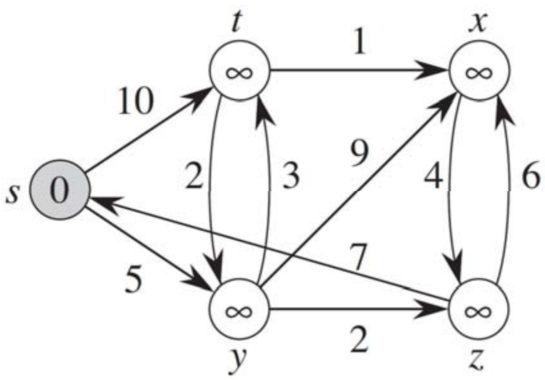
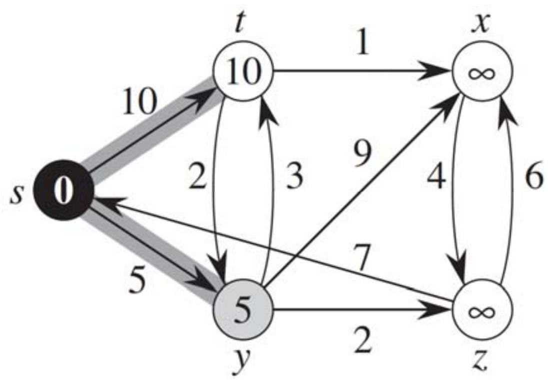
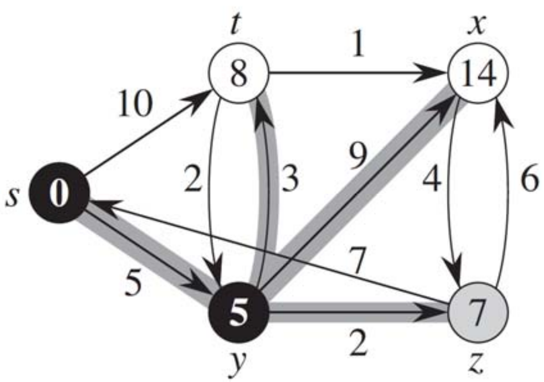
Example (1/6)



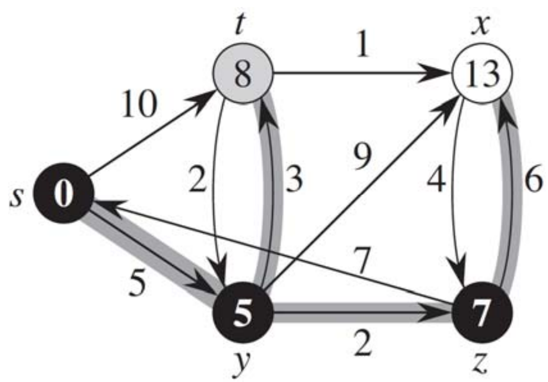
Example (2/6)



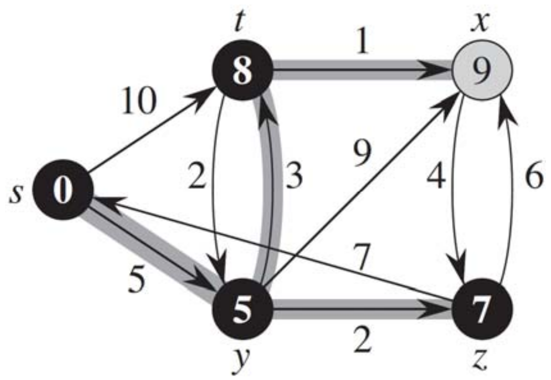
Example (3/6)



Example (4/6)



Example (5/6)



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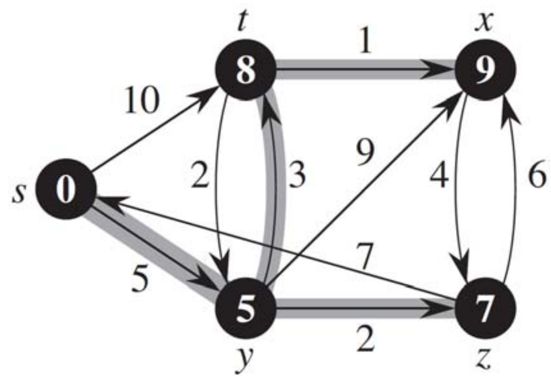
Example (6/6)



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Analysis

Like Prim's algorithm, depends on implementation of priority queue.

- If binary heap, each operation takes $O(\lg V)$ time
  ➡ $O(E \lg V)$.
- If a Fibonacci heap:
  - Each EXTRACT-MIN takes $O(1)$ amortized time.
  - There are $O(V)$ other operations, taking $O(\lg V)$ amortized time each.
  - Therefore, time is $O(V \lg V + E)$.

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Outline

- Shortest Paths
- Shortest-Paths Properties
- The Bellman-Ford Algorithm
- Single-Source Shortest Paths in Directed Acyclic Graphs
- Dijkstra's Algorithm
- ***Difference Constraints and Shortest Paths***

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Difference Constraints (1/3)

- Given a set of inequalities of the form $x_j - x_i \le b_k$.
  - x's are variables, $1 \le i, j \le n$,
  - b's are constants, $1 \le k \le m$.
- Want to find a set of values for the x's that satisfy all m inequalities, or determine that no such values exist. Call such a set of values a feasible solution.

Difference Constraints (2/3)

- **Example:**

$x_1 - x_2 \le 5$

$x_1 - x_3 \le 6$

$x_2 - x_4 \le -1$

$x_3 - x_4 \le -2$

$x_4 - x_1 \le -3$

Solution: $x = (0, -4, -5, -3)$

Also: $x = (5, 1, 0, 2)$ = [above solution] + 5

Difference Constraints (3/3)

- **Lemma 24.8** Let $x = (x_1, x_2, \ldots, x_n)$ be a solution to a system $Ax \le b$ of difference constraints, and let d be any constant. Then $x + d = (x_1 + d, x_2 + d, \ldots, x_n + d)$ is a solution to $Ax \le b$ as well.

  Proof. x is a feasible solution $\Rightarrow x_j - x_i \le b_k$ for all $i, j, k \Rightarrow (x_j + d) - (x_i + d) \le b_k$.

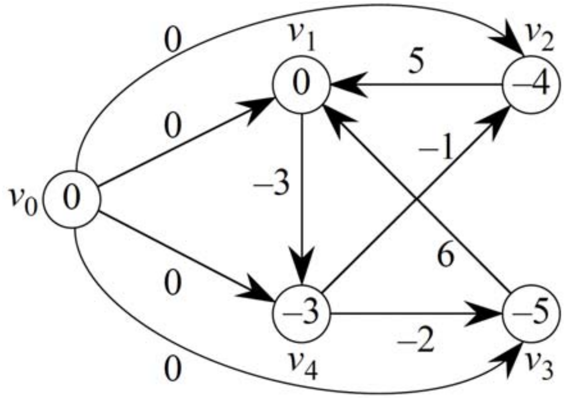
Constraint Graph (1/6)

$G = (V, E)$, weighted, directed.

- $V = (v_0, v_1, v_2, \ldots, v_n)$: one vertex per variable + v_0
- $E = \{(v_i, v_j) : x_j - x_i \le b_k$ is a constraint$\} \cup \{(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_n)\}$
- $w(v_0, v_j) = 0$ for all j
- $w(v_i, v_j) = b_k$ if $x_j - x_i \le b_k$

Constraint Graph (2/6)

$$
\begin{aligned}
x_1 - x_2 &\le 5 \\
x_1 - x_3 &\le 6 \\
x_2 - x_4 &\le -1 \\
x_3 - x_4 &\le -2 \\
x_4 - x_1 &\le -3
\end{aligned}
$$



Constraint Graph (3/6)

- **Theorem 24.9** Given a system $Ax \le b$ of difference constraints, let $G = (V, E)$ be the corresponding constraint graph. If G contains no negative-weight cycles, then

  $$x = (\delta(v_0, v_1), \delta(v_0, v_2), \delta(v_0, v_3), \ldots, \delta(v_0, v_n))$$

  is a feasible solution for the system. If G contains a negative-weight cycle, then there is no feasible solution for the system.

Constraint Graph (4/6)

Proof.

- Show no negative-weight cycles $\Rightarrow$ feasible solution.
  - Need to show that $x_j - x_i \le b_k$ for all constraints. Use

    $$
    \begin{aligned}
    x_j &= \delta(v_0, v_j) \\
    x_i &= \delta(v_0, v_i) \\
    b_k &= w(v_i, v_j) .
    \end{aligned}
    $$

  - By the triangle inequality,

    $$
    \begin{aligned}
    \delta(v_0, v_j) &\le \delta(v_0, v_i) + w(v_i, v_j) \\
    x_j &\le x_i + b_k \\
    x_j - x_i &\le b_k .
    \end{aligned}
    $$

  ➡ Therefore, feasible.

Constraint Graph (5/6)

- Show negative-weight cycles $\Rightarrow$ no feasible solution.
  - Without loss of generality, let a negative-weight cycle be $c = \langle v_1, v_2, \ldots, v_k \rangle$, where $v_1 = v_k$. (v_0 can't be on c, since v_0 has no entering edges.) c corresponds to the constraints

    $$
    \begin{aligned}
    x_2 - x_1 &\le w(v_1, v_2) , \\
    x_3 - x_2 &\le w(v_2, v_3) , \\
    &\vdots \\
    x_{k-1} - x_{k-2} &\le w(v_{k-2}, v_{k-1}) , \\
    x_k - x_{k-1} &\le w(v_{k-1}, v_k) .
    \end{aligned}
    $$

Constraint Graph (6/6)

– If x is a solution satisfying these inequalities, it must satisfy their sum.

So add them up.

Each x_i is added once and subtracted once. ($v_1 = v_k \Rightarrow x_1 = x_k$.)

We get $0 \le w(c)$.

But $w(c) < 0$, since c is a negative-weight cycle.

Contradiction $\Rightarrow$ no such feasible solution x exists.

How to Find a Feasible Solution?

- Form constraint graph.
  - $n + 1$ vertices.
  - $m + n$ edges.
  - $\Theta(m + n)$ time.
- Run BELLMAN-FORD from v_0.
  - $O((n + 1)(m + n)) = O(n^2 + nm)$ time.
- If BELLMAN-FORD returns FALSE $\Rightarrow$ no feasible solution.

  If BELLMAN-FORD returns TRUE $\Rightarrow$ set $x_i = \delta(v_0, v_i)$ for all i.

Homework Assignment #7

Exercise 24.3-6

- TAs will announce the detailed Input/Output format in Moodle.
- Please submit your program to e-Tutor.
- Please submit your README document to Moodle.
- **Due Date:**