Important Property (5/7)

Basis. Initially, when the queue contains only s, the lemma certainly holds ($v_r.d \le v_1.d + 1$ and $v_i.d \le v_{i+1}.d$).

Inductive Step: For the inductive step, we must prove that the lemma holds after both dequeuing and enqueuing a vertex.

If the head v_1 of the queue is dequeued, v_2 becomes the new head.

– By the inductive hypothesis, $v_1.d \le v_2.d$.

– Then we have $v_r.d \le v_1.d + 1 \le v_2.d + 1$, and the remaining inequalities are unaffected.

Important Property (6/7)

When we enqueue a vertex v in line 17 of BFS, it becomes v_{r+1}. At that time, we have already removed vertex u, whose adjacency list is currently being scanned.

– By the inductive hypothesis, the new head v_1 has $v_1.d \ge u.d$.

– Thus, $v_{r+1}.d = v.d = u.d + 1 \le v_1.d + 1$.

– From the inductive hypothesis, we also have $v_r.d \le u.d + 1$, and so $v_r.d \le u.d + 1 = v.d = v_{r+1}.d$.

Important Property (7/7)

- The following corollary shows that the d values at the time that vertices are enqueued are monotonically increasing over time.
- **Corollary 4.** Suppose that vertices v_i and v_j are enqueued during the execution of BFS, and that v_i is enqueued before v_j. Then $v_i.d \le v_j.d$ at the time that v_j is enqueued.

  Proof. Immediate from **Lemma 3** and the property that each vertex receives a finite d value at most once during the course of BFS.

Correctness of Breadth-First Search (1/3)

- **Theorem 5.** Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then, during its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$. Moreover, for any vertex $v \ne s$ that is reachable from s, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.