The Optimal Substructure of the Activity-Selection Problem (3/3)

- If we denote the size of an optimal solution for the set S_{ij} by $c[i, j]$, then we would have the recurrence
$$c[i,j] = c[i,k] + c[k,j] + 1 .$$
- Of course, if we did not know that an optimal solution for the set S_{ij} includes activity a_k, we would have to examine all activities in S_{ij} to find which one to choose, so that
$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset , \\ \max\limits_{a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\} & \text{if } S_{ij} \neq \emptyset . \end{cases}$$

Making the Greedy Choice (1/4)

- What if we could choose an activity to add to our optimal solution without having to first solve all the subproblems?

➡ For the activity-selection problem, we need consider only one choice: the greedy choice.

- What does our intuition suggest?
  - We should choose an activity that leaves the resource available for as many other activities as possible.
  - We should choose the activity in S with the earliest finish time, since that would leave the resource available for as many of the activities that follow it as possible.

Making the Greedy Choice (2/4)

- Since the activities are sorted in monotonically increasing order by finish time, the greedy choice is activity a_1.
- We have only one remaining subproblem to solve: finding activities that start after a_1 finishes.
- Let $S_k = \{a_i \in S : s_i \geq f_k\}$ be the set of activities that start after activity a_k finishes.

➡ S_1 remains as the only subproblem to solve.

- One big question remains: is our intuition correct?

Making the Greedy Choice (3/4)

- **Theorem 16.1:** Consider any nonempty subproblem S_k, and let a_m be an activity in S_k with the earliest finish time. Then a_m is included in some maximum-size subset of mutually compatible activities of S_k.

Proof Let A_k be a maximum-size subset of mutually compatible activities in S_k, and let a_j be the activity in A_k with the earliest finish time. If $a_j = a_m$, we are done, since we have shown that a_m is in some maximum-size subset of mutually compatible activities of S_k. If $a_j \neq a_m$, let the set $A'_k = A_k - \{a_j\} \cup \{a_m\}$ be A_k but substituting a_m for a_j. The activities in A'_k are disjoint, which follows because the activities in A_k are disjoint, a_j is the first activity in A_k to finish, and $f_m \leq f_j$. Since $|A'_k| = |A_k|$, we conclude that A'_k is a maximum-size subset of mutually compatible activities of S_k, and it includes a_m. ■