OPER 527 Final Project

Jonathan Miller, Dan Huang, Will Leach

December 6, 2017

Contents

1. INTRODUCTION

“Winter is coming.” Jon Snow, Daenerys Targaryen and Cersei Lannister have agreed to an alliance to fight the White Walkers. It is urgent to ensure their available resources are utilized effectively and efficiently to protect the forts of Westeros. However, resources are not available to locate armies in every fort. Therefore, it is important to maximize the advantages of the available armies to protect the most essential forts as well as possible. This project attempts to decide how to allocate these armies according to different criteria through solving network optimization problems.

In this project, there are 18 different forts and 4 available armies to allocate among the forts and roads of Westeros. Given the specific context in the problem description, there are several important aspects that we could concentrate on, such as:

- The appropriate strategies for army deployment
- The number of armies allocated to each fort within a given distance
- The response time between armies and forts covered
- The distance up to which armies could serve other forts
- The maximum delayed shortest path between forts

Given data including distance between forts, number of armies to allocate, coverage threshold, and delay factor, we create four models which each focus on a different aspect of the fort coverage system through implementing integer programming models. Each model focuses on different criterions, including maximum weighted coverage, total weighted response time, maximum delayed shortest path, and maximum total delayed shortest paths.

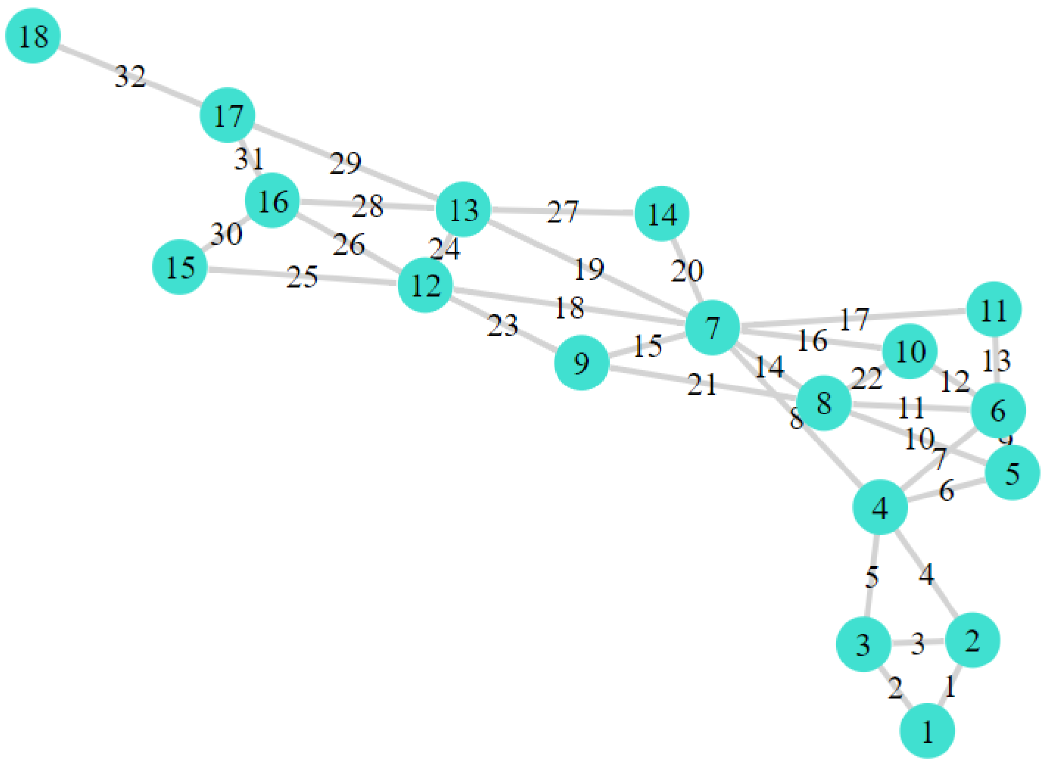

Figure 1: Map of Westeros

We use the Julia programing language and JuMP software package to implement and solve the four mathematical models. We can analyze how the optimal solutions change using sensitivity analysis by altering several different parameters, which let us understand the relative importance of army allocations, distance between forts, and delay factor. Finally, the pros and cons of these different models can be shown as the parameters change.

2. MATHEMATICAL MODELS AND SOLUTIONS

2.1 Model 1a- Maximizing Weighted Coverage

2.1.1 Parameter and Decision Variable Definitions

Before building our models, we need to define several parameters and decision variables:

Parameters ==

V A set containing the indices for all locations of forts in Figure 1

E A set containing the indices of all edges representing roads in Westeros in Figure 1

J A set containing the indices of all potential army locations, J $\in V$

I A set containing the indices of all coverage forts, $I \in V$

Distanceij A matrix denoting the shortest distance between fort i and fort j , $\forall$ $i, j \in$ V

D Coverage distance threshold

N Number of armies to allocate among forts

coverageij A coverage binaries matrix generated using the Distance matrix denoting whether a fort at location i is covered by an army located at j

priority(i) The priority level of each fort, $i \in V$

p=1, the forts include Castle Black, Nightsong;

p=2, the forts include White Harbor, Trident, Harrehal, Summerhall, Hornhill;

p=3, the forts include Moat Cailin, The Twins, Riverrun, Storm's End;

p=4, the forts include Winterfall, The Eyrie, Castely Rock, Sunspear;

p=5, the forts include Kings Landing, Highgarden, Oldtown.

Variables ==

x_j A yes-or-no decision for whether or not to station an army at a particular fort j, $j \in$ V

y_i Whether or not a particular fort i is covered, $i \in$ V

Details about how we set binary variables x_j and y_i

$$x_j = \begin{cases} 1, if\ we\ set\ one\ army\ in\ fort\ j; \\ 0, otherwise; \end{cases}$$

$$y_i = \begin{cases} 1, if\ fort\ i\ is\ covered\ by\ at\ least\ onc\ army \\ 0, otherwise; \end{cases}$$

2.1.2 Model Description

The purpose of this model is to maximize the weighted coverage for all 18 forts of Westeros. Hence, it is necessary to define both the weights and the coverage for these four armies' locations before we formulate the mathematical model. The weights in model 1a are defined by the priority level of these forts, which is $priority_i$ for each fort i. The coverage for each fort is defined as the number of forts covered y_i. Therefore, the total coverage for these four army locations is $\sum_i^I y_i.priority(i)$, which is defined as the objective function.

2.1.3 Model Formulation

With the parameters and decision variables defined above, the model 1a can be formulated as follows:

$$max\ weighted\ coverage = \sum_{i \in I} y_i * priority_i$$

s.t. $\sum_{j \in J} x_j \leq armies$

$$y_i \leq \sum_{j \in J} x_j * coverage[j,i]\ ; \quad \forall\ i\ \in\ I$$

$$x_j\ is\ binary\ variable\ \forall\ j \in J;$$

$$y_i\ is\ binary\ varibale\ \forall\ i\ \in\ I;$$

The objective function is described in the model description section, which is to maximize the total weighted coverage.

- The first constraint limits the number of armies allocated based on the total number of armies available
- The second constraint gives the information that if fort j has an army and $distance_{ij}$ less than D then fort j and fort i is connected and fort i is covered, $\forall\ j\ \in\ J\ and\ \forall\ i\ \in\ I$
- The other constraints restrict the variables to take binary values.

2.1.4 Solution and Explanation

The optimal solution we get from model 1a is 42, which means the maximum weighted coverage is 42 and we should locate our 4 armies at White Harbor, The Twins, King's Landing and Highgarden. In Figure 1, the four army locations are at nodes 3, 5, 7 and 12.

2.1.5 Sensitivity Analysis

By changing the parameters N (the number of armies) and D (coverage threshold) in model 1a, we get to know how the weighted coverage and the decision of locating armies at forts vary.

Parameter N: the number of armies to allocate

The original value of N in the model 1a is given by 4. Therefore, we change N from 1 to 18 (the total number of forts) to see the fluctuations in the objective function values, optimal solutions and the covered forts. The detailed data is displayed in Table 1.

Table 1: Results for model 1a with changing parameter N only

N	Objective	The number of armies at forts	Army allocated in forts	Other covered forts
1	15	16	Horn Hill	2,13,15,17
2	27	10,16	Harrenhal, Horn Hill	6,7,8,12,13,15,17
3	36	3,10,16	White Harbor, Harrenhal, Horn Hill	2,4,6,7,8,12,13,15,17
4	**42**	**3,5,7,12**	**White Harbor, The Twins, King's Landing, Highgarden**	**2,4,6,8,10,13,14,15,16,17**
5	46	3,5,7,9,12	White Harbor, The Twins, King's Landing, Casterly Rock, Highgarden	2,4,6,8,9,13,14,15,16,17
6	50	3,5,7,11,12,18	White Harbor, The Twins,King's Landing, The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
7	54	3,5,7,9,11,12,18	White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17

8	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
9	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
10	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
11	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
12	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
13	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
14	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
15	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17

16	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
17	55	1,3,5,7,9,11,12,18	Castle Black, White Harbor, The Twins,King's Landing, Casterly Rock,The Eyrie, Highgarden, Sunspear	2,4,6,8,9,13,14,15,16,17
18	55	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	All forts	All forts

As we increase value of N, the objective value changes from 15 to 54 when N grows from 1 to 7. After N increases from 8 to 18, the objective value is fixed to 55, the maximum possible weighted coverage score. It is worth noting that the weighted coverage increases as we increase N from 1 to 8 but the covered forts and objective value don't change when the number of armies grows from 8 to 18, which reveals only 8 armies are needed to cover all forts. In conclusion, the limitation is not on parameter N and could be dependent on distance D.

Parameter D: the coverage threshold distance

The original value for D in the model 1a is 500. Therefore, we change D from 100 to 1000 to see how the objective function value, optimal solutions and the coverage varies. The details are shown in table 2.

Table 2: Results for model 1b with changing parameter D only

D	Objective	The number of Armies at forts	Army allocated at forts	Other covered forts
100	19	7,12,15,18	King's Landing, Highgarden,Oldtown, Sunspear	Null

200	22	3,7,12,15	White Harbor, King's Landing,Highgarden, Oldtown	4,16
300	25	7,10,15,16	King's Landing, Harrenhal, Oldtown, Horn Hill	6,8,12,17
400	35	3,7,10,16	White Harbor, King's Landing,Harrenhal, Horn Hill	4,6,8,12,13,14,15,17
500	**42**	**3,5,7,12**	**White Harbor, The Twins,King's Landing, Highgarden**	**2,4,6,8,10,13,14,15,16,17**
600	47	2,6,16,18	Winterfell, The Trident, Horn Hill, Sunspear	3,4,5,7,8,10,11,12,13,15,17
700	48	2,8,17,18	Winterfell, Riverrun, Nightsong, Sunspear	1,3,4,5,6,9,10,12,13,15,16
800	55	2,10,12,18	Winterfell, Harrenhal, Highgarden, Sunspear	1,3,4,6,7,8,11,13,15,16,17
900	55	2,6,13,18	Winterfell, The Trident, Summerhall, Sunspear	1,3,4,5,7,8,9,10,11,12,14,15,16,17
1000	55	1,5,12,18	Castle Black, The Twins, Highgarden, Sunspear	2,3,4,6,7,8,9,10,11,13,14,15,16,17

As we know from Table 2 above, the objective value is increasing and the number of covered forts goes up as the distance between two forts connected becomes longer until it gets to 800 and the objective is fixed to the maximum of 55. When we increase the distance to 900 and armies are located at the forts from the table, all forts can be covered using 4 armies.

2.2 Model 1b –Minimizing the total weighted response time

2.2.1 Parameter and Decision Variable Definitions

We need to define our parameters and decision variables used in model 1b:

Parameters ==

V A set containing the indices for all locations of forts in Figure 1

E A set containing the indices of all edges representing roads in Westeros in Figure 1

J A set containing the indices of all potential army locations, J ∈ V

I A set containing the indices of all coverage forts, I ∈ V

N Number of armies that could be located at forts

$shortest\_path_{ij}$ A matrix containing the shortest path distance between node i and node j

priority(i) The priority level of each fort, $i \in I$

p=1, the forts include Castle Black, Nightsong;

p=2, the forts include White Harbor, Trident, Harrehal, Summerhall, Hornhill;

p=3, the forts include Moat Cailin, The Twins, Riverrun, Storm's End;

p=4, the forts include Winterfall, The Eyrie, Castely Rock, Sunspear;

p=5, the forts include Kings Landing, Highgarden, Oldtown.

Variables ==

x_j A yes-or-no decision for whether or not to station an army at a particular fort j, j in J

y_{ij} Whether or not a particular fort i is covered by a particular army j at a location, $i \in I, j \in J$

Details about how we set binary variables x_j and y_{ij}:

$$x_j = \begin{cases} 1, if\ we\ set\ one\ army\ in\ fort\ j; \\ 0, otherwise; \end{cases}$$

$$y_{ij} = \begin{cases} 1, if\ a\ fort\ i\ is\ covered\ by\ an\ army\ at\ location\ j; \\ 0, otherwise; \end{cases}$$

2.2.2 Model Description

For this model, the goal is to minimize the total weighted response time for all forts. Hence, in this model we define y_{ij} as the variable for the army j located at the fort to cover a particular fort i. Based on model description, we write the objective function and constraints as following.

2.2.3 Model Formulation

Model 1b can be formulated by all parameters and variables defined above.

$$min\ weightedtime = \sum_{i\in I}^{j\in J} y_{ij} * shortest\_path[ij] * priority[i]$$

s.t. $\sum_{j\in J} x_j \leq armies$

$\sum_{i\in I} y_{ij} \geq 1\ \forall\ j \in\ J$

$y_{ij} \leq x_j\ \forall\ i \in\ I, j\ \in\ J$

x_j is binary variable, $\forall\ j \in\ J$

y_{ij} is binary variable, $\forall\ i \in\ I, j\ \in\ J$

In the model description section, we know the objective function is described to minimize the total weighted response time to protect the forts in any emergency.

- The first constraint means that allocations of armies are no more than the total number of armies' N allowed;
- The second constraint indicates that all forts must be coved by an army;
- The third shows that if a fort is covered by an army at a location, there must be actually an army located there, used to link x and y;
- The rest are the general format constraints for binary variables.

2.2.4 Solution and Explanation

By building model 1b in Julia, we get an optimal minimum total weighted response time of 20,620 when we choose to allocate four armies at Winterfell, Harrenhal, Highgarden and Sunspear, labelled 2, 10, 12, 18 in Figure 1.

2.2.5 Sensitivity Analysis

From the problem, it is given that the original value of total number of armies is by 4. We set the values of N changing from 1 to 18 (the total number of forts in Westeros) to test the variance in objective function values and optimal solution. The details are shown in Table 3.

Parameter N: The number of armies

Table 3: Result for Model 1b with changing parameter N only

N	Objective	Army Allocations	Covered Forts
1	55430	7	10,13,14
2	34470	6,16	5,8,10,12,13,15,17
3	26210	2,6,16	3,5,8,10,12,13,15,17
4	**20620**	**2,10,12,18**	**3,6,8,7,15,16,17**
5	15605	2,6,7,12,18	3,5,8,10,13,14,15,16,17
6	12685	2,6,7,9,12,18	3,5,8,10,13,14,15,16,17
7	10295	2,4,6,7,9,12,18	3,5,8,10,13,14,15,16,17
8	8115	2,4,6,7,9,11,12,18	3,5,8,10,13,14,15,16,17
9	6165	2,4,6,7,9,11,12,15,18	3,5,8,10,13,14,16,17
10	4830	2,4,5,7,8,9,11,12,15,18	3,6,10,13,14,16,17
11	3630	2,4,5,7,8,9,11,12,14,15,18	3,6,10,13,16,17
12	2930	2,4,5,7,8,9,11,12,13,14,15,18	3,6,10,16,17
13	2240	2,4,5,7,8,9,10,11,12,13,14,15,18	3,6,16,17
14	1560	1,2,4,5,7,8,9,10,11,12,13,14,15,16	3,6,17
15	975	1,2,4,5,7,8,9,10,11,12,13,14,15,16,18	3,6,17
16	585	1,2,4,5,6,7,8,9,10,11,12,13,14,15,16,18	3,17
17	225	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,18	17
18	0	1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18	Null

From the table, it indicates that as N decreases, the total weighted response time grows dramatically to 55430 when we set N equaling to 1. If we make N ascend from 1 to 18, the objective value becomes smaller to 0, which means the optimal solution is dependent on the number of armies to be allocated among forts.

2.3 Model 2a – Maximizing the Delayed Shortest Distance from Castle Black to King's Landing

2.3.1 Parameter and Decision Variable Definitions

First, we need to define our parameters and decision variables in the model:

Parameters ==

V A set containing the indexes for all locations of forts in Figure 1

E A set containing the indexes of all edges representing roads in Westeros in Figure 1

$Distance_{ij}$ A matrix denoting the distance between fort i and fort j, ∀ j, i in V

N Number of armies to be allocated

delay Delay factor that is applied to any road with an army

s The starting fort of the path

t The destination fort of the path

Variables ==

x_{ij} A yes-or-no decision for whether or not to station an army on a particular road from fort i and fort j

y_i A node label for the shortest distance from s to fort i

Details about how we set binary variables x_{ij}:

$$x_{ij} = \begin{cases} 1, if\ we\ set\ one\ army\ on\ a\ particular\ road\ from\ fort\ i\ to\ fort\ j; \\ 0, otherwise; \end{cases}$$

2.3.2 Model Description

In comparison with the first two models in this project to maximize the weighted coverage and minimize the response time, for this model we mainly focus on maximizing the shortest delayed distance traveling from Castle Black to King's Landing, however the model can be generalized to find the maximum shortest delayed distance between fort i and fort j. We utilize a delay factor, which is a penalty imposed on the distance between two forts.

2.3.3 Model Formulation

After defining all parameters and decision variables above, we can formulate the model 2a as following:

$$max\ shortest\_distance = y_s - y_t$$

s.t. $\sum_{i \in I}^{j \in J} X_{ij} \leq 2 * armies$

$$X_{ij} = X_{ji} \quad \forall\, i \in I, j \in J$$

$$distance[i,j] \geq X_{ij} \quad \forall\, i \in I, j \in J$$

$$y_i - y_j \leq distance[i,j] + distance[i,j] * X_{ij} * delay$$

x_{ij} is binary variable, $\forall\, i \in I, j \in J$

y_i, y_j >= 0

The objective function is defined in the model description section, which is to maximize the shortest distance between two forts.

- The first constraint indicates that allocation of armies cannot be exceed the total number of armies. Note that it is multiplied by 2 due to the undirected nature of the graph and the fact that armies will be allocated on edges instead of nodes;
- The second constraint means that an army on an edge (i, j) implies that an army is on edge (j, i);
- The third constraint shows that at the same time, the edge must exist if an army is allocated for it;
- The fourth constraint is an extension of the optimal sub-path property of the shortest path, indicating that the shortest distance between connected nodes is the length of their shortest connecting edge. This constraint is able to be made a function of the decision x in order to maximize the imposed delay;
- The rest are the general format constraints for binary and integer variables.

2.3.4 Solution and Explanation

This aims to maximizing the shortest distance traveling from Castle Black to King's Landing. The optimal solution is 3190 with locating the armies on the roads between forts (4,5), (4,6), (4,7) and (7,10). These roads connect Moat Cailin with The Twins, Moat Cailin with Trident, Moat Cailin with King's Landing and King's Landing with Harrehal.

2.3.5 Sensitivity Analysis

In this model, we examine how the number of armies and delay factor change to influence the optimal solution and objective value.

Parameter N: The number of armies to locate

Table 4: Result for Model 2a with Parameter N changing only

N	Objective	Army Allocations on Roads
0	2615	0
1	2690	[4, 6]
2	2955	[1, 2], [1, 3]
3	3035	[1, 2], [1, 3], [2, 4]
4	**3190**	**[4, 5], [4, 6], [4, 7], [7, 10]**
5	3340	[1, 2], [1, 3], [4, 5], [4, 6], [4, 7]
6	3530	[1, 2], [1, 3], [4, 5], [4, 6], [4, 7], [7, 10]
7	3627.5	[1, 2], [1, 3], [4, 5], [4, 6], [4, 7], [6, 10], [7, 10]
8	3757.5	[1, 2], [1, 3], [2, 3], [2, 4], [4, 5], [4, 6], [4, 7], [7, 10]
9	3855	[1, 2], [1, 3], [2, 3], [2, 4], [4, 5], [4, 6], [4, 7], [6, 10], [7, 10]
10	3922.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [6, 10], [7, 10]
11	3922.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 10], [7, 10]
12	3922.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 8], [6, 10], [7, 10]
13	3922.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 8], [6, 10], [6, 11], [7, 10]

14	3922.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 8], [6, 10], [6, 11], [7, 10]
15	3922.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 8], [6, 10], [6, 11], [7, 10]
16	3922.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 8], [6, 10], [6, 11], [7, 10]
17	3922.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 8], [6, 10], [6, 11], [7, 10]
18	3922.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 8], [6, 10], [6, 11], [7, 10]

It is shown that we change the parameter N from 0 to 18 (the total number of the forts in Westeros). When we decrease the value of N from 4 to 0, the army allocations reduce from 4 to 0 along with objective value going down, which implies the shortest distance between Castle Black and King's Landing 2615 without any army allocations. However, when the number of armies goes up from 10 to 18, the objective value is fixed to 3922.5, which indicates that only 10 armies could get maximum shortest distance to delay the white walker's progress, indicating the number of armies is not the limitation factor anymore after we increase the number of armies to 10.

Parameter α: The delay factor

Table 5: Result for Model 2a with Parameter α Changing only

α	Objective Value	Army Allocation on Roads
0.1	2815.5	[1, 2], [1, 3], [2, 4], [4, 6]
0.2	2906	[1, 2], [1, 3], [2, 4], [4, 6]
0.3	2999	[1, 2], [1, 3], [4, 6], [4, 7]

0.4	3075	[4, 5], [4, 6], [4, 7], [7, 10]
0.5	**3190**	**[4, 5], [4, 6], [4, 7], [7, 10]**
0.6	3305	[4, 5], [4, 6], [4, 7], [7, 10]
0.7	3416	[4, 5], [4, 6], [4, 7], [7, 10]
0.8	3514	[1, 2], [1, 3], [2, 3], [2, 4]
0.9	3624.5	[1, 2], [1, 3], [2, 3], [2, 4]
1	3735	[1, 2], [1, 3], [2, 3], [2, 4]

From the table 5, it is obvious that the objective value goes up as delay factor α increases from 0.1 to 1, which tells us that delay factor is a limitation factor as long as the number of armies is kept the same.

2.4 Model 2b – Maximizing the Summation of the Shortest Delayed Distance from Castle Black to All Other Forts

2.4.1 Parameter and Decision Variable Definitions

First, we need to define our parameters and decision variables in the model 2b, as follows:

Parameters ==

V A set containing the indexes for all locations of forts in Figure 1

E A set containing the indexes of all edges representing roads in Westeros in Figure 1

J A set containing the indexes of all potential army locations, J in V

I A set containing the indexes of all coverage forts, I in V

$Distance_{ij}$ A matrix denoting the distance between fort i and fort j, ∀ j,i in V

N Number of armies that could be located at forts

delay Delay factor that is applied to any road with an army

s The starting fort of the path

t The destination fort of the path

Variables ==

x_{ij} A yes-or-no decision for whether or not to station an army on a particular road from fort i and fort j

y_i A node label for the shortest distance from s to fort i

Details about how we set binary variables x_{ij}

$$x_{ij} = \begin{cases} 1, if\ we\ set\ one\ army\ on\ a\ particular\ road\ from\ fort\ i\ to\ fort\ j; \\ 0, otherwise; \end{cases}$$

2.4.2 Model Description

Comparing with the Model 2a to maximize the shortest distance, here we are going to maximize the summation of the shortest distance traveling from Castle Black to all other forts.

2.4.3 Model Formulation

With all the parameters and decision variables defined above, we can formulate model 2b as follows:

$$max\ shortest\_distance = \sum(y_s - y_i)\ \forall\ i \in\ I$$

s.t. $\sum_{i\in I}^{j\in J} X_{ij} \leq\ 2 * armies$

$X_{ij}\ =\ X_{ji}\quad \forall\ i \in\ I, j \in\ J$

$distance[i,j]\ \geq\ X_{ij}\quad \forall\ i \in\ I, j \in\ J$

$y_i - y_j \leq distance[i,j]\ +\ distance[i,j] * X_{ij} * delay$

x_{ij} is binary variable, $\forall\ i \in\ I, j \in\ J$

y_i, y_j >= 0

The objective function is defined in the model description section, which is to maximize the summation of the distance of the shortest path from a starting fort to all other forts.

- The first constraint indicates that allocation of armies cannot be exceed the total number of armies;
- The second constraint means that an army on an edge (i,j) implied an army is on edge (j,i);

- The third constraint shows that at the same time, the edge must exist by satisfying the second constraint;
- The fourth constraint is an extension of the optimal sub-path property of the shortest path, indicating that the shortest distance between connected nodes is the length of their shortest connecting edge. This constraint is able to be made a function of the decision x in order to maximize the imposed delay;
- The rest are the general format constraints for binary and integer variables.

2.3.4 Solution and Explanation

This model maximizes the summation of the shortest distance traveling from Castle Black to all other forts in Westeros. The optimal solution is 53755, obtained by locating the armies on the roads between forts (1,2), (1,3), (2,3) and (2,4). These roads connect Castle Black with Winterfell, Castle Black with White Harbor, Winterfell with White Harbor and Winterfell with Moat Cailin.

2.4.5 Sensitivity Analysis

In this model, we examine how the number of armies and delay factor change to influence the optimal solution and objective value.

Parameter N: The number of armies to locate

Table 6: Result for Model 2a with Parameter N changing only

N	Objective	Army Allocations on Roads
0	44350	0
1	45490	[4, 6]

2	50195	[1, 2], [1, 3]
3	51395	[1, 2], [1, 3], [2, 4]
4	**53755**	**[1, 2], [1, 3], [2, 3], [2, 4]**
5	55190	[1, 2], [1, 3], [4, 5], [4, 6], [4, 7]
6	56400	[1, 2], [1, 3], [4, 5], [4, 6], [4, 7], [7, 10]
7	58750	[1, 2], [1, 3], [2, 3], [2, 4], [4, 5], [4, 6], [4, 7]
8	59960	[1, 2], [1, 3], [2, 3], [2, 4], [4, 5], [4, 6], [4, 7], [7, 10]
9	60972	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [7, 10]
10	61612	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [7, 10], [8, 9]
11	62477.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [7, 10], [7, 13], [8, 9]
12	63300	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [6, 10], [7, 10], [7, 13], [8, 9]
13	63885	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [6, 10], [7, 10], [7, 13], [8, 9], [17, 18]
14	64310	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [6, 10], [7, 10], [7, 13], [8, 9], [13, 17], [17, 18]

15	64605	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 10], [7, 10], [7, 13], [8, 9], [13, 17], [17, 18]
16	64877.5	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 10], [6, 11], [7, 10], [7, 13], [8, 9], [13, 17], [17, 18]
17	65205	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [6, 10], [7, 10], [7, 12], [7, 13], [8, 9], [9, 12], [13, 16], [13, 17], [17, 18]
18	65550	[1, 2], [1, 3], [2, 3], [2, 4], [3, 4], [4, 5], [4, 6], [4, 7], [5, 8], [6, 8], [6, 10], [7, 10], [7, 12], [7, 13], [8, 9], [13, 16], [13, 17], [17, 18]

From the table 6, it is obvious that the objective value goes up as the number of armies increases from 0 to 18, which tells us that the parameter N is limitation factor in this model as long as the delay factor α=0.5 is kept the same.

Parameter α: The delay factor

Table 7: Result for Model 2a with Parameter α Changing only

α	Objective Value	Army Allocation on Roads
0.1	47242	[1, 2], [1, 3], [2, 4], [4, 6]
0.2	48989	[1, 2], [1, 3], [2, 4], [4, 6]
0.3	50223	[1, 2], [1, 3], [2, 4], [4, 6]
0.4	51919	[1, 2], [1, 3], [2, 3], [2, 4]
0.5	**53755**	**[1, 2], [1, 3], [2, 3], [2, 4]**
0.6	55591	[1, 2], [1, 3], [2, 3], [2, 4]
0.7	57427	[1, 2], [1, 3], [2, 3], [2, 4]
0.8	59263	[1, 2], [1, 3], [2, 3], [2, 4]
0.9	61099	[1, 2], [1, 3], [2, 3], [2, 4]
1	62935	[1, 2], [1, 3], [2, 3], [2, 4]

From the table 7, it is obvious that the objective value goes up as delay factor α increases from 0.1 to 1, which tells us that delay factor is limitation factor as long as the number of armies is kept the same.

3 Model Comparison and Discussion

In this project, our mission is to allocate our armies at the forts or roads to protect the continent of Westeros from White Walkers. A set of edges and a set of labelled forts is created with a network system G (V, E) in Figure 1, which represents a road map system. For this problem, we are required to maximize the weighted coverage for all forts and minimize the total weighted response time in model 1a and 1b by allocating our armies at the forts. In model 2a and 2b, we have to maximize the shortest distance from one fort to another and the summation of shortest distance from one fort to all other forts by allocating armies along roads.

In the same system G (V, E), the given resources in the models are limited. Therefore, formulating different mathematical models is necessary to get optimal solutions with different goals. All of these models have their pros and cons.

Model 1a focuses on maximizing a weighted coverage score by using the available armies and a specified coverage threshold to cover as many forts as possible, weighted by priority. The advantages of model 1a include using available resources efficiently to protect the most important locations. However, it does not have the requirement of needing to protect all locations, hence less important forts by priority level or distance will go uncovered in the event of emergency.

Model 1b only involves the number of armies as a limitation factor, and attempts to minimize the total weighted response time to all other forts. The advantage of using model 1b is that it includes protection for all forts in the solution. However, model 1b does not protect the most important forts by priority level and distance as efficiently as the solution of model 1a does. The objective value stops changing when the number of armies increases to a specific number.

Model 2a examines allocation of armies along roads, rather than at forts, to maximize the shortest delayed distance between a starting fort and ending fort. The advantage of using this model is it is able to protect a specific fort as well as possible, by delaying the most efficient path used to reach that fort from a known starting location. The disadvantages of this model include not protecting the fort as well from a multiple-pronged attack (the model assumes the attack comes from a single, centralized starting location), and the fact that it only protects one specific fort as well as possible while neglecting others.

Model 2b extends the solution of model 2a by maximizing the summation of the shortest delayed distance from a starting fort to all other forts in Westeros. This model mitigates the disadvantage of only protecting a single fort as well as possible as outlined in discussion of model 2a above, but tends to leave your resources spread thin as the model values allocating all available armies on specific choke points, such as the roads between Castle Black and Winterfell/White Harbor, or Moat Cailin to The Twins/The Trident/King's Landing.

Overall, all models examined have particular advantages and disadvantages, and the model and allocation decision ultimately selected depends heavily on the specific needs of the desired result.