Divide-and-Conquer Solution

- We assume that n is an exact power of 2 in each of the $n \times n$ matrices.
- Suppose that we partition each of A, B, and C into four $n/2 \times n/2$ matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

so that we rewrite the equation $C = A \cdot B$ as

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

1 $n = A.rows$
2 let C be a new $n \times n$ matrix
3 **if** $n == 1$
4 $\quad c_{11} = a_{11} \cdot b_{11}$
5 **else** partition A, B, and C as in equations (4.9)
6 $\quad C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
$\qquad\quad +$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
7 $\quad C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
$\qquad\quad +$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
8 $\quad C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
$\qquad\quad +$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
9 $\quad C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
$\qquad\quad +$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
10 **return** C

Implementation Detail

- How do we partition the matrices in line 5?
  - We can partition the matrices without copying entries.
  - The trick is to use index calculations. We identify a submatrix by a range of row indices and a range of column indices of the original matrix.

Strassen's Method (1/6)

- The key to Strassen's method is to make the recursion tree slightly less bushy.
- ➡ Instead of performing 8 recursive multiplications of $n/2 \times n/2$ matrices, it performs only 7.
- Strassen's method has four steps:
  1. Divide the input matrices A and B and output matrix C into $n/2 \times n/2$ submatrices. (It takes $\Theta(1)$ time.)
  2. Create 10 matrices $S_1, S_2, \ldots, S_{10}$, each of which is $n/2 \times n/2$ and is the sum or difference of two matrices created in step 1. (It takes $\Theta(n^2)$ time.)

Strassen's Method (2/6)

3. Using the submatrices created in step 1 and the 10 matrices created in step 2, recursively compute seven matrix products P_1, P_2, …, P_7. Each matrix P_i is $n/2 \times n/2$.
4. Compute the desired submatrices C_{11}, C_{12}, C_{21}, C_{22} of the result matrix C by adding and subtracting various combinations of the P_i matrices. (We can compute all four submatrices $\Theta(n^2)$ time.)

Strassen's Method (3/6)

- In step 2, we create the following 10 matrices:

$$\begin{aligned}
S_1 &= B_{12} - B_{22}\,,\\
S_2 &= A_{11} + A_{12}\,,\\
S_3 &= A_{21} + A_{22}\,,\\
S_4 &= B_{21} - B_{11}\,,\\
S_5 &= A_{11} + A_{22}\,,\\
S_6 &= B_{11} + B_{22}\,,\\
S_7 &= A_{12} - A_{22}\,,\\
S_8 &= B_{21} + B_{22}\,,\\
S_9 &= A_{11} - A_{21}\,,\\
S_{10} &= B_{11} + B_{12}\,.
\end{aligned}$$

Strassen's Method (4/6)

- In step 3, we recursively multiply $n/2 \times n/2$ matrices seven times to compute the following $n/2 \times n/2$ matrices, each of which is the sum or difference of products of A and B submatrices:

$$\begin{aligned}
P_1 &= A_{11} \cdot S_1 &&= A_{11} \cdot B_{12} - A_{11} \cdot B_{22}\,,\\
P_2 &= S_2 \cdot B_{22} &&= A_{11} \cdot B_{22} + A_{12} \cdot B_{22}\,,\\
P_3 &= S_3 \cdot B_{11} &&= A_{21} \cdot B_{11} + A_{22} \cdot B_{11}\,,\\
P_4 &= A_{22} \cdot S_4 &&= A_{22} \cdot B_{21} - A_{22} \cdot B_{11}\,,\\
P_5 &= S_5 \cdot S_6 &&= A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22}\,,\\
P_6 &= S_7 \cdot S_8 &&= A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22}\,,\\
P_7 &= S_9 \cdot S_{10} &&= A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12}\,.
\end{aligned}$$

Strassen's Method (5/6)

- Step 4 adds and subtracts the P_i matrices created in step 3 to construct the four $n/2 \times n/2$ submatrices of the product C.

$$C_{11} = P_5 + P_4 - P_2 + P_6$$

$$\begin{array}{lllllll}
A_{11}\cdot B_{11} & + A_{11}\cdot B_{22} & + A_{22}\cdot B_{11} & + A_{22}\cdot B_{22} & & & \\
 & & - A_{22}\cdot B_{11} & & + A_{22}\cdot B_{21} & & \\
 & - A_{11}\cdot B_{22} & & & & - A_{12}\cdot B_{22} & \\
 & & & - A_{22}\cdot B_{22} & - A_{22}\cdot B_{21} & + A_{12}\cdot B_{22} & + A_{12}\cdot B_{21} \\
\hline
A_{11}\cdot B_{11} & & & & & & + A_{12}\cdot B_{21}
\end{array}$$

Strassen's Method (6/6)

$$C_{12} = P_1 + P_2$$

$$\begin{array}{l} A_{11} \cdot B_{12} - A_{11} \cdot B_{22} \\ \qquad\quad + A_{11} \cdot B_{22} + A_{12} \cdot B_{22} \\ \hline A_{11} \cdot B_{12} \qquad\qquad\quad + A_{12} \cdot B_{22} \end{array}$$

$$C_{21} = P_3 + P_4$$

$$\begin{array}{l} A_{21} \cdot B_{11} + A_{22} \cdot B_{11} \\ \qquad\quad - A_{22} \cdot B_{11} + A_{22} \cdot B_{21} \\ \hline A_{21} \cdot B_{11} \qquad\qquad\quad + A_{22} \cdot B_{21} \end{array}$$

$$C_{22} = P_5 + P_1 - P_3 - P_7$$

$$\begin{array}{l} A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\ \qquad - A_{11} \cdot B_{22} \qquad\qquad\qquad\qquad\qquad + A_{11} \cdot B_{12} \\ \qquad\qquad\qquad - A_{22} \cdot B_{11} \qquad\qquad\qquad\qquad - A_{21} \cdot B_{11} \\ - A_{11} \cdot B_{11} \qquad\qquad\qquad\qquad\qquad - A_{11} \cdot B_{12} + A_{21} \cdot B_{11} + A_{21} \cdot B_{12} \\ \hline \qquad\qquad\qquad\qquad\qquad A_{22} \cdot B_{22} \qquad\qquad\qquad + A_{21} \cdot B_{12} \end{array}$$