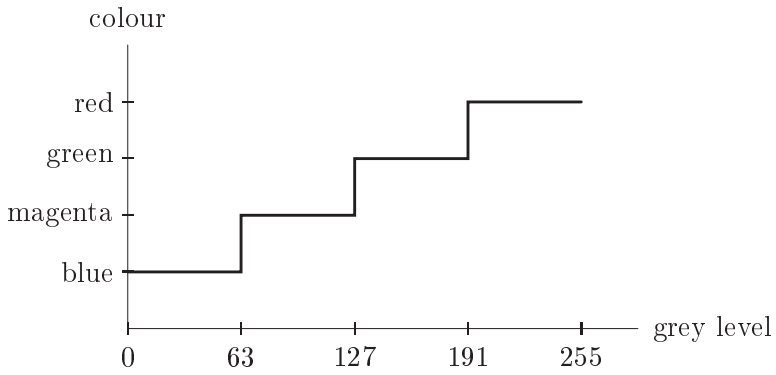
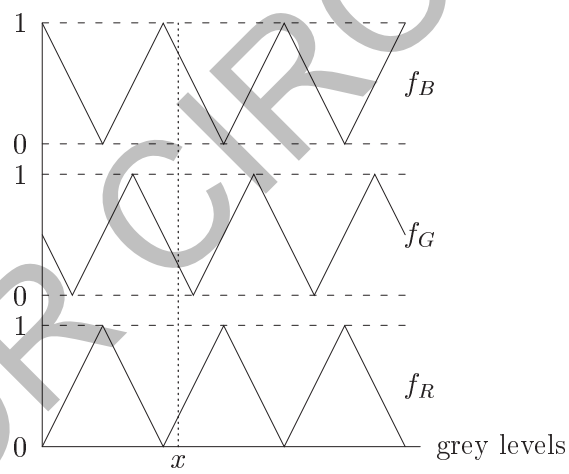


Figure 13.12: Intensity slicing as a mapping

13.4.2 Grey—Colour transformations

We have three functions $f_R(x)$, $f_G(x)$, $f_B(x)$ which assign red, green and blue values to each grey level x. These values (with appropriate scaling, if necessary) are then used for display. Using an appropriate set of functions can enhance a grey-scale image with impressive results.



The grey level x in the diagram is mapped onto red, green and blue values of 0.375, 0.125 and 0.75 respectively.

In MATLAB, a simple way to view an image with added colour is to use `imshow` with an extra `colormap` parameter. For example, consider the image `blocks.tif`. We can add a colour map with the `colormap` function; there are several existing colour maps to choose from. Figure 13.13 shows the children's blocks image (from figure 1.4) after colour transformations. We created the colour image (a) with:

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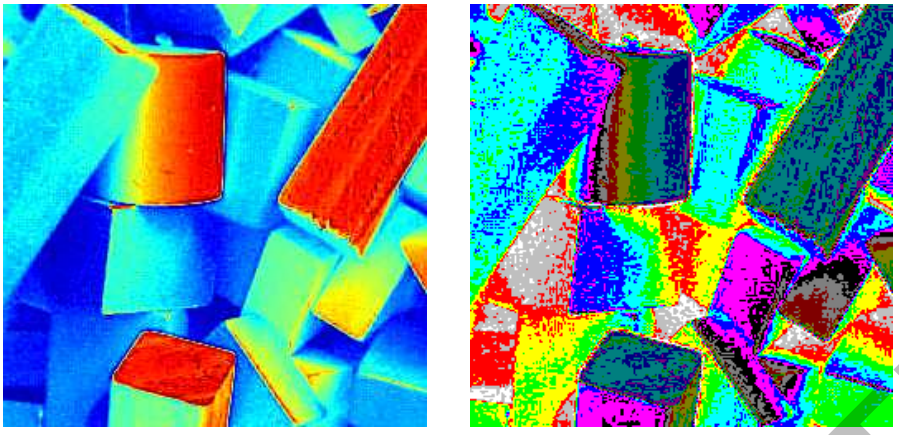

(a) (b)

Figure 13.13: Applying a colour map to a greyscale image

```
>> b=imread('blocks.tif');
>> imshow(b,colormap(jet(256))
```

However, a bad choice of colour map can ruin an image. Image (b) in figure 13.13 is an example of this, where we apply the `vga` colour map. Since this only has 16 rows, we need to reduce the number of greyscales in the image to 16. This is done with the `grayslice` function:

```
>> b16=grayslice(b,16);
>> figure,imshow(b16,colormap(vga))
```

The result, although undeniably colourful, is not really an improvement on the original image. The available colour maps are listed in the help file for `graph3d`:

```
hsv       - Hue-saturation-value color map.
hot       - Black-red-yellow-white color map.
gray      - Linear gray-scale color map.
bone      - Gray-scale with tinge of blue color map.
copper    - Linear copper-tone color map.
pink      - Pastel shades of pink color map.
white     - All white color map.
flag      - Alternating red, white, blue, and black color map.
```

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```
lines      - Color map with the line colors.
colorcube  - Enhanced color-cube color map.
vga        - Windows colormap for 16 colors.
jet        - Variant of HSV.
prism      - Prism color map.
cool       - Shades of cyan and magenta color map.
autumn     - Shades of red and yellow color map.
spring     - Shades of magenta and yellow color map.
winter     - Shades of blue and green color map.
summer     - Shades of green and yellow color map.
```

There are help files for each of these colour maps, so that

```
>> help hsv
```

will provide some information on the `hsv` colour map.

We can easily create our own colour map: it must by a matrix with 3 columns, and each row consists of RGB values between 0.0 and 1.0. Suppose we wish to create a blue, magenta, green, red colour map as shown in figure 13.12. Using the RGB values:

Colour	Red	Green	blue
Blue	0	0	1
Magenta	1	0	1
Green	0	1	0
Red	1	0	0

we can create our colour map with:

```
>> mycolourmap=[0 0 1;1 0 1;0 1 0;1 0 0];
```

Before we apply it to the blocks image, we need to scale the image down so that there are only the four greyscales 0, 1, 2 and 3:

```
>> b4=grayslice(b,4);
>> imshow(b4,mycolourmap)
```

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and the result is shown in figure 13.14.

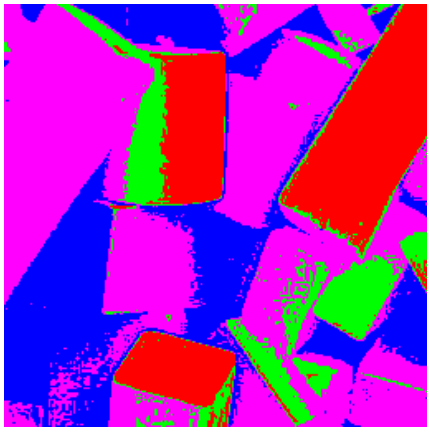

Figure 13.14: An image coloured with a "handmade" colour map

13.5 Processing of colour images

There are two methods we can use:

1. we can process each R, G, B matrix separately,
2. we can transform the colour space to one in which the intensity is separated from the colour, and process the intensity component only.

Schemas for these are given in figures 13.15 and 13.16.

We shall consider a number of different image processing tasks, and apply either of the above schema to colour images.

Contrast enhancement

This is best done by processing the intensity component. Suppose we start with the image `cat.tif`, which is an indexed colour image, and convert it to a truecolour (RGB) image.

```
>> [x,map]=imread('cat.tif');
>> c=ind2rgb(x,map);
```

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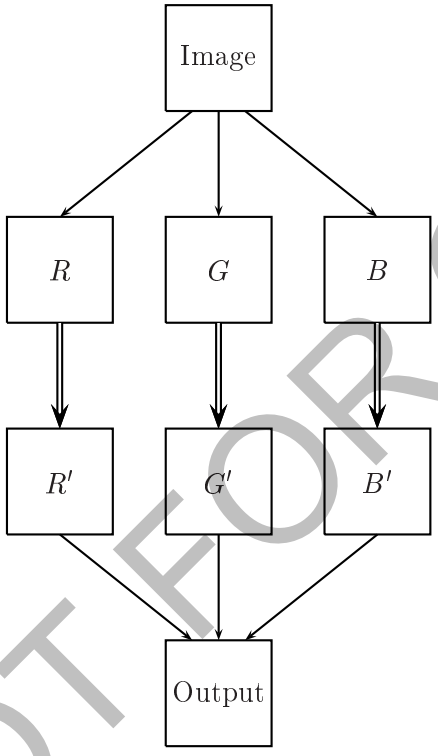

Figure 13.15: RGB processing

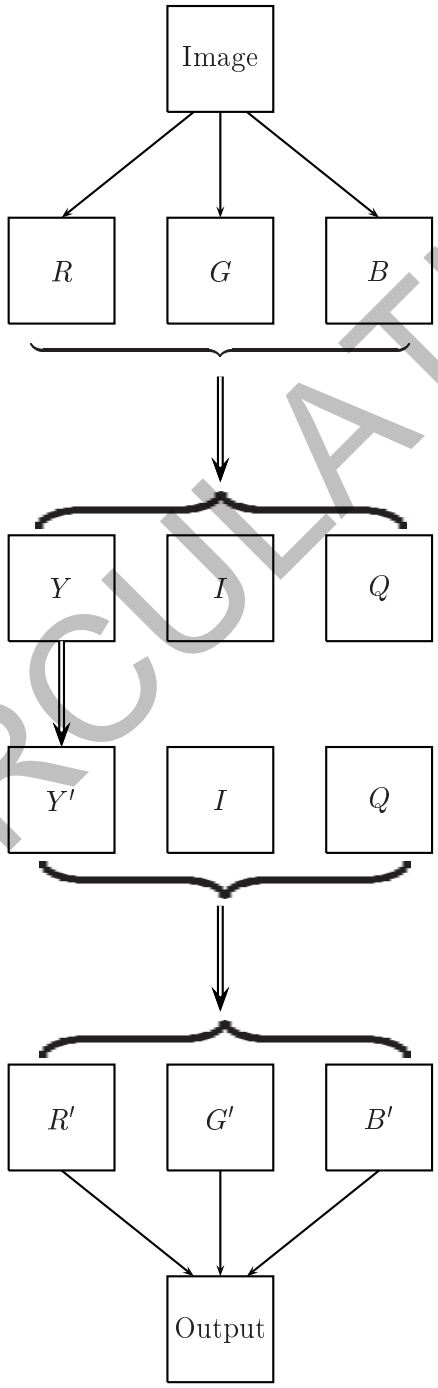

Figure 13.16: Intensity processing

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Now we have to convert from RGB to YIQ, so as to be able to isolate the intensity component:

```
>> cn=rgb2ntsc(c);
```

Now we apply histogram equalization to the intensity component, and convert back to RGB for display:

```
>> cn(:,:,1)=histeq(cn(:,:,1));
>> c2=ntsc2rgb(cn);
>> imshow(c2)
```

The result is shown in figure 13.17. Whether this is an improvement is debatable, but it has had its contrast enhanced.

But suppose we try to apply histogram equalization to each of the RGB components:

```
>> cr=histeq(c(:,:,1));
>> cg=histeq(c(:,:,2));
>> cb=histeq(c(:,:,3));
```

Now we have to put them all back into a single 3 dimensional array for use with `imshow`. The `cat` function is what we want:

```
>> c3=cat(3,cr,cg,cb);
>> imshow(c3)
```

The first variable to `cat` is the dimension along which we want our arrays to be joined. The result is shown for comparison in figure 13.17. This is not acceptable, as some strange colours have been introduced; the cat's fur has developed a sort of purplish tint, and the grass colour is somewhat washed out.

Spatial filtering

It very much depends on the filter as to which schema we use. For a low pass filter, say a blurring filter, we can apply the filter to each RGB component:

```
>> a15=fspecial('average',15);
```

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Intensity processing

Using each RGB component

Figure 13.17: Histogram equalization of a colour image

```
>> cr=filter2(a15,c(:,:,1));
>> cg=filter2(a15,c(:,:,2));
>> cb=filter2(a15,c(:,:,3));
>> blur=cat(3,cr,cg,cb);
>> imshow(blur)
```

and the result is shown in figure 13.18. We could also obtain a similar effect by applying the filter to the intensity component only. But for a high pass filter, for example an unsharp masking filter, we are better off working with the intensity component only:

```
>> cn=rgb2ntsc(c);
>> a=fspecial('unsharp');
>> cn(:,:,1)=filter2(a,cn(:,:,1));
>> cu=ntsc2rgb(cn);
>> imshow(cu)
```

and the result is shown in figure 13.18. In general, we will obtain reasonable results using the intensity component only. Although we can sometimes apply a filter to each of the RGB components, as we did for the blurring example above, we cannot be guaranteed a good result. The problem is that any filter will change the values of the pixels, and this may introduce unwanted colours.

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Low pass filtering

High pass filtering

Figure 13.18: Spatial filtering of a colour image

Noise reduction

As we did in chapter 8, we shall use the image `twins.tif`: but now in full colour!

```
>> tw=imread('twins.tif');
```

Now we can add noise, and look at the noisy image, and its RGB components:

```
>> tn=imnoise(tw,'salt & pepper');
>> imshow(tn)
>> figure,imshow(tn(:,:,1))
>> figure,imshow(tn(:,:,2))
>> figure,imshow(tn(:,:,3))
```

These are all shown in figure 13.19. It would appear that we should apply median filtering to each of the RGB components. This is easily done:

```
>> trm=medfilt2(tn(:,:,1));
>> tgm=medfilt2(tn(:,:,2));
>> tbm=medfilt2(tn(:,:,3));
>> tm=cat(3,trm,tgm,tbm);
>> imshow(tm)
```

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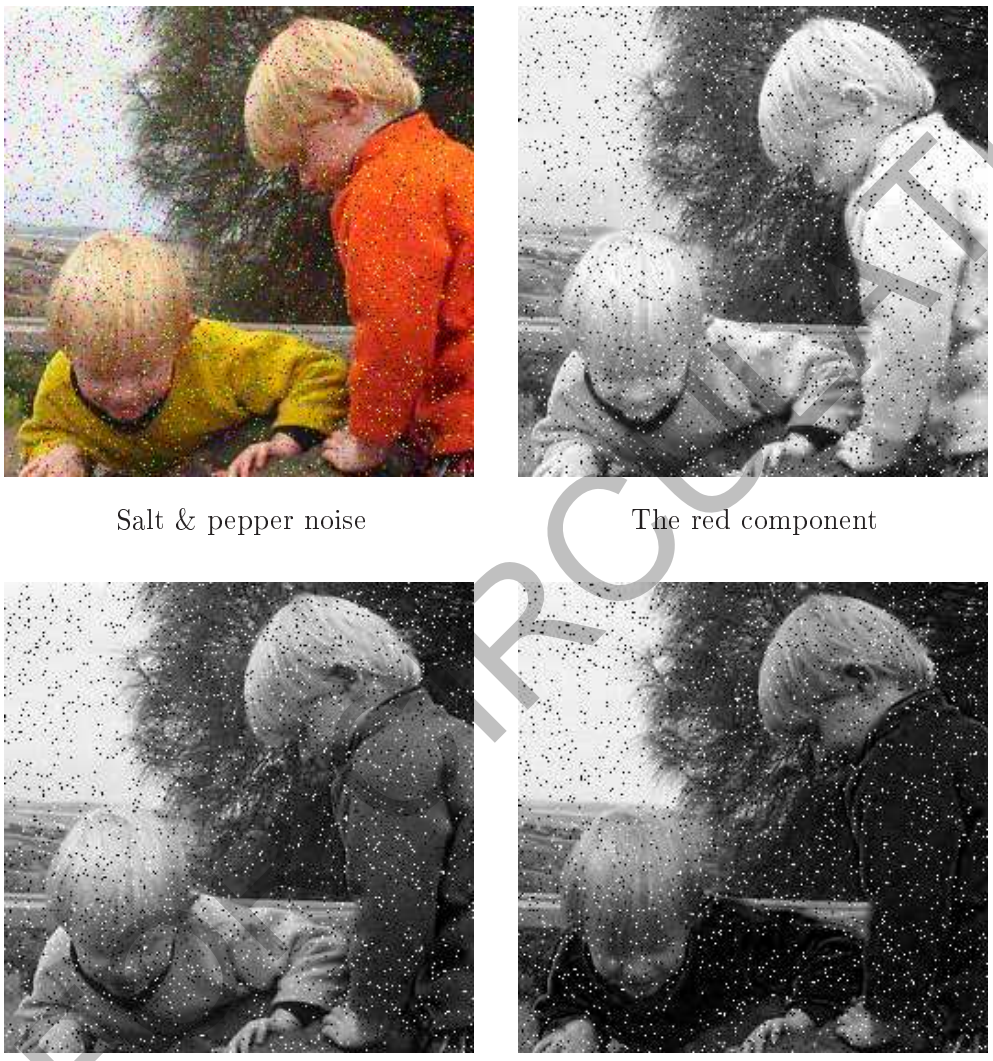

Salt & pepper noise

The red component

The green component

The blue component

Figure 13.19: Noise on a colour image

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and the result is shown in figure 13.20. We can't in this instance apply the median filter to the intensity component only, because the conversion from RGB to YIQ spreads the noise across all the YIQ components. If we remove the noise from Y only:

```
>> tnn=rgb2ntsc(tn);
>> tnn(:,:,1)=medfilt2(tnn(:,:,1));
>> tm2=ntsc2rgb(tnn);
>> imshow(tm2)
```

then the noise has been slightly diminished as shown in figure 13.20, but it is still there. If the noise

Denoising each RGB component Denoising Y only

Figure 13.20: Attempts at denoising a colour image

applies to only one of the RGB components, then it would be appropriate to apply a denoising technique to this component only.

Also note that the method of noise removal must depend on the generation of noise. In the above example we tacitly assumed that the noise was generated after the image had been acquired and stored as RGB components. But as noise can arise anywhere in the image acquisition process, it is quite reasonable to assume that noise might affect only the brightness of the image. In such a case denoising the Y component of YIQ will produce the best results.

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Edge detection

An edge image will be a binary image containing the edges of the input. We can go about obtaining an edge image in two ways:

1. we can take the intensity component only, and apply the `edge` function to it,

2. we can apply the `edge` function to each of the RGB components, and join the results.

To implement the first method, we start with the `rgb2gray` function:

```
>> fg=rgb2gray(f);
>> fe1=edge(fg);
>> imshow(fe1)
```

Recall that `edge` with no parameters implements Sobel edge detection. The result is shown in figure 13.21. For the second method, we can join the results with the logical "or":

```
>> f1=edge(f(:,:,1));
>> f2=edge(f(:,:,2));
>> f3=edge(f(:,:,3));
>> fe2=f1 | f2 | f3;
>> figure,imshow(fe2)
```

and this is also shown in figure 13.21. The edge image `fe2` is a much more complete edge image. Notice that the rose now has most of its edges, where in image `fe1` only a few were shown. Also note that there are the edges of some leaves in the bottom left of `fe2` which are completely missing from `fe1`. The success of these methods will also depend on the parameters of the edge function chosen; for example the threshold value used. In the examples shown, the `edge` function has been used with its default threshold.

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`fe1`: Edges after `rgb2gray`

`fe2`: Edges of each RGB component

Figure 13.21: The edges of a colour image

Exercises

1. By hand, determine the saturation and intensity components of the following image, where the RGB values are as given:

$(0,1,1)$	$(1,2,3)$	$(7,7,7)$	$(5,1,2)$	$(1,1,7)$
$(2,1,2)$	$(1,7,7)$	$(2,0,2)$	$(3,3,2)$	$(5,5,0)$
$(4,4,4)$	$(4,6,7)$	$(4,5,6)$	$(1,5,7)$	$(3,6,7)$
$(3,0,3)$	$(5,2,2)$	$(1,1,1,$	$(6,6,0)$	$(2,2,2)$
$(1,2,1)$	$(0,4,4)$	$(3,1,6)$	$(3,3,3)$	$(2,4,6)$

2. Suppose the intensity component of an HSV image was thresholded to just two values. How would this affect the appearance of the image?

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3. By hand, perform the conversions between RGB and HSV or YIQ, for the values:

R	G	B	H	S	V
0.5	0.5	0			
0	0.7	0.7			
0.5	0	0.5			
			0.33	0.5	1
			0.67	0.7	0.7
			0	0.2	0.8

R	G	B	Y	I	Q
0.3	0.3	0.7			
0.7	0.9	0			
0.8	0.8	0.7			
			1	0.3	0.3
			0.5	0.5	0.5
			0	1	1

You may need to normalize the RGB values.

4. Check your answers to the conversions in question 3 by using the MATLAB functions `rgb2hsv`, `hsv2rgb`, `rgb2ntsc` and `ntsc2rgb`.

5. Threshold the intensity component of a colour image, say `flowers.tif`, and see if the result agrees with your guess from question 2 above.

6. The image `spine.tif` is an indexed colour image; however the colours are all very close to shades of grey. Experiment with using `imshow` on the index matrix of this image, with varying colour maps of length 64.

   Which colour map seems to give the best results? Which colour map seems to give the worst results?

7. View the image `autumn.tif`. Experiment with histogram equalization on:

   (a) the intensity component of HSV,

   (b) the intensity component of YIQ.

   Which seems to produce the best result?

8. Create and view a random "patchwork quilt" with:

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```
>> r=uint8(floor(256*rand(16,16,3)));
>> r=imresize(r,16);
>> imshow(r),pixval on
```

What RGB values produce (a) a light brown colour? (b) a dark brown colour?

Convert these brown values to HSV, and plot the hues on a circle.

9. Using the flowers image, see if you can obtain an edge image from the intensity component alone, that is as close as possible to the image `fe2` in figure 13.21. What parameters to the `edge` function did you use? How close to `fe2` could you get?

10. Add Gaussian noise to an RGB colour image `x` with

```
>> xn=imnoise(x,'gaussian');
```

View your image, and attempt to remove the noise with

(a) average filtering on each RGB component,

(b) Wiener filtering on each RGB component.

11. Take the twins image and add salt & pepper noise to the intensity component. This can be done with

```
>> ty=rgb2ntsc(tw);
>> tn=imnoise(ty(:,:,1).'salt & pepper');
>> ty(:,:,1)=tn;
```

Now convert back to RGB for display.

(a) Compare the appearance of this noise with salt & pepper noise applied to each RGB component as shown in figure 13.19. Is there any observable difference?

(b) Denoise the image by applying a median filter to the intensity component.

(c) Now apply the median filter to each of the RGB components.

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(d) Which one gives the best results?

(e) Experiment with larger amounts of noise.

(f) Experiment with Gaussian noise.

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Chapter 14

Image coding and compression

14.1 Lossless and lossy compression

We have seen that image files can be very large. It is thus important for reasons both of storage and file transfer to make these file sizes smaller, if possible. In section 1.9 we touched briefly on the topic of compression; in this section we investigate some standard compression methods. It will be necessary to distinguish between two different classes of compression methods: *lossless compression*, where all the information is retained, and *lossy compression* where some information is lost.

Lossless compression is preferred for images of legal, scientific or political significance, where loss of data, even of apparent insignificance, could have considerable consequences. Unfortunately this style tends not to lead to high compression ratios. However, lossless compression is used as part of many standard image formats.

14.2 Huffman coding

The idea of Huffman coding is simple. Rather than using a fixed length code (8 bits) to represent the grey values in an image, we use a variable length code, with smaller length codes corresponding to more probable grey values.

A small example will make this clear. Suppose we have a 2-bit greyscale image with only four grey levels: 0, 1, 2, 3, with the probabilities 0.2, 0.4, 0.3 and 0.1 respectively. That is, 20% of pixels in the image have grey value 50; 40% have grey value 100, and so on. The following table shows

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fixed length and variable length codes for this image:

Grey value	Probability	Fixed code	Variable code
0	0.2	00	000
1	0.4	01	1
2	0.3	10	01
3	0.1	11	001

Now consider how this image has been compressed. Each grey value has its own unique identifying code. The average number of bits per pixel can be easily calculated as the expected value (in a probabilistic sense):

$$(0.2 \times 3) + (0.4 \times 1) + (0.3 \times 2) + (0.1 \times 3) = 1.9.$$

Notice that the longest codewords are associated with the lowest probabilities. This average is indeed smaller than 2.

This can be made more precise by the notion of *entropy*, which is a measure of the amount of information. Specifically, the entropy H of an image is the theoretical minimum number of bits per pixel required to encode the image with no loss of information. It is defined by

$$H = -\sum_{i=0}^{L-1} p_i \log_2(p_i)$$

where the index i is taken over all greyscales of the image, and p_i is the probability of grey level i occurring in the image. Very good accounts of the basics of information theory and entropy are given by Roman [26] and Welsh [36]. In the example given above,

$$H = -\Big(0.2\log_2(0.2) + 0.4\log_2(0.4) + 0.3\log_2(0.3) + 0.1\log_2(0.1)\Big) = 1.8464.$$

This means that no matter what coding scheme is used, it will never use less than 1.8464 bits per pixel. On this basis, the Huffman coding scheme given above, giving an average number of bits per pixel much closer to this theoretical minimum than 2, provides a very good result.

To obtain the Huffman code for a given image we proceed as follows:

1. Determine the probabilities of each grey value in the image.

2. Form a binary tree by adding probabilities two at a time, always taking the two lowest available values.

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3. Now assign 0 and 1 arbitrarily to each branch of the tree from its apex.

4. Read the codes from the top down.

To see how this works, consider the example of a 3-bit greyscale image (so the grey values are 0–7) with the following probabilities:

Grey value	0	1	2	3	4	5	6	7
Probability	0.19	0.25	0.21	0.16	0.08	0.06	0.03	0.02

For these probabilities, the entropy can be calculated to be 2.6508. We can now combine probabilities two at a time as shown in figure 14.1.

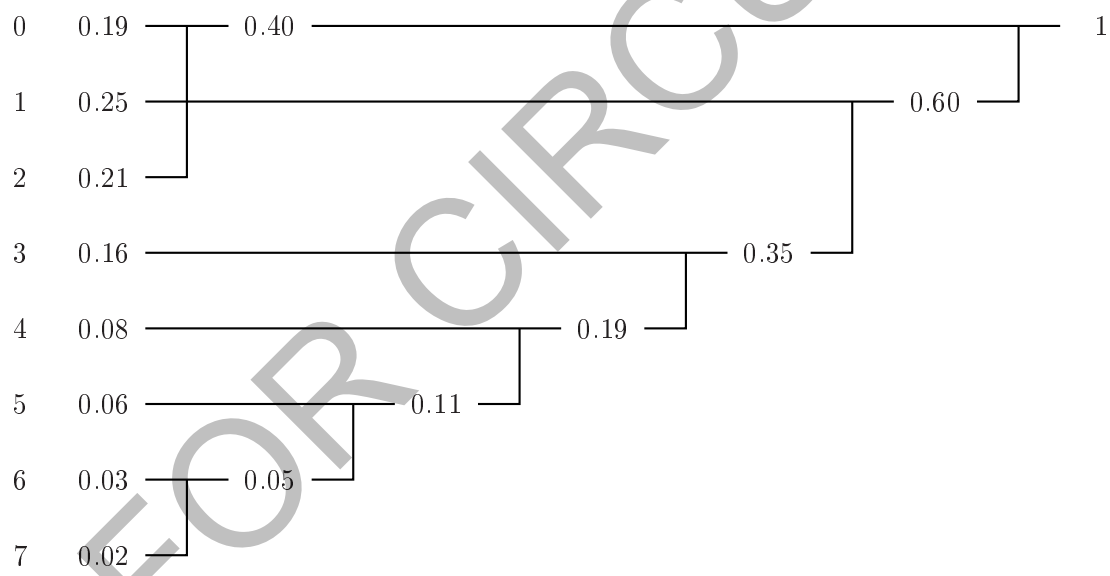

Figure 14.1: Forming the Huffman code tree

Note that if we have a choice of probabilities we choose arbitrarily. The second stage consists of arbitrarily assigning 0's and 1's to each branch of the tree just obtained. This is shown in figure 14.2.

To obtain the codes for each grey value, start at the 1 on the top right, and work back towards the grey value in question, listing the numbers passed on the way. This produces:

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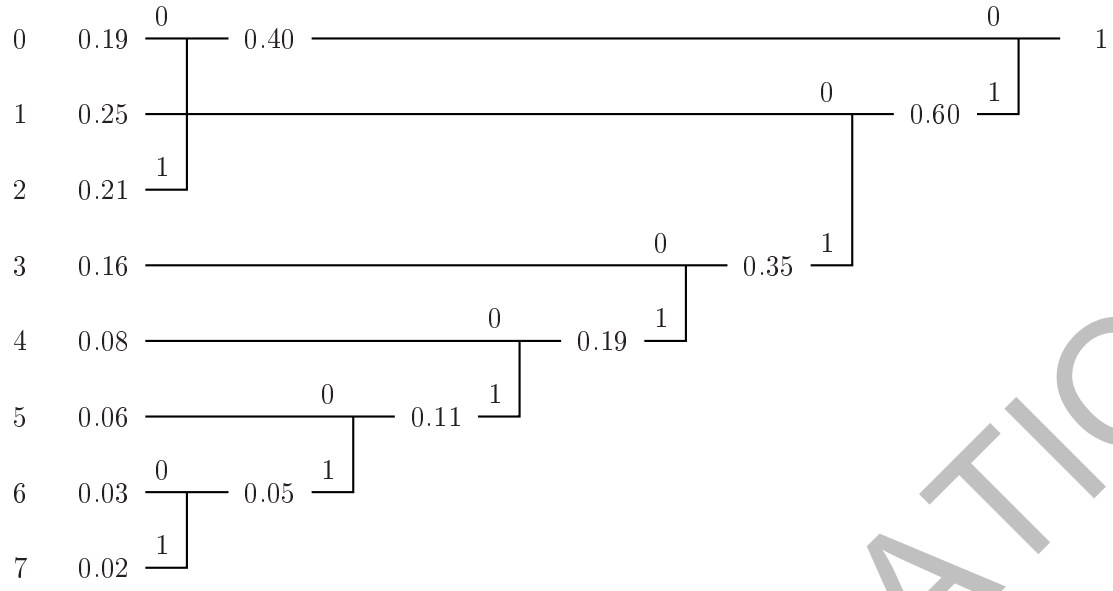

Figure 14.2: Assigning 0's and 1's to the branches

Grey value	Huffman code
0	00
1	10
2	01
3	110
4	1110
5	11110
6	111110
7	111111

As above, we can evaluate the average number of bits per pixel as an expected value:

$$(0.19 \times 2) + (0,25 \times 2) + (0.21 \times 2) + (0.16 \times 3) +$$
$$(0.08 \times 4) + (0.06 \times 5) + (0.03 \times 6) + (0.02 \times 6) = 2.7$$

which is a significant improvement over 3 bits per pixel, and very close to the theoretical minimum of 2.6508 given by the entropy.

Huffman codes are *uniquely decodable*, in that a string can be decoded in only one way. For

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example, consider the string

$$1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0$$

to be decoded with the Huffman code generated above. There is no code word 1, or 11, so we may take the first three bits 110 as being the code for grey value 3. Notice also that no other code word begins with this string. For the next few bits, 1110 is a code word; no other begins with this string, and no other smaller string is a codeword. So we can decode this string as grey level 4. Continuing in this way we obtain:

$$\underbrace{1\ \ 1\ \ 0}_{3}\ \ \underbrace{1\ \ 1\ \ 1\ \ 0}_{4}\ \ \underbrace{0\ \ 0}_{0}\ \ \underbrace{0\ \ 0}_{0}\ \ \underbrace{1\ \ 0}_{1}\ \ \underbrace{0\ \ 1}_{2}\ \ \underbrace{1\ \ 1\ \ 1\ \ 1\ \ 0}_{5}$$

as the decoding for this string.

For more information about Huffman coding, and its limitations and generalizations, see [7, 24].

14.3 Run length encoding

Run length encoding (RLE) is based on a simple idea: to encode strings of zeros and ones by the number of repetitions in each string. RLE has become a standard in facsimile transmission. For a binary image, there are many different implementations of RLE; one method is to encode each line separately, starting with the number of 0's. So the following binary image:

```
0 1 1 0 0 0
0 0 1 1 1 0
1 1 1 0 0 1
0 1 1 1 1 0
0 0 0 1 1 1
1 0 0 0 1 1
```

would be encoded as

(123)(231)(0321)(141)(33)(0132)

Another method [32] is to encode each row as a list of pairs of numbers; the first number in each pair given the starting position of a run of 1's, and the second number its length. So the above

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binary image would have the encoding

$$(22)(33)(1361)(24)(43)(1152)$$

Greyscale images can be encoded by breaking them up into their *bit planes*; these were discussed in chapter 3.

To give a simple example, consider the following 4-bit image and its binary representation:

$$\begin{matrix} 10 & 7 & 8 & 9 \\ 11 & 8 & 7 & 6 \\ 9 & 7 & 5 & 4 \\ 10 & 11 & 2 & 1 \end{matrix} \longrightarrow \begin{matrix} 1010 & 0111 & 1000 & 1001 \\ 1011 & 1000 & 0111 & 0110 \\ 1001 & 0111 & 0101 & 0100 \\ 1010 & 1011 & 0010 & 0001 \end{matrix}$$

We may break it into bit planes as shown:

$$\begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix}$$

0th plane 1st plane 2nd plane 3rd plane

and then each plane can be encoded separately using our chosen implementation of RLE.

However, there is a problem with bit planes, and that is that small changes of grey value may cause significant changes in bits. For example, the change from value 7 to 8 causes the change of all four bits, since we are changing the binary strings 0111 to 1000. The problem is of course exacerbated for 8-bit images. For RLE to be effective, we should hope that long runs of very similar grey values would result in very good compression rates for the code. But this may not be the case. A 4-bit image consisting of randomly distributed 7's and 8's would thus result in uncorrelated bit planes, and little effective compression.

To overcome this difficulty, we may encode the grey values with their binary *Gray codes*. A Gray code is an ordering of all binary strings of a given length so that there is only one bit change

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between a string and the next. So a 4-bit Gray code is:

15	1	0	0	0
14	1	0	0	1
13	1	0	1	1
12	1	0	1	0
11	1	1	1	0
10	1	1	1	1
9	1	1	1	0
8	1	1	0	0
7	0	1	0	0
6	0	1	0	1
5	0	1	1	1
4	0	1	1	0
3	0	0	1	0
2	0	0	1	1
1	0	0	0	1
0	0	0	0	0

See [24] for discussion and detail. To see the advantages, consider the following 4-bit image with its binary and Gray code encodings:

$$\begin{matrix} 8 & 8 & 7 & 8 \\ 8 & 7 & 8 & 7 \\ 7 & 7 & 8 & 7 \\ 7 & 8 & 7 & 7 \end{matrix} \longrightarrow \begin{matrix} 1000 & 1000 & 0111 & 1000 \\ 1000 & 0111 & 1000 & 0111 \\ 0111 & 0111 & 1000 & 0111 \\ 0111 & 1000 & 0111 & 0111 \end{matrix} \qquad \begin{matrix} 1100 & 1100 & 0100 & 1100 \\ 1100 & 0100 & 1100 & 0100 \\ 0100 & 0100 & 1100 & 0100 \\ 0100 & 1100 & 0100 & 0100 \end{matrix}$$

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where the first binary array is the standard binary encoding, and the second array the Gray codes. The binary bit planes are:

0 0 1 0	0 0 1 0	0 0 1 0	1 1 0 1
0 1 0 1	0 1 0 1	0 1 0 1	1 0 1 0
1 1 0 1	1 1 0 1	1 1 0 1	0 0 1 0
1 0 1 1	1 0 1 1	1 0 1 1	0 1 0 0
0th plane	1st plane	2nd plane	3rd plane

and the bit planes corresponding to the Gray codes are:

0 0 0 0	0 0 0 0	1 1 1 1	1 1 0 1
0 0 0 0	0 0 0 0	1 1 1 1	1 0 1 0
0 0 0 0	0 0 0 0	1 1 1 1	0 0 1 0
0 0 0 0	0 0 0 0	1 1 1 1	0 1 0 0
0th plane	1st plane	2nd plane	3rd plane

Notice that the Gray code planes are highly correlated except for one bit plane, whereas all the binary bit planes are uncorrelated.

Run length encoding in MATLAB

We can experiment with run length encoding by writing a simple function to implement it. To make it easy on ourselves, we shall just stick with single binary images. Our output will be a single vector, giving the numbers of zeros and ones, alternating, through our image row by row. We start by putting our image into a single row. For a binary image `im`, this can be done with the two commands

```
L=prod(size(im));
im=reshape(im',1,L);
```

To find the number of beginning zeros, we obtain the position of the first `1`, thus:

```
min(find(im==1))
```

We append one less than this result to our output vector. It may well be that there are no further ones, in which case we have reached the end of the file, and we stop by appending the current length

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of our image to the output vector. We now change to looking for the place of the next zero; we can use the `min(find)` command again, but we first reduce our image by the zeroes we have already found.

The following table shows how we can implement run length encoding:

Image	Looking for	Place	RLE output
			`[]`
`[0 0 1 1 1 0 0 0 1]`	1	3	`[2]`
`[1 1 1 0 0 0 1]`	0	4	`[2 3]`
`[0 0 0 1]`	1	4	`[2 3 3]`
`[1]`	0	*Not found*	`[2 3 3 1]`

One more example:

Image	Looking for	Place	RLE output
			`[]`
`[1 1 1 0 0 0 0 1 1]`	1	1	`[0]`
`[1 1 1 0 0 0 0 1 1]`	0	4	`[0 3]`
`[0 0 0 0 1 1]`	1	4	`[0 3 4]`
`[1 1]`	0	*Not found*	`[0 3 4 2]`

Notice that in this second example, since the length of our initial run of zeros was found to be zero, we do not reduce the length of our image in the next step.

Figure 14.3 shows the implementation of this algorithm in `Matlab`.

Now we can test this on a few images:

```
>> c=imread('circles.tif');
>> cr=rle(c);
>> whos c cr
  Name      Size            Bytes  Class


  c       256x256           65536  uint8 array (logical)
  cr        1x693            5544  double array
```

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```
function out=rle(image)
%
%  RLE(IMAGE) produces a vector containing the run-length encoding of
%  IMAGE, which should be a binary image.   The image is set out as a long
%  row, and the conde contains the number of zeros,  followed by the number
%  of ones, alternating.
%
%  Example:
%
%    rle([1 1 1 0 0;0 0 1 1 1;1 1 0 0 0])
%
%    ans =
%
%        0    3    4    5    3
%
L=prod(size(image));
im=reshape(image',1,L);
x=1;
out=[];
while L ~= 0,
  temp=min(find(im == x));
  if isempty(temp),
    out=[out L];
    break
  end;
  out=[out temp-1];
  x=1-x;
  im=im(temp:L);
  L=L-temp+1;
end;
```

Figure 14.3: A MATLAB function for obtaining the run length code of a binary image

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We can reduce the size of the output by storing it using the data type `uint16`: unsigned 16-bit integers.

```
>> cr=uint16(cr);
>> whos cr
  Name      Size            Bytes  Class


  cr        1x693            1386  uint16 array
```

Even if the original circles image was stored as one bit per pixel, or eight pixels per byte, we would have a total of

$$\frac{65536}{8} = 8192$$

bytes: still more than the run length code. So in this example, run length encoding provides a reasonable amount of compression.

```
>> t=imread('text.tif');
>> tr=rle(t);
>> whos t tr
  Name      Size            Bytes  Class


  t       256x256           65536  uint8 array (logical)
  tr        1x2923          23384  double array
```

Again better compression can be obtained by changing the data type.

```
>> tr=uint16(tr);
>> whos tr
  Name      Size            Bytes  Class


  tr        1x2923           5846  uint16 array
```

Although this is not as good as for the previous image, it is still better than the minimum of 8192 bytes for the original image.

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14.4 The JPEG algorithm

Lossy compression trades some acceptable data loss for greater rates of compression. Of the many compression methods available, the algorithm developed by the Joint Photographic Experts Group (JPEG) has become one of the most popular. It uses *transform coding*, where the coding is done not on the pixel values themselves, but on a transform.

The heart of this algorithm is the *discrete cosine transform* (DCT), which is defined similarly to the Fourier transform. Although it can be applied to an array of any size, in the JPEG algorithm it is applied only to 8×8 blocks. If $f(j,k)$ is one such block, the the forward (two-dimensional) DCT is defined as:

$$F(u,v) = \frac{C(u)C(v)}{4} \sum_{j=0}^{7} \sum_{k=0}^{7} f(j,k) \cos\left(\frac{(2j+1)u\pi}{16}\right) \cos\left(\frac{(2k+1)v\pi}{16}\right)$$

and the corresponding inverse DCT as

$$f(i,j) = \sum_{u=0}^{7} \sum_{v=0}^{7} f(u,v)C(u)c(v) \cos\left(\frac{(2j+1)u\pi}{16}\right) \cos\left(\frac{(2k+1)v\pi}{16}\right)$$

where $C(w)$ is defined as

$$C(w) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } w = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The DCT has a number of properties which make it particularly suitable for compression:

1. it is real-valued, so there is no need to manipulate complex numbers,
2. it has a high information packing ability, in that it packs large amounts of information into a small number of coefficients,
3. it can be implemented very efficiently in hardware,
4. like the FFT, there is a "fast" version of the transform which maximizes efficiency,
5. the basis values are independent of the data [7].

Like the 2-dimensional FFT, the 2-dimensional DCT (given above), is separable, and so can be calculated by a sequence of 1-dimensional DCT's; first we apply the 1-d DCT to the rows, then the 1-d DCT to the columns of the result.

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To see an example of the information packing capabilities, we consider the simple linear sequence given by

```
>> a=[10:15:115]

a =

    10    25    40    55    70    85   100   115
```

We shall apply both the FFT and DCT to this sequence, remove half the result, and invert the remainder. First the FFT:

```
>> fa=fft(a);
>> fa(5:8)=0;
>> round(abs(ifft(fa)))

ans =

    49    41    56    57    71    70    85    90
```

Now the DCT (using the `dct` and `idct` functions):

```
>> da=dct(a);
>> da(5:8)=0;
>> round(idct(da))

ans =

    11    23    41    56    69    84   102   114
```

Notice how much closer the DCT result is to the original, in spite of the loss of information from the transform. This example is illustrated in figure 14.4.

The JPEG baseline compression scheme is applied as follows:

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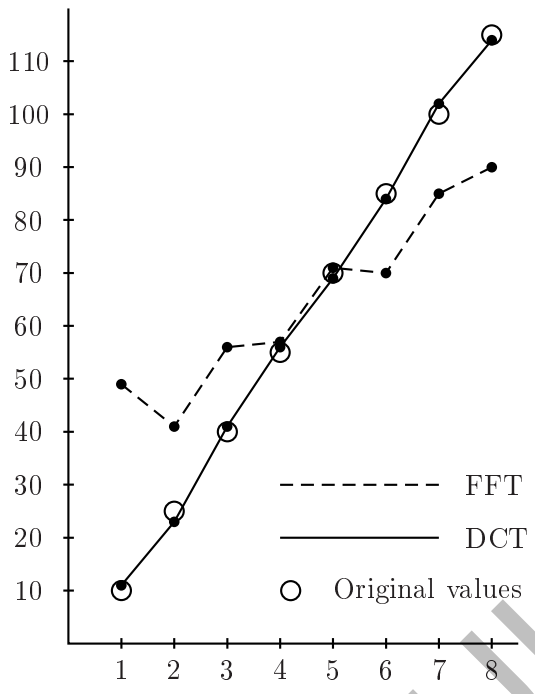

Figure 14.4: Comparison of the FFT and the DCT

1. The image is divided up into 8×8 blocks; each block is transformed and compressed separately.

2. For a given block, the values are "shifted" by subtracting 128 each value.

3. The DCT is applied to this shifted block.

4. The DCT values are "normalized" by dividing through by a "normalization matrix" Q.

   It is this normalization which provides the compression, by making most of the elements of the block zero.

5. This matrix is formed into a vector by reading off all non-zero values from the top left in a zig-zag fashion:

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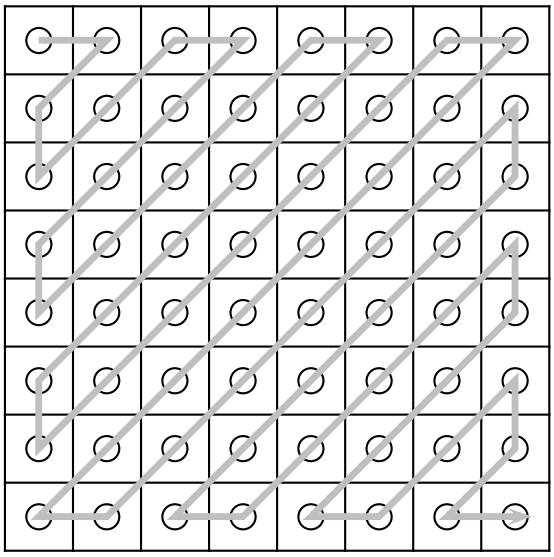

6. The first coefficients of each vector, which will be the largest elements in each vector, and which are known as the *DC coefficients*, are encoded by listing the *difference* between each value and the values from the previous block. This helps keep all values (except for the first) small.

7. These are then compressed using run length encoding.

8. All other values (known as the *AC coefficients*) are compressed using a Huffman coding.

The amount of lossy compression can be changed by scaling the normalization matrix Q in step 4 above,

To decompress, the above steps are applied in reverse; the Huffman encoding and run length encoding can be decoded with no loss of information. Having done that, then:

1. The vector is read back into an 8×8 matrix.

2. The matrix is multiplied by the normalization matrix.

3. The inverse DCT is applied to the result.

4. The result is shifted back by 128 to obtain the original image block.

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The normalization matrix which has been used by the JPEG group is

$$\begin{bmatrix} 16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\ 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\ 18 & 22 & 37 & 56 & 68 & 109 & 103 & 77 \\ 24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\ 49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\ 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{bmatrix}$$

We can experiment with the DCT and quantization using standard MATLAB functions. The two-dimensional DCT and inverse DCT are implemented with the functions `dct2` and `idct2` respectively.

To see all this in action, we shall take an 8×8 block from an image.

```
>> c=imread('caribou.tif');
>> x=151;y=90;
>> block=c(x:x+7,y:y+7)

block =

    87    95    92    73    59    57    57    55
    74    71    68    59    54    54    51    57
    64    58    57    55    58    65    66    65
    57    63    68    66    74    89    98   104
    95   109   117   114   119   134   145   140
   128   139   146   139   140   148   151   143
   137   135   125   118   137   156   154   132
   122   119   113   110   128   144   140   142
```

Now we subtract 128 from each value of the block:

```
>> b=double(block)-128
```

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```
b =

   -41   -33   -36   -55   -69   -71   -71   -73
   -54   -57   -60   -69   -74   -74   -77   -71
   -64   -70   -71   -73   -70   -63   -62   -63
   -71   -65   -60   -62   -54   -39   -30   -24
   -33   -19   -11   -14    -9     6    17    12
     0    11    18    11    12    20    23    15
     9     7    -3   -10     9    28    26     4
    -6    -9   -15   -18     0    16    12    14
```

Now we apply the DCT:

```
>> bd=dct2(b)

bd =

 -225.3750  -30.7580   17.3864    5.6543  -22.3750   -1.8591    3.7575    1.7196
 -241.5333   52.0722    0.8745  -21.2434    8.1434    1.8639    0.9420   -1.3369
   -2.5427   50.9316    5.0847    9.1573    1.5820   -3.8454    1.5706   -0.6043
  102.5557   23.3927  -11.5151  -12.7655  -10.6629    2.8179   -3.6743    1.2462
   -2.3750  -20.7081    3.5090  -10.3182   -1.3750   -2.4723    0.3054   -0.7308
  -12.7510    1.5740    2.7664    8.1034   -5.2779    1.0922   -1.6694    1.0561
    6.6005    7.8668   -4.9294   -7.0092    2.1860    0.8872    0.6653   -0.1783
   10.6630    0.4486   -0.1019    7.9728   -4.0241    2.4364   -2.3823    0.6011
```

We now need to enter our normalization matrix, and divide by it:

```
>> q = [16 11 10 16 24 40 51 61;...
12 12 14 19 26 58 60 55;...
14 13 16 24 40 57 69 56;...
```

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```
14 17 22 29 51 87 80 62;...
18 22 37 56 68 109 103 77;...
24 35 55 64 81 104 113 92;...
49 64 78 87 103 121 120   101;...
72 92 95 98 112 100 103 99];
>> bq=round(bd./q)

bq =

   -14    -3     2     0    -1     0     0     0
   -20     4     0    -1     0     0     0     0
     0     4     0     0     0     0     0     0
     7     1    -1     0     0     0     0     0
     0    -1     0     0     0     0     0     0
    -1     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0
```

At this stage we have turned our block from above into a block containing mostly zeros. If we were to output a vector from this block, it would be:

```
-14 -3 -20 0 4 2 0 0 4 7 0 1 0 -1 -1 0 0 0 -1 0 -1 EOB
```

where `EOB` signifies the end of the block. So by this stage we have reduced an 8×8 block to a vector of length 21, containing only small values.

To uncompress, we take the vector and reorder its elements into the matrix `bq` above. Multiplying by the normalization matrix produces:

```
>> bq2=bq.*q

bq2 =
```

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```
  -224   -33    20     0   -24     0     0     0

  -240    48     0   -19     0     0     0     0

     0    52     0     0     0     0     0     0

    98    17   -22     0     0     0     0     0

     0   -22     0     0     0     0     0     0

   -24     0     0     0     0     0     0     0

     0     0     0     0     0     0     0     0

     0     0     0     0     0     0     0     0
```

Now we apply the inverse DCT:

```
>> bd2=idct2(bq2)


bd2 =


  -48.1431  -39.4257  -39.8246  -53.5852  -65.6253  -68.5089  -70.4960  -74.9017

  -52.5762  -46.8345  -50.6187  -65.1825  -74.3228  -71.8983  -68.6282  -69.8981

  -70.6699  -66.1335  -69.6750  -80.0362  -81.2435  -69.9392  -59.8115  -57.6095

  -68.4457  -61.8134  -60.1283  -62.2478  -54.7898  -37.7751  -26.0007  -24.2342

  -29.6526  -21.6215  -16.7263  -14.7648   -5.2632    9.2177   14.8476   11.3981

    1.6297    7.3329    9.2805    8.8036   15.4106   24.9217   24.0240   15.7152

    3.0533    4.5797    1.2799   -2.1680    4.6076   15.8252   16.2949    8.4867

   -2.8366   -4.0640  -10.3394  -14.0550   -3.8001   13.2610   19.3977   14.9470
```

and finally add 128 to each value and round off:

```
>> b2=round(bd2+128)


b2 =


    80    89    88    74    62    59    58    53

    75    81    77    63    54    56    59    58
```

NOT FOR CIRCULATION

```
    57    62    58    48    47    58    68    70

    60    66    68    66    73    90   102   104

    98   106   111   113   123   137   143   139

   130   135   137   137   143   153   152   144

   131   133   129   126   133   144   144   136

   125   124   118   114   124   141   147   143
```

It can be seen that these values are very close, if not quite exactly the same, as the values in the original block. The differences between original and reconstructed values are:

```
>> double(block)-double(b2)


ans =


     7     6     4    -1    -3    -2    -1     2

    -1   -10    -9    -4     0    -2    -8    -1

     7    -4    -1     7    11     7    -2    -5

    -3    -3     0     0     1    -1    -4     0

    -3     3     6     1    -4    -3     2     1

    -2     4     9     2    -3    -5    -1    -1

     6     2    -4    -8     4    12    10    -4

    -3    -5    -5    -4     4     3    -7    -1
```

The algorithm works best on regions of low frequency; in such cases the original block can be reconstructed with only very small errors.

We can experiment with JPEG compression by using the `blkproc` function, which applies a function to each block in the image; the size of the blocks being given as parameters to `blkproc`. We shall design two functions: `jpg_in`, which applies JPEG compression to each 8×8 block, and `jpg_out`, which reverses the compression. To each function we shall include a further parameter `n`, which will be used to scale the normalization matrix. These functions are shown in figure 14.5. Now we can apply them to the matrix of the caribou image.

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```
function out=jpg_in(x,n)
q=[16 11 10 16 24 40 51 61;...
   12 12 14 19 26 58 60 55;...
   14 13 16 24 40 57 69 56;...
   14 17 22 29 51 87 80 62;...
   18 22 37 56 68 109 103 77;...
   24 35 55 64 81 104 113 92;...
   49 64 78 87 103 121 120 101;...
   72 92 95 98 112 100 103 99];
bd=dct2(double(x)-128);
out=round(bd./(q*n));
```

```
function out=jpg_out(x,n)
q=[16 11 10 16 24 40 51 61;...
   12 12 14 19 26 58 60 55;...
   14 13 16 24 40 57 69 56;...
   14 17 22 29 51 87 80 62;...
   18 22 37 56 68 109 103 77;...
   24 35 55 64 81 104 113 92;...
   49 64 78 87 103 121 120 101;...
   72 92 95 98 112 100 103 99];
out=round(idct2(x.*q*n)+128);
```

Figure 14.5: MATLAB functions for experimenting with JPEG compression

```
>> cj1=blkproc(c,[8,8],'jpg_in',1);

>> length(find(cj1==0))


ans =


       51940
```

So we have applied the compression scheme to each 8×8 block of the image, up to the point of dividing by the quantization matrix and round off. The point of the second command is to show how much information has been lost at this stage. The original image contained 65536 different items of information (the pixel values); each one between 0 and 128. But now we have only $65536 - 51940 = 13596$ items of information, and their maxima and minima are:

```
>> max(cj1(:)),min(cj1(:))


ans =


    60



ans =
```

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```
   -45
```

So the information is not only far less in number, but also in range. Now we can go backwards:

```
>> c1=jpg_out(cj1,2);
>> c1=uint8(c1);
```

The original image and result `c2` are shown in figure 14.6. There is no apparent difference between

Figure 14.6: An image before and after JPEG compression and decompression

the images—they look identical. However, they are not identical:

```
>> max(double(c(:))-double(c1(:)))

ans =

    31

>> min(double(c(:))-double(c1(:)))

ans =

   -26
```

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and we can see the difference (scaled for viewing):

```
>> imshow(mat2gray(double(c)-double(c1)))
```

in figure 14.7. We can experiment with different levels of quantization. Suppose we take our extra

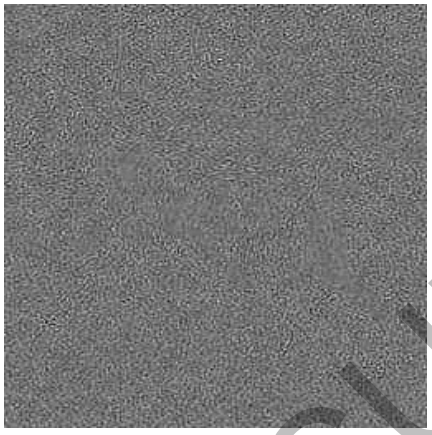

Figure 14.7: The difference between an image and its JPEG decompression

parameter `n` to be 2. This has the effect of doubling each value in the normalization matrix, so should set more of the DCT values to zero:

```
>> cj2=blkproc(c,[8,8],'jpg_in',2);
>> length(find(cj2==0))


ans =


      56729
```

This means that only 8807 values are non-zero. We can also find that the maximum and minimum values of `cj` are 30 and −22. We can now decompress, and display the result and the difference with the same commands as above. The results are shown in figure 14.8. Now we shall try a scale of 5:

```
>> cj5=blkproc(c,[8,8],'jpg_in',5);
```

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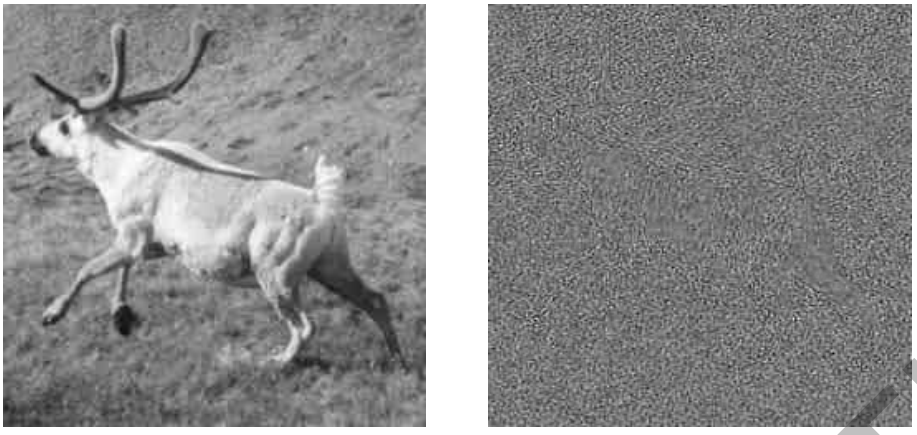

Figure 14.8: JPEG compression with a scale factor of 2

```
>> length(find(c5==0))


ans =


       61512
```

and the result and difference image are shown in figure 14.9. At this stage we may be losing some

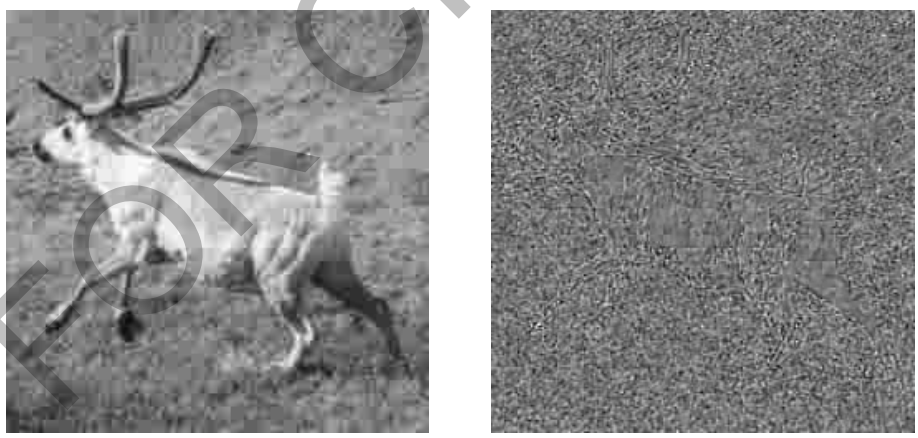

Figure 14.9: JPEG compression with a scale factor of 5

fine detail, but the image is still remarkably good. We also note that as the scale factor increases, the range of values after the quantization decreases. For the matrix `cj` above, the maximum and minimum are 12 and -9.

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Finally we shall try a scale factor of 10:

```
>> cj10=blkproc(c,[8,8],'jpg_in',10);
>> length(find(cj10==0))


ans =


       63684
```

The maximum and minimum are 6 and −4, and we have only 1852 items of information. The results are shown in figure 14.10. This image certainly shows considerable degradation; however,

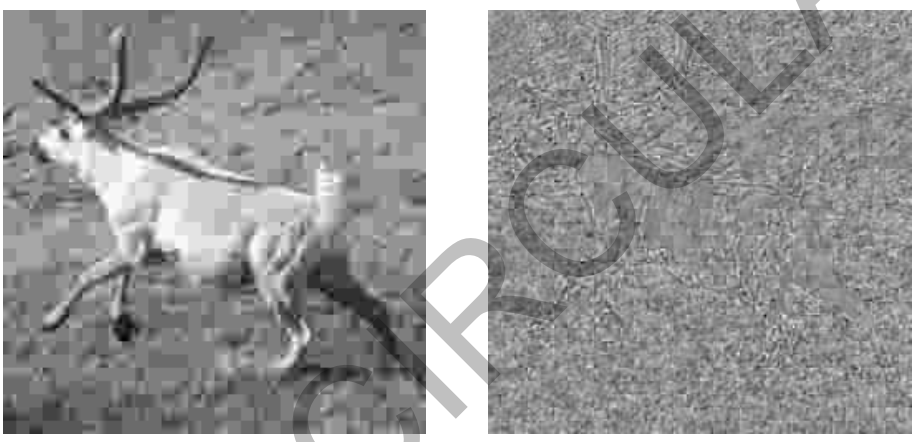

Figure 14.10: JPEG compression with a scale factor of 10

the animal is still quite clear.

We can see the results of JPEG compression and decompression by looking at a closeup of the image; we shall investigate that portion of the animal around the top of the head:

```
>> imshow(imresize(c(68-31:68+32,56-31:56+32),4))
```

which is shown in figure 14.11. The same areas, with the scales of 1 and 2, are shown in figure 14.12. And the closeups, with the scales of 5 and 10, are shown in figure 14.13. Notice that "blockiness" becomes more apparent as the scale factor increases. This is due to the working of the algorithm; that each 8×8 block is processed independently of the others. This tends to produce discontinuities in the output, and is one of disadvantages of the JPEG algorithm for high levels of compression.

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Figure 14.11: An image closeup

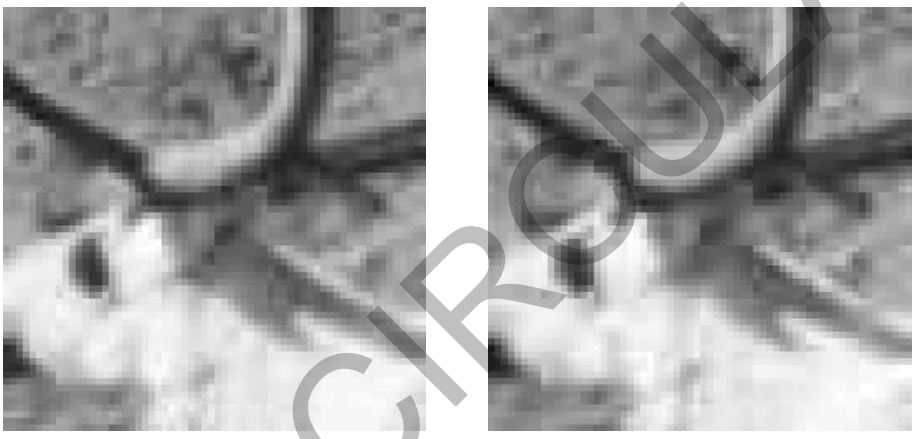

Figure 14.12: Closeups after the scale factors of 1 and 2

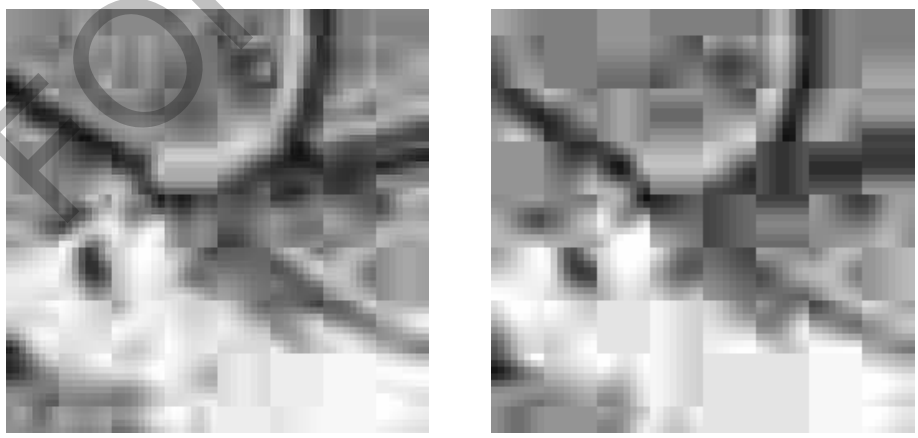

Figure 14.13: Closeups after the scale factors of 5 and 10

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In chapter 15 we shall see that some of these disadvantages can be over come by using wavelets for compression.

We have seen how changing the compression rate may affect the output. But the JPEG algorithm is particularly designed for storage. For example, suppose we take the original block from the caribou image, but divide it by double the quantization matrix. After reordering, the output vector will be

```
7 -1 -10 0 2 1 0 0 2 4 0 1 0 -1 EOB
```

This vector is further encoded using Huffman coding. To do this, each element of the vector (except for the first); that is, each AC value, is defined to be in a particular *category* depending on its absolute value. In general, the value 0 is given category 0, and category k contains all elements x whose absolute value satisfies

$$2^k \leq |x| \leq 2^{k+1} - 1.$$

The categories are also used for the differences of the first (DC) value. The first few categories are:

Range	DC category	AC category
0	0	Not applicable
$-1, 1$	1	1
$-3, -2, 2, 3$	2	2
$-7, \ldots, -4, 4, \ldots, 7$	3	3
$-15, \ldots, -8, 8, \ldots, 15$	4	4

To encode the vector, the category is used with all non-zero terms, along with the number of preceding zeros. For example, for the vector above:

Values:	7	-1	10	0	2	1	0	0	2	4	0	1	0	-1
Category:		1	4		2	1			2	3		1		1
Preceding zeros:		0	0		1	0			2	0		1		1

For each non-zero AC value then, a binary vector is produced containing the Huffman code for its particular category and run of preceding zeros, followed by a sign bit (0 for negative, 1 for positive), and a then for category k a run of $k-1$ bits indicating the position within that category. The

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Huffman code table is provided as part of the JPEG baseline standard; for the first few categories and runs they are:

Run	Category	Code
0	0	1010
0	1	00
0	2	01
0	3	100
0	4	1011
1	1	1100
1	2	111001
1	3	1111001
1	4	111110110
2	1	11011
2	2	11111000
2	3	1111110111

A full table is given by Gonzalez and Woods [7]. For the non-zero AC values in the above vector, the output binary strings will be created from:

Run	Category	Code	Sign	Position
0	1	00	0	0
0	4	1011	1	011
1	2	111001	1	0
0	1	00	1	1
2	2	11111000	1	0
0	3	100	1	00
1	1	1100	1	0
1	1	1100	0	0

The output string of bits for this block will consist of the code for the difference of the DC coefficient, followed by:

0000/10111011/11100110/0011/1111100010/100100/110010/110000

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where the lines just show the individual strings. Assuming 8 bits for the first code, the entire block has been encoded with only 60 bits,

$$60/64 \approx 0.94$$

bit per pixel: a compression rate of over 8.5.

For a detailed account of the JPEG algorithm, see Pennebaker [37].

Exercises

1. Construct a Huffman code for each of the probability tables given:

grey scale		0	1	2	3	4	5	6	7
probability	(a)	.07	.11	.08	.04	.5	.05	.06	.09
	(b)	.13	.12	.13	.13	.12	.12	.12	.13
	(c)	.09	.13	.15	.1	.14	.12	.11	.16

   In each case determine the average bits/pixel given by your code.

2. From your results of the previous question, what do think are the conditions of the probability distribution which give rise to a high compression rate using Huffman coding?

3. Encode each of the following binary images using run length encoding:

(a)

1	0	0	1	1	1
0	1	0	1	1	1
1	0	0	1	1	1
0	1	1	1	0	1
1	0	1	0	1	1
0	1	1	1	1	0

(b)

1	0	1	0	0	0
0	0	1	1	0	1
1	1	0	0	0	0
0	0	0	0	1	1
1	1	1	1	0	0
1	1	1	0	0	0

4. Using run length encoding, encode each of the following 4-bit images:

(a)

1	1	3	3	1	1
1	7	10	10	7	1
6	13	15	15	13	6
6	13	15	15	13	6
1	7	10	10	7	1
1	1	3	3	1	1

(b)

0	0	0	6	12	12	1	9
1	1	1	6	12	11	9	13
2	2	2	6	11	9	13	13
8	10	15	15	7	5	5	5
14	8	10	15	7	4	4	4
14	14	5	10	7	3	3	3

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5. Check your answers to the previous two questions with MATLAB. You can isolate the bit planes by using the technique discussed in section 3.4.

6. Encode the preceding images using the 4-bit Gray code, and apply run length encoding to the bit planes of the result.

   Compare the results obtained using Gray codes, and standard binary codes.

7. Write a MATLAB function for restoring a binary image from a run length code. Test it on the images and codes from the previous questions.

8. The following are the run-length encodings for a 4×4 4-bit image from most to least important bit-planes:

```
3  1  2  2  1  4  1  2
1  2  1  2  1  2  1  2  1  3
2  1  2  1  2  2  1  5
0  3  1  3  2  3  1  2  1
```

   Construct the image.

9. (a) Given the following 4-bit image:

```
0   4   4   4   4   4   6   7
0   4   5   5   5   4   6   7
1   4   5   5   5   4   6   7
1   4   5   5   5   4   6   7
1   4   4   4   4   4   6   7
2   2   8   8   8  10  10  11
2   2   9   9   9  12  13  13
3   3   9   9   9  15  14  14
```

   transform it to a 3-bit image by removing the least most significant bit plane. Construct a Huffman code on the result and determine the average number of bits/pixel used by the code.

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(b) Now apply Huffman coding to the original image and determine the average number of bits/pixel used by the code.

(c) Which of the two codes gives the best rate of compression?

10. Apply JPEG compression to an 8×8 block consisting of (a) all the same value, (b) the left half one value, and the right half another, (c) random values uniformly distributed in the 0–255 range.

    Compare the length of the code vector in each case, and the results of decompression.

11. Open the image `engineer.tif`.

    Using the JPEG compression commands described above, attempt to compress this image using greater and greater compression rates. What is the largest quantization scale factor for which the image is still recognizable? How many zeros are in the DCT block matrix?

12. Apply the given JPEG Huffman codes to the vector:

```
-14 -3 -7 0 4 2 0 0 4 3 0 1 0 -1 -1 0 0 0-1 0 -1 EOB
```

    and determine the compression rate.

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Chapter 15

Wavelets

15.1 Waves and wavelets

We have seen in chapters 7 and 14 that the discrete Fourier transform and its cousin, the discrete cosine transform, are of enormous use in image processing and image compression. But as we saw with the Fourier transform, this power does not come without cost. In particular, as the Fourier transform is based on the idea of an image being decomposed into periodic sine or cosine functions, we have to assume some sort of periodicity in the image to make the theory fit the practice. This periodicity was obtained by assuming that the image "bent round" to meet itself at the top and bottom, and at the left and right sides. But this means that we have introduced unnecessary discontinuities in the image, and these may affect the transform, and our use of it.

The idea of wavelets is to keep the wave idea, but drop the periodicity. So we may consider a *wavelet* to be a little part of a wave: a wave which is only non-zero in small region. Figure 15.1 illustrate the idea. Figure 15.1(a) is just the graph of

$$y = \sin(x)$$

for $-10 \leq x \leq 10$, and figure 15.1(b) is the graph of

$$y = \sin(x)e^{-x^2}$$

over the same interval.

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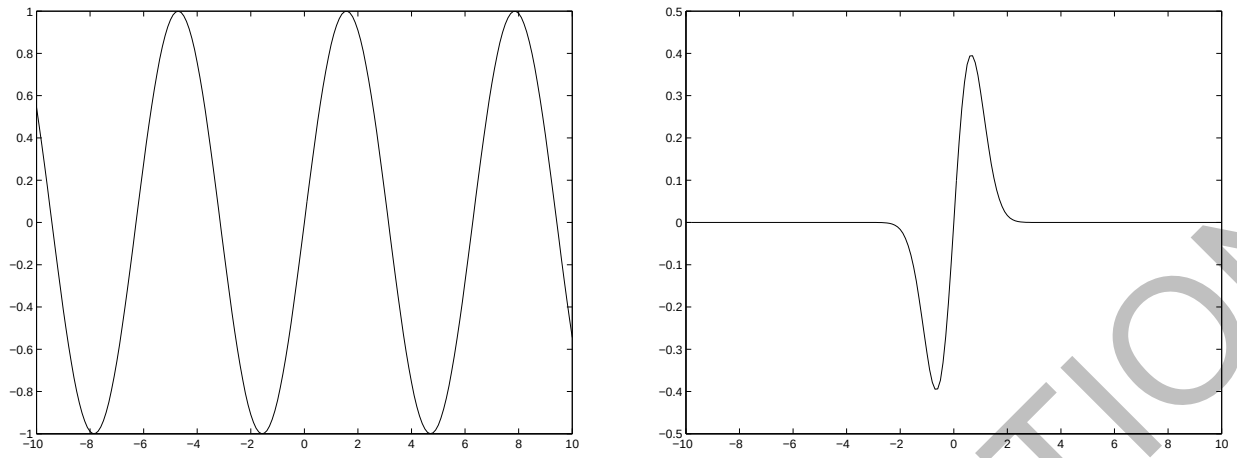

(a) A wave (b) A wavelet

Figure 15.1: Comparing a wave and a wavelet

Suppose we are given a wavelet. What can we do with it? Well, if

$$f = w(x)$$

is the function which defines our wavelet, we can:

dilate it by applying a scaling factor to x: $f(2x)$ would "squash" the wavelet; $f(x/2)$ would expand it,

translate it by adding or subtracting an appropriate value from x: $f(x-2)$ would shift the wavelet 2 to the right; $f(x+3)$ would shift the wavelet 3 to the left,

change its height by simply multiplying the function by a constant.

Of course we can do any or all at once:

$$6f(x/2-3), \quad \frac{1}{3}f(16x+17), \quad f(x/128-33.5), \ldots$$

Let

$$w(x) = \sin(x)e^{-x^2}$$

be the function shown in figure 15.1(b). Figure 15.2 shows some dilations and shifts of this wavelet.

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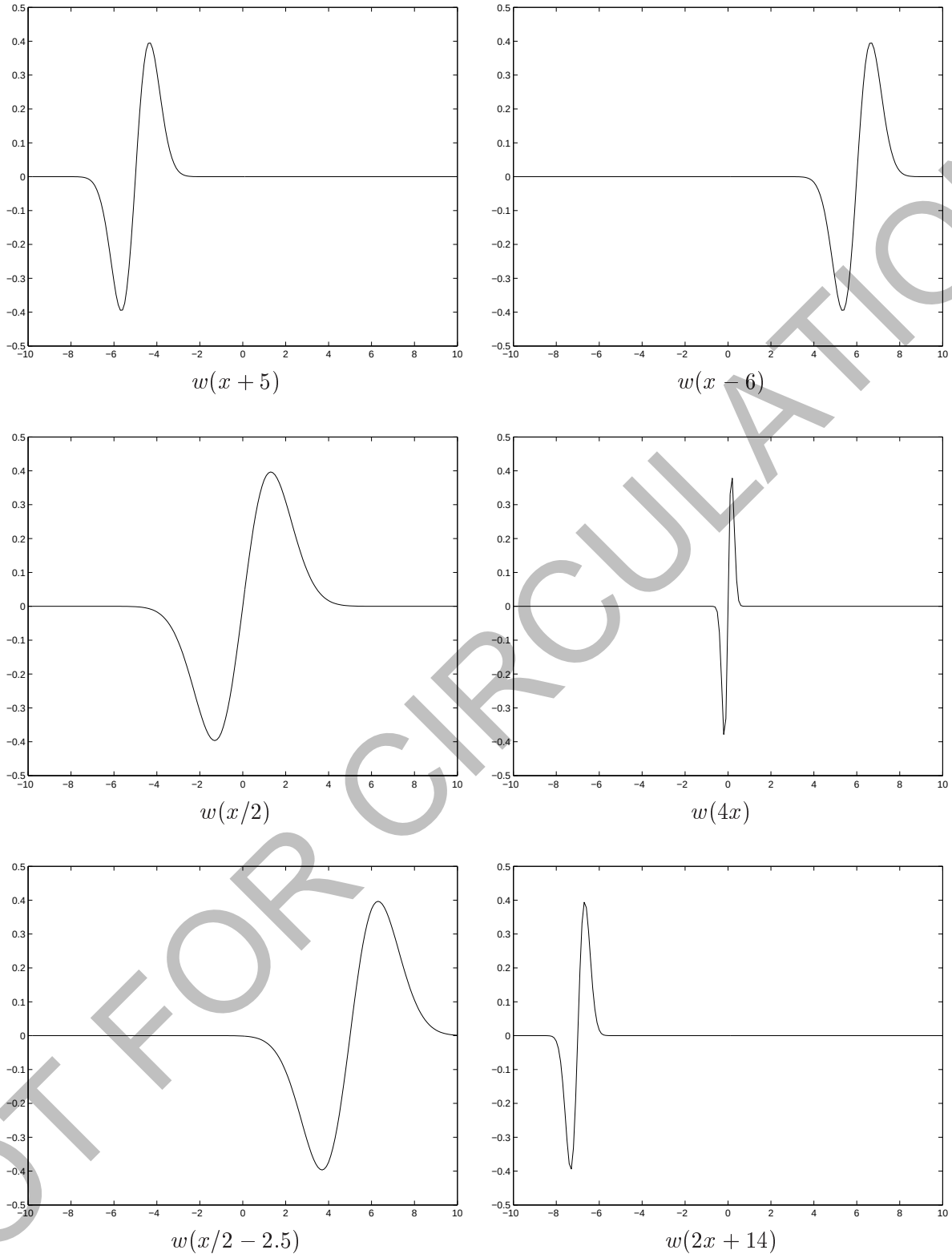

Figure 15.2: Shifts and dilations of the wavelet $w(x)$

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Given our knowledge of the working of the Fourier transform, it should come as no surprise that given a suitable starting wavelet $w(x)$, we can express any function $f(x)$ as a sum of wavelets of the form

$$aw(bx + c)$$

Moving into two dimensions, we can apply wavelets to images in the same way we applied sine and cosines with the Fourier transform.

Using wavelets has provided a new class of very powerful image processing algorithms: wavelets can be used for noise reduction, edge detection, and compression. The use of wavelets has superseded the use of the DCT for image compression in the JPEG2000 image compression algorithm.

A possible problem with wavelets is that the theory, which has been very well researched, can be approached from many different directions. There is thus no "natural" method of presenting wavelets: it depends on your background, and on the use to which they will be put. Also, much writing on wavelets tends to dive into the theory very quickly. In this chapter, though, we shall just look at the very simplest examples of the use of wavelets, and their application to images.

A simple wavelet transform

To obtain a feeling for how a wavelet transform works and behaves, we shall look at a very simple example. All wavelet transforms work by taking weighted averages of input values, and providing any other necessary information to be able to recover the original input.

For our example, we shall perform just two operations: averaging of two values, and differencing. For example, suppose we are given two numbers 14 and 22. We can easily create their average:

$$\frac{14 + 22}{2} = 18.$$

To recover the original two values from their average, we need a second value: the *difference*, obtained by subtracting the average from the first value:

$$14 - 18 = -4.$$

From the average and the difference, we can obtain the original two numbers by adding and sub-

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tracting the difference and the average:

$$\begin{aligned} 18 + (-4) &= 14 \\ 18 - (-4) &= 22. \end{aligned}$$

In general, if a and b are our two numbers, we create an average s and a difference d by

$$\begin{aligned} s &= \frac{a+b}{2}, \\ d &= a - s. \end{aligned}$$

We can then recover the two original numbers with

$$\begin{aligned} a &= s + d, \\ b &= s - d. \end{aligned}$$

Let's take a simple vector v of eight elements, and create two new vectors v_1 and v_2 of four elements each. Vector v_1 will consist of the averages of each of the pairs of elements from v, and vector v_2 will consist of the differences of the first four elements of v with the elements of v_1. We start with

$$v = [71, \quad 67, \quad 24, \quad 26, \quad 36, \quad 32, \quad 14, \quad 18].$$

Then

$$\begin{aligned} v_1 &= \left[\frac{71+67}{2}, \quad \frac{24+26}{2}, \quad \frac{36+32}{2}, \quad \frac{14+18}{2}\right] \\ &= [69, \quad 25, \quad 34, \quad 16] \end{aligned}$$

and

$$\begin{aligned} v_2 &= [71-69, \quad 24-25, \quad 36-34, \quad 14-16] \\ &= [2, \quad -1, \quad 2, \quad -2] \end{aligned}$$

The concatenation of v_1 and v_2 is the *Discrete wavelet transform at 1 scale* of the original vector:

$$d_1 = [69, \quad 25, \quad 34, \quad 16, \quad 2, \quad -1, \quad 2, \quad -2].$$

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We can continue by performing this same averaging and differencing the first four elements of the result; this will involve the concatenation of two vectors of two elements each:

$$\begin{aligned} v_1 &= \left[\frac{69+25}{2}, \quad \frac{34+16}{2}\right] \\ &= [47, \quad 25] \\ v_2 &= [69-47, \quad 34-25] \\ &= [22, \quad 9] \end{aligned}$$

Replacing the first four elements of d_1 above with this new v_1 and v_2 produces

$$d_2 = [47, \quad 25, \quad 22, \quad 9, \quad 2, \quad -1, \quad 2, \quad -2]$$

which is the *Discrete wavelet transform at 2 scales* of the original vector. We can go one more step, replacing the first two elements of d_2 with their average and difference:

$$d_3 = [36, \quad 11, \quad 22, \quad 9, \quad 2, \quad -1, \quad 2, \quad -2]$$

and this is the *Discrete wavelet transform at 3 scales* of the original vector.

To recover the original vector, we simply add and subtract, first using only the first two elements, then the first four, and finally the lot:

$$\begin{aligned} &[36+11, \quad 36-11, \quad 22, \quad 9, \quad 2, \quad -1, \quad 2, \quad -2] \\ &\quad = [47, \quad 25, \quad 22, \quad 9, \quad 2, \quad -1, \quad 2, \quad -2] \\ &[47+22, \quad 47-22, \quad 25+9, \quad 25-9, \quad 2, \quad -1, \quad 2, \quad -2] \\ &\quad = [69, \quad 25, \quad 34, \quad 16, \quad 2, \quad -1, \quad 2, \quad -2] \\ &[69+2, \quad 69-2, \quad 25+(-1), \quad 25-(-1), \quad 34+2, \quad 34-2, \quad 16+(-2), \quad 16-(-2)] \\ &\quad = [71, \quad 67, \quad 24, \quad 26, \quad 36, \quad 32, \quad 14, \quad 18]. \end{aligned}$$

At each stage, the averaging vector produces a "lower resolution" version of the original vector. Wavelet transforms produce a mix of lower resolutions of the input, and the extra information required for inversion.

We notice that the differences may be small if the input values are close together. This leads to an idea for compression: we apply a threshold by setting to zero all values in the transform which

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are less than a predetermined value. Suppose we take a threshold of 0, so that after thresholding d_3 becomes:

$$d_3' = [36, \quad 11, \quad 22, \quad 9, \quad 2, \quad 0, \quad 2, \quad 0].$$

If we now use this as the starting place for our adding and subtracting, we end up with

$$v' = [71, \quad 67, \quad 25, \quad 25, \quad 36, \quad 32, \quad 16, \quad 16].$$

Notice how close this is to the original vector, in spite of the loss of some information from the transform.

15.2 A simple wavelet: the Haar wavelet

The Haar wavelet has been around for a long time, and has been used with images as the *Haar transform*. Only recently has the Haar transform been viewed as a simple wavelet transform. In fact the Haar wavelet is the simplest possible wavelet, and for that reason is a good starting place.

The Haar wavelet is defined by the function

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x < 1/2 \\ -1 & \text{if } 1/2 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and is shown in figure 15.3. As with our wavelet function $w(x)$, we can compress and expand this

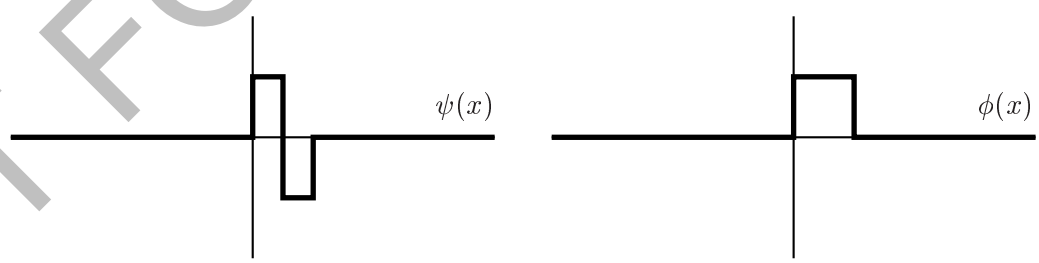

Figure 15.3: The Haar wavelet and pulse function

wavelet horizontally or vertically, and shift it.

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Applying the Haar wavelet

Given the Haar wavelet, what do we do with it? That is, how do we use the Haar wavelet in a wavelet transform? We haven't yet defined what a wavelet transform is, or what it might look like. Without going into too many details, a wavelet transform can be defined in much the same way as the DFT or the DCT: as a sum of function values multiplied by wavelet values. In comparison, for the DFT we multiplied function values by complex exponentials, and in the DCT we multiplied our function values by cosines.

The discrete wavelet transform (DWT) can be written [7] as:

$$W_\phi(j_0, k) = \frac{1}{\sqrt{M}} \sum_x f(x)\phi_{j_0,k}x \tag{15.1}$$

$$W_\psi(j, k) = \frac{1}{\sqrt{M}} \sum_x f(x)\psi_{j,k}(x) \tag{15.2}$$

where the subscripts on $\phi(x)$ and $\psi(x)$ represent different dilations and shifts of the basic functions. Then we can recover $f(x)$ with

$$f(x) = \frac{1}{\sqrt{M}} \sum_k W_\phi(j_0, k)\phi_{j_0,k}x + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} W_\psi(j, k)\psi_{j,k}(x).$$

(See Gonzalez and Woods [7].) But all these expressions really show is that the DWT has the same basic appearance as the DFT and DCT: in each case we form a sum of input values multiplied by the values from a particular class of functions. The form of the above equations indicates that the discrete wavelet transform can be written as a matrix multiplication, as we saw for the DFT. We shall show below how this is done.

Notice that the Haar wavelet can be written in terms of the simpler *pulse function*:

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

using the relation

$$\psi(x) = \phi(2x) - \phi(2x - 1). \tag{15.3}$$

We can see that this is true by noting that $\phi(2x)$ is equal to 1 for $0 \le x \le 1/2$, and that $\phi(2x - 1)$ is equal to 1 for $1/2 \le x < 1$.

NOT FOR CIRCULATION

The pulse function also satisfies the equation

$$\phi\left(\frac{x}{2}\right) = \phi(x) - \phi(x-1). \tag{15.4}$$

In the theory of wavelets, the "starting wavelet", in this case our function $\psi(x)$, is called the *mother wavelet*, and the corresponding function $\phi(x)$ is called the *scaling function* (and sometimes called a *father wavelet*).

We can also write our scaling expression 15.4 as

$$\phi(x) = \phi(2x) + \phi(2x-1). \tag{15.5}$$

Equations 15.4 and 15.5 are important because they tell us how the wavelets are rescaled at different resolutions. Equation 15.5 is a very important equation: it is called the *dilation equation*, as it relates the scaling function to dilated versions of itself. Generalizations of this equation, as we shall see below, have led to other wavelets than the Haar wavelet. Equation 15.3 is the *wavelet equation* for the Haar wavelet. Note that the dilation and wavelet equations have the same right hand side, except for a change of sign. These can be generalized to produce

$$\begin{aligned}\phi(x) &= \cdots + h_{-2}\phi(2x+2) + h_{-1}\phi(2x+1) + h_0\phi(2x) + h_1\phi(2x-1) \\ &\qquad + h_2\phi(2x-2) + h_3\phi(2x-3) + \cdots \qquad (15.6)\\ \psi(x) &= \cdots - h_{-2}\phi(2x-3) + h_{-1}\phi(2x-2) - h_0\phi(2x-1) + h_1\phi(2x) \\ &\qquad - h_2\phi(2x+1) + h_3\phi(2x+2) - \cdots \qquad (15.7)\end{aligned}$$

where the values h_i are called the *filter coefficients* or the *taps* of the wavelet. A wavelet is completely specified by its taps.

We thus have, for the Haar wavelet:

$$\phi(x) = h_0\phi(2x) + h_1\phi(2x-1) \tag{15.8}$$

$$\psi(x) = h_1\phi(2x) - h_0\phi(2x-1) \tag{15.9}$$

where $h_0 = h_1 = 1$. It is actually these h_i values which we use in a calculation of the DWT. We

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can put them into a DWT matrix:

$$H_{2^n} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 \end{bmatrix}$$

If v is the vector we investigated earlier, then $vH_8 = d_1$. We can obtain d_2 and d_3 by multiplying the first values of the vector by the appropriate sized H matrix.

Let's try this in MATLAB:

```
>> h8=[1 1 0 0 0 0 0 0;0 0 1 1 0 0 0 0;...
       0 0 0 0 1 1 0 0;0 0 0 0 0 0 1 1;...
       1 -1 0 0 0 0 0 0;0 0 1 -1 0 0 0 0;...
       0 0 0 0 1 -1 0 0;0 0 0 0 0 0 1 -1];
>> h4=[1 1 0 0;0 0 1 1;1 -1 0 0;0 0 1 -1];
>> h2=[1 1;1 -1];
>> v=[71 67 24 26 36 32 14 18];
>> d1=v*h8'/2

d1 =

    69    25    34    16     2    -1     2    -2
```

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```
>> d2=d1;d2(1:4)=d1(1:4)*h4'/2

d2 =

    47    25    22     9     2    -1     2    -2

>> d3=d2;d3(1:2)=d2(1:2)*h2'/2

d3 =

    36    11    22     9     2    -1     2    -2
```

The extra divisions by 2 are simply because the Haar matrix, as we have given it, simply adds two numbers, instead of taking their average, and similarly for the difference.

We have suppressed some numeric detail in our previous discussion for simplicity. To be precise, the values of h_0 and h_1 for the Haar wavelet are

$$h_0 = h_1 = \frac{1}{\sqrt{2}}$$

and the matrix H_{2^n} given above should have the factor $1/\sqrt{2}$ in front of it.

Any wavelet transform can be computed by matrix multiplications. But as we have seen in our discussion of the DFT, this can be very inefficient. Happily many wavelet transforms can be quickly computed by a fast algorithm similar in style to the average/difference method we saw earlier. These methods are called *lifting methods* [17].

If we go back to our averaging and differencing example, we can see that the averaging part of the transform corresponds to low pass filtering, in that we are "coarsening" or "blurring" our input. Similarly, the differencing part of the transform corresponds to a high pass filter, of the sort we looked at in our discussion of edges in chapter 9. Thus a wavelet transform contains within it both high and low pass filtering of our input, and we can consider a wavelet transform entirely in term of filters. This approach will be discussed below in section 15.4.

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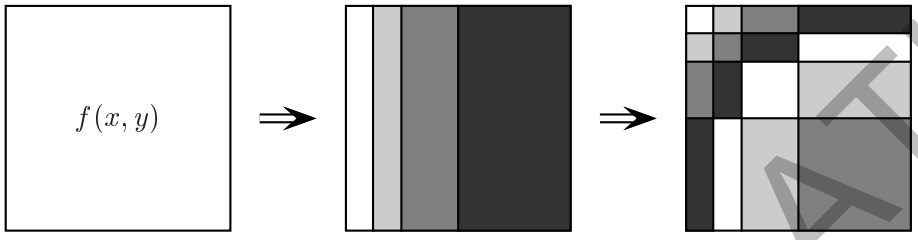
Two dimensional wavelets

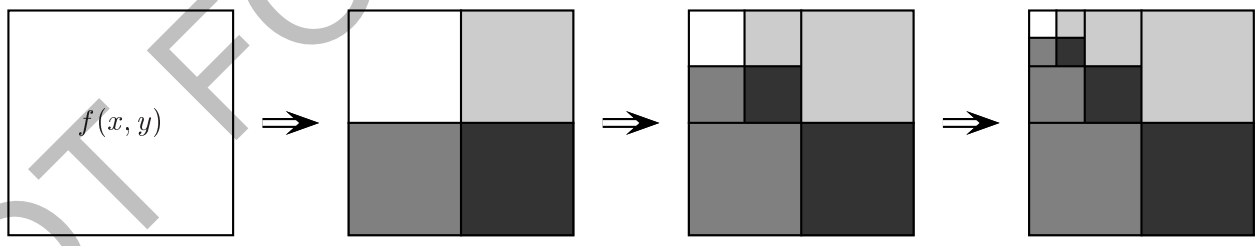
The two-dimensional wavelet transform is *separable*, which means we can apply a one dimensional wavelet transform to an image in the same way as for the DFT: we apply a one-dimensional DWT to all the columns, and then one-dimensional DWT's to all the rows of the result. This is called the *standard decomposition*, and is illustrated in figure 15.4.



Original image DWT of rows DWT of columns

Figure 15.4: The standard decomposition of the two dimensional DWT

We can also apply a wavelet transform differently. Suppose we apply a wavelet transform (say, using the Haar wavelet) to an image by columns then rows, but using our transform at 1 scale only. This will produce a result in four quarters: the top left will be a half-sized version of the image; the other quarters high pass filtered images. These quarters will contain horizontal, vertical and diagonal edges of the image. We then apply a one-scale DWT to the top left quarter, creating smaller images. And so on. This is called the *nonstandard decomposition*, and is illustrated in figure 15.5.



Original image 1 scale DWT 2 scale DWT 3 scale DWT

Figure 15.5: The non-standard decomposition of the two dimensional DWT

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15.3 Wavelets in MATLAB

Although there is a Wavelets toolbox produced by the Mathworks[1], we shall not assume access to it. However, there are a number of publicly available wavelet toolboxes available, which if not as full featured, contain enough functionality for our purposes.

We shall use the `UviWave` toolbox, developed at the University of Vigo, in Spain. Its home page is

`http://www.tsc.uvigo.es/~wavelets/uvi_wave.html`

but it can also be found at other places on the Web. Assuming that you've downloaded and installed it, let's try it, using the same vector as above:

```
>> >> [h,g,rh,rg]=daub(2)
h =
    0.7071    0.7071
g =
   -0.7071    0.7071
rh =
    0.7071    0.7071
rg =
    0.7071   -0.7071
```

Here `h` and `g` are the low pass and high pass filter coefficients for the forward transform; `rh` and `rg` are the low pass and high pass filter coefficients for the inverse transform. The `daub` function produces the filter coefficients for a class of wavelets called *Daubechies wavelets*, of which the Haar wavelet is the simplest. Now we can apply the DWT to our vector:

```
>> w=wt(v,h,g,3)


w =
```

[1]See `http://www.mathworks.com/products/wavelet/` for more information.

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```
  101.8234  31.1127  44.0000  18.0000   2.8284  -1.4142   2.8284  -2.8284
```

This is in fact the same as our vector d_3 from above, except for scaling factors: if we divide the first two elements by $(\sqrt{2})^3$, the next two by $(\sqrt{2})^2$, and the last four by $\sqrt{2}$:

```
>> w./sqrt(2).^[3 3 2 2 1 1 1 1]

ans =

   36.0000  11.0000  22.0000   9.0000   2.0000  -1.0000   2.0000  -2.0000
```

we have d_3 from above. We can reconstruct the original vector now:

```
>> iwt(w,rh,rg,3)

ans =

   71.0000  67.0000  24.0000  26.0000  36.0000  32.0000  14.0000  18.0000
```

We could construct our original d_3 by adjusting `h` and `g` to remove the scaling values:

```
>> h=[1 1]/2,g=[-1 1]/2

h =

    0.5000    0.5000


g =

   -0.5000    0.5000

>> wt(v,h,g,3)
```

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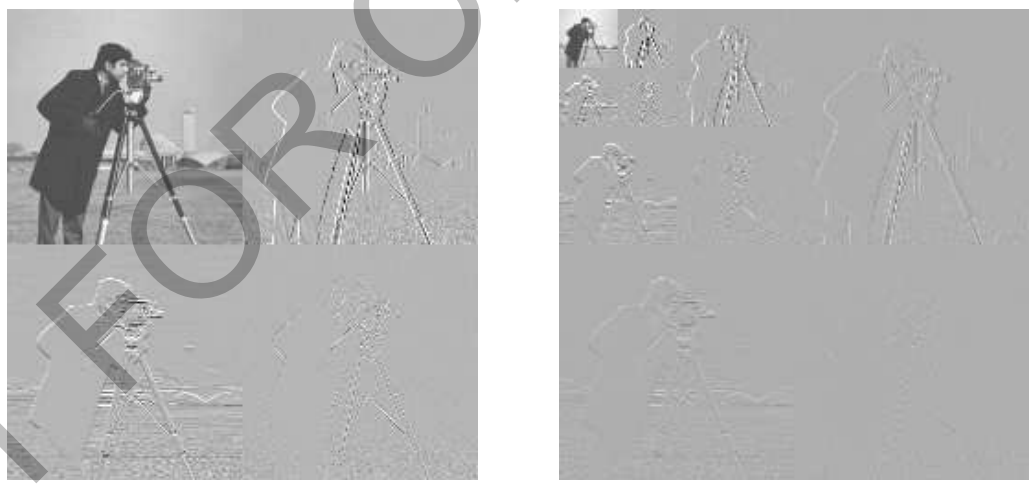
```
ans =

    36    11    22     9     2    -1     2    -2
```

Let's try an image. We shall apply the Haar wavelet to an image of size 256×256, at one and three scales. For display, it will be necessary to scale parts of the image for viewing.

```
>> c=imread('cameraman.tif');
>> [h,g,rh,rg]=daub(2);
>> cw1=wt2d(double(c),h,g,1);
>> m1=2.4*ones(2,2);m1(1,1)=1;m1=imresize(m1,128,'nearest');
>> imshow(mat2gray(cw1).*m1)
>> cw3=wt2d(double(c),h,g,3);
>> m3=2.4*ones(8,8);m3(1,1)=1;m3=imresize(m3,32,'nearest');
>> figure,imshow(mat2gray(cw3).*m3)
```

and the result is shown in figure 15.6. In each case the scaling matrix `m1` or `m3` simply has ones over



(a) One scale (b) Three scales

Figure 15.6: Different scales of DWT applied to an image

the image in the upper left, and 2.4's everywhere else.

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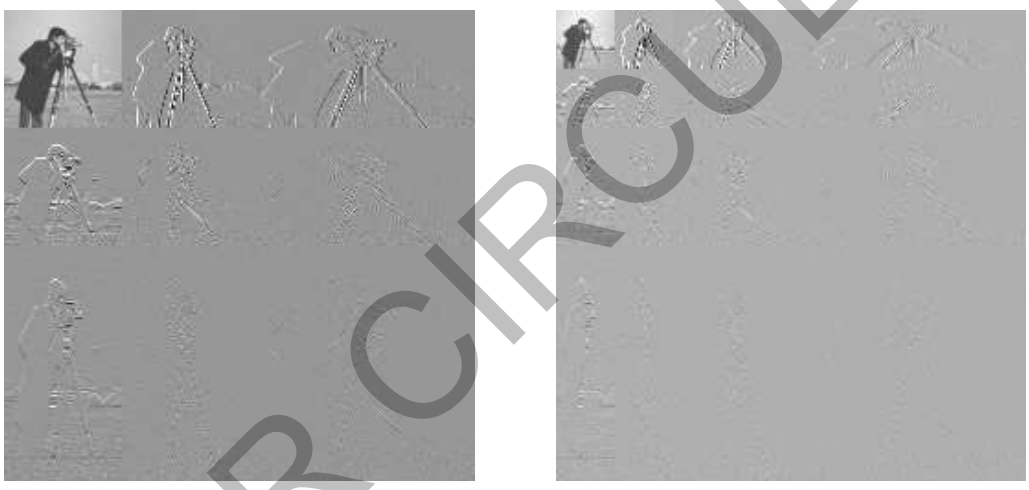
We notice that this wavelet transform uses the non-standard decomposition. To see what a standard decomposition looks like, we use the `wt` function, which when applied to a matrix, produces the DWT's of all the rows. So we perform a `wt` operation, followed by a transposition (to bring the columns into row positions), then another `wt`, and finally another transposition.

```
>> cw2=wt(wt(double(c),h,g,2)',h,g,2)';
>> m2=2.4*ones(4,4);m2(1,1)=1;m2=imresize(m2,64,'nearest');
>> cw3=wt(wt(double(c),h,g,3)',h,g,3)';
>> imshow(mat2gray(cw2).*m2)
>> figure,imshow(mat2gray(cw3).*m3)
```

This is shown in figure 15.7. Although it is not easy to see, you can just notice that in the standard



(a) Two scales (b) Three scales

Figure 15.7: Different scales of DWT applied to an image, using the standard decomposition

decomposition, the filtered images in the transform are squashed, rather than all retaining their shape as in the non-standard decomposition.

15.4 The Daubechies wavelets

We have seen that the dilation and wavelet equations for the Haar wavelet are particular examples of the general equations 15.6 and 15.7. However, it is not at all obvious that there are any other

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solutions. One of the many contributions of Ingrid Daubechies to the theory of wavelets was to define an entire class of wavelets which may be considered as solutions to these equations.

The Daubechies-4 wavelet has a scaling function $\phi(x)$ and wavelet function $\psi(x)$ which satisfy the equations:

$$\phi(x) = h_0\phi(2x) + h_1\phi(2x-1) + h_2\phi(2x-2) + h_1\phi(2x-2) \tag{15.10}$$

$$\psi(x) = h_0\phi(2x-1) - h_1\phi(2x) + h_2\phi(2x+1) - h_3\phi(2x+2) \tag{15.11}$$

where the values of the filter coefficients are:

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}} \approx 0.48296$$

$$h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}} \approx 0.83652$$

$$h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}} \approx 0.22414$$

$$h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}} \approx -0.12941$$

These can be obtained with the `daub` function of `UviWave`:

```
>> [h,g,rh,rg]=daub(4)
h =
   -0.1294    0.2241    0.8365    0.4830
g =
   -0.4830    0.8365   -0.2241   -0.1294
rh =
    0.4830    0.8365    0.2241   -0.1294
rg =
   -0.1294   -0.2241    0.8365   -0.4830
```

The four vectors contain the same values, but in different orders and with different signs. As for the Haar wavelet, we can apply the Daubechies-4 wavelet by a matrix multiplication; the matrix

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for a 1-scale DWT on a vector of length 8 is

$$\begin{bmatrix} h_0 & h_1 & h_2 & h_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_0 & h_1 & h_2 & h_3 \\ h_2 & h_3 & 0 & 0 & 0 & 0 & h_0 & h_1 \\ h_3 & -h_2 & h_1 & -h_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_3 & -h_2 & h_1 & -h_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_3 & -h_2 & h_1 & -h_0 \\ h_1 & -h_0 & 0 & 0 & 0 & 0 & h_3 & -h_2 \end{bmatrix}$$

Notice that the filter coefficients overlap between rows, which is not the case for the Haar matrix H_{2^n}. This means that the use of the Daubechies-4 wavelet will have smoother results than using the Haar wavelet. The form of the matrix above is similar to circular convolution with the one-dimensional filters

$$\begin{bmatrix} h_0 & h_1 & h_2 & h_3 \end{bmatrix}$$

and

$$\begin{bmatrix} h_3 & -h_2 & h_1 & -h_0 \end{bmatrix}.$$

The discrete wavelet transform can indeed be approached in terms of filtering; the above two filters are then known as the low pass and high pass filters respectively. Steps for performing a 1-scale wavelet transform are given by Umbaugh [34]:

1. Convolve the image rows with the low pass filter.

2. Convolve the columns of the result of step 1 with the low pass filter, and rescale this to half its size by subsampling.

3. Convolve the result of step 1 with the high pass filter, and again subsample to obtain an image of half the size.

4. Convolve the original image rows with the high pass filter.

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5. Convolve the columns of the result of step 4 with the low pass filter, and rescale this to half its size by subsampling.

6. Convolve the result of step 4 with the high pass filter, and again subsample to obtain an image of half the size.

At the end of these steps there are four images, each half the size of the original. They are:

1. the "low pass/low pass" image (LL); result of step 2.

2. the "low pass/high pass" image (LH); result of step 3.

3. the "high pass/low pass" image (HL); result of step 5.

4. the "high pass/high pass" image (HH); result of step 6.

These can then be placed into a single image grid as shown in figure 15.8.

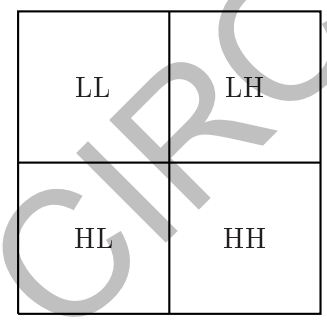

Figure 15.8: The 1-scale wavelet transform in terms of filters

The filter coefficients of a wavelet are such that the transform may be inverted precisely to recover the original image. Using filters, this is done by taking each subimage, zero-interleaving to produce image of double the size, and convolving with the inverse low pass and high pass filters. Finally the results of all the filterings are added. For the Daubechies-4 wavelet, the inverse low pass and high pass filters are:

$$\begin{bmatrix} h_2 & h_1 & h_0 & h_3 \end{bmatrix}$$

and

$$\begin{bmatrix} h_3 & -h_0 & h_1 & -h_2 \end{bmatrix}$$

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respectively.

A further approach to the discrete wavelet transform may be considered as a generalization to filtering; it is called *lifting*, and is discussed by Jensen and la Cour-Harbo [17]. Lifting starts with a sequence $s_j[n], n = 0 \ldots 2^j - 1$ and produces two sequences $s_{j-1}[n], n = 0 \ldots 2^{j-1} - 1$ and $d_{j-1}[n], n = 0 \ldots 2^{j-1} - 1$, each half the length of the original. The lifting scheme for the Haar wavelet may be described as

$$\begin{aligned} d_{j-1}[n] &= s_j[2n+1] - s_j[2n] \\ s_{j-1}[n] &= s_j[2n] + d_{j-1}[n]/2, \end{aligned}$$

and a lifting scheme for the Daubechies-4 wavelet is:

$$\begin{aligned} s^{(1)}_{j-1}[n] &= s_j[2n] + \sqrt{3}s_j]2n+1] \\ d^{(1)}_{j-1}[n] &= s_j[2n+1] - \frac{1}{4}\sqrt{3}s^{(1)}_{j-1}[n] - \frac{1}{4}(\sqrt{3}-2)s^{(1)}_{j-1}[n-1] \\ s^{(2)}_{j-1} &= s^{(1)}_{j-1}[n] - d^{(1)}_{j-1}[n+1] \\ s_{j-1}[n] &= \frac{\sqrt{3}-1}{\sqrt{2}} s^{(2)}_{j-1}[n] \\ d_{j-1}[n] &= \frac{\sqrt{3}+1}{\sqrt{2}} d^{(1)}_{j-1}[n]. \end{aligned}$$

Both the Haar and Daubechies-4 schemes can be reversed; see [17] for details. A single lifting, as shown above, produces a 1-scale wavelet transform. To produce transforms of higher scales, lifting can be applied to s_{j-1} to produce s_{j-2} and d_{j-2}, and then to s_{j-2}, and so on. In each lifting scheme, the sequence s_{j-1} may be considered as the subsampled low pass version of s_j, and the sequence d_{j-1} may be considered as the subsampled high pass version of s_j. Thus a single lifting scheme provides, in effect, both the low pass and high pass filter results. Applying a lifting to an image, first to all the rows, and then to all the columns of the result, will produce a 1-scale wavelet transform.

We might hope that since the scaling and wavelet functions $\phi(x)$ and $\psi(x)$ have such simple forms for the Haar wavelet, as we saw in figure 15.3, the same would be true for the Daubechies-4 wavelet. We can plot a wavelet by inverting a "basis vector": a vector consisting all of zeros except for a single one. Following [17] we shall choose a vector whose sixth element is one. We shall

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experiment first with the Haar wavelet:

```
>> [h,g,rh,rg]=daub(2);
>> t=zeros(1,512);t(6)=1;
>> td=iwt(t,rh,rg,9);
>> plot([1:512]/512,td)
```

The result is shown in figure 15.9. Other basis vectors produce similar results, but with the wavelet

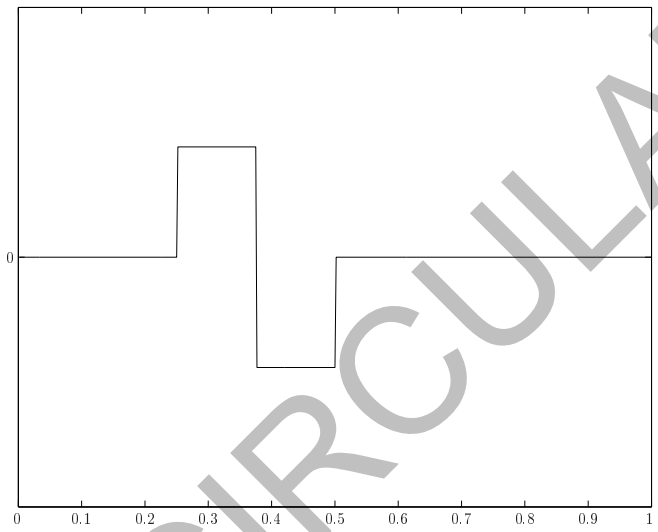

Figure 15.9: A plot of the Haar wavelet

scaled or translated. The vector we have chosen seems to produce the best plot. For the Daubechies-4 wavelet, we repeat the above four commands, but starting with

```
>> [h,g,rh,rg]=daub(4);
```

The result is shown in figure 15.10. This is a very spiky and unusual graph! It is in fact a fractal, and unlike the Haar wavelet, the function cannot be described in simple terms.

15.5 Image compression using wavelets

Wavelets provide some of the most powerful tools known for image compression. As we stated earlier, they have replaced the use of the DCT in the JPEG2000 algorithm. In this section we shall

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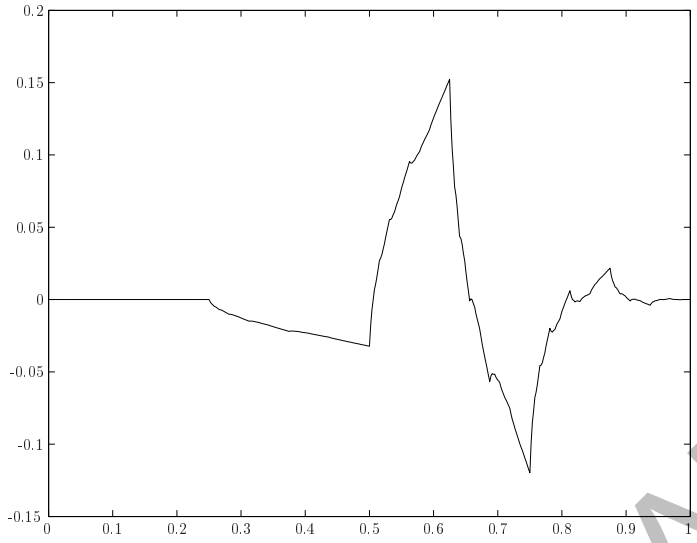

Figure 15.10: A plot of the Daubechies-4 wavelet

play around with investigating what information we can remove from the DWT of an image, and still retain most of the original information.

Thresholding and quantization

The idea is to take the DWT of an image, and for a given value d, set all values x in the DWT for which $|x| \leq d$ to zero. This is at the basis of the JPEG2000 compression [33]. We shall try this using the Daubechies-4 wavelets, and $d = 10$.

```
>> c=imread('caribou.tif');
>> imshow(c);
>> cw=wt2d(double(c),h,g,8);
>> length(find(abs(cw)<=10))

ans =

       47427
>> cw(find(abs(cw)<=10))=0;
>> cw=round(cw);
```

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```
>> ci=iwt2d(cw,rh,rg,8);
>> imshow(mat2gray(ci))
```

We are thus removing nearly three quarters of the information from the DWT. We further simplify the result by quantizing; in this case we just use the `round` function to turn fractions into integers. The original caribou image and the result of the above commands are shown in figure 15.11. The

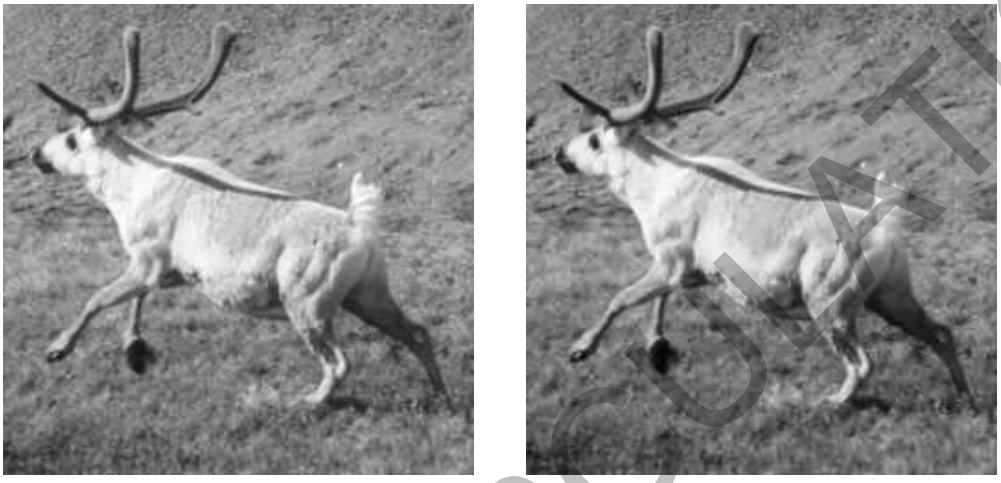

Figure 15.11: An image after DWT thresholding and inversion

result is indistinguishable from the original image. We can use different wavelets, by using either `daub(2)` or `daub(6)` to obtain the Haar filter coefficients, or the filter coefficients of the Daubechies-6 wavelet. The results of thresholding (again with $d = 10$) are shown in figures 15.12(a) and (b). We could clearly obtain higher compression rates by removing more information; this can be obtained by increasing the threshold value d. We will try with $d = 30$ and $d = 50$.

```
>> cw=wt2d(double(c),h,g,8);
>> length(find(abs(cw)<=30))

ans =

       61182

>> cw(find(abs(cw)<=30))=0;
```

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(a) Using the Haar wavelet

(b) Using the Daubechies-6 wavelet

Figure 15.12: Use of different wavelets

```
>> cw=round(cw);
>> ci=iwt2d(cw,rh,rg,8);
>> imshow(mat2gray(ci))
>> cw=wt2d(double(c),h,g,8);
>> length(find(abs(cw)<=50))

ans =

       63430

>> cw(find(abs(cw)<=50))=0;
>> cw=round(cw);
>> ci=iwt2d(cw,rh,rg,8);
>> figure,imshow(mat2gray(ci))
```

The results are shown in figure 15.13. Notice that with a threshold value of 50 we are throwing away 63430 items of information, leaving only 2106 non-zero values. However, the result is extremely good, when compared with the results of JPEG compression as shown in figure 14.10 in chapter 14.

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Figure 15.13: Results of wavelet compression with $d = 30$ and $d = 50$

As we did in chapter 14, we shall investigate the appearance of the image with a closeup. Figure 15.14 shows the closeup after thresholding at $d = 10$ using both the Haar wavelets and the Daubechies-6 wavelets. Figure 15.15 shows the closeup after thresholding at $d = 30$ and $d = 50$

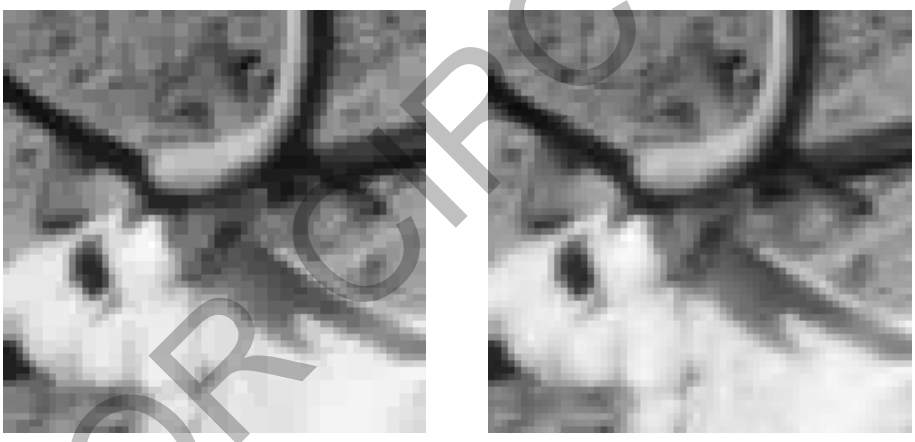

Figure 15.14: Closeups of wavelet compression using $d = 10$

using the Daubechies-6 wavelets. Even though we can see that we are losing some fine detail, the result is far superior to the result obtained by using the JPEG compression algorithm.

Extraction

Here we cut off a portion of the DWT; setting all values to zero, and invert the rest. This is not a standard method of compression, but it does gives reasonably good results. It also has the advantage

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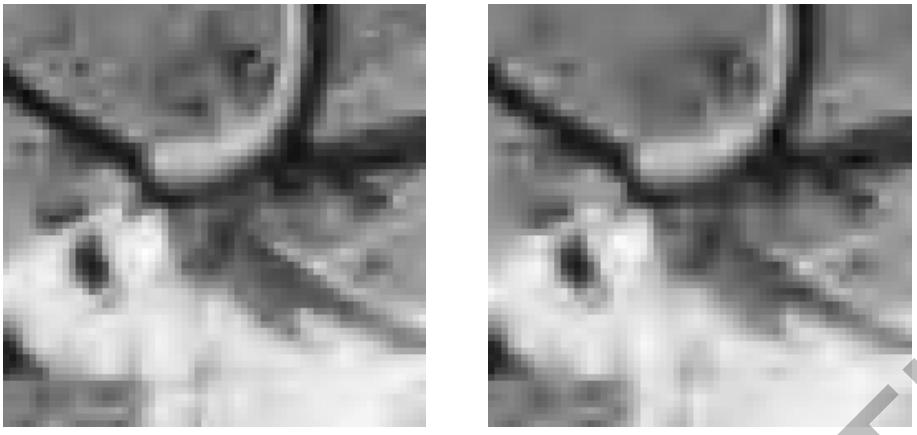

Figure 15.15: Closeups of wavelet compression using $d = 30$ and $d = 50$

of being computationally easier than thresholding and quantization. For example, suppose we keep just the upper left 100×100 elements of the transform:

```
>> [h,g,rh,rg]=daub(4);
>> cw=wt2d(double(c),h,g,8);
>> temp=zeros(size(c));
>> temp(1:100,1:100)=cw(1:100,1:100);
>> ci=iwt2d(temp,rh,rg,8);
>> imshow(mat2gray(ci))
```

and the result is shown in figure 15.16. Notice that whereas we took only 10,000 elements of the

Figure 15.16: An image after DWT extraction and inversion

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65,536 elements of the transform—less than a sixth—the output is remarkably good. Figure 15.17 shows the results after two more extractions. On the left, we used the top left 50×50 elements of

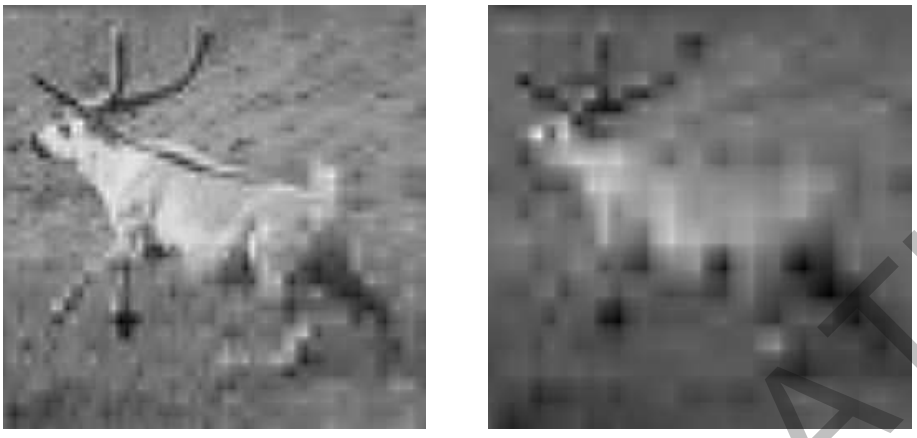

Figure 15.17: Compression by DWT extraction

the DWT; on the right only 20×20 elements. The left hand figure is getting quite blurry: but as we have used only 2500 of the possible 65,536 elements of the transform, this is to be expected. On the right, with only 400 elements used, the figure is basically unrecognizable. Even so, we can still distinguish size, shapes, and greyscales.

15.6 High pass filtering using wavelets

If we look at a wavelet transform, then apart from the rescaled image in the top left, the rest is high frequency information. We would expect then, that if we were to eliminate the top left corner, by setting all the transform values to zero, the result after inversion would be a high-pass filtered version of the original image.

We will perform a 2-scale decomposition (with Daubechies-4) on the caribou image, and remove the image from the top left.

```
>> cw=wt2d(double(c),h,g,2);
>> cw(1:64,1:64)=0;
>> ci=iwt2d(cw,rh,rg,2);
>> imshow(mat2gray(ci))
```

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This is shown on the left in figure 15.18. We can do the same thing but using a 3-scale decomposition; this will mean that the corner image will be half the size.

```
>> cw=wt2d(double(c),h,g,3);
>> cw(1:32,1:32)=0;
>> ci=iwt2d(cw,rh,rg,3);
>> imshow(mat2gray(ci))
```

This is shown on the right in figure 15.18. These images can be thresholded to produce edge images.

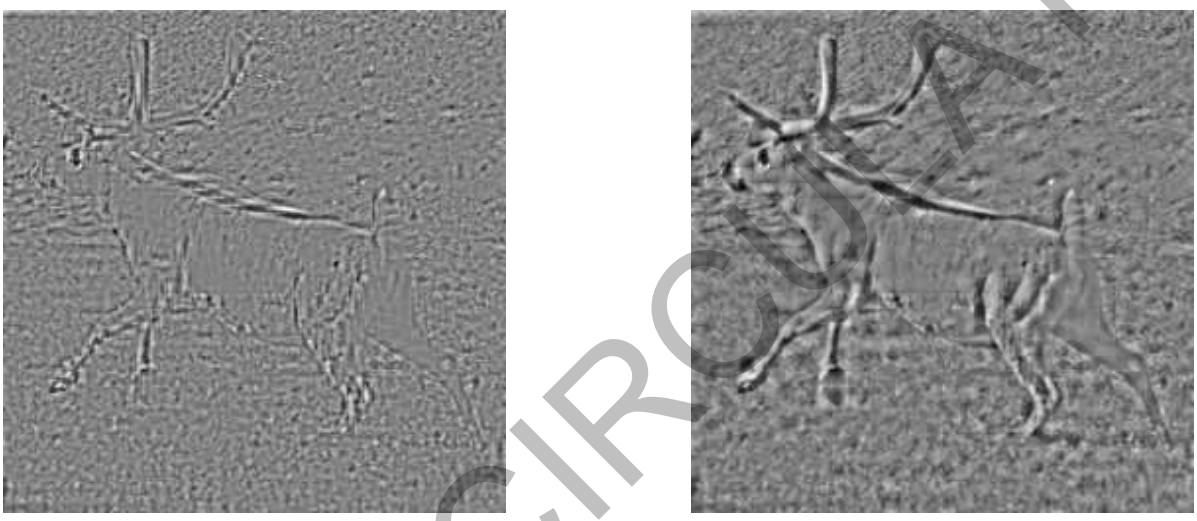

Figure 15.18: High Pass filtering with wavelets

We can be more specific in the part of the transform we eliminate or keep, and we can extract the horizontal or vertical edges only.

15.7 Denoising using wavelets

Since noise can be considered as a high frequency component, some form of low pass filtering should remove it. As we did for compression, we can remove such components by thresholding. To compare with the results obtained in chapter 8, we shall use a grey version of the twins image, converted to type `double`.

```
>> tw=imread('twins.tif');
```

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```
>> t=im2double(rgb2gray(tw));
>> tg=imnoise(t,'gaussian');
```

Now we can attempt some noise removal. We shall assume that all the filter coefficients are those of the Daubechies-4 wavelet.

```
>> imshow(tg);
>> tw=wt2d(tg,h,g,4);
>> length(find(abs(tw)<0.1))

ans =

       42384

>> tw(find(abs(tw)<0.15))=0;
>> ti=iwt2d(tw,rh,rg,4);
>> figure,imshow(mat2gray(ti))
```

The noisy image and the result after wavelet filtering are shown in figure 15.19. At this level

Figure 15.19: Denoising using wavelets

of thresholding, not much of the noise has been removed, although there has been some slight reduction. Figure 15.20 shows the results using (a) a threshold of 0.2 and (b) a threshold of 0.5.

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The results compare very favourable to those we investigated in chapter 8; however we can see that

(a) (b)

Figure 15.20: Denoising using different thresholds

too high a threshold results in unacceptable blurring of the image.

Exercises

1. For each of the sequences

   (a) `20 38 6 25 30 29 21 32`

   (b) `30 11 0 38 0 15 22 32`

   (c) `7 38 19 11 24 14 32 14`

   apply, by hand, the simple wavelet transform discussed in section 15.1 at 1, 2 and 3 scales.

2. Check your answers with MATLAB using the filter coefficients

```
>> h=[1 1]/2,g=[-1 1]/2
```

3. Work backward from the transforms and obtain the original sequence.

4. Determine the filter coefficients `rh` and `rg` to perform the calculations in the previous question.

5. Repeat the above calculations in MATLAB using the Haar wavelet transform; obtaining the filter coefficients with the command

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```
>> [h,g,rh,rg]=daub(2)
```

6. Obtain a 256×256 greyscale image. Obtain the DWT at 1, 2 and 3 scales using `daub(2)`, `daub(4)` and `daub(6)`.

   Are there any observable differences in the outputs of the transform for the different wavelets?

7. Write a MATLAB function `showdwt` which will scale the result of a two-dimensional wavelet transform for easier viewing.

8. Obtain the following greyscale image:

```
>> f=imread('flowers.tif');
>> fg=rgb2gray(f);
>> f=im2uint8(f(30:285,60:315));
```

   Perform the forward DWT at 8 scales using the Daubechies-4 wavelet. For each of the following values of d:

   (a) 10, (b) 25, (c) 50, (d) 60

   - make all elements of the transform whose absolute values are less than d equal to zero,
   - determine the number of zeros in the result,
   - quantize the result by using the `floor` or `round` functions,
   - invert the result and display.

   At what values of d do you notice degradation in the result?

9. Repeat the above question for some very large values of d: 100, 200, 300,...

   What size of d renders the image completely unrecognizable? How many zeros are in the transform?

10. Repeat the above two questions with an image of a face. If you haven't got one, you can obtain some faces with:

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```
>> load gatlin.mat
>> g=im2uint8(ind2gray(X(31:286,41:296),map));
```

11. Choose any of your 256×256 greyscale images. Add Gaussian noise with mean 0, and with standard variations 0.01, 0.02, 0.05, and 0.1. For each noisy image, attempt to denoise it with wavelets.

    How reasonable a result can you obtain for the noise with the higher standard deviations?

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Chapter 16

Special effects

We have seen a great many different methods of changing the appearance of an image, and usually our need for the particular change was motivated by a problem: we may need to remove something from an image (noise, for example); we may need to make the image "look better"; we may need to calculate some aspect of the image (size, position, number of components...)

But there is another totally different aspect of image processing, that of adding some sort of special effect to an image. We may want to do this for effect alone, or simply for fun. There are a great many different effects possible, and in this chapter we shall look at a few of them. This is a sort of grey area of image processing; we are encroaching toward the area of computer graphics. However, our algorithms involve the changing either of the value or the position of pixels, and we can achieve many affects with very simple means.

16.1 Polar coordinates

Many effects have a "radial" nature, their appearance "radiates" outward from the centre of the image. In order to achieve these effects, we need to transform our image from cartesian coordinates to polar coordinates, and back again.

The origin. Suppose we have an image of size `rows` $\times$ `cols`. If the dimensions are odd, we can choose as our origin the very centre pixel. If the dimensions are even, we choose as origin a pixel at the top left of the bottom right quadrant. so in a 7×9 image, our origin would be at pixel $(4, 5)$,

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and in a 6×8 image, our origin would be at position $(4, 4)$. We can express the origin formally by

$$\begin{aligned} x_0 &= \lceil (r+1)/2 \rceil \\ y_0 &= \lceil (c+1)/2 \rceil \end{aligned}$$

where r and c are the numbers of rows and columns respectively, and $\lceil x \rceil$ is the *ceiling function*, which returns the smallest integer not less than x. Of course if one dimension is odd and the other even the origin will be at the centre of the odd dimension, and in the first place of the second half of the even dimension.

Thus we can find coordinates of the origin by:

```
>> ox=ceil((rows+1)/2)
>> oy=ceil((cols+1)/2)
```

Polar coordinates. To find the polar coordinates, we apply the usual formulas

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}(y/x), \end{aligned}$$

which can be implemented for an array by using the the `meshgrid` function, where the indices are offset by the values of the origin:

```
>> [y,x]=meshgrid([1:cols]-oy,[1:rows]-ox);
>> r=sqrt(x.^2+y.^2);
>> theta=atan2(y,x);
```

We can easily go back, using the fact that the (x, y) coordinates corresponding to the polar coordinates (r, θ) are

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta. \end{aligned}$$

Thus:

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```
>> x2=round(r.*cos(theta))+ox
>> y2=round(r.*sin(theta))+oy
```

and these arrays will contain the same indices as the original arrays.

An example: radial pixelization

We have seen in chapter 3 that an image can be "pixelated"; that is, be shown in large blocks of low resolution, by use of the `imresize` function. We can achieve the same effect, however by using the `mod` function. In general, $\mathrm{mod}(x, n)$ is the remainder when x is divided by n. For example:

```
>> x=1:12

x =

     1     2     3     4     5     6     7     8     9    10    11    12

>> mod(x,4)

ans =

     1     2     3     0     1     2     3     0     1     2     3     0
```

So if x is divisible by 4, then there is no remainder, and so the `mod` function returns 0. Now if we *subtract* the mod values form the original, we obtain repetition, which in an image would produce a pixelated effect:

```
>> x-mod(x,4)

ans =

     0     0     0     4     4     4     4     8     8     8     8    12
```

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So a subtraction of moduli from the radius and argument values should produce a radial pixelated effect.

We can try it with the flowers image, which for simplicity we shall turn into a greyscale image.

```
>> f=imread('flowers.tif');
>> fg=rgb2gray(f);
>> [rows cols]=size(fg)

rows =

   362



cols =

   500

>> ox=ceil((rows+1)/2)

ox =

   182

>> oy=ceil((cols+1)/2)

oy =

   251

>> [y,x]=meshgrid([1:cols]-oy,[1:rows]-ox);
```

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```
>> r=sqrt(x.^2+y.^2);
>> theta=atan2(y,x);
```

This is just the standard setup, now we perform the pixelation:

```
>> r2=r-mod(r,5);
>> theta2=theta-mod(theta,0.087);
```

Since the angles are given as radians, we use a small value in the modulus; here $0.087 = 5\pi/180$.

Now we transfer back into cartesian coordinates:

```
>> x2=r2.*cos(theta2);
>> y2=r2.*sin(theta2);
```

To obtain an image for display, we have to adjust the indices `x2` and `y2` by adding `ox` and `oy` to them, rounding off, and making sure that the row indices are between 1 and `rows`, and that the column indices are between 1 and `cols`:

```
>> xx=round(x2)+ox;
>> yy=round(y2)+oy;
>> xx(find(xx>rows))=rows;
>> xx(find(xx<1))=1;
>> yy(find(yy>cols))=cols;
>> yy(find(yy<1))=1;
```

Now we can use these new indices `xx` and `yy` to obtain an image from the original flower image matrix `fg`. To do this, we could use a double loop:

```
>> for i=1:rows,...
     for j=1:cols,...
        f2(i,j)=fg(xx(i,j),yy(i,j));...
     end;...
   end;
```

or more simply the `sub2ind` function, which does exactly the same thing:

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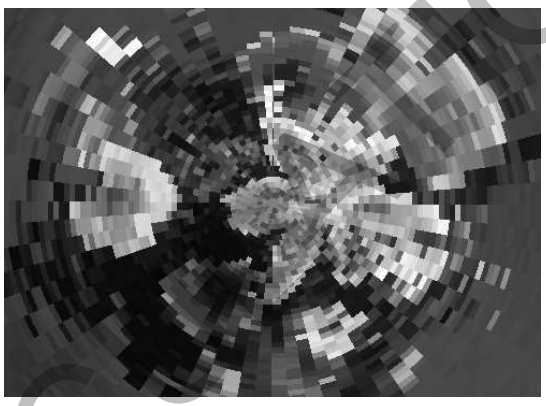
```
>> f2=fg(sub2ind([rows,cols],xx,yy));
```

Now we can display the original and the result:

```
>> imshow(fg),figure,imshow(f2)
```

and the results are shown in figure 16.1.

The original `fg`



The pixelated `f2`

Figure 16.1: Radial pixelation

16.2 Ripple effects

We will investigate two different ripple effects: "bathroom glass" ripples, which give the effect of an image seen through wavy glass, such as is found in bathrooms; and "pond ripples" which approximate a reflection in the surface of a pond.

We have seen that subtracting moduli produces a pixelated effect. A bathroom glass effect can be obtained by *adding* moduli:

```
>> x=1:12;
>> x+mod(x,4)


ans =
```

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```
     2     4     6     4     6     8    10     8    10    12    14    12
```

Notice that although the values increase from left to right, they do in small runs, which seem to overlap. On an image, the appearance will be of a ripple across the image. We shall experiment first using cartesian coordinates:

```
>> [y,x]=meshgrid(1:cols,1:rows);
>> y2=y+mod(y,32);
>> y2(find(y2<1))=1;
>> y2(find(y2>cols))=cols;
```

Here we have just added moduli across the columns, then adjusted the result to ensure that the values stay with the column range. Now we can use `sub2ind` to create a new image:

```
>> ripple1=fg(sub2ind([rows cols],x,y2));
>> imshow(ripple1)
```

We can of course do the same thing down the rows:

```
>> x2=x+mod(x,32);
>> x2(find(x2<1))=1;
>> x2(find(x2>rows))=rows;
>> ripple2=fg(sub2ind([rows cols],x2,y));
>> imshow(ripple2)
```

or even both at once:

```
>> ripple3=fg(sub2ind([rows cols],x2,y2));
>> imshow(ripple3)
```

To obtain a radial ripple, corresponding to ripples on a pond, we create our polar coordinate matrices `r` and `theta` as above, and then:

```
>> r2=r+mod(r,10);
>> x2=r2.*cos(theta);
```

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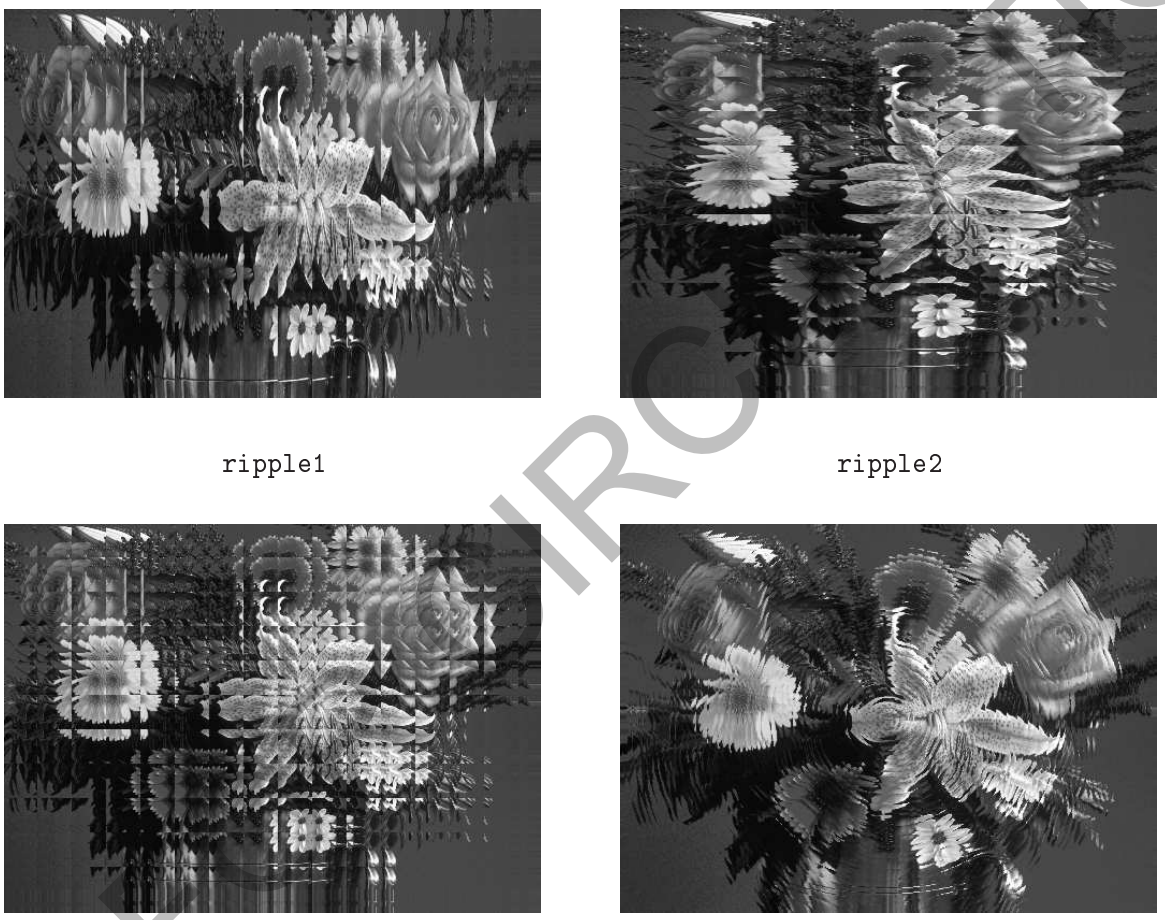
```
>> y2=r2.*sin(theta);
```

Now we create the indices `xx` and `yy` as in the pixelation example above, and:

```
>> ripple4=fg(sub2ind([rows,cols],xx,yy));
>> imshow(ripple4)
```

All results are shown in figure 16.2.



`ripple1` `ripple2`

`ripple3` Radial ripple: `ripple4`

Figure 16.2: "Bathroom glass" rippling on an image

To obtain the pond ripple effect, we move the pixels around by using a sine wave. For cartesian effects:

```
>> [y,x]=meshgrid(1:cols,1:rows);
```

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```
>> y2=round(y+3*sin(x/2));
>> x2=round(x+3*sin(y/2));
```

As above, we ensure that the values of `x2` and `y2` fit within the row and column bounds. Then:

```
>> ripple5=fg(sub2ind([rows cols],x,y2));
>> ripple6=fg(sub2ind([rows cols],x2,y));
>> ripple7=fg(sub2ind([rows cols],x2,y2));
```

For a radial sine wave ripple, we adjust `r` first:

```
>> r2=r+sin(r/2)*3;
```

After that we create `x2` and `y2` as above, and turn them back into indices `xx` and `yy` with the appropriate bounds. Then

```
>> ripple8=fg(sub2ind([rows cols],xx,yy));
```

These new ripples are shown in figure 16.3. The effects can be varied by changing the parameters to the sine functions in the commands above.

16.3 General distortion effects

The rippling effects in the previous section are examples of more general effects called *distortion effects*. Here it is not so much the *value* of the pixels which are changed as their *position*. All distortion effects work in much the same way, which we shall describe in a sequence of steps. Suppose we have an effect which involves cartesian coordinates only.

1. Start with an image matrix $p(i, j)$, of size $m \times n$.

2. Produce two new indexing arrays (the same size as the matrix): $x(i, j)$ and $y(i, j)$.

3. Create a new image $q(i, j)$ by

$$q(i, j) = p(x(i, j), y(i, j)).$$

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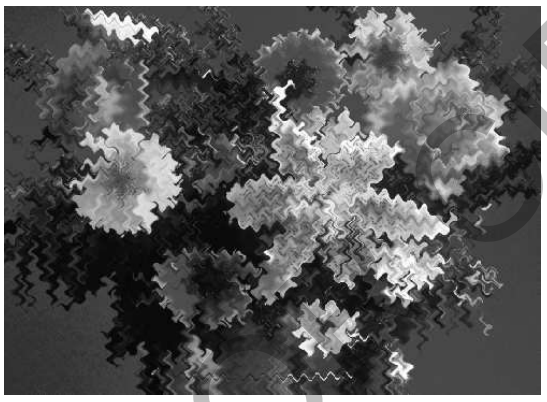
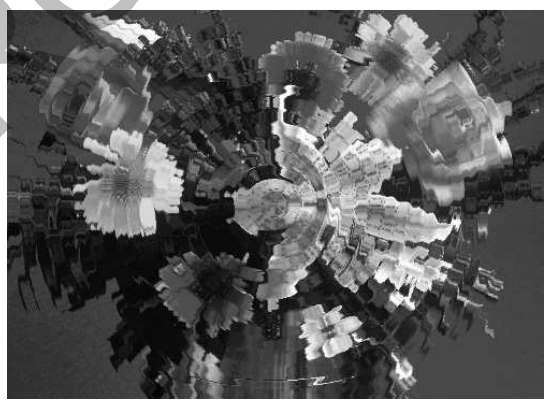
`ripple5`

`ripple6`



`ripple7`



`ripple8`

Figure 16.3: "Pond rippling" on an image

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We have seen above that the arrays $x(i,j)$ and $y(i,j)$ may be created by a procedure which produces values which may be fractional, or be outside the bounds of the image matrix. So we will need to ensure that the indexing arrays contain integers only, and that for all i and j, that $1 \leq x(i,j) \leq m$, and $1 \leq y(i,j) \leq n$.

If we have an effect which requires polar coordinates, we add a few steps to the procedure above:

1. Start with an image matrix $p(i,j)$, of size $m \times n$.

2. Create the arrays $r(i,j)$ and $\theta(i,j)$ containing the polar coordinates of each position (i,j).

3. Produce new arrays $s(i,j)$ and $\phi(i,j)$ by adjusting the values of r and θ.

4. Write these new arrays out to cartesian indexing arrays (the same size as the matrix):

$$\begin{aligned} x(i,j) &= s(i,j)\cos(\phi(i,j)) + o_x \\ y(i,j) &= s(i,j)\sin(\phi(i,j)) + o_y \end{aligned}$$

   where (o_x, o_y) are the coordinates of the polar origin.

5. Create a new image $q(i,j)$ by

$$q(i,j) = p(x(i,j), y(i,j)).$$

As above, we need to ensure that the indexing arrays contain only integers, and have the appropriate bounds.

We have seen above how to implement various distortion effects in MATLAB, so just by changing the functions used to distort the indexing arrays or the polar coordinates, we can achieve many different effects.

Fisheye. The "fisheye lens" effect can be obtained very easily with polar coordinates. Here are the relevant commands:

```
>> R=max(r(:));
>> r=r.^2/R;
```

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To put these into context, we shall list the entire sequence of commands for use with the flower matrix `fg`:

```
>> [rows,cols]=size(fg);
>> ox=ceil((rows+1)/2);
>> oy=ceil((cols+1)/2);
>> [y,x]=meshgrid([1:cols]-oy,[1:rows]-ox);
>> r=sqrt(x.^2+y.^2);
>> theta=atan2(y,x);
>> R=max(r(:));                % Here is where we implement the fisheye effect
>> s=r.^2/R;
>> x2=round(s.*cos(theta))+ox; % Now we write out to the indexing arrays...
>> y2=round(s.*sin(theta))+oy;
>> x2(find(x2<1))=1;           % ...and ensure that their bounds are correct
>> x2(find(x2>rows))=rows;
>> y2(find(y2<1))=1;
>> y2(find(y2>cols))=cols;
>> fisheye=fg(sub2ind([rows cols],x2,y2));  % Create the new image...
>> imshow(fisheye)                          % ...and view it
```

The result is shown in figure 16.4.

Twirl. A "twirl" or " swirl" effect, as of the top of a cup of coffee being swirled with a spoon, is also very easily done. Again, it is a polar effect. A twirl effect can be obtained by

$$s(i,j) = \theta(i,j) + r(i,j)/K$$

where K can be changed to adjust the amount of twirl. A small value provides more twirl, a large value less. So give the arrays `r` and `theta`, and a value `K`, the relevant commands are:

```
>> phi=theta+(r/K);
>> x2=round(r.*cos(phi))+ox;
>> y2=round(r.*sin(phi))+oy;
```

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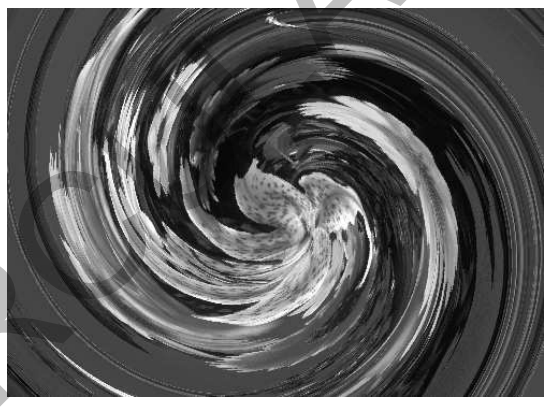
Figure 16.4 shows the results of twirls with different values of K.

`fisheye`

A twirl with $K = 200$

A twirl with $K = 100$



A twirl with $K = 50$

Figure 16.4: Fisheye and twirls

Jitter. The "jitter" effect is more easily implemented that described. It is like a radial bathroom glass effect, but instead of manipulating the `r` matrix to obtain a ripple effect, we manipulate the `theta` matrix.

```
>> phi=theta+mod(theta,0.1396)-0.0698;
```

Here $0.1396 = 8\pi/180$, and $0.0698 = 4\pi/180$. The result is shown in figure 16.5.

Circular slice. This is another effect obtained by manipulating the angle. It's implementation is similar to that of the jitter effect:

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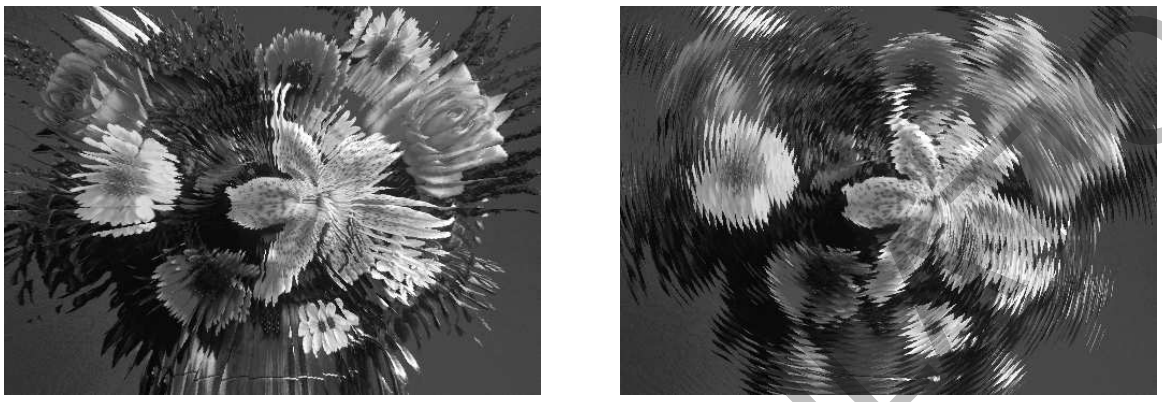
```
>> phi=theta+mod(r,6)*pi/180
```

We include the extra $\pi/180$ factor to convert to radians. The result, which has a rippling appearance, is shown in figure 16.5.



Jitter Circular slice

Figure 16.5: Jitter and circular slice

Square slice This is a cartesian effect. It works by cutting the image into small squares and perturbing them a little bit. It uses the `sign` function, which is defined as:

$$\texttt{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

We start with

```
>> [y,x]=meshgrid(1:cols,1:rows);
```

and then implement the slice distortion with

```
>> K=8;
>> Q=10;
>> x2=round(x+K*sign(cos(y/Q)));
>> y2=round(y+K*sign(cos(x/Q)));
```

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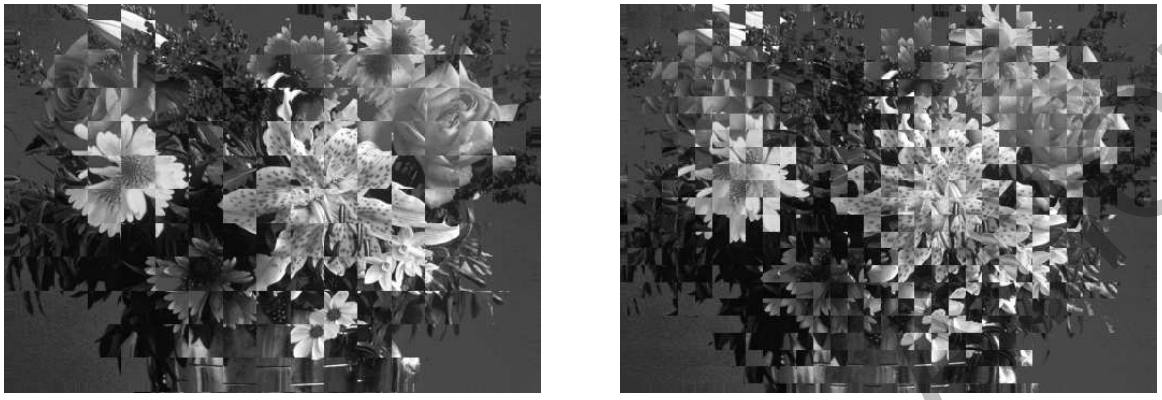
The values `K` and `Q` influence the size of the squares and the amount of perturbation. A large value of `K` produces a very perturbed image; a large value of `Q` produces large squares. After rounding, fixing the bounds and writing to an new image, results are shown in figure 16.6.



`K=8;Q=10` `K=10;Q=5`

`K=20;Q=10` `K=10;Q=5`

Figure 16.6: The "square slice" effect

Fuzzy effect. This is obtained by replacing each pixel with a randomly selected pixel from its neighbourhood. To do this, we simply add random values to the index matrices `x` and `y`. If we are choosing values from a 7×7 neighbourhood, then we randomize the indices with integer values chosen from $-3, -2, -1, 0, 1, 2, 3$. We can achieve this with

```
>> x2=x+floor(7*rand(rows,cols)-3);
>> y2=y+floor(7*rand(rows,cols)-3);
```

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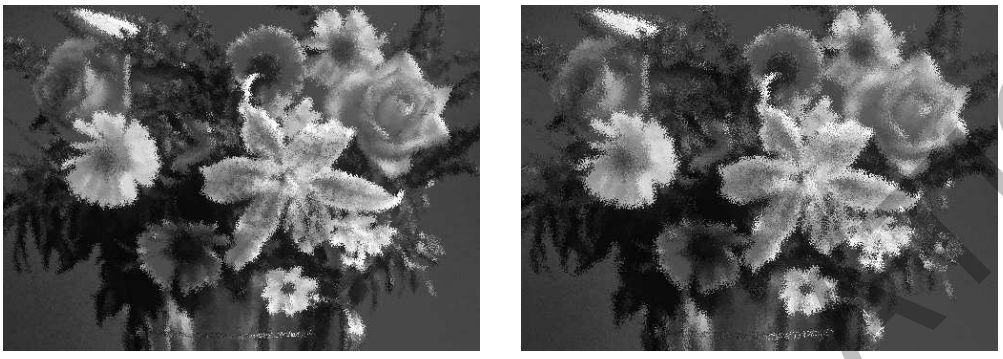
For an $(2N+1) \times (2N+1)$ neighbourhood, our randomization commands would be

```
>> x2=x+floor((2*N+1)*rand(rows,cols)-N);
```

and similarly for `y2`. Again, results are shown in figure 16.7. For a large value of N, the result



5×5 7×7

Figure 16.7: The fuzzy effect using different neighbourhoods

begins to take on a frosted glass effect.

16.4 Pixel effects

In a sense, *all* image effects are "pixel effects", since we are dealing with pixels. However, in the previous sections, we didn't change the values of the greyscales explicitly; we merely copied them to different positions in the output arrays. However, many effects take the pixel values and apply some sort of processing routines to them.

Oil painting. This is a very popular effect, and is included in most image manipulation and photo handling software. In conception it is very simple; it works by means of a non-linear filter; the output of the filter is the most common pixel value in the filter mask.

The most commonly occurring value in a neighbourhood is called the statistical *mode*. To implement this we need to write a small function. We use the `hist` function to create a list of the frequencies of each value in our neighbourhood, and find the values which occur most often. Then we pick a random one. Figure 16.8 shows our function. Now we apply it, say to the flowers image.

```
>> fo=nlfilter(double(fg),[5,5],'mode');
```

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```
function out=mode(mat)
%
% Finds a most commonly occuring value in a matrix
%
h=hist(mat(:),0:255);
temp=find(h==max(h(:)))-1;
n=floor(rand*length(temp))+1;
out=temp(n);
```

Figure 16.8: A simple program for computing the mode

```
>> imshow(uint8(fg))
```

The result is shown in figure 16.9. We can obtain a more "painterly" effect by choosing a larger block size in `nlfilter`. The use of `nlfilter` can be very slow; if you have access to the MATLAB

5×5 9×9

Figure 16.9: The oil painting effect with different filter sizes

compiler, you will achieve faster results by writing a program called, say `oilify.m`, and compiling it. Such a function is given in figure 16.10. Given the slowness of `nlfilter`, it would be much faster to use this function.

Solarization. This is a photographic effect which is obtained by applying diffuse light to a developing photograph, and then continuing with the development. The result, which seems somehow magical, is that the image can be partly positive and partly negative[1]. This effect has been used by many photographers, but can be a fairly hit or miss affair in the photographic darkroom. However, it is very easy to create solarization effects digitally.

[1] A good account of solarization can be found at `http://www.cchem.berkeley.edu/wljeme/SOUTLINE.html`

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```
function out=oilify(im,s)
%
% Produces an "oilpaint" effect by taking the pixel value most commonly
% occurring in a (2s+1)x(2s+1) neighbourhood of each pixel.
%
[rows,cols]=size(im);
out=zeros(rows,cols);
la=(2*s+1)^2;
for i=s+1:rows-s,
  for j=s+1:cols-s,
    a=double(im(i-s:i+s,j-s:j+s))+1;
    h=zeros(1,256);
    for k=1:la,
      h(a(k))=h(a(k))+1;
    end;
    [p,q]=sort(h);
    out(i,j)=q(256)-1;
  end
end
if isa(im,'uint8'),
  out=uint8(out);
end;
```

Figure 16.10: A MATLAB function for creating oil paintings

We have seen examples of solarization functions in chapter 4; now we explore how they can be applied.

A simple solarization effect can be obtained by taking the complement of all pixels in an image $p(i,j)$ whose greyscales are less than 128:

$$\mathrm{sol}(i,j) = \begin{cases} p(i,j) & \text{if } p(i,j) > 128, \\ 255 - p(i,j) & \text{if } p(i,j) \leq 128. \end{cases}$$

This is very straightforward:

```
>> fgd=double(fg);

>> u=double(fg>128);

>> sol=u.*fgd+(1-u).*(255-fgd);

>> imshow(uint8(sol)),figure,imshow(mat2gray(sol))
```

We display the result twice; the second time the image is shown with better contrast. These are shown in figure 16.11. We can make more interesting solarization effects by varying the comple-

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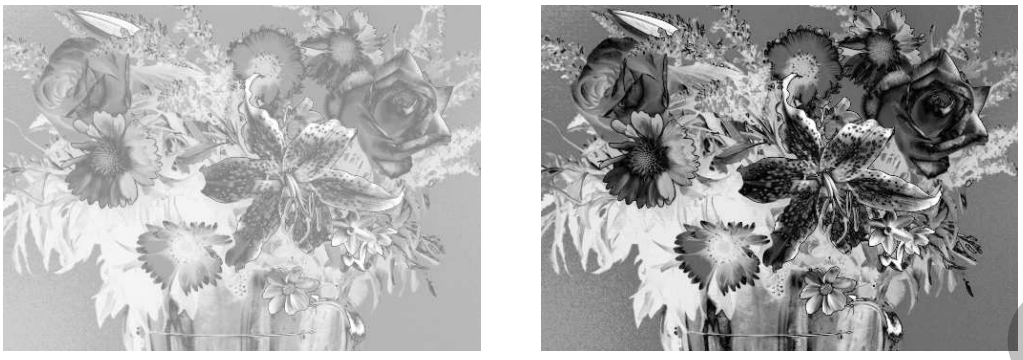

Figure 16.11: The solarization effect

mentation according to the value of the row or column. All we need to do is to change the matrix `u` above:

```
>> u1=double(fg > 255*x/(2*rows));
>> sol1=u1.*fgd+(1-u1).*(255-fgd);
>> u2=double(fg > 255*y/(2*cols));
>> sol2=u2.*fgd+(1-u2).*(255-fgd);
>> imshow(uint8(sol1)),figure,imshow(uint8(sol2))
```

The results are shown in figure 16.12.

`sol1` `sol2`

Figure 16.12: More solarization effects

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16.5 Colour images

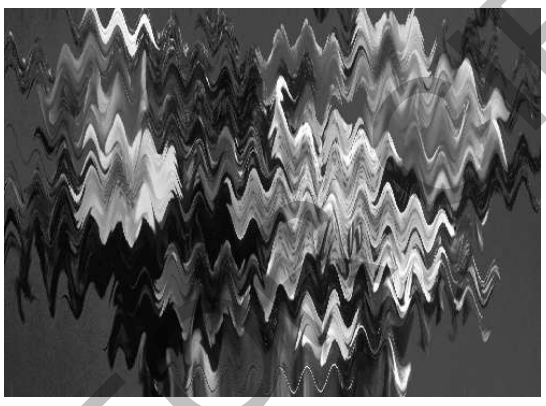
Most of the above effects look far more dramatic when applied to colour images. For all the distortion effects, application to a colour image means applying the effect to each of the RGB components separately. Recall that distortion effects don't change the values of the pixels so much as their position. Thus in order to maintain the colours, we need to shift all the RGB components.

This is most easily done with a MATLAB function. Figure 16.13 shows, as an example, how to turn the twirl process into a function which can handle both greyscale and colour images.

Oil painting is again done by applying the effect to each of the colour components separately.

Exercises

1. Experiment with changing the parameters to the effects described in this chapter. For example, see if you can use the pond ripple effect to obtain pictures like this:



2. Write some MATLAB functions to implement all the effects given. Include some parameters to change the output.

3. Extend your functions so that they can be applied to colour images.

4. If you can get hold of Holzmann's book [13], many other effects are described. For example, the following pictures show the result of a random "tile effect", where the image is broken up into squares which are then shifted by a random amount.

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```
function out=twirl(img,K)

% TWIRL(IMG,K) creates a twirl effect on an image IMG, where K provides the
% amount of twirl.  The larger the value of K, the smaller the amount of twirl.
%
% IMG can be an image of type UINT8 or DOUBLE, and greyscale or RGB.
%
%  Example:
%
%    f=imread('flowers.tif');
%    f2=twirl(f,100);
%    imshow(f2)

sf=size(size(img));
if sf(2)==2,
  out=makeTwirl(img,K);
elseif sf(2)==3,
  out=cat(3,makeTwirl(img(:,:,1),K),...
  makeTwirl(img(:,:,2),K),...
  makeTwirl(img(:,:,3),K));
end;

function res=makeTwirl(im,K)
[rows,cols]=size(im);
ox=ceil((rows+1)/2);
oy=ceil((cols+1)/2);
[y,x]=meshgrid([1:cols]-oy,[1:rows]-ox);
r=sqrt(x.^2+y.^2);
theta=atan2(y,x);
phi=theta+(r/K);
x2=round(r.*cos(phi))+ox;
y2=round(r.*sin(phi))+oy;
%
% Now fix the bounds on  x2 and y2 so that
% 1 <= x2 <= rows and 1 <= y2 <= cols.
%
x2(find(x2>rows))=rows;
x2(find(x2<1))=1;
y2(find(y2>cols))=cols;
y2(find(y2<1))=1;
%
% Use x2 and y2 to write out to the new image.
%
res=zeros(size(im));
res=im(sub2ind([rows,cols],x2,y2));
if isa(im,'uint8'),
  res=uint8(res);
end;
```

Figure 16.13: A MATLAB function for performing general twirls

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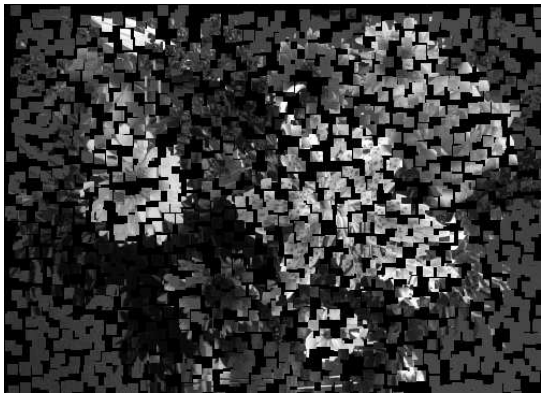

See if you can write a function to implement this effect, and then extend your function to work with colour images.

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Appendix A

Basic use of MATLAB

A.1 Introduction

MATLAB is a data analysis and visualization tool which has been designed with powerful support for matrices and matrix operations. As well as this, MATLAB has excellent graphics capabilities, and its own powerful programming language. One of the reasons that MATLAB has become such an important tool is through the use of sets of MATLAB programs designed to support a particular task. These sets of programs are called *toolboxes*, and the particular toolbox of interest to us is the *image processing toolbox*.

Rather than give a description of all of MATLAB's capabilities, we shall restrict ourselves to just those aspects concerned with handling of images. We shall introduce functions, commands and techniques as required. A MATLAB *function* is a keyword which accepts various parameters, and produces some sort of output: for example a matrix, a string, a graph or figure. Examples of such functions are `sin`, `imread`, `imclose`. There are *many*functions in MATLAB, and as we shall see, it is very easy (and sometimes necessary) to write our own. A *command* is a particular use of a function. Examples of commands might be

```
>> sin(pi/3)
>> c=imread('cameraman.tif');
>> a=imclose(b);
```

As we shall see, we can combine functions and commands, or put multiple commnds on a single

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input line.

MATLAB's standard data type is the matrix—all data are considered to be matrices of some sort. Images, of course, are matrices whose elements are the grey values (or possibly the RGB values) of its pixels. Single values are considered by MATLAB to be 1×1 matrices, while a string is merely a $1 \times n$ matrix of characters; n being the string's length.

In this chapter we will look at the more generic MATLAB commands, and discuss images in further chapters.

When you start up MATLAB, you have a blank window called the "Command Window" in which you enter commands. This is shown in figure A.1. Given the vast number of MATLAB's functions, and the different parameters they can take, a command line style interface is in fact much more efficient than a complex sequence of pull-down menus.

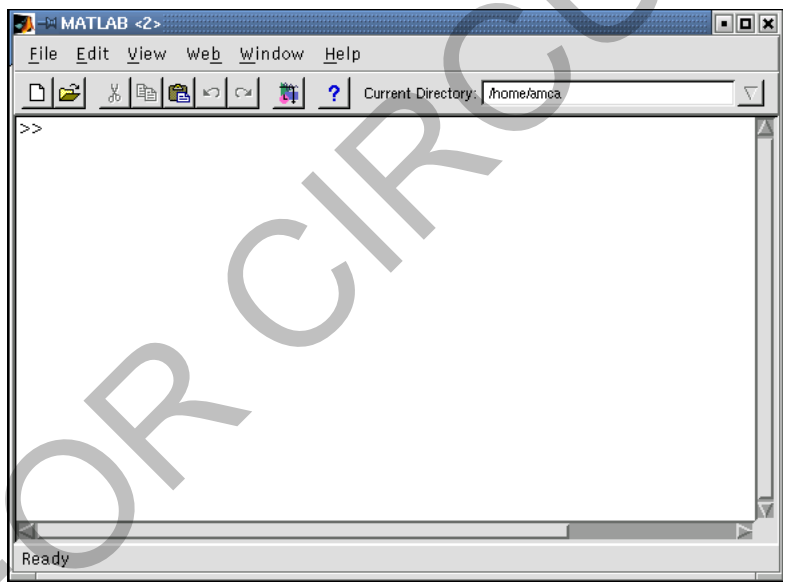

Figure A.1: The MATLAB command window ready for action

The prompt consists of two right arrows:

```
>>
```

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A.2 Basic use of MATLAB

If you have never used MATLAB before, we will experiment with some simple calculations. We first note that MATLAB is *command line driven*; all commands are entered by typing them after the prompt symbol. Let's start off with a mathematical classic:

```
>> 2+2
```

What this means is that you type in

```
2
```

at the prompt, and then press your Enter key. This sends the command to the MATLAB kernel. What you should now see is

```
ans =

     4
```

Good, huh?

MATLAB of course can be used as a calculator; it understands the standard arithmetic operations of addition (as we have just seen), subtraction, multiplication, division and exponentiation. Try these:

```
>> 3*4

>> 7-3

>> 11/7

>> 2^5
```

The results should not surprise you. Note that for the output of `11/7` the result was only given to a few decimal places. In fact MATLAB does all its calculations internally to *double precision*.

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However, the default display format is to use only eight decimal places. We can change this by using the `format` function. For example:

```
>> format long
>> 11/7

ans =

   1.57142857142857
```

Entering the command `format` by itself returns to the default format.

MATLAB has all the elementary mathematical functions built in:

```
>> sqrt(2)

ans =

    1.4142

>> sin(pi/8)

ans =

    0.3827

>> log(10)

ans =

    2.3026
```

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```
>> log10(2)


ans =


    0.3010
```

The trigonometric functions all take radian arguments; and `pi` is a built-in constant. The functions `log` and `log10` are the natural logarithm and logarithms to base 10.

A.3 Variables and the workspace

When using any sort of computer system, we need to store things with appropriate names. In the context of MATLAB, we use *variables* to store values. Here are some examples:

```
>> a=5^(7/2)


a =


  279.5085


>> b=sin(pi/9)-cos(pi/9)


b =


   -0.5977
```

Note that although `a` and `b` are displayed using the `short` format, MATLAB in fact stores their full values. We can see this with:

```
>> format long;a,format


a =
```

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```
     2.795084971874737e+02
```

We can now use these new variables in further calculations:

```
>> log(a^2)/log(5)

ans =

     7

>> >> atan(1/b)

ans =

   -1.0321
```

A.3.1 The workspace

If you are using a windowed version of MATLAB, you may find a `Workspace` item in the `View` menu. This lists all your currently defined variables, their numeric data types, and their sizes in bytes. To open the workspace window, use the `View` menu, and choose `Workspace`. The same information can be obtained using the `whos` function:

```
>> whos
  Name      Size         Bytes  Class

  a         1x1                8  double array
  ans       1x1                8  double array
  b         1x1                8  double array

Grand total is 3 elements using 24 bytes
```

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Note also that `ans` is variable: it is automatically created by MATLAB to store the result of the last calculation. A listing of the variable names only is obtained using `who`:

```
>> who

Your variables are:

a    ans  b
```

The numeric date type `double` is `Matlab`'s standard for numbers; such numbers are stored as double-precision 8-byte values.

Other data types will be discussed below.

A.4 Dealing with matrices

MATLAB has an enormous number of commands for generating and manipulating matrices. Since a greyscale image is an matrix, we can use some of these commands to investigate aspects of the image.

We can enter a small matrix by listing its elements row by row, using spaces or commas as delimiters for the elements in each row, and using semicolons to separate the rows. Thus the matrix

$$a = \begin{bmatrix} 4 & -2 & -4 & 7 \\ 1 & 5 & -3 & 2 \\ 6 & -8 & -5 & -6 \\ -7 & 3 & 0 & 1 \end{bmatrix}$$

can be entered as

```
>> a=[4 -2 -4 7;1 5 -3 2;6 -8 -5 -6;-7 3 0 1]
```

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A.4.1 Matrix elements

Matrix elements can be obtained by using the standard row, column indexing scheme. So for our image matrix `a` above, the command

```
>> a(2,3)


ans =


     -3
```

returns the element of the matrix in row 2 and column 3.

MATLAB also allows matrix elements to be obtained using a single number; this number being the position when the matrix is written out as a single column. Thus in a 4×4 matrix as above, the order of elements is

$$\begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}.$$

So the element `a(2,3)` can also be obtained as `a(10)`:

```
>> a(10)


ans =


    -3
```

In general, for a matrix M with r rows and c columns, element $m(i,j)$ corresponds to $m(i+r(j-1))$. Using the single indexing allows us to extract multiple values from a matrix:

```
>> a([1 6 11 16])


ans =
```

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```
     4     5    -5     1
```

To obtain a row of values, or a block of values, we use MATLAB's *colon* operator (`:`). This generates a vector of values; the command

```
a:b
```

where `a` and `b` are integers, lists all integers from `a` to `b`. The more general version of this command:

```
i:b
```

lists all values from `a` by increment `i` up to `b`. So for example:

```
>> 2:3:16
```

generates the list

```
ans =


      2 5 8 11 14
```

Applied to our matrix `a`, for example:

```
>> a(2,1:3)


ans =


     6    -8     -5
```

lists all values in row 2 which are between columns 1 and 3 inclusive.

Similarly

```
>> a(2:4,3)


ans =
```

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```
    -3
    -5
     0
```

lists all the values in column 3 which are between rows 2 to 4 inclusive. And we can choose a block of values such as

```
>> a(2:3,3:4)

ans =

    -3     2
    -5    -6
```

which lists the 2 by 2 block of values which lie between rows 2 to 3 and columns 3 to 4.

The colon operator by itself lists *all* the elements along that particular row or column. So, for example, all of row 3 can be obtained with:

```
>> a(3,:)

ans =

     6    -8    -5    -6
```

and all of column 2 with

```
>> a(:,2)

ans =

    -2
     5
```

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```
    -8

     3
```

Finally, the colon on its own lists all the matrix elements as a single column:

```
a(:)
```

shows all the 16 elements of `a`.

A.4.2 Matrix operations

All the standard operations are supported. We can add, subtract, multiply and invert matrices, and take matrix powers. For example, with the matrix `a` from above, and a new matrix `b` defined by

```
>> b=[2 4 -7 -4;5 6 3 -2;1 -8 -5 -3;0 -6 7 -1]
```

we can have, for example:

```
>> 2*a-3*b


ans =


      2   -16    13    26

    -13    -8   -15    10

      9     8     5    -3

    -14    24   -21     1
```

As an example of matrix powers:

```
>> a^3*b^4


ans =


      103788     2039686     1466688      618345
```

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```
      964142     2619886     2780222      345543

    -2058056    -2327582      721254     1444095

     1561358     3909734    -3643012    -1482253
```

Inversion is performed using the `inv` function:

```
>> inv(a)


ans =


   -0.0125    0.0552   -0.0231   -0.1619

   -0.0651    0.1456   -0.0352   -0.0466

   -0.0406   -0.1060   -0.1039   -0.1274

    0.1082   -0.0505   -0.0562    0.0064
```

A transpose is obtained by using the apostrophe:

```
>> a'


ans =


     4     1     6    -7

    -2     5    -8     3

    -4    -3    -5     0

     7     2    -6     1
```

As well as these standard arithmetic operations, MATLAB supports some geometric operations on matrices; `flipud` and `fliplr` flip a matrix up/down and left/right respectively, and `rot90` rotates a matrix by 90 degrees:

```
>> flipud(a)


ans =
```

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```
    -7     3     0     1
     6    -8    -5    -6
     1     5    -3     2
     4    -2    -4     7

>> fliplr(a)

ans =

     7    -4    -2     4
     2    -3     5     1
    -6    -5    -8     6
     1     0     3    -7
>> rot90(a)

ans =

     7     2    -6     1
    -4    -3    -5     0
    -2     5    -8     3
     4     1     6    -7
```

The **reshape** function produces a matrix with elements taken column by column from the given matrix. Thus:

```
>> c=[1 2 3 4 5;6 7 8 9 10;11 12 13 14 15;16 17 18 19 20]

c =
```

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```
     1     2     3     4     5

     6     7     8     9    10

    11    12    13    14    15

    16    17    18    19    20


>> reshape(c,2,10)


ans =


     1    11     2    12     3    13     4    14     5    15

     6    16     7    17     8    18     9    19    10    20


>> reshape(c,5,4)


ans =


     1     7    13    19

     6    12    18     5

    11    17     4    10

    16     3     9    15

     2     8    14    20
```

`Reshape` produces an error if the product of the two values is not equal to the number of elements of the matrix. Note that we could have produced the original matrix above with

```
>> c=reshape([1:20],5,4])'
```

All these commands work equally well on vectors. In fact, MATLAB makes no distinction between matrices and vectors; a vector merely being a matrix with number of rows or columns equal to 1.

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The dot operators

A very distinctive class of operators in MATLAB are those which use dots; these operate in an element-wise fashion. For example, the command

```
b
```

performs the usual matrix multiplication of `a` and `b`. But the corresponding dot operator:

```
*b
```

produces the matrix whose elements are the products of the corresponding elements of `a` and `b`. That is, if

```
a.*b
```

then $c(i,j) = a(i,j) \times b(i,j)$:

```
>> a.*b

ans =

     8    -8    28   -28
     5    30    -9    -4
     6    64    25    18
     0   -18     0    -1
```

We have dot division, and dot powers. The command `a.^2` produces a matrix each element of which is a square of the corresponding elements of `a`:

```
>> a.^2

ans =

    16     4    16    49
     1    25     9     4
```

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```
    36    64    25    36

    49     9     0     1
```

Similarly we can produce a matrix of reciprocals by writing `1./a`:

```
>> 1./a


ans =


    0.2500   -0.5000   -0.2500    0.1429

    1.0000    0.2000   -0.3333    0.5000

    0.1667   -0.1250   -0.2000   -0.1667

   -0.1429    0.3333       Inf    1.0000
```

The value `Inf` is MATLAB's version of infinity; it is returned for the operation 1/0.

Operators on matrices

Many functions in MATLAB, when applied to a matrix, work by applying the function to each element in turn. Such functions are the trigonometric and exponential functions, and logarithms. The use of functions in this way means that in MATLAB many iterations and repetitions can be done with *vectorization* rather than by using loops. We will explore this below.

A.4.3 Constructing matrices

We have seen that we can construct matrices by listing all their elements. However, this can be tedious if the matrix is large, or if it can be generated by a function of its indices.

Two special matrices are the matrix consisting of all zeros, and the matrix consisting of all ones. These are generated by the `zero` and `ones` functions respectively. Each function can be used in several different ways:

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`zeros(n)`	if `n` is a number, will produce a zeros matrix of size $n \times n$
`zeros(m,n)`	if `m` and `n` are numbers, will produce a zeros matrix of size $m \times n$
`zeros(m,n,p,...)`	where `m`, `n`, `p` and so on are numbers, will produce an $m \times n \times p \times \cdots$ multidimensional array of zeros
`zeros(a)`	where `a` is a matrix, will produce a matrix of zeros of the same size as `a`.

Matrices of random numbers can be produced using the `rand` and `randn` functions. They differ in that the numbers produced by `rand` are taken from a uniform distribution on the interval $[0, 1]$, and those produced by `randn` are take from a normal distribution with mean zero and standard deviation one. For creating matrices the syntax of each is the same as the first three options of `zeros` above. The `rand` and `randn` functions on their own produce single numbers taken from the appropriate distribution.

We can construct random integer matrices by multiplying the results of `rand` or `randn` by an integer and then using the `floor` function to take the integer part of the result:

```
>> floor(10*rand(3))


ans =


     8     4     6
     8     8     8
     5     8     6


>> floor(100*randn(3,5))


ans =


  -134   -70  -160   -40    71
    71    85  -145    68   129
   162   125    57    81    66
```

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The `floor` function will be automatically applied to every element in the matrix.

Suppose we wish to create a matrix every element of which is a function of one of its indices. For example, the 10×10 matrix A for which $A_{ij} = i + j - 1$. In most programming languages, such a task would be performed using nested loops. We can use nested loops in `Matlab`, but it is easier here to use dot operators. We can first construct two matrices: one containing all the row indices, and one containing all the column indices:

```
>> rows=(1:10)'*ones(1,10)


rows =


     1     1     1     1     1     1     1     1     1     1
     2     2     2     2     2     2     2     2     2     2
     3     3     3     3     3     3     3     3     3     3
     4     4     4     4     4     4     4     4     4     4
     5     5     5     5     5     5     5     5     5     5
     6     6     6     6     6     6     6     6     6     6
     7     7     7     7     7     7     7     7     7     7
     8     8     8     8     8     8     8     8     8     8
     9     9     9     9     9     9     9     9     9     9
    10    10    10    10    10    10    10    10    10    10


>> cols=ones(10,1)*(1:10)


cols =


     1     2     3     4     5     6     7     8     9    10
     1     2     3     4     5     6     7     8     9    10
     1     2     3     4     5     6     7     8     9    10
     1     2     3     4     5     6     7     8     9    10
```

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```
     1     2     3     4     5     6     7     8     9    10

     1     2     3     4     5     6     7     8     9    10

     1     2     3     4     5     6     7     8     9    10

     1     2     3     4     5     6     7     8     9    10

     1     2     3     4     5     6     7     8     9    10

     1     2     3     4     5     6     7     8     9    10
```

Now we can construct our matrix using `rows` and `cols`:

```
>> A=rows+cols-1


A =


     1     2     3     4     5     6     7     8     9    10

     2     3     4     5     6     7     8     9    10    11

     3     4     5     6     7     8     9    10    11    12

     4     5     6     7     8     9    10    11    12    13

     5     6     7     8     9    10    11    12    13    14

     6     7     8     9    10    11    12    13    14    15

     7     8     9    10    11    12    13    14    15    16

     8     9    10    11    12    13    14    15    16    17

     9    10    11    12    13    14    15    16    17    18

    10    11    12    13    14    15    16    17    18    19
```

The construction of `rows` and `cols` can be done automatically with the `meshgrid` function:

```
[cols,rows]=meshgrid(1:10,1:10)
```

will produce the two index matrices above.

The size of our matrix `a` can be obtained by using the `size` function:

```
>> size(a)
```

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```
ans =


     4     4
```

which returns the number of rows and columns of `a`.

A.4.4 Vectorization

Vectorization, in MATLAB, refers to an operation carried out over an entire matrix or vector. We have seen examples of this already, in our construction of the 10×10 matrix A above, and in our use of the dot operators. In most programming languages, applying an operation to elements of a list or array will require the use of a loop, or a sequence of nested loops. Vectorization in MATLAB allows us to dispense with loops in almost all instances, and is a very efficient replacement for them.

For example, suppose we wish to calculate the sine values of all the integer radians one to one million. We can do this with a `for` loop:

```
>> for i=1:10^6,sin(i);end
```

and we can measure the time of the operation with `Matlab`'s `tic`, `toc` timer: `tic` starts a stop watch timer, and `toc` stops it and prints out the elapsed time in seconds. Thus, on my computer:

```
>> tic,for i=1:10^6,sin(i);end,toc


elapsed_time =


   27.4969
```

We can perform the same calculation with:

```
>> i=1:10^6;sin(i);
```

and print out the elapsed time with:

```
>> tic,i=1:10^6;sin(i);toc
```

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```
elapsed_time =


    1.3522
```

Note that the second command applies the sine function to `all` the elements of the vector `1:10^6`, whereas with the `for` loop, sine is only applied to each element of the loop in turn.

As another example, we can easily generate the first 10 square nunmbers with:

```
>> [1:10].^2


ans =


1 4 9 16 25 36 49 64 81 100
```

What happens here is that `[1:10]` generates a vector consisting of the numbers 1 to 10; and the dot operator `.^2` squares each element in turn.

Vectorization can also be used with logical operators; we can obtain all positive elements of the matrix `a` above with:

```
>> a>0


ans =


     1     0     0     1
     1     1     0     1
     1     0     0     0
     0     1     0     1
```

The result consists of 1's only in the places where the elements are positive.

MATLAB is designed to perform vectorized commands very quickly, and whenever possible such a command should be used instead of a `for` loop.

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A.5 Plots

MATLAB has outstanding graphics capabilities. But we shall just look at some simple plots. The idea is straightforward: we create two vectors `x` and `y` of the same size, then the command

```
plot(x,y)
```

will plot `y` against `x`. If `y` has been created from `x` by using a vectorized function $f(x)$ the plot will show the graph of $y = f(x)$. Here's a simple example:

```
>> x=[0:0.1:2*pi];
>> plot(x,sin(x))
```

and the result is shown in figure A.2. The `plot` function can be used to produce many different

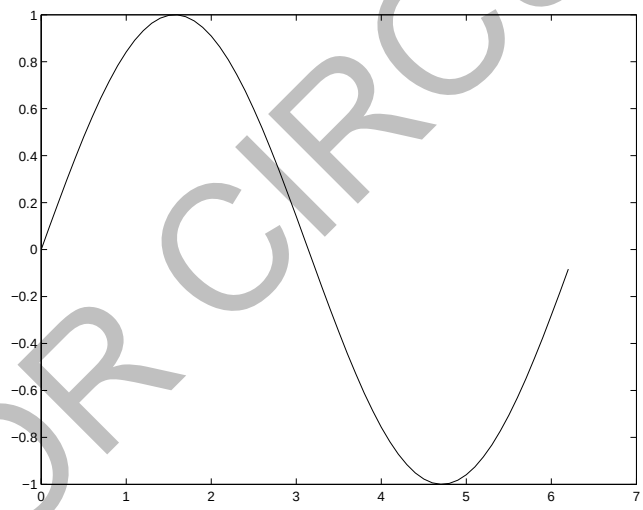

Figure A.2: A simple plot in MATLAB

plots. We can, for example, plot two functions simultaneously with different colours or plot symbols. For example:

```
>> plot(x,sin(x),'.',x,cos(x),'o')
```

produces the graph shown in figure A.3.

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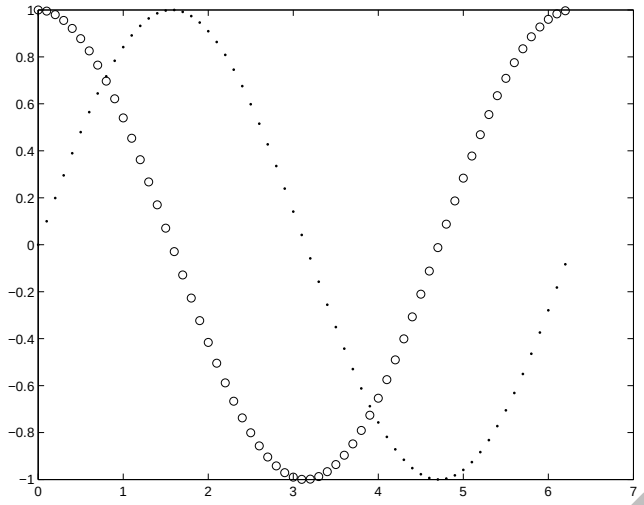

Figure A.3: A different plot in MATLAB

A.6 Help in MATLAB

MATLAB comes with a vast amount of online help and information. So much, in fact, that it's quite easy to use MATLAB without a manual. To obtain information on a particular command, you can use `help`. For example:

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```
>> help for


 FOR    Repeat statements a specific number of times.
    The general form of a FOR statement is:


       FOR variable = expr, statement, ..., statement END


    The columns of the expression are stored one at a time in
    the variable and then the following statements, up to the
    END, are executed. The expression is often of the form X:Y,
    in which case its columns are simply scalars. Some examples
    (assume N has already been assigned a value).


         FOR I = 1:N,
             FOR J = 1:N,
                 A(I,J) = 1/(I+J-1);
             END
         END


    FOR S = 1.0: -0.1: 0.0, END steps S with increments of -0.1
    FOR E = EYE(N), ... END  sets E to the unit N-vectors.


    Long loops are more memory efficient when the colon expression appears
    in the FOR statement since the index vector is never created.


    The BREAK statement can be used to terminate the loop prematurely.


    See also IF, WHILE, SWITCH, BREAK, END.
```

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If there is too much information, it may scroll past you too fast to see. In such case you can turn

on the `Matlab` pager with the command

```
>> more on
```

For more help on `help`, try:

```
>> help help
```

Better formatted help can be obtained with the `doc` function, which opens up a help browser which interprets HTML-formatted help files. The result of the command

```
>> doc help
```

is the window shown in figure A.4.

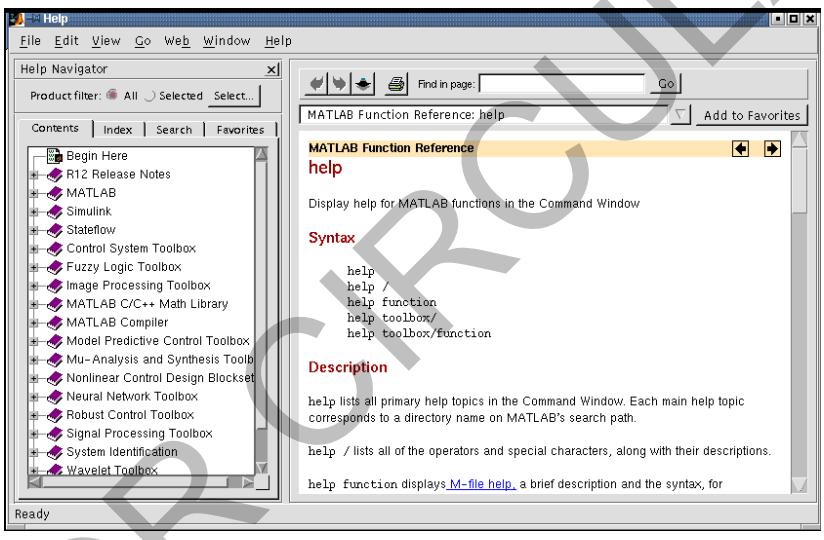

Figure A.4: The MATLAB help browser

You can find more about the `doc` function with any of

```
>> doc doc
>> help doc
```

If you want to find help on a particular topic, but don't know the function to use, the `lookfor` function is extremely helpful. The command

```
lookfor topic
```

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lists all commands for which the first line of the help text contains the string `topic`. For example, suppose we want to find out if MATLAB supports the exponential function

$$e^x.$$

Here's how:

```
>> lookfor exponential
EXP    Exponential.
EXPINT Exponential integral function.
EXPM   Matrix exponential.
EXPM1  Matrix exponential via Pade approximation.
EXPM2  Matrix exponential via Taylor series.
EXPM3  Matrix exponential via eigenvalues and eigenvectors.
BLKEXP Defines a function that returns the exponential of the input.
```

and we now know that the function is implemented using `exp`. We could have used

```
>> lookfor exp
```

and this would have returned many more functions.

Note that MATLAB convention is to use uppercase names for functions in help texts, even though the function itself is called in lowercase.

A.7 Programming in MATLAB

MATLAB has a very rich programming language. Only a small set of functions is actually "built in" to MATLAB, others are written in MATLAB's own programming language. We consider two distinct programs: script files, and functions.

A.7.1 Script files

A script file is simply a list of commands to be executed. It may be that we will execute the same sequence of commands many times; in which case it is more efficient to write a file containing those

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commands. If the file is called `script.m` and placed somewhere on the path, then simply entering `script` at the prompt will execute all the commands in it. Of course you can use any name you like for your script file! However, it is convention to end MATLAB files with the extension `.m`.

A.7.2 Functions

As we have seen, a *function* is a MATLAB command which takes an input (one or several variables) and returns one or several values. Let's look at a simple example: writing a function which returns the number of positive values of a matrix. This function will take a matrix as input, and return a single number as output. We have seen that

```
>> a>0
```

produces a matrix with 1's in the positions of positive elements. So the sum of all elements in this new matrix is the number we require. We can obtain the sum of matrix elements using the `sum` function. If applied to a vector, `sum` produces the sum of all its elements. If applied to a matrix, however, `sum` produces a vector whose elements are the sums of the matrix columns:

```
>> sum(a)

ans =

     4    -2   -12     4

>> sum(a>0)

ans =

     3     2     0     3
```

So we have two options here: we can use `sum` twice:

```
>> sum(sum(a>0))
```

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```
ans =


     8
```

or we could convert the matrix to a vector by using the colon operator before finding the positive elements:

```
>> sum(a(:)>0)


ans =


     8
```

Our function must have a name; let's call it `countpos`. A function file starts with the word `function`. The first line defines how the function is called. After that come some lines of help text, and finally the code. Our `countpos` function is shown in figure A.5.

```
function num=countpos(a)

% COUNTPOS finds the number of positive elements in a matrix.  The matrix can
% be of any data type.
%
% Usage:
%
%   n=countpos(a)

num=sum(a(:)>0);
```

Figure A.5: A simple MATLAB function

If this file is saved as `countpos.m` somewhere on the path, we can use `countpos` exactly as we use any other MATLAB function or function:

```
>> countpos(a)


>> help countpos
```

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```
>> doc countpos
```

and the command

```
>> lookfor count
```

will include a reference to `countpos`.

Finally, if we want to explore functions in more detail, we can use `type`, which lists the entire program corresponding to that function or command. So we can enter

```
>> type countpos.m
```

to see a listing of our new function.

Exercises

1. Perform the following calculations in MATLAB:

$$132 + 45, \quad 235 \times 645, \quad 12.45/17.56. \quad \sin(\pi/6), \quad e^{0.5}, \quad \sqrt{2}$$

2. Now enter `format long` and repeat the above calculations.

3. Read the help file for `format`, and experiment with some of the other settings.

4. Enter the following variables: $a = 123456$, $b = 3^{1/4}$, $c = \cos(\pi/8)$. Now calculate:

$$(a+b)/c, \quad 2a - 3b, \quad c^2 - \sqrt{a-b}, \quad a/(3b+4c), \quad \exp(a^{1/4} - b^{10})$$

5. Find the MATLAB functions for the inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$. Then calculate:

$$\sin^{-1}(0.5), \quad \cos^{-1}(\sqrt{3}/2), \quad \tan^{-1}(2)$$

Convert your answers from radians to degrees.

6. Using vectorization and the colon operator, use a single command each to generate:

   (a) the first 15 cubes,

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(b) the values $\sin(n\pi/16)$ for n from 1 to 16,

(c) the values $\sqrt{n}$ for n from 10 to 20.

7. Enter the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & -1 \\ -3 & -4 & 5 \\ 2 & 3 & -4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -2 & 1 \\ -3 & 5 & 2 \\ 1 & 1 & -7 \end{bmatrix}$$

Now calculate:

$$2A - 3B, \quad A^T, \quad AB - BA, \quad BC^{-1}, \quad (AB)^T, \quad B^T A^T, \quad A^2 + B^3$$

8. Use the `det` function to determine the determinant of each of the matrices in the previous question. What happens if you try to find the inverse of matrix A?

9. Write a little function `issquare` which will determine whether a given integer is a square number. Thus:

```
>> issquare(9)

ans =

     1

>> issquare(9.000)

ans =

     1

>> issquare(9.001)
```

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```
ans =

     0

>> issquare([1:10])

ans =

     1     0     0     1     0     0     0     0     1     0
```

10. Enter the command

```
>> imshow(issquare(reshape([1:65536],256,256)))
```

What are you seeing here?

11. Plot the function $tan(x)$ with the following commands:

```
>> x=[0:0.1:10];
>> plot(x,tan(x))
>> figure,plot(x,tan(x)),axis([0,10,-10,10])
```

What does the `axis` function do? Read the help file for it. Experiment with changing the last two numbers in `axis` for the above command.

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Appendix B

The fast Fourier transform

We have discussed the Fourier transform and its uses in chapter 7. However, as we know, the Fourier transform gains much of its usefulness by the existence of a fast algorithm to compute it. We look briefly at one version of the fast Fourier transform here. More details can be found in [7] or [35].

To start, we shall look at the very simple DFT for a two element vector, where for convenience we shall omit the scaling factor:

$$\begin{aligned}\begin{bmatrix} X_0 \\ X_1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \\ &= \begin{bmatrix} x_0 + x_1 \\ x_0 - x_1 \end{bmatrix}.\end{aligned}$$

We can express this combination with a "butterfly diagram" as shown in figure B.1. We shall see how

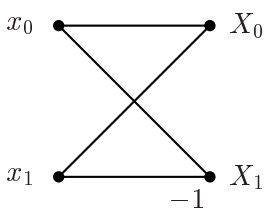

Figure B.1: A butterfly diagram

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this butterfly diagram can be extended to vectors of greater length. In general, a butterfly diagram will consist of nodes joined as shown in figure B.1, with scaling factors as shown in figure B.2. In

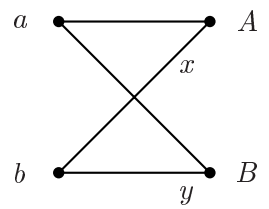

Figure B.2: A general butterfly

this figure, we have

$$\begin{aligned} A &= a + xb, \\ B &= a + yb. \end{aligned}$$

By convention, if a scaling factor is not specified, it is assumed to be 1.

We can extend this to the DFT for a four element vector, starting with its matrix definition.

$$\begin{aligned}
\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \begin{bmatrix} x_0 + x_1 + x_2 + x_3 \\ x_0 - ix_1 - x_2 + ix_3 \\ x_0 - x_1 + x_2 - x_3 \\ x_0 + ix_1 - x_2 - ix_3 \end{bmatrix} \\
&= \begin{bmatrix} (x_0 + x_2) + (x_1 + x_3) \\ (x_0 - x_2) - i(x_1 - x_3) \\ (x_0 + x_2) - (x_1 + x_3) \\ (x_0 - x_2) + i(x_1 - x_3) \end{bmatrix}
\end{aligned}$$

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$$= \begin{bmatrix} x'_0 + x'_1 \\ x'_2 - ix'_3 \\ x'_0 - x'_1 \\ x'_2 + ix'_3 \end{bmatrix}$$

This means we can obtain a four element DFT by a two stage process; first we create "intermediate" values x'_i:

$$\begin{aligned} x'_0 &= x_0 + x_2 \\ x'_1 &= x_1 + x_3 \\ x'_2 &= x_0 - x_2 \\ x'_3 &= x_1 - x_3 \end{aligned}$$

ans use these new values to calculate the final values:

$$\begin{aligned} X_0 &= x'_0 + x'_1 \\ X_1 &= x'_2 - ix'_3 \\ X_2 &= x'_0 - x'_1 \\ X_3 &= x'_2 + ix'_3 \end{aligned}$$

The butterfly diagram for this is shown in figure B.3. Note that it consists of two simple butterfly diagrams, on the left, joined together with more butterflies on the right.

Note that the order of the original elements x_i has been changed; we shall discuss this more below.

To apply the same idea to an eight element vector, we consider the general term of the transform:

$$\begin{aligned} X_k &= \omega^0 x_0 + \omega^k x_1 + \omega^{2k} x_2 + \omega^{3k} x_3 + \cdots + \omega^{7k} x_7 \qquad \text{where } \omega = e^{-2\pi i/8} \\ &= \underbrace{\omega^0 x_0 + \omega^{2k} x_2 + \omega^{4k} x_4 + \omega^{6k} x_6}_{\text{even values}} + \underbrace{\omega^k x_1 + \omega^{3k} x_3 + \omega^{5k} x_5 + \omega^{7k} x_7}_{\text{odd values}} \\ &= \omega^0 x_0 + \omega^{2k} x_2 + \omega^{4k} x_4 + \omega^{6k} x_6 + \omega^k (x_1 + \omega^{2k} x_3 + \omega^{4k} x_5 + \omega^{6k} x_7). \end{aligned}$$

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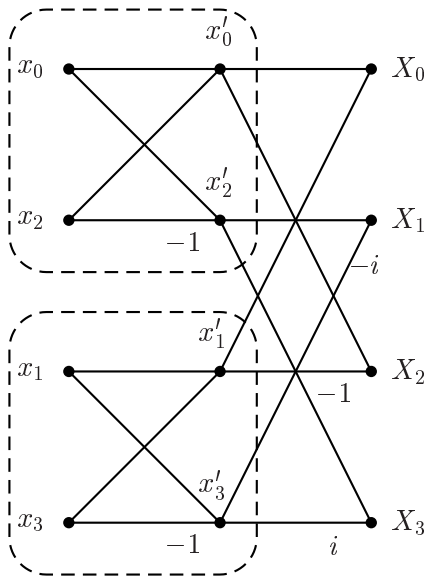

Figure B.3: A butterfly diagram for a four element FFT

Now, let $z = \omega^2 = e^{-2\pi i/4}$. Then

$$X_k = (z^0 x_0 + z^k x_2 + z^{2k} x_4 + z^{3k} x_6) + \omega^k (z^0 x_1 + z^k x_3 + z^{2k} x_5 + z^{3k} x_7) \tag{B.1}$$

If we examine at the brackets in this expression, we note that the first bracket is the kth term of the DFT of (x_0, x_2, x_4, x_6), and the second bracket is the k-th term of the DFT of (x_1, x_3, x_5, x_7). So we can write:

$$X_k = \mathrm{DFT}(x_0, x_2, x_4, x_6)_k + \omega^k \mathrm{DFT}(x_1, x_3, x_5, x_7)_k.$$

To make the argument easier to follow, let us write

$$\begin{aligned}(Y_0, Y_1, Y_2, Y_3) &= \mathrm{DFT}(x_0, x_2, x_4, x_6)\\ (Y_0', Y_1', Y_2', Y_3') &= \mathrm{DFT}(x_1, x_3, x_5, x_7)\end{aligned}$$

Thus we have

$$X_k = Y_k + \omega^k Y_k'.$$

There's a slight problem here: the DFT of a four element vector only has four terms, so that k here only take values 0 to 3, and we need values 0 to 7. However, if we look back at equation B.1, we

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can see that if k takes values 4 to 7, the powers of z just cycle round again, since $z^4 = 1$. Thus for these values of k we have:

$$X_k = Y_{k-4} + \omega^k Y'_{k-4}.$$

This means that the indices for X_k can take all values between 0 and 7, but the indices of Y_k and Y'_k only take on values 0 to 3.

We can check this in MATLAB; from first principles. Recall that the `fft` function, to obtain a correct result, must be applied to a *column* vector.

Let us first create an eight element vector, and obtain its Fourier transform:

```
>> x=2:9


x =


     2     3     4     5     6     7     8     9


>> fx=fft(x')


fx =


  44.0000
  -4.0000 + 9.6569i
  -4.0000 + 4.0000i
  -4.0000 + 1.6569i
  -4.0000
  -4.0000 - 1.6569i
  -4.0000 - 4.0000i
  -4.0000 - 9.6569i
```

Now we shall split up the vector into even and odd parts, and put their Fourier transforms together using the above formula:

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```
>> even=[2 4 6 8];odd=[3 5 7 9];
>> feven=fft(even');
>> fodd=fft(odd');
>> X=zeros(8,1);
>> omega=exp(-2*pi*sqrt(-1)/8);
>> for i=0:3,X(i+1)=feven(i+1)+omega^i*fodd(i+1);end
>> for i=4:7,X(i+1)=feven(i-3)+omega^i*fodd(i-3);end
>> X

X =

  44.0000
  -4.0000 + 9.6569i
  -4.0000 + 4.0000i
  -4.0000 + 1.6569i
  -4.0000
  -4.0000 - 1.6569i
  -4.0000 - 4.0000i
  -4.0000 - 9.6569i
```

and this result agrees with the Fourier transform obtained from above. Note that in the `for` loops we used indices $i+1$ and $i-3$. This is because the theory starts indices at 0, but MATLAB starts indices at 1.

As with the four element transform, we can express the eight element transform by using a butterfly diagram. Such a diagram, treating the four element FFT as a "black box", is shown in figure B.4. Note that as in figure B.3 the order of the original elements is changed.

If we now replace the black box FFT's from figure B.4 with the four element butterfly diagram of figure B.3 we obtain a "complete" butterfly diagram for eight elements, which is given in figure B.5. To show the structure of the diagram, we have left out the multiplication factors, which are simply

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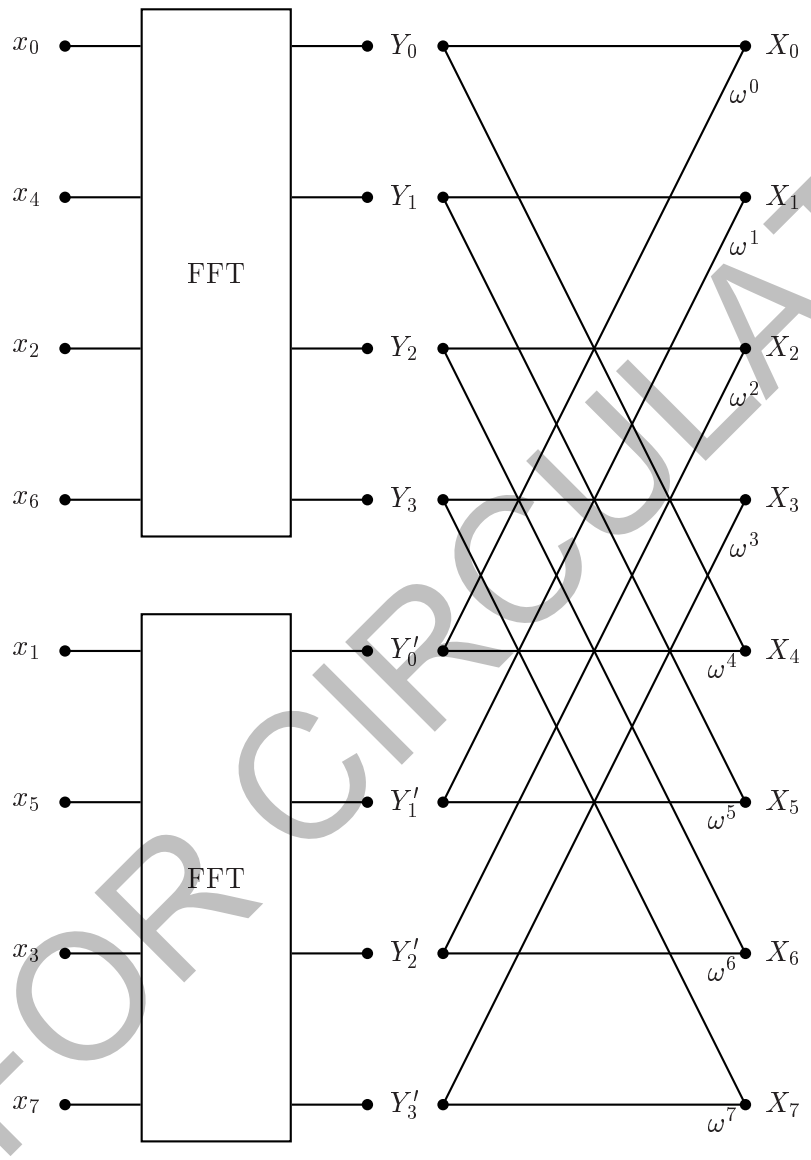

Figure B.4: A butterfly diagram for an eight element FFT

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inherited from their respective diagrams.

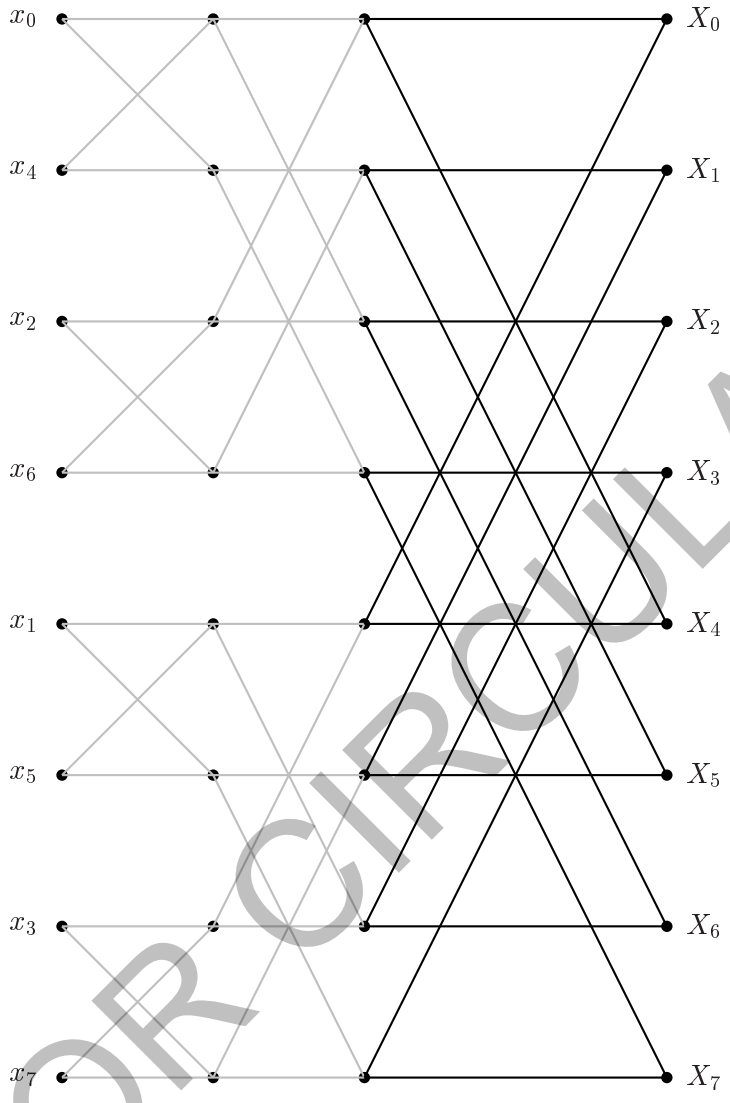

Figure B.5: A complete butterfly diagram for an eight element FFT

We have noted that the input values in our butterfly diagrams are not given in their correct order. The ordering we require can be obtain by *binary bit reversal*. Suppose we list the input elements x_i in order but replace the indices with their binary expansions:

$$\begin{array}{cccccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ = x_{000} & x_{001} & x_{010} & x_{011} & x_{100} & x_{101} & x_{110} & x_{111} \end{array}$$

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Now we replace each index with the reversal of its bits, and go back to decimal:

$$\begin{array}{cccccccc} & x_{000} & x_{100} & x_{010} & x_{110} & x_{001} & x_{101} & x_{011} & x_{111} \\ = & x_0 & x_4 & x_2 & x_6 & x_1 & x_5 & x_3 & x_7 \end{array}$$

and now we have the new order for input into the FFT.

The particular form of the FFT we have developed here is known as the "decimation in time, 2 radix" FFT. We can describe its general workings as follows:

1. re-order the initial 2^n vector elements by binary bit reversal,
2. butterfly the elements two at a time, using scaling factors 1 and -1,
3. butterfly the resulting elements four at a time, using the scaling factors 1, ω, ω^2 and ω^3, with $\omega = \exp(2i\pi/4)$,
4. butterfly the resulting elements eight at a time, using the scaling factors 1, ω, $\omega^2, \ldots, \omega^7$, with $\omega = \exp(2i\pi/8)$,
5. and so on, until we butterfly all elements together using scaling factors ω^k, with $\omega = \exp(2i\pi/2^n)$.

In practice, an FFT program will use a divide and conquer strategy: the initial vector is broken up into smaller vectors, the algorithm is applied recursively to these smaller vectors, and the results "butterflied" together.

There are many other forms of the FFT, but they all work by the same basic principle of dividing the vector into shorter lengths. Clearly we will obtain the greatest speed for vectors whose lengths are a power of 2 (such as the examples above), but we can also apply similar schemes to other lengths.

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NOT FOR CIRCULATION

Index

affine transformations, 145
anamorphosis, 164
    undoing, 164
artistc effects
    solarization, 547
averaging filter, *see* filter, average

binary morphology
    closing, 318
binary bit reversal, 592
binary image, *see* digital image, binary
binary morphology
    boundary detection, 314
    connected components, 328
    dilation, 307, 337
    erosion, 310, 336
    external boundary, 314
    filtering, 323
    hit-or-miss transform, 324
    internal boundary, 314
    kernel, 309
    morphological gradient, 314
    noise removal, 323
    opening, 317
    reflection, 306
    region filling, 327
    skeletonization, 332
    structuring element, 309
    translation, 305
bit planes, 64, 472
    least significant, 64
    most significant, 64
BMP image, *see* file types, Microsoft BMP
boundary, 314, 362, 409, 428
boundary following, 411
    in Matlab, 412–418
butterfly diagram, 585
Butterworth filter
    functions, 204
    high pass, 207
    in Matlab, 204
    low pass, 206
    order, 204

Canny edge detector, 280
    hysteresis thresholding, 284
    non-maximum suppression, 281
cartesian to polar coordinates, 532
ceiling function, 532
chain code, 409
    normalization, 418
chromaticity coordinates, 440
chromaticity diagram, 440
circular convolution, 179
    polynomial definition, 179
colour

NOT FOR CIRCULATION

hue, 443
perception, 438
physical properties, 437
processing, 455–462
pseudocolouring, 451
RGB gamut, 442
RGB, HSV conversion, 444
saturation, 443
value, 443
XYZ model, 439
colour cube, 35
colour gamut, 442
colour matching, 438
colour model, 35, 442
HSV, 443
RGB, 442
YIQ, 447
colour processing
contrast enhancement, 455
edge detection, 462
histogram equalization, 457
noise removal, 459
spatial filtering, 457
complement, 88, 548
component, 353
labelling, 355
compression
lossless, 51, 477
lossy, 51, 495
wavelet, 519
connectivity, 379
convolution, 179
convolution theorem, 181, 187
crossing number, 382

Daubechies wavelets, 511
Daubechies wavelets, 514, 519
DCT, *see* discrete cosine transform
deletable pixels, *see* pixels, deletability
DFT, *see* discrete Fourier transform
DICOM image, *see* file types, DICOM
digital camera, 17
digital image, 12
binary, 25
file types, 42
greyscale, 25, 33
indexed, 27, 37
colour map, 38
index, 38
raster image, 46
RGB, *see* digital image, true colour
true colour, 25, 35, 448
vector image, 46
discrete cosine transform, 478, 499
discrete Fourier transform, 172
discrete wavelet transform (DWT), 503
distance transform, 364
fast method, 368
in Matlab, 370–376
mask, 367
skeletonization, 399
dithering, 74
dither matrix, 74

edge
definition, 267
detection filters, 270

NOT FOR CIRCULATION

gradient, 269
ideal, 268
ramp, 268
sharpening, 128
step, 268
edge detection
Canny, 280
edge detection
colour images, 462
Laplacian filter, 275
LoG, 280
Marr-Hildreth, 279
Prewitt filters, 270
Roberts cross-gradient filters, 272
Sobel filters, 274
zero crossings, 279
electromagnetic spectrum, 19
entropy, 468
equivalence relations, 354
Euclidean distance, 364
external boundary, 314

fast Fourier transform, 183, 585–593
father wavelet, 507
FFT, *see* fast Fourier transform
file types
Compuserve GIF, 49
DICOM, 53
JPEG, 51
Microsoft BMP, 47
PGM, 46
PNG, 50
TIFF, 52
filter
adaptive, 232
alpha-trimmed mean, 136
average, 110, 116
band reject, 237
Butterworth, 202, *see* Butterworth filter
edge detection, 270
edges of image, 113
frequency domain, 198
Gaussian, *see* Gaussian
geometric mean, 136
high boost, 131
high pass, 119, 509, 525
homomorphic, 211
ideal, *see* ideal filter
implementation of linear, 108
in MATLAB, 113
inverse, 239
isotropic, 275
Laplacian, 275
Laplacian of Gaussian, 279
linear, 107
low pass, 119, 509
mask, 107
matrix notation, 112
maximum, 134
median, 136, 225
minimum, 134
minimum mean-square error, 232
non-linear spatial, 134
Prewitt, 270
pseudo-median, 248
rank-order, 134, 227
Roberts cross-gradient, 272

NOT FOR CIRCULATION

separable, 117
Sobel, 274
spatial, 139
unsharp mask, 130
Wiener, 234, 245
flat-bed scanner, 17
Fourier descriptor, 428
Fourier series, 171
Fourier transform, 499
comparison of DFT and FFT, 183
convolution, 179
convolution theorem, 181, 187
corrugation functions, 184
DC coefficient, 188
definition 2d DFT, 184
discrete decomposition, 172
displaying, 189, 194
FFT, 183
filtering, 198
ideal filtering, 199
inverse one dimensional DFT, 175
Matlab functions, 176
one dimensional DFT, 174
properties 1d, 177
properties 2d, 185
ringing, 200
scaling factor, 184
single edge, 193
spectrum, 190
trigonometric decomposition, 169
two dimensional DFT, 183
Fourier transform ringing, 198
frame-grabbing card, 17
frequency, 119

Gaussian
frequency filter, 208
high pass filter, 209
low pass filter, 209
noise, 221, 246
spatial filter, 124
Gaussian distribution, 124
GIF image, *see* file types, Compuserve GIF
Gray codes, 472
greyscale image, *see* digital image, greyscale
greyscale morphology, 336
closing, 345
dilation, 340
edge detection, 345
erosion, 338
noise removal, 346
opening, 345
Guo-Hall, *see* skeleton, Guo-Hall algorithm

Haar transform, 505
Haar wavelet, 505, 513
pulse function, 506
halftoning, 74
high pass filter, *see* filter, high pass
histogram, 88–101
definition, 88
equalization, 95
piecewise linear stretching, 95
stretching, 90
Hough transform
in Matlab, 299
Hough transform, 285

NOT FOR CIRCULATION

accumulator array, 285
drawing lines, 296
implementation, 289
in Matlab, 291
line parameterization, 288
vertical lines, 288
Huffman coding, 467
for JPEG compression, 494

ideal filter
cutoffs, 200, 202
functions, 204
high pass, 201
low pass, 199
image, 11
acquisition, 15
perception, 29
image arithmetic
subtraction, 84
image minimization, 159
image arithmetic
addition, 84
clipping output values, 84
complement, 88
multiplication, 86
scaling, 61
image enlargement
interpolation, 150
spatial filter, 155
image interpolation, *see* interpolation
image rotation, 159
image types, 25
impulse noise, 323, *see* noise, salt and pepper
indexed colour image, *see* digital image, indexed
internal boundary, 314
interpolation
bicubic, 154
bilinear, 150
cubic, 153
data, 145
general function for, 152
image, 148
image rotation, 159
image scaling, 150
linear, 147
nearest neighbour, 146
scaling smaller, 159

JPEG algorithm, 478, 479
JPEG image, *see* file types, JPEG
JPEG2000, 502, 519

lexicographical sorting, 420
lifting, 518
lookup table, 102
lookup tables, 335, 360
image boundaries, 362
lossless compression, 467
lossy compression, 467
low pass filter, *see* filter, low pass
luminance, 439, 447

mask
distance transform, 367
spatial filter, 107
Mathematical morphology, 305
mathematical morphology, *see* morphology
Matlab, 553
Matlab

NOT FOR CIRCULATION

colon operator, 561
colour maps, 453
commands, 553
data types, 41
dot operators, 567
entering matrices, 559
functions, 553
matrix indexing, 560
measuring time, 572
multidimensional array, 36
obtaining help, 575
plotting, 574
programming, 578
variables, 557
vectorization, 568, 572
workspace, 557, 558
MATLAB functions
ordfilt2, 136, 401
roipoly, 137
wiener2, 234
MATLAB data types, 41
double, 559
logical, 63
uint16, 477
uint8, 42
MATLAB functions
axis, 90
blkproc, 266, 486
bwlabel, 358
bwmorph, 335
cat, 457
conv, 180
dct2, 482
doc, 577
edge, 267, 462
fft, 176
fft2, 190
fftshift, 190
figure, 34
filter2, 113
find, 95
fliplr, 164, 564
flipud, 163, 564
floor, 569
format, 556
fspecial, 116
gray2ind, 42
gray2rgb, 42
grayslice, 71, 453
graythresh, 261
help, 575
hist, 546
histeq, 99
hsv2rgb, 447
idct2, 482
ifft, 176
ifft2, 190
im2bw, 254
im2double, 62
im2uint8, 62
imadd, 85
imadjust, 92
image, 58
imclose, 319, 345
imcomplement, 88
imdilate, 309, 341

NOT FOR CIRCULATION

imerode, 312, 341
imfinfo, 38
imhist, 90
immultiply, 86
imnoise, 221
imopen, 319, 345
impixel, 37
imread, 33, 43
imresize, 67, 150, 533
imrotate, 162
imshow, 34, 60, 449
imsubtract, 86
imwrite, 43, 54
ind2gray, 42
ind2rgb, 42
inv, 564
log, 557
log10, 557
lookfor, 577
mat2gray, 116
medfilt2, 225
meshgrid, 571
mod, 65, 533
nlfilter, 135
ntsc2rgb, 448
ones, 568
ordfilt2, 227
pixval, 34
plot, 574
rand, 569
randn, 569
reshape, 565
rgb2gray, 42
rgb2hsv, 446
rgb2ind, 42
rgb2ntsc, 448
roifilt2, 139
rot90, 564
round, 521
sign, 544
sin, 556
size, 571
sortrows, 420
sqrt, 556
strel, 341
sub2ind, 535, 537
sum, 579
surf, 124
tic, 572
toc, 572
transpose, 564
whos, 558
zero, 568
MATLABfunctions
colfilt, 136
maximum filter, *see* filter, maximum
metric, 363
minimum filter, *see* filter, minimum
Minkowski addition, 307
Minkowski subtraction, 312
mode, 546
morphological gradient, 314
morphology
binary, *see* binary morphology
greyscale, *see* greyscale morphology
mother wavelet, 507

NOT FOR CIRCULATION

motion deblurring, 241

neighbourhood, 21
neighbourhood processing, 107–139
noise
    definition, 220
    Gaussian, 221, 246
    periodic, 223, 236
    salt and pepper, 221
    speckle, 222
noise removal
    adaptive filtering, 234
    colour images, 459
    Gaussian, 230
    image averaging, 230
    median filtering, 225
    morphological filtering, 323, 346
    outliers, 228
    periodic, 237
    spatial averaging, 224, 231
    using wavelets, 526
non-linear filter, *see* filter, non-linear
normalization matrix, 480
Nyquist criterion, *see* sampling, Nyquist criterion

one dimensional DFT, 174
orthogonal matrix, 160
Otsu, *see* thresholding, Otsu's method

path, 353
PGM image, *see* file types, PGM
photosites, 17
pixelization, 68
    radial, 533
pixels
    background, 355
    deletability, 379
    foreground, 355
    simple, 379
PNG image, *see* file types, PNG
point processing, 83–104
    arithmetic operations, *see* image arithmetic
polar coordinates, 531
    origin, 531
polar to cartesian coordinates, 532
primary colours, 438
pulse function, 506

quantization, 57, 520
    error diffusion, 77
        Floyd-Steinberg, 77
        Jarvis-Judice-Ninke, 80
        Stucki, 80
    false contours, 72
    uniform, 70

radial pixelization, 533
region of interest, 137
    spatial filtering, 139
RGB image, *see* digital image, true colour
ringing, *see* Fourier transform, ringing
ROI, *see* region of interest
run length encoding (RLE), 471
    in Matlab, 474

sampling, 15
    Nyquist criterion, 16
    theorem, 16
second derivatives, 275
secondary colours, 438

NOT FOR CIRCULATION

shape descriptors, 409
shape number, 422
definition, 425
shape representation, 409
shot noise, *see* noise, salt and pepper
simple pixels, *see* pixels, simple
skeleton, 376–402
examples, 376
Guo-Hall algorithm, 392
implementation, 394
Lantuéjoul's method, 334
medial axis, 377
morphology, *see* binary morphology, skeletonization
using distance transform, 399
Zhang-Suen algorithm, 385
implementation, 388
solarization, 88, 547
spatial resolution, 57
spatial resolution, 67
special effects, 531–550
"bathroom glass" ripples, 536
"pond ripples", 536
circular slice, 543
colour images, 550
distortion, 539
fisheye lens, 541
fuzzy, 545
jitter, 543
oil painting, 546
pixel effects, 546
radial ripple, 537
ripples, 536
square slice, 544
swirl, 542
tile, 550
twirl, 542
subfield algorithms, 385
subiteration algorithms, 385
subsampling, 159, 516, 517

thresholding, 65, 253–267, 520
adaptive, 264
applications, 257
choosing a threshold value, 258
definition, 253
double, 256
Otsu's method, 261
use of histograms, 259
TIFF image, *see* file types, TIFF
tomogram, 20
tomography, 19
topology, 351–402
translation, 305
tristimulus values, 439
true colour image, *see* digital image, true colour
two dimensional DFT, 183

undersampling, 15
unsharp masking, 128

visible light, 17, 19

wavelets
2D DWT, 510
basic idea, 499
compression, 519
Daubechies, 511

NOT FOR CIRCULATION

dilation, 500
dilation equation, 507
father wavelet, 507
filter coefficients, 507
Haar, 505
in Matlab, 511
lifting methods, 509
mother wavelet, 507
noise removal, 526
nonstandard decomposition, 510
scaling function, 507, 515
simple transform, 502
standard decomposition, 510
taps, 507
translation, 500
wavelet equation, 507

x-rays, 19
XYZ colour model, 439

zero crossings, 277
zero padding, 113
zero-interleaving, 155
Zhang-Suen, *see* skeleton, Zhang-Suen algorithm

NOT FOR CIRCULATION