



COURSE: DIGITAL SIGNAL PROCESSING



Instructor: Ninh Khanh Duy

CHAPTER 2:

DISCRETE-TIME

SIGNALS & SYSTEMS



Lecture 2: Discrete-time (DT) signals

Lecture 3: Discrete-time (DT) systems

Duration: 6 periods

Lecture 2

Discrete-time (DT) Signals

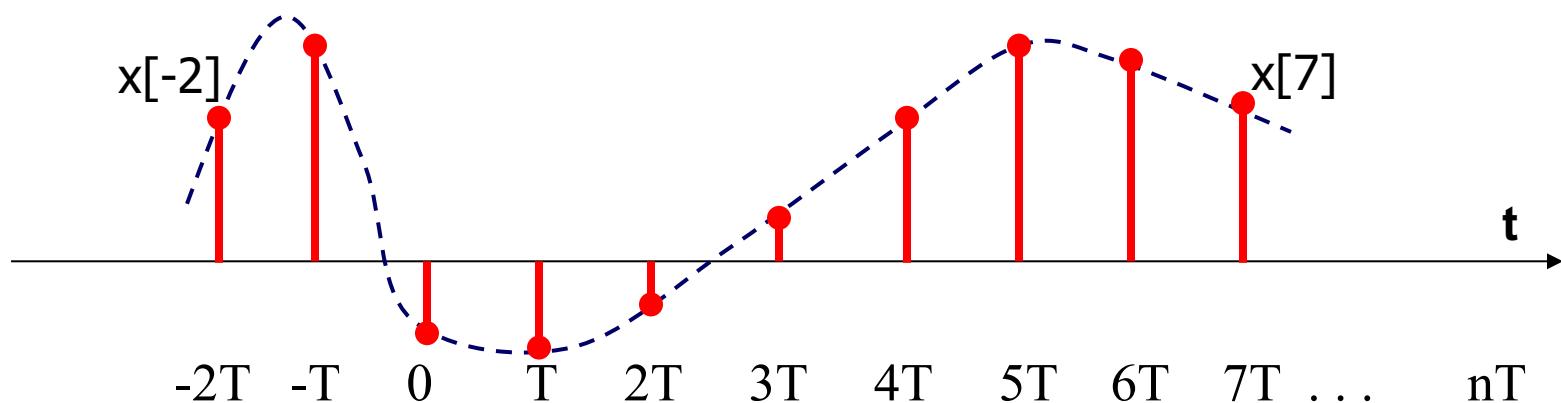
- **Duration:** 3 periods
- **Outline:**
 1. Representations of DT signals
 2. Some elementary DT signals
 3. Simple manipulations of DT signals

Sampled signals (tin hieu lay mau)

CT: contiguous-time

DT: discrete-time

Converting a CT signal into a DT signal by **sampling**: given $x_a(t)$ to be a CT signal, $x_a(nT)$ is the value of $x_a(t)$ at $t = nT \rightarrow$ DT signal $x_a(nT)$ is defined only **for n an integer**



$$x_a(t) \Big|_{t=nT} = x_a(nT) \equiv x(n), \quad -\infty < n < \infty$$

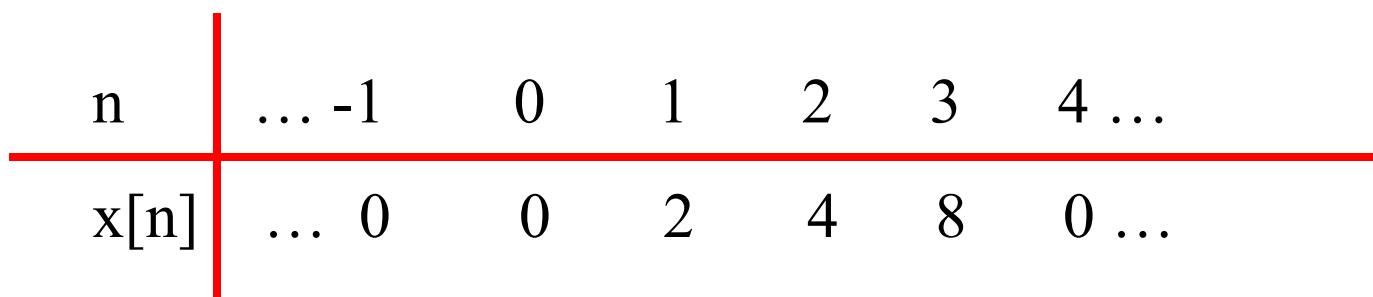
T: sampling period (s), t: contiguous time (s), n: discrete time = sample index

Representations of DT signals

1. Functional representation

$$x[n] = \begin{cases} 2^n, & 1 \leq n \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

2. Tabular representation

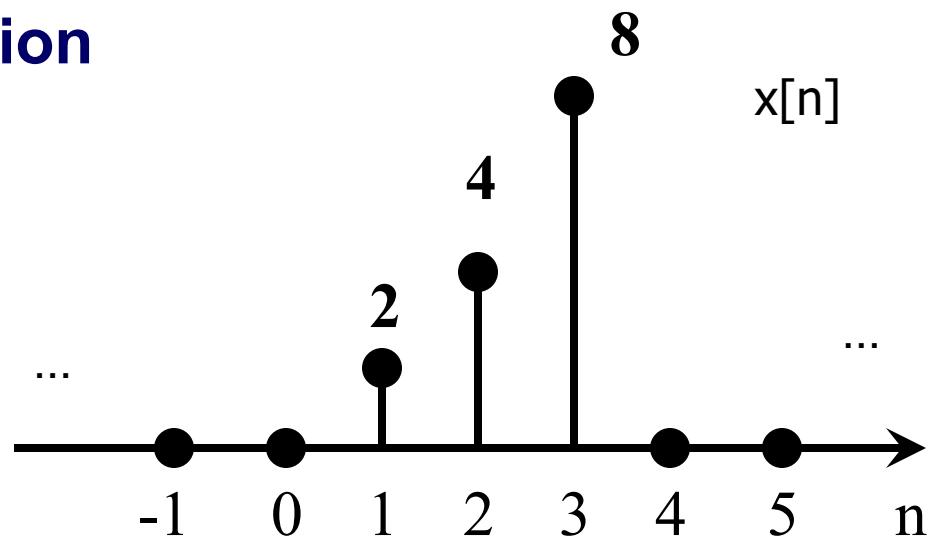


Representations of DT signals

3. Sequence representation

$$x[n] = \left\{ \begin{array}{l} 0, \\ \uparrow \\ n=0 \end{array}, 2, 4, 8 \right\}$$

4. Graphical representation



Lecture 2

Discrete-time (DT) Signals

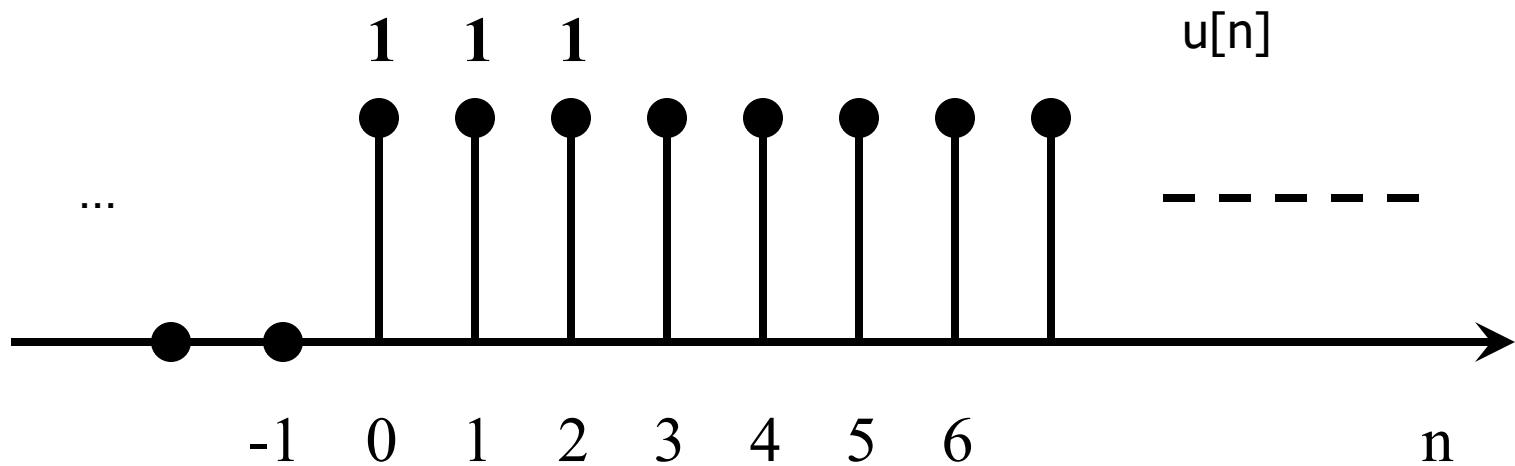
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Some elementary DT signals

1. Unit step sequence
2. Unit impulse signal
3. Sinusoidal signal
4. Exponential signal

Unit step sequence

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

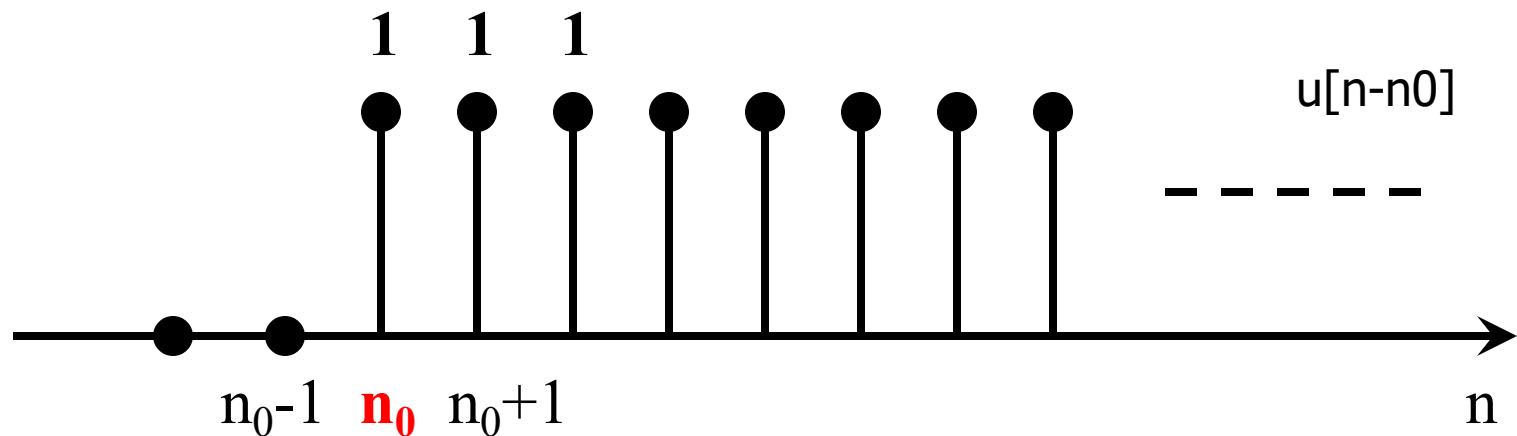


Q: $x[n]=5.u[n]?$

Time-shifted unit step

Replace n with $n-n_0 \rightarrow u[n - n_0] = \begin{cases} 1, & n \geq n_0 \\ 0, & n < n_0 \end{cases}$ n_0 : shift (an integer)

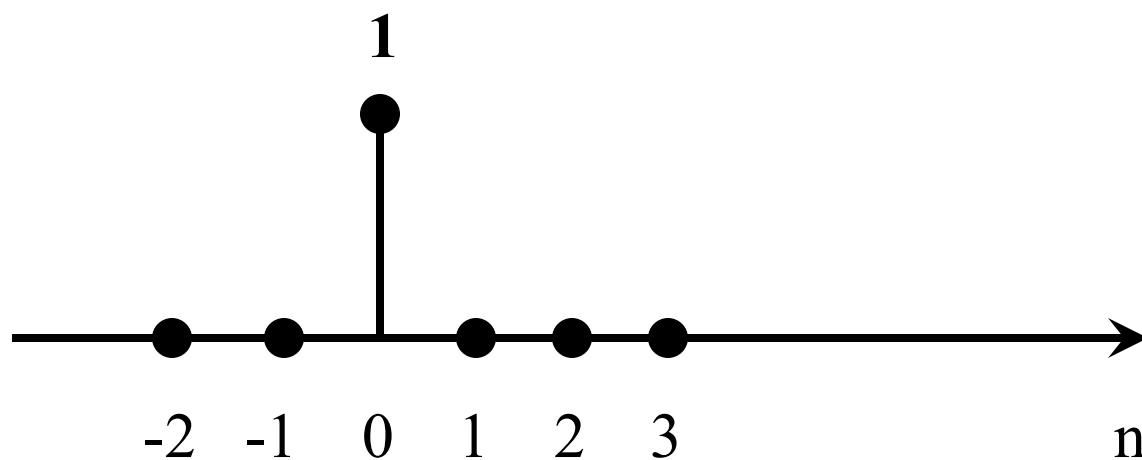
(is a n_0 -samples shifted version of the signal $u[n]$)



Q: $x[n] = -5 \cdot u[n-2]?$

Unit impulse sequence

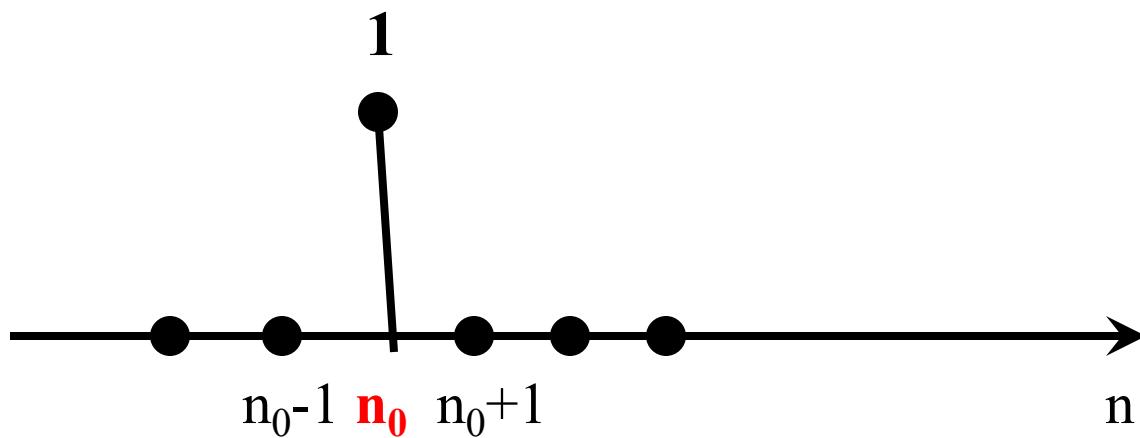
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Q: $x[n] = -5 \delta[n]$?

Time-shifted unit impulse

$$\delta[n - n_0] = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$



$$Q: x[n] = -5 \delta[n+2]$$

Relation between unit step and unit impulse

$$u[n] = \sum_{k=-\infty}^n \delta[k] = d[n] + d[n-1] + d[n-2] + \dots + d[n-M] + \dots$$

: running sum

$$\delta[n] = u[n] - u[n-1]$$

: first difference

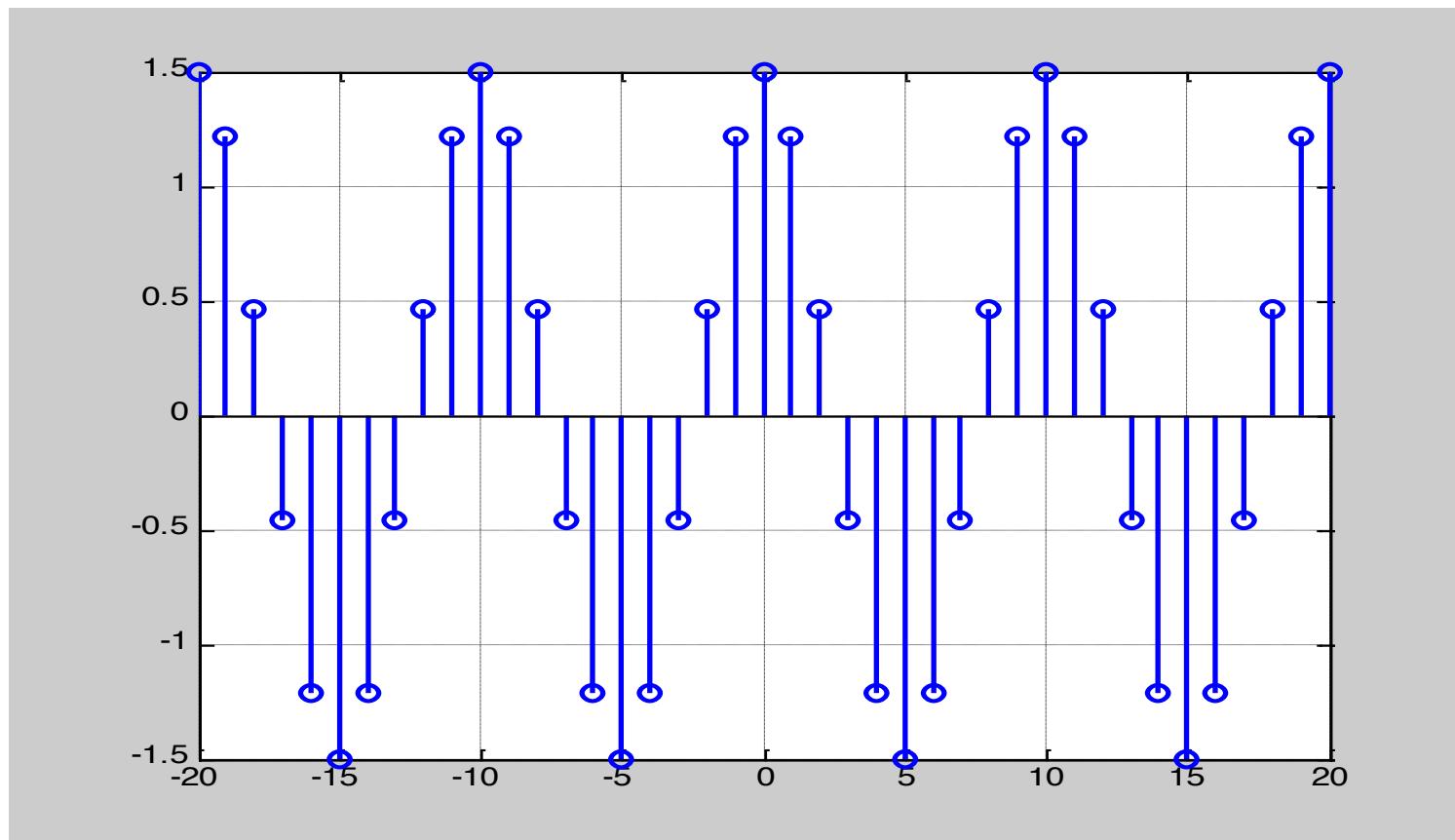
$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

: sample & hold

Sinusoidal signal

$$x(n) = A \cos(\Omega n + \theta), \quad -\infty < n < +\infty$$

$$= A \cos(2\pi F_n + \theta), \quad -\infty < n < +\infty$$



Exponential signal

$$x[n] = a^n$$

1. If a is real, then $x[n]$ is a real exponential

$a > 1 \rightarrow$ growing exponential

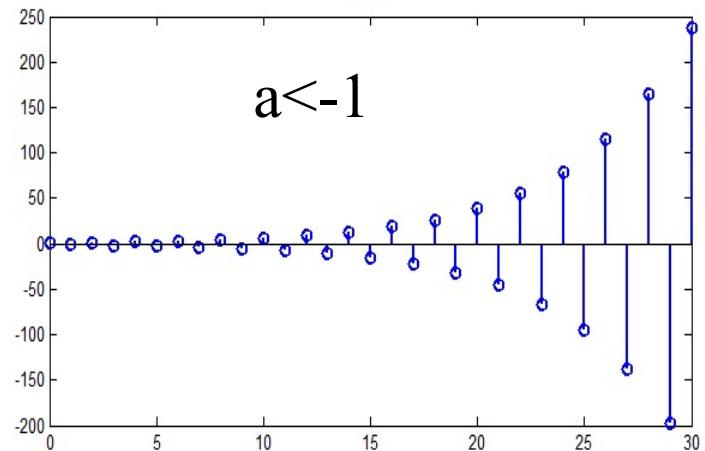
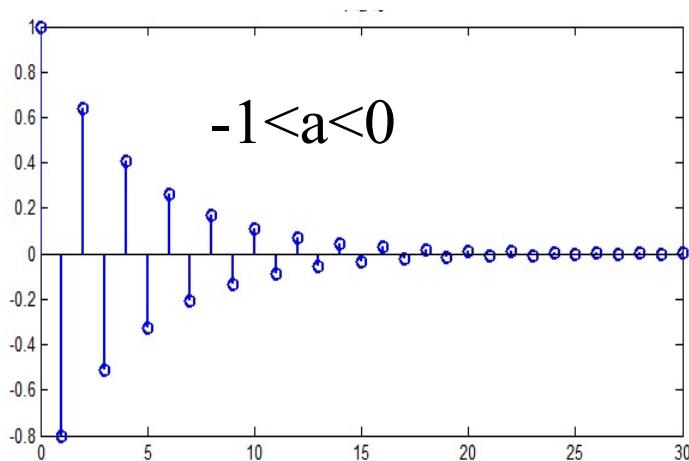
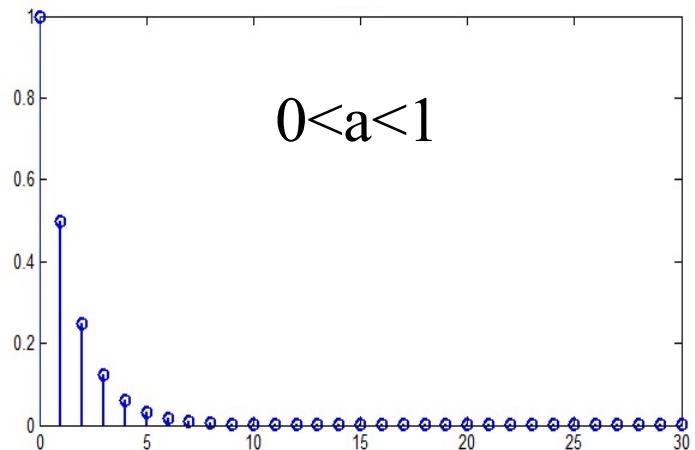
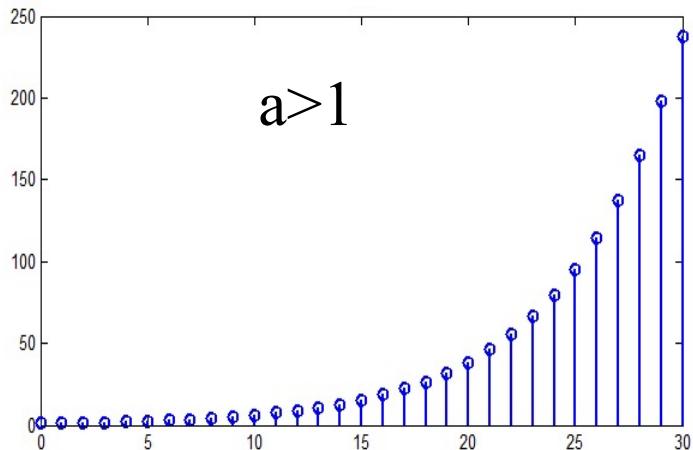
$0 < a < 1 \rightarrow$ shrinking exponential

$-1 < a < 0 \rightarrow$ alternate and decay

$a < -1 \rightarrow$ alternate and grows

2. If a is complex, then $x[n]$ is a complex exponential

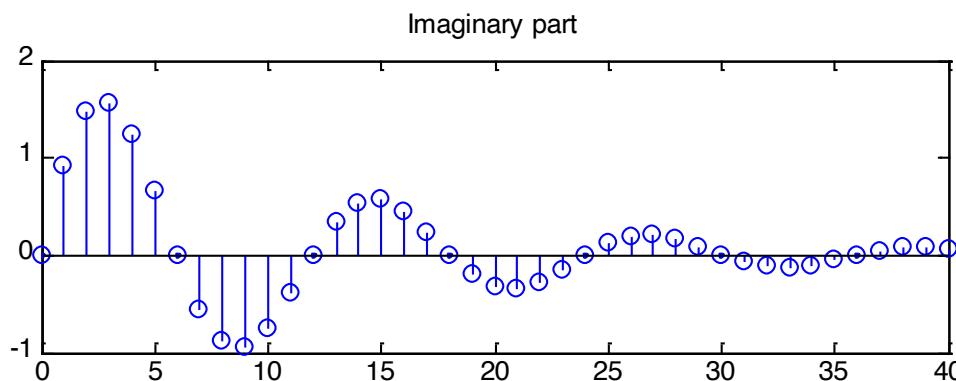
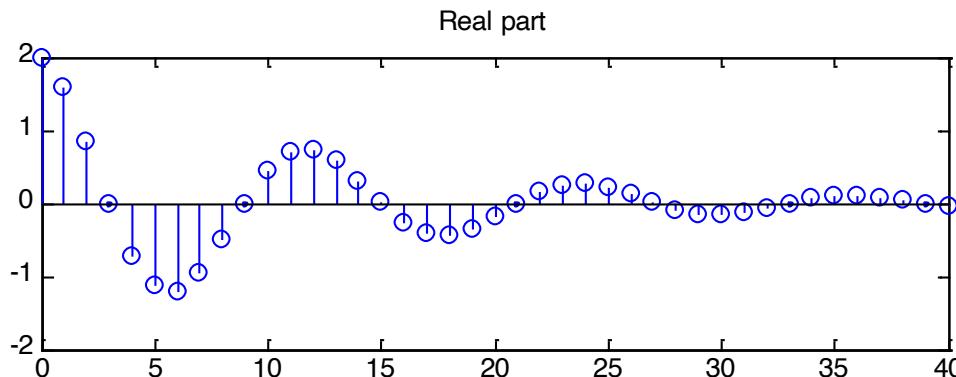
Exponential signal - Real examples



Exponential signal - Complex example

$$x[n] = 2e^{\left(-\frac{1}{12} + j\frac{\pi}{6}\right)n}$$

j: imaginary unit



Lecture 2

Discrete-time (DT) Signals

- **Duration:** 3 periods
- **Outline:**
 1. Representations of DT signals
 2. Some elementary DT signals
 - 3. Simple manipulations of DT signals**

Simple manipulations of DT signals

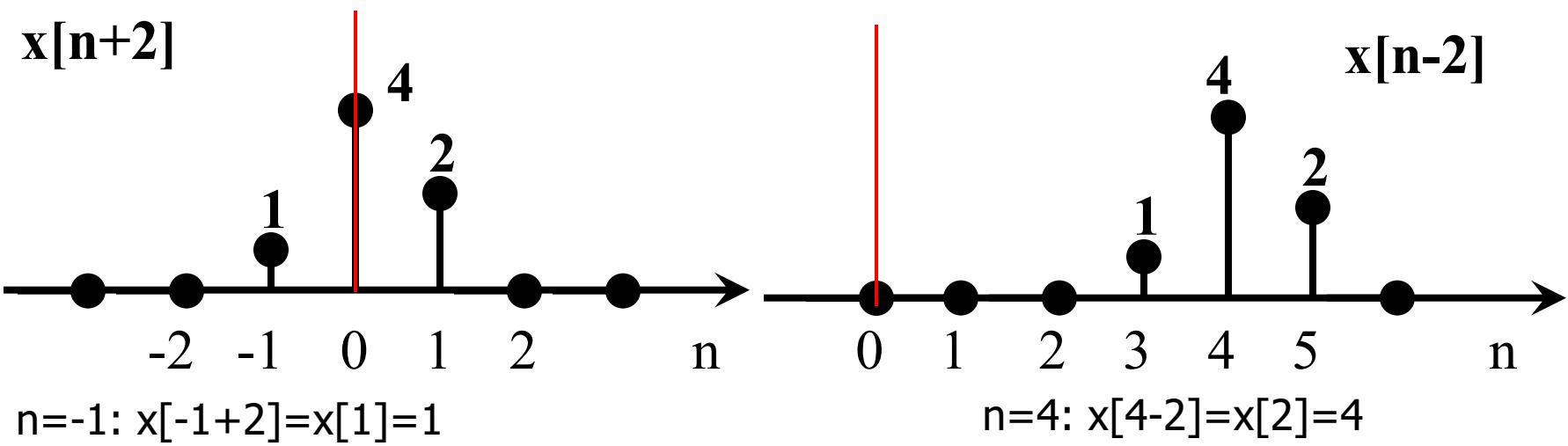
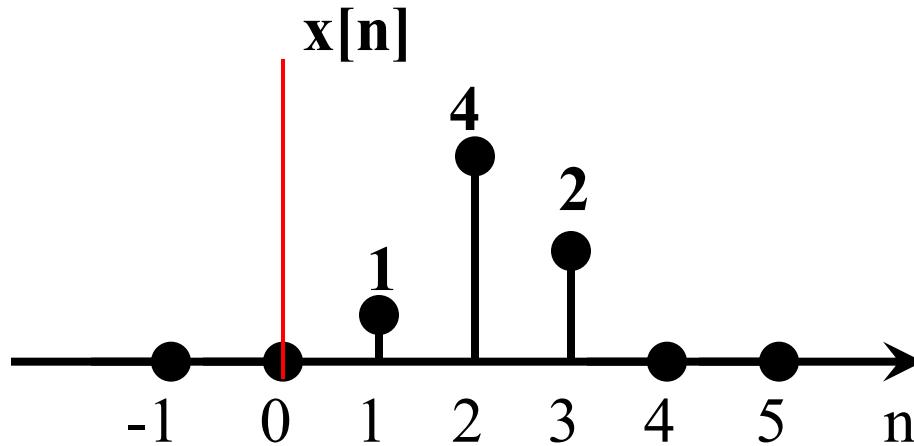
- Transformation of time:
 - Time shifting
 - Time reversal
- Adding and subtracting signals

Time shifting a DT signal

$x[n] \rightarrow x[n - k]$; k is an integer

- $k > 0$: right-shift $x[n]$ by $|k|$ samples
(delay of signal)
- $k < 0$: left-shift $x[n]$ by $|k|$ samples
(advance of signal)

Examples of time shifting

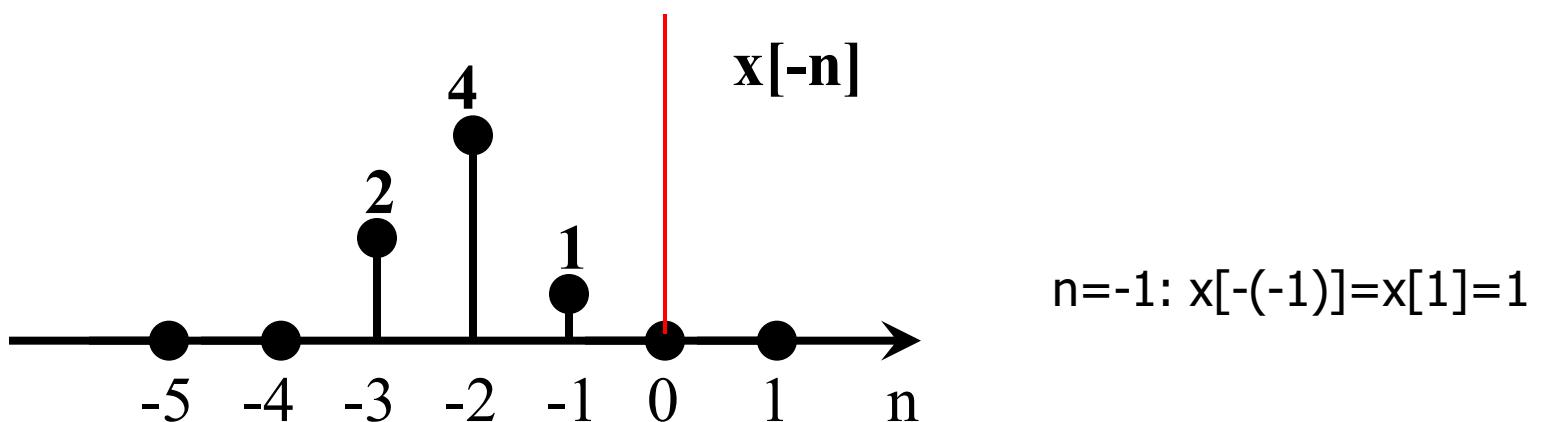
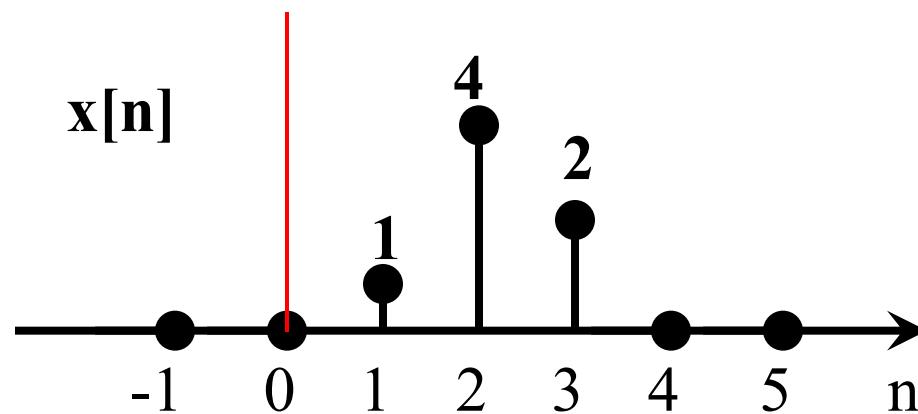


Time reversal a DT signal

$$x[n] \rightarrow x[-n]$$

Flip a signal $x[n]$ about the vertical axis at $n=0$

Example of time reversal



Combining time reversal and time shifting

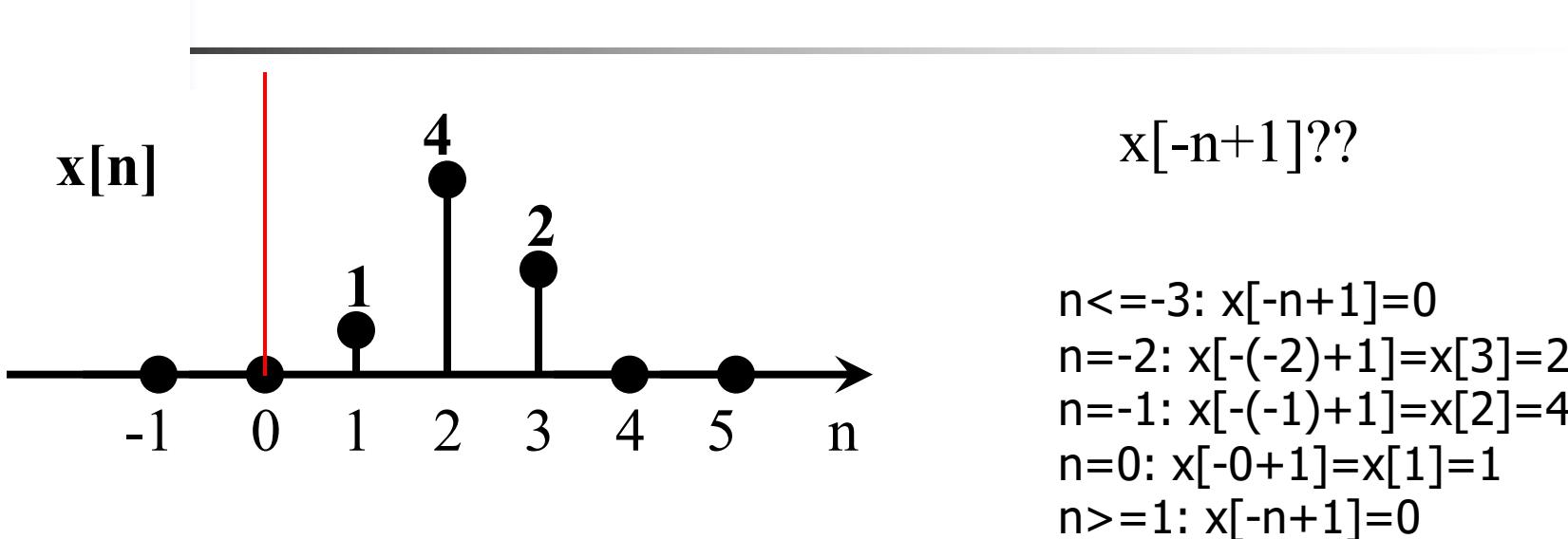
$$x[n] \rightarrow x[-n-k]$$

(k: an integer)

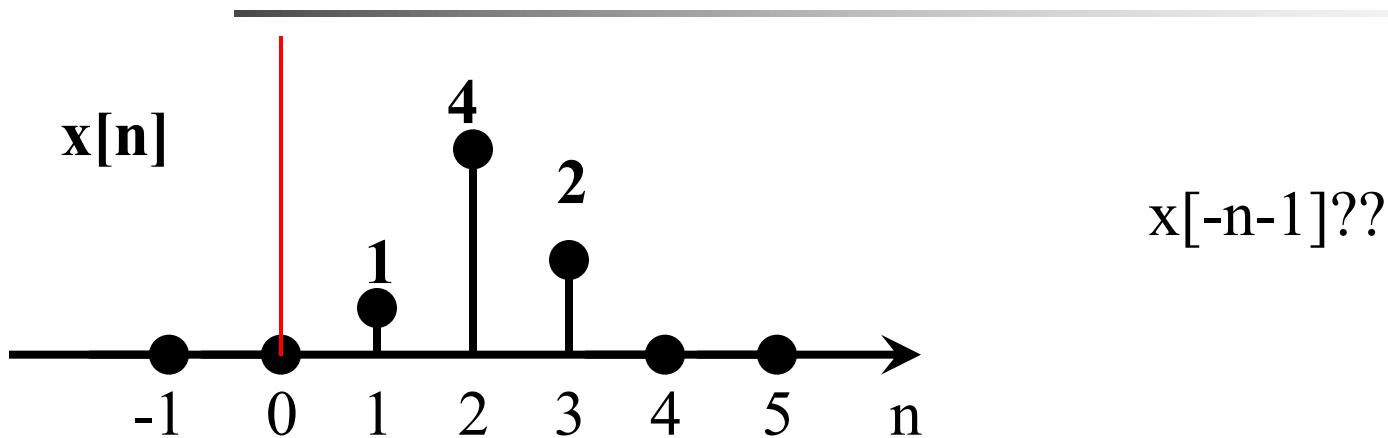
Method 1: Flip first, then shift

Method 2: Shift first, then flip

Examples of combining time reversal and time shifting



Examples of combining time reversal and time shifting



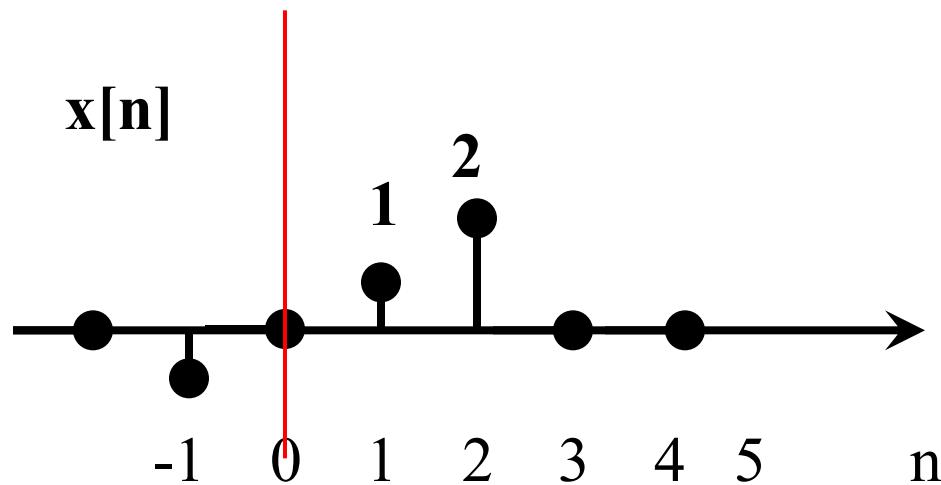
Adding and subtracting signals

- Do it “point by point”
- Can do using a table, or graphically, or by computer program
- Example: $x[n] = u[n] - u[n-4]$

n	≤ -1	0	1	2	3	≥ 4
x[n]	0	1	1	1	1	0

Exercise

- Find $x[n] = (u[n+1] - u[n-5])(nu[2-n])$



(Hint: $x_1[n] = u[n+1] - u[n-5]$, $x_2[n] = n \cdot u[2-n] \rightarrow x[n] = x_1[n] \cdot x_2[n]$)

Homework (HW)

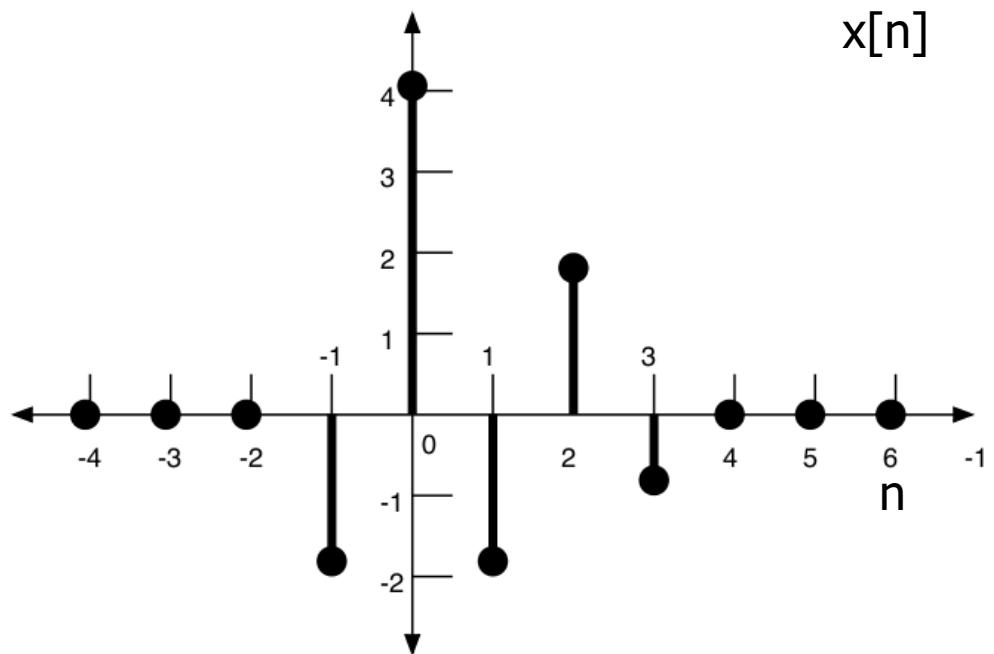
Prob.1 The following graph is of signal $x[n]$.

Plot the following:

a) $y[n] = 3x[-n-1]$

b) $y[n] = x[2n] - 1$

c) $y[n] = -x[n] + 2$



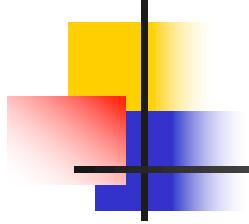
HW

Prob.2 Sometimes signals can be decomposed into combinations of simple unit step sequences such as this one:

$$y[n] = 2u[n - 2] - 2u[n - 7] - 2u[-n] + 2u[4 - n]$$

Sketch $y[n]$ and the following signals:

- a) $2-3y[n]$
- b) $3y[n-2]$
- c) $2-2y[-2+n]$



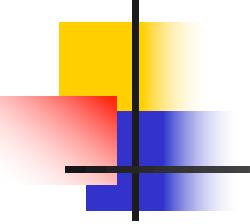
HW

Prob.3 Write a program to do the followings:

- Read a recorded .wav file to a data vector x & a sample rate value F_s
- Plot the signal over time (s) and show its length in samples and in seconds
- Playback the signal with 3 different sampling rate: F_s , $F_s/2$, $2*F_s$ & give your remarks

(Hint: read the textbook

"Applied Digital Signal Processing -Theory and Practice_Manolakis-Ingle_2011"
at pages 15, 28 & 30.)



HW

Write a program to do the followings:

Prob.4

- Given an analog sine signal $x(t)$ having its parameters (amplitude, frequency in Hz (denoted as F_0), and phase)
- Sampling the signal with 2 dif. sampling frequencies: $F_s1 = 3*F_0$ ($\rightarrow x1[n]$), $F_s2 = 1.5*F_0$ ($\rightarrow x2[n]$)
- Plot 2 resulting discrete signals $x1[n]$ & $x2[n]$ on the same time axis
- Playback the 2 discrete signals with its corresponding sampling rates : $x1[n]$ with F_s1 , $x2[n]$ with F_s2
- Check if we can recover the original analog signal in 2 cases

(Hint: read the textbook

“Applied Digital Signal Processing -Theory and Practice_Manolakis-Ingle_2011”
at pages 15, 28 & 30.

- F_0 : an audible frequency in (1000Hz, 15000Hz)
- sampling $x(t)=\text{Acos}(2*\pi*F_0*t+\phi)$ to receive $x[n]$): $t=n*T_s$

Lecture 3

DT systems

- **Duration:** 3 periods
- **Outline:**
 1. Input-output description of systems
 2. DT system properties
 3. Linear-time invariant (LTI) systems

Input-output description of DT systems

Tín hiệu vào --> Hệ --> Tín hiệu ra $y[n]$
 $x[n]$

Think of a DT system as an operator on DT signals:

- It processes DT input signals, to produce DT output signals
- **Notation: $y[n] = T\{x[n]\}$:** $y[n]$ is the response of the system T to the excitation $x[n]$
 $y[n]$ là hệ đáp ứng của hệ $T(\cdot)$ đ/v kích thích $x[n]$ (tín hiệu vào)
- Ex1:
$$y[n] = \frac{1}{3}\{x[n] + x[n - 1] + x[n - 2]\}$$
→ a three-point moving average filter, which is often used to smooth a signal corrupted by additive noise
- Ex2:
$$y[n] = \text{median}\{x[n - 1], x[n - 2], x[n], x[n + 1], x[n + 2]\}.$$
→ a five-point median filter, used to remove spikes from experimental data

Lecture 3

DT systems

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 3. LTI systems

DT system properties

- Causality Nhân quả
- Stability Ổn định
- Linearity Tuyến tính
- Time-invariance Bất biến

Causality

- The output of a causal system (at each time) **does not depend on** future inputs
(Không thể dự đoán được tín hiệu ra tại thời điểm tương lai)
(n_0 : current time instant $\rightarrow n_0 - 1$: past, $n_0 + 1$: future)
 - $y[n_0]$ (result) only depends on $x[n_0], x[n_0 - 1], x[n_0 - 2], \dots$ (cause)
for all n_0 integer
 - **Imply if the system is physically implementable in online mode**
Cho biết hệ thống có thể cài đặt (triển khai) về mặt vật lý hay không

Examples for causality

Determine which of the systems below are causal:

a) $y[n] = x[-n]$: Non-causal (eg. $y[-1]=x[1]$)

b) $y[n] = (n+1)x[n-1]$: causal since $y[n_0]$ only depends on $x[n_0-1]$

c) $y[n] = x[(n-1)^2]$: Non-causal (eg. $y[0]=x[1]$)

d) $y[n] = \cos(\omega_0 n + x[n])$: causal since $y[n]$ only depends on $x[n]$

e) $y[n] = 0.5y[n-1] + x[n-1]$:

causal since $y[n]$ only depends on $x[n-1]$

Stability

- If a system “blow up” it is **not stable**

In particular, if a “well-behavior” signal (all values have finite amplitude) results in infinite magnitude output, the system is **unstable**

- BIBO stability: “**bounded input – bounded output**” – if you put finite signals in, you will get finite signals out

biên độ (giá trị) hữu hạn

Examples for stability

Determine which of the systems below are BIBO stable:

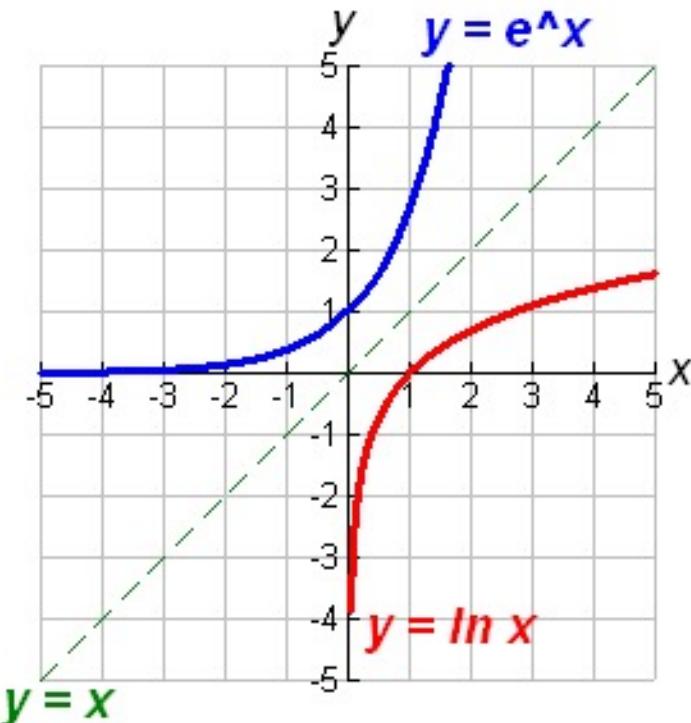
a) A unit delay system:

b) An accumulator:

c) $y[n] = \cos(x[n])$:

d) $y[n] = \ln(x[n])$:

e) $y[n] = \exp(x[n])$:



Linearity

Scaling signals and adding them, then processing through the system

same as

Processing signals through system, then scaling and adding them

If $T(x_1[n]) = y_1[n]$ and $T(x_2[n]) = y_2[n]$

$\rightarrow T(ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n]$

Only has theoretical meanings, but simplify modeling & analysis of real systems

Time-invariance

$T(\cdot)$

$ax[n] \xrightarrow{T(\cdot)} aT(x[n])$ với mọi a thuộc \mathbb{R}

- If you time shift the input, get the **same** output, but with the **same** time shift
- The behavior of the system **doesn't change** with time

$T(\cdot)$: a liner system

If $T(x[n]) = y[n]$

then $T(x[n-n_0]) = y[n-n_0]$

Với mọi $x_1[n], x_2[n]$
 a, b thuộc \mathbb{R}

$$x_3[n] = ax_1[n] + bx_2[n] \rightarrow y_3[n] = T(x_1[n]) + T(x_2[n])$$

Only has theoretical meanings, but simplify modeling & analysis of real systems

Examples for linearity and time-invariance

Prob 0. Determine if the systems below are linear and/or time-invariant. Prove!

a) $y[n] = nx[n]$

Linear

Time-variant

Examples for linearity and time-invariance

Determine if the systems below are linear and/or time-invariant. Prove!

b) $y[n] = x^2[n]$

Non-linear
Time-invariant

Examples for linearity and time-invariance

Determine if the systems below are linear and/or time-invariant. Prove!

c)

$$y[n] = \sum_{r=0}^M b_r x[n - r]$$

Linear

Time-invariant (**LTI**)

Lecture 3

DT systems

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 3. **LTI systems**

Computing the response of DT LTI systems to arbitrary input

Method 1: based on the direct solution of the input-output equation for the system → not often used

Method 2:

- Decompose the input signal into a sum of elementary signals
 - Find the response of system to each elementary signal
 - Add those responses to obtain the total response of the system to the given input signal
- often used

$$x[n] = \sum_k c_k x_k[n]$$

$$x_k[n] \rightarrow y_k[n]$$

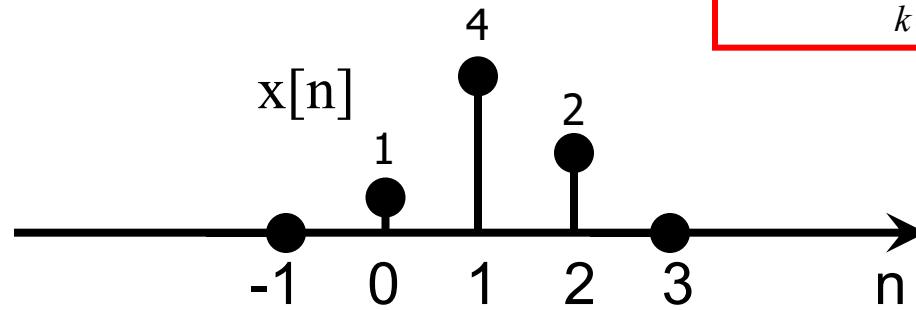
$$x[n] \rightarrow y[n] = \sum_k c_k y_k[n]$$

Impulse representation of DT signals

We can describe any DT signal $x[n]$ as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$$

Example:



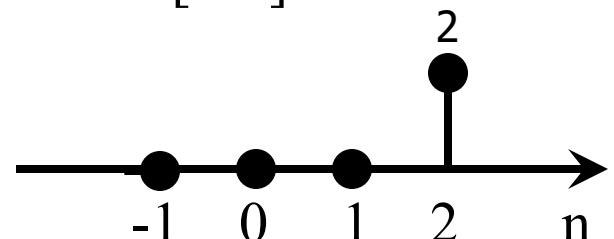
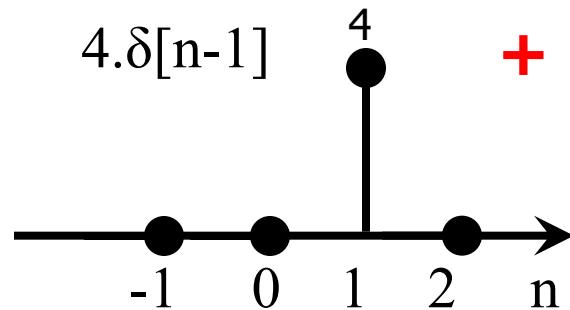
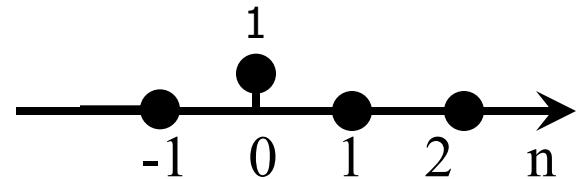
$$1.\delta[n-0]$$

+

$$4.\delta[n-1]$$

+

$$2.\delta[n-2]$$



Impulse response of DT LTI systems

- Any DT LTI system is characterized by its **impulse response** $h[n]$.
- Notation:

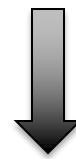


- Impulse response = response of the system to the (unit) impulse



$$h[n] = y[n] \mid x[n] = \delta[n]$$

Response of LTI system to delayed impulse



Time-invariant property



Response of LTI system to a DT signal



Time-invariant property

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



Linear property

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

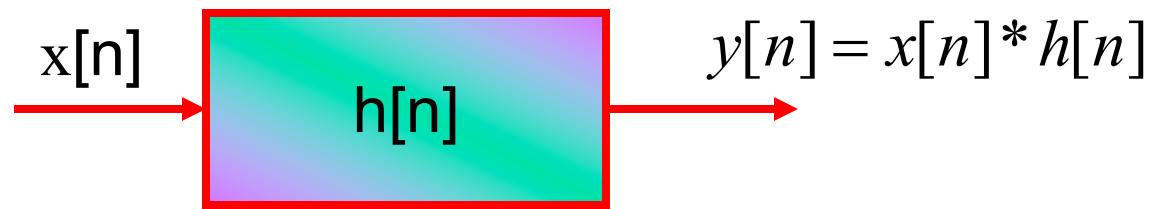
$$y[n] \stackrel{\triangle}{=} x[n] * h[n]$$

Convolution sum

(an operation on 2 signals)

DT convolution formula

Convolution: an operation between the **input signal** to a system and its **impulse response**, resulting in the **output signal**



In DT systems: convolution of 2 signals involves **summing** the product of the 2 signals – where one of signals is flipped and shifted

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Computing the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Suppose to compute the output $y[n]$ at time $n = n_0$.

1. **Flip** $h[k]$ about $k = 0$, to obtain $h[-k]$
2. **Shift** $h[-k]$ by n_0 to the right (left) if n_0 is positive (negative), to obtain $h[n_0-k]$
3. **Multiply** $x[k]$ and $h[n_0-k]$ for all k , to obtain the product
 $x[k].h[n_0-k]$
4. **Sum** up the product for all k , to obtain $y[n_0]$

Repeat from 2-4 for all of n

The length of the convolution sum result

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Suppose:

Length of $x[k]$ is $N_x \rightarrow N_1 \leq k \leq N_1 + N_x - 1$

Length of $h[n-k]$ is $N_h \rightarrow N_2 \leq n-k \leq N_2 + N_h - 1$

$\rightarrow N_1 + N_2 \leq n \leq N_1 + N_2 + N_x + N_h - 2$

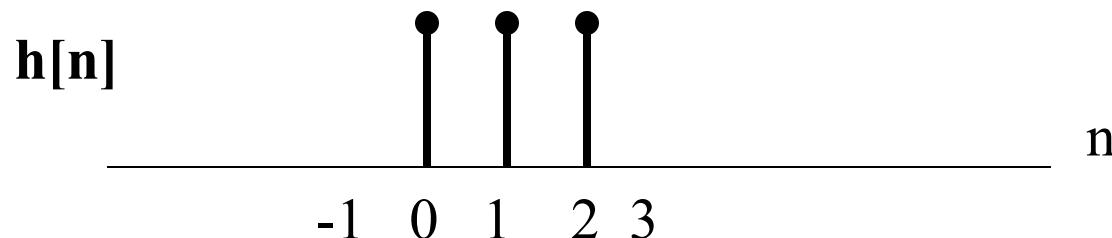
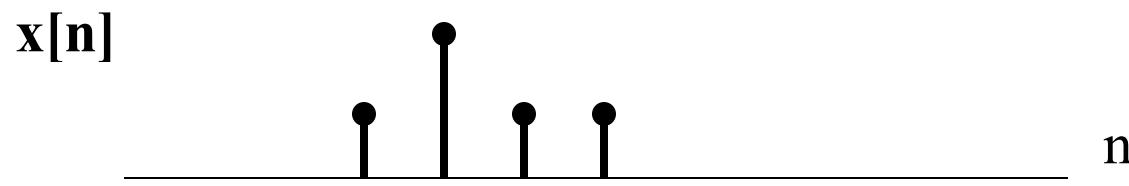
Length of $y[n]$:

$$N_y = N_x + N_h - 1$$

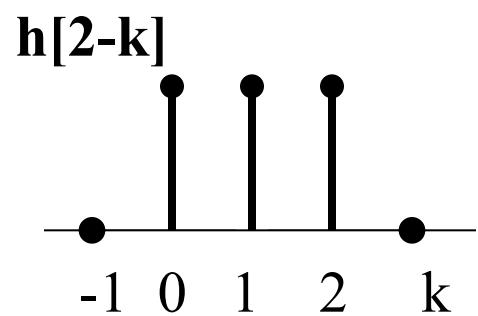
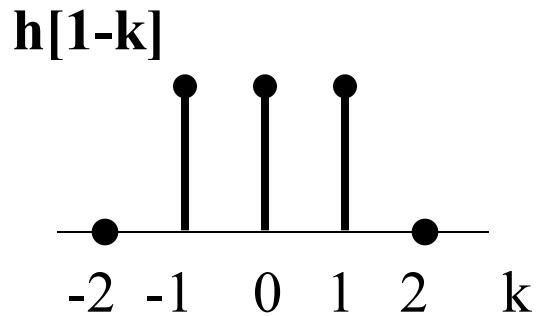
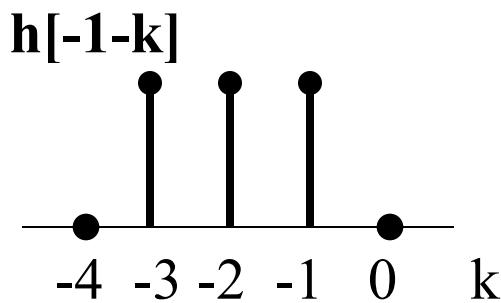
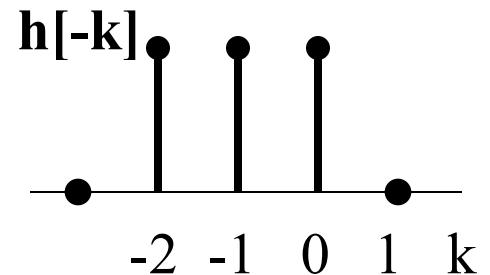
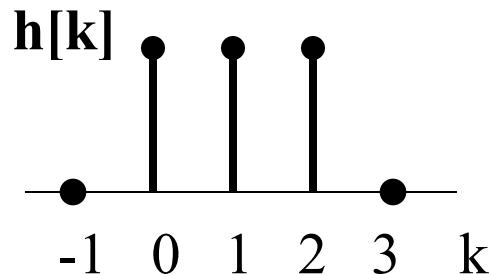
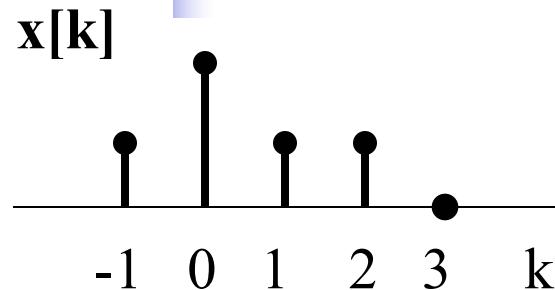
Examples of computing the convolution sum

Ex1. Find $y[n] = x[n]*h[n]$ where

$$x[n] = u[n+1] - u[n-3] + \delta[n] \quad h[n] = 2(u[n] - u[n-3])$$



Ex1 (cont.)



$$\begin{aligned}y[n < -1] &= 0; y[-1] = 2; y[0] = 6; y[1] = 8; \\y[2] &= 8; y[3] = 4; y[4] = 2; y[n > 4] = 0\end{aligned}$$

Examples of computing the convolution sum

Ex2. Find $y[n] = x[n]*h[n]$ where $x[n] = a^n u[n]$ $h[n] = u[n]$

Examples of computing the convolution sum

Ex3. Find $y[n] = x[n]*h[n]$ where $x[n] = b^n u[n]$ and $h[n] = a^n u[n+2]$

$$|a| < 1, |b| < 1, a \neq b$$

DT LTI properties based on impulse response

1. Causal system: $h[n]$ is zero for all time $n < 0$

2. BIBO stable system:

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

- Finite impulse response (FIR) systems \rightarrow always stable
- Infinite impulse response (IIR) systems \rightarrow can be stable or not

Examples of DT LTI properties

1. Is $h[n] = 0.5^n u[n]$ BIBO stable? Causal?

2. Is $h[n] = 3^n u[n]$ BIBO stable? Causal?

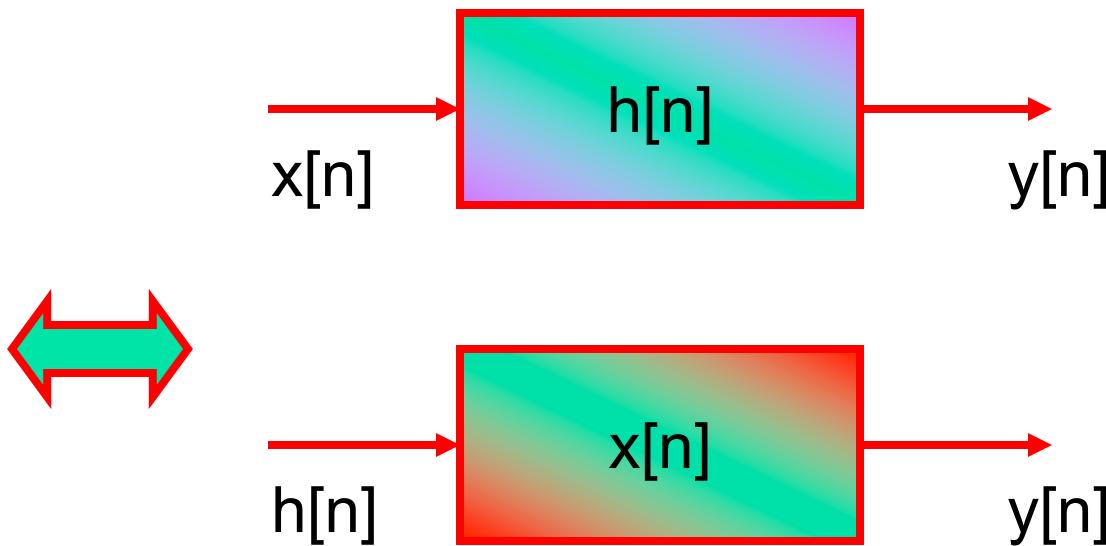
3. Is $h[n] = 3^n u[-n]$ BIBO stable? Causal?

Convolution sum properties

- Commutative law
- Associative law
- Distributive law

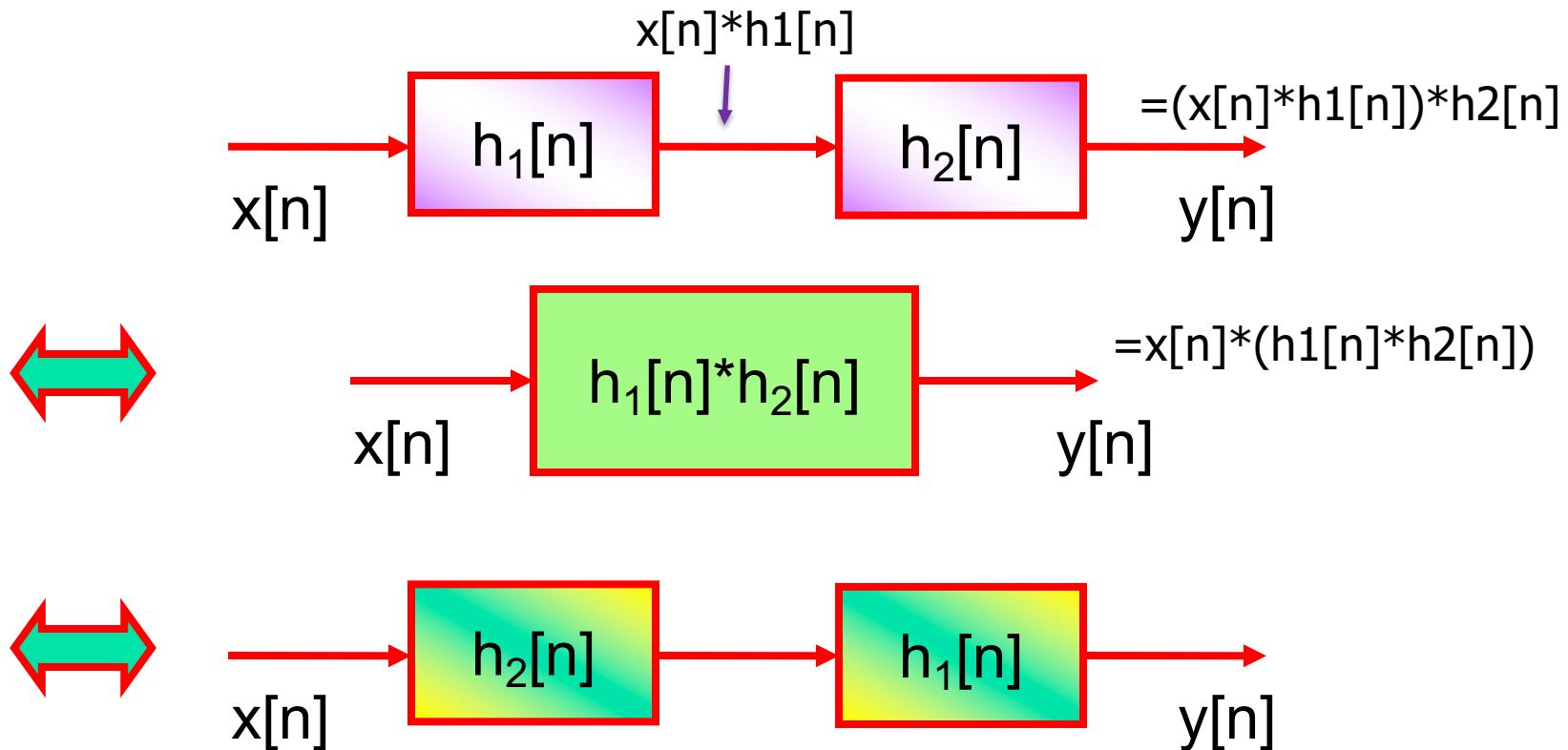
Commutative law

$$x[n] * h[n] = h[n] * x[n]$$



Associative law

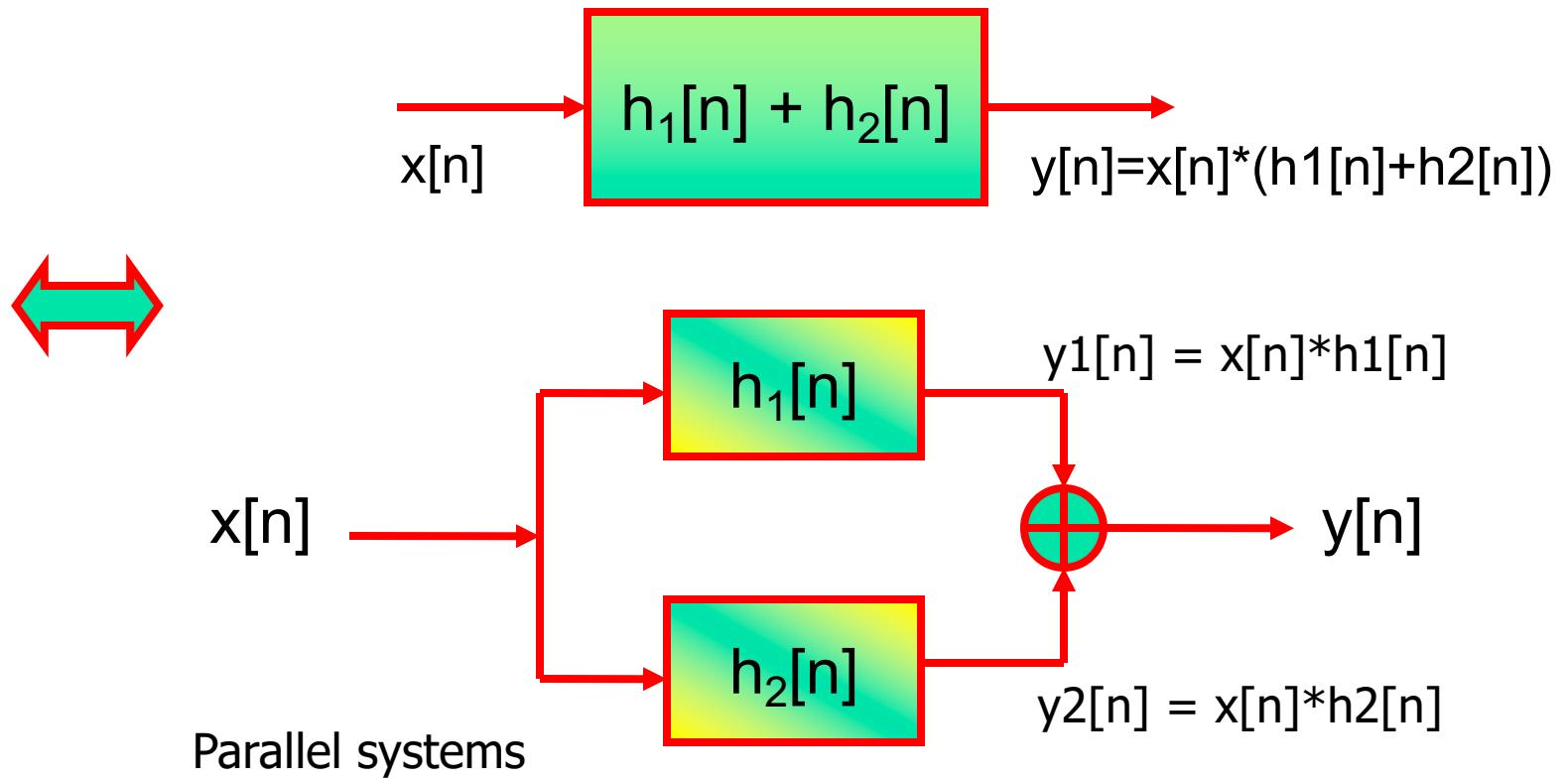
$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$



Directly connected systems

Distributive law

$$x[n] * (h_1[n] + h_2[n]) = (x[n] * h_1[n]) + (x[n] * h_2[n])$$



LTI systems characterized by linear constant coefficient difference equations

General form:

$$y[n] + a_1 y[n-1] + \dots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]$$

$$\Leftrightarrow \sum_{k=0}^N a_k y[n-k] = \sum_{r=0}^M b_r x[n-r], \quad a_0 = 1$$

N, M: non-negative integers

N: order of equation

a_k, b_r: constant real coefficients

HW

Prob.3 Conclude about the linearity, time-invariance, stability, causality of the following systems. Prove!

(a) $y[n] = \frac{1}{3}\{x[n] + x[n - 1] + x[n - 2]\}$

(b) $y[n] = \text{median}\{x[n - 1], x[n - 2], x[n], x[n + 1], x[n + 2]\}.$

(Hint: read pp. 32-35 textbook)

Trung vị (Median) là gì?

<https://vietnambiz.vn/trung-vi-median-la-gi-vi-du-ve-trung-vi-2019110713491368.htm>

HW

Prob.4 Determine the causality and the BIBO stability for the systems with the following impulse responses:

a) $h[n] = \sin(-n)u[n]$

b) $h[n] = e^{-n}u[-n]$

c) $h[n] = e^n u[n]$

d) $h[n] = \sin(n)u[-n]$

e) $h[n] = n e^{-n}u[n]$

f) $h[n] = e^{-n}\sin(n)u[n]$

HW

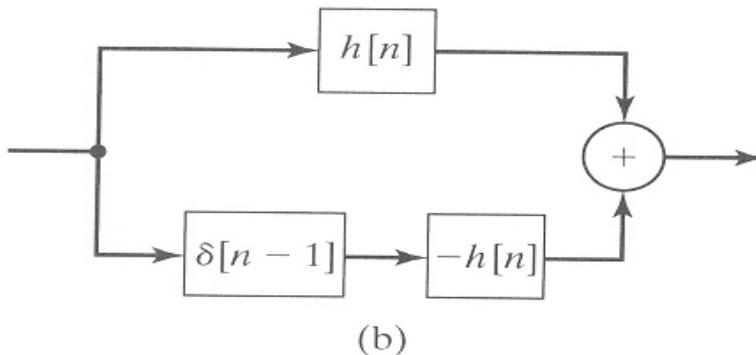
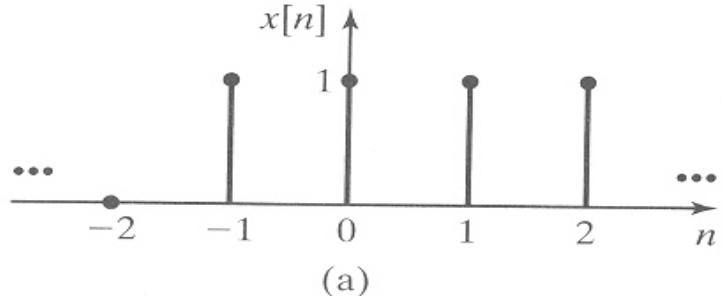
Prob.5 Find $y[n] = x[n]^*h[n]$ where:

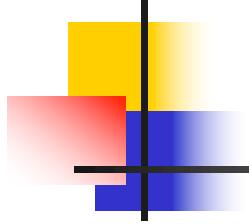
- a) $x[n] = a^n u[n]$ and $h[n] = u[n] - u[n-10]$
- b) $x[n] = u[-n]$ and $h[n] = a^n u[n-2]$, $|a| < 1$
- c) $x[n] = 2\delta[n+2] + 2\delta[n+1] + 2\delta[n-1] + 2\delta[n-2] + 2\delta[n-3] + 2\delta[n-4]$
and $h[n] = \delta[n] - \delta[n-1] + \delta[n-2]$
- d) $x[n] = u[-n+2]$ and $h[n] = a^n u[-n]$
- e) $x[n] = 0.2^n u[n]$ and $h[n] = \delta[n] - 0.2\delta[n-1]$

HW

Prob.6 Consider the LTI system with the input and output related by: $y[n] = 0.5x[n-1] + 0.7x[n]$

- Find the impulse response $h[n]$
- Is this system causal? Stable? Why?
- Determine the system response $y[n]$ for the input shown in Fig. (a)
- Consider the interconnections of the LTI systems given in Fig. (b). Find the impulse response of the total system
- Solve for the response of the system of part (d) for the input of part (c)





HW

Denoising signal using moving averaging system:

Modify the `moving_average_smoothing.m` program so that it implements the system in Prob. (3a) (slide 66, Chapter 2) in 3 different ways:

- Averaging of time-shifted input signals
- Using `conv()` function
- Using `filter()` function