GRA 6227 Business Optimisation

Multi-period planning models

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Multi-period planning problems

- In many companies, planning future activities in a good way is crucial for the company's profitability
- Many companies have invested in ERP «Enterprise Resource Planning» software and dedicated planning software to aid the planning
- Common for these tools is that the future is divided into a number of discrete time periods, within a given time horizon
- Basically, the planning task is to decide upon various planning variables (workforce, capacities, production, investments, loans, purchasing, inventories, transportation, sales, prices, sales campaigns, etc etc…) for each future period

Optimisation support for planning

- The planning task is often inherently complex
  - Time consuming
  - Frequent need for re-planning
  - Hard to find near-optimal solutions
- Optimisation based planning software is gradually becoming more common
- Many challenges
  - Understand the business situation
  - Understand the basics of modelling and optimisation
  - Have updated «real-time» data
  - Create a modelling tool that is user friendly enough so that is actually used

Production planning tool at a Scandinavian producer of construction material

Date	Production						Demand						Inventory					
	Product 1	Product 2	Product 3	Product 4	Product 5	Product 6	Product 1	Product 2	Product 3	Product 4	Product 5	Product 6	Product 1	Product 2	Product 3	Product 4	Product 5	Product 6
01.09.2019	3963	2283				0							25266	10647	7122	1607	1511	1236
02.09.2019	3670	92			2444	175	7504			1607	3505	1127	21432	10739	7034	0	450	284
03.09.2019	1065	775		2065	1571	233							22497	11514	6918	2065	2021	517
04.09.2019	3535	530		1207		198	7515			1819	2200		**19317**	12044	6819	1453	-179	715
05.09.2019	4317			1669		334	3500						20134	12044	6652	3122	-179	1049
06.09.2019	**3105**			1483		212	2300		3700			1200	**21239**	12044	2846	4605	-179	61
07.09.2019	4242		988	993		0	11900						13581	12044	3834	5598	-179	61
08.09.2019	3651		2237			0							17232	12244	6071	5598	-179	61
09.09.2019	1140	1244	1087		1107	220	4400	7500					13972	5988	7048	5598	928	281
10.09.2019		2676			2275	141							13972	8664	6977	5598	3203	422
11.09.2019	2206				2414	489					3540		16178	8664	6733	5598	2077	911
12.09.2019	2223				2412	317	12544		2526				**7020**	8664	4048	5598	4489	1228
13.09.2019	2263	850			1463	308					3500		9283	9514	3894	5598	2452	1536
14.09.2019	2163	1846				323	6200			2500			5246	11360	3733	3098	2452	1859
15.09.2019	2024	997	1175			0	3500						4070	12857	5308	2698	2452	1859
16.09.2019	2460	1407	832			338		4400					6530	9864	5971	2698	2452	2197
17.09.2019	3859	1185	1304			151							10389	11049	7199	2698	2452	2348
18.09.2019	2200	2700	154		1200	240		7500			3500		12589	6249	7233	2698	**502**	2588
19.09.2019		5500			1800	240	4400	7500				2500	8189	4249	7113	2698	2302	328
20.09.2019		5500			2400	240	3200			2800			4989	9749	6993	-102	4702	568
21.09.2019		5500			2400	240	3500	7500			4400		1489	7749	6873	-102	**3142**	808
22.09.2019	4400				2400	240	2900	4000					2989	3749	6753	-102	5542	1048
23.09.2019	4400			1600		240	7900				4100		-511	3749	4533	1498	**1442**	1288
24.09.2019	4400			1600		240	4900	3500		2800		1200	-1011	249	4413	298	1442	328
25.09.2019	4400			1600		240							3389	249	4293	1898	1442	568
26.09.2019	4400		1100		1200	240							7789	249	5273	1898	2642	808
27.09.2019	4400				2400	240	10700				2900		1489	249	5153	1898	2142	1048
28.09.2019		5500			2400	240	4400						-2911	5749	5033	1898	4542	1288
29.09.2019		5700		1400		240		7500			2900		-2911	3949	4913	3298	1642	1528
30.09.2019		5500		1600		240		11900		4000			-2911	-2451	4793	898	1642	1768

Link between periods

- Normally, the periods are linked in some way, for instance

Cash balance (t) =
Cash balance (t-1) + Incoming cash flow (t) - Payments (t)

or

Workforce (t) =
Workforce (t-1) + PeopleHired (t) - PeopleFired (t)

or

Inventory (t) =
Inventory (t-1) + Production (t) - Demand (t)

Date	Production						Demand						Inventory					
	Product 1	Product 2	Product 3	Product 4	Product 5	Product 6	Product 1	Product 2	Product 3	Product 4	Product 5	Product 6	Product 1	Product 2	Product 3	Product 4	Product 5	Product 6
01.09.2019	3963	2283				0							25266	10647	7122	1607	1511	1236
02.09.2019	3670	92			2444	175	7504			1607	3505	1127	21432	10739	7034	0	450	284
03.09.2019	1065	775		2065	1571	233							22497	11514	6918	2065	2021	517
04.09.2019	3535	530		1207		198	7515			1819	2200		**19317**	12044	6819	1453	-179	715
05.09.2019	4317			1669		334	3500						20134	12044	6652	3122	-179	1049
06.09.2019	**3105**			1483		212	2300		3700			1200	**21239**	12044	2846	4605	-179	61
07.09.2019	4242		988	993		0	11900										-179	61
08.09.2019	3651		2237			0											-179	61
09.09.2019	1140	1244	1087		1107	220	4400	7500					13972	5988	7048	5598	928	281
10.09.2019		2676			2275	141							13972	8664	6977	5598	3203	422
11.09.2019	2206				2414	489					3540		16178	8664	6733	5598	2077	911
12.09.2019	2223				2412													1228
13.09.2019	2263	850			1463													1536
14.09.2019	2163	1846																1859
15.09.2019	2024	997	1175															1859
16.09.2019	2460	1407	832															2197
17.09.2019	3859	1185	1304			151									7199	2698	2452	2348
18.09.2019	2200	2700	154		1200	240									7233	2698	**502**	2588
19.09.2019		5500			1800	240	4400	7500				2500	8189	4249	7113	2698	2302	328
20.09.2019		5500			2400	240	3200			2800			4989	9749	6993	-102	4702	568
21.09.2019		5500			2400	240	3500	7500			4400		1489	7749	6873	-102	**3142**	808
22.09.2019	4400				2400	240	2900	4000					2989	3749	6753	-102	5542	1048
23.09.2019	4400			1600		240	7900				4100		-511	3749	4533	1498	**1442**	1288
24.09.2019	4400			1600		240	4900	3500		2800		1200	-1011	249	4413	298	1442	328
25.09.2019	4400			1600		240							3389	249	4293	1898	1442	568
26.09.2019	4400		1100		1200	240							7789	249	5273	1898	2642	808
27.09.2019	4400				2400	240	10700				2900		1489	249	5153	1898	2142	1048
28.09.2019		5500			2400	240	4400						-2911	5749	5033	1898	4542	1288
29.09.2019		5700		1400		240		7500			2900		-2911	3949	4913	3298	1642	1528
30.09.2019		5500		1600		240		11900		4000			-2911	-2451	4793	898	1642	1768

2065 + 1207 – 1819 = 1453

Inventory (t) =
Inventory (t-1) + Production (t) - Demand (t)

(inventory balance constraint

Example – Production planning problem

• 1 product	3 periods		
• Demand:	4 in period 1	9 in period 2	14 in period 3
• Initial inventory:	3		
• Production capacity:	10 in each period		
• Production cost per unit:	4 in period 1	7 in period 2	8 in period 3
• Inventory holding cost per unit per period:	2 in period 1	2 in period 2	2 in period 3
• Minimum inventory at the end of period 3:	2		

- Optimisation problem:

Minimize	total costs = sum production costs + sum inventory holding costs
Subject to	production less than or equal to production capacity in each period
Subject to	inventory balance constraint
Subject to	inventory must be greater than or equal to the minimum inventory
Subject to	production and inventory must be non-negative in each period

Example – Production planning problem (mathematical model)

- Optimisation problem:

Minimize	total costs = sum production costs + sum inventory holding costs
Subject to	production less than or equal to production capacity in each period
Subject to	production and inventory must be non-negative in each period
Subject to	inventory must be greater than or equal to the minimum inventory

Variables: X_t = Production quantity in period t

I_t = Inventory by the end of period t

Model:

minimize	$4 X_1 + 7 X_2 + 8 X_3 \; + \; 2 I_1 + 2 I_1 + 2 I_3$	(minimise total costs)
s.t.	$X_1 \leq 10 \quad X_2 \leq 10 \quad X_3 \leq 10$	(production capacity constraints)
	$I_1 = 3 + X_1 - 4$	(inventory balance period 1)
	$I_2 = I_1 + X_2 - 9$	(inventory balance period 2)
	$I_3 = I_2 + X_3 - 14$	(inventory balance period 3)
	$I_3 \geq 2$	(minimum inventory end of period 3)
	$X_1 \geq 0 \quad X_2 \geq 0 \quad X_3 \geq 0$	(non-negativity constraints)
	$I_1 \geq 0 \quad I_2 \geq 0 \quad I_3 \geq 0$	(non-negativity constraints)

AMPL model

```
var X1 >= 0;
var X2 >= 0;
var X3 >= 0;
var I1 >= 0;
var I2 >= 0;
var I3 >= 0;

minimize totalcosts:
4*X1 + 7*X2 + 8*X3 + 2*I1 + 2*I2 + 2*I3;

s.t. Con1: X1 <= 10;
s.t. Con2: X2 <= 10;
s.t. Con3: X3 <= 10;
s.t. Con4: I1 =  3 + X1 -  4;
s.t. Con5: I2 = I1 + X2 -  9;
s.t. Con6: I3 = I2 + X3 - 14;
s.t. Con7: I3 >= 2;
```

Set-based mathematical model

Set-based AMPL model

```
param T;
var X{1..T} >= 0;
var I{1..T} >= 0;

param ic{1..T};          (inventory holding cost per unit)
param pc{1..T};          (production cost per unit)
param demand{1..T};
param cap{1..T};
param initinv;
param endinv;

minimize totalcosts:
  sum{t in 1..T} pc[t] * X[t]
+ sum{t in 1..T} ic[t] * I[t];
s.t. Con1 {t in 1..T}:  X[t] <= cap[t];
s.t. Con2 {t in 2..T}:  I[t] = I[t-1] + X[t] - demand[t];
s.t. Con3:              I[1] = initinv + X[1] - demand[1];
s.t. Con4:              I[T] >= endinv;

data;
param T := 3;
param pc :=     1  4    2  7    3  8;
param ic :=     1  2    2  2    3  2;
param demand := 1  4    2  9    3 14;
param cap :=    1 10    2 10    3 10;
param initinv :=  3;
param endinv :=  2;
```

Planning models with multiple products

Date	Production						Demand						Inventory					
	Product 1	Product 2	Product 3	Product 4	Product 5	Product 6	Product 1	Product 2	Product 3	Product 4	Product 5	Product 6	Product 1	Product 2	Product 3	Product 4	Product 5	Product 6
01.09.2019	3963	2283				0							25266	10647	7122	1607	1511	1236
02.09.2019	3670	92			2444	175	7504			1607	3505	1127	21432	10739	7034	0	450	284
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09.09.2019	1140	1244	1087		1107	220	4400	7500					13972	5988	7048	5598	928	281
10.09.2019		2676			2275	141							13972	8664	6977	5598	3203	422
11.09.2019	2206				2414	489					3540		16178	8664	6733	5598	2077	911
12.09.2019	2223				2412	317	12544		2526				**7020**	8664	4048	5598	4489	1228
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14.09.2019	2163	1846				323	6200			2500			5246	11360	3733	3098	2452	1859
15.09.2019	2024	997	1175			0	3500						4070	12857	5308	2698	2452	1859
16.09.2019	2460	1407	832			338		4400					6530	9864	5971	2698	2452	2197
17.09.2019	3859	1185	1304			151							10389	11049	7199	2698	2452	2348
18.09.2019	2200	2700	154		1200	240		7500			3500		12589	6249	7233	2698	**502**	2588
19.09.2019		5500			1800	240	4400	7500				2500	8189	4249	7113	2698	2302	328
20.09.2019		5500			2400	240	3200			2800			4989	9749	6993	-102	4702	568
21.09.2019		5500			2400	240	3500	7500			4400		1489	7749	6873	-102	**3142**	808
22.09.2019	4400				2400	240	2900	4000					2989	3749	6753	-102	5542	1048
23.09.2019	4400			1600		240	7900				4100		-511	3749	4533	1498	**1442**	1288
24.09.2019	4400			1600		240	4900	3500		2800		1200	-1011	249	4413	298	1442	328
25.09.2019	4400			1600		240							3389	249	4293	1898	1442	568
26.09.2019	4400		1100		1200	240							7789	249	5273	1898	2642	808
27.09.2019	4400				2400	240	10700				2900		1489	249	5153	1898	2142	1048
28.09.2019		5500			2400	240	4400						-2911	5749	5033	1898	4542	1288
29.09.2019		5700		1400		240		7500			2900		-2911	3949	4913	3298	1642	1528
30.09.2019		5500		1600		240		11900		4000			-2911	-2451	4793	898	1642	1768

Set-based mathematical model
for production planning with multiple products

Numerical example

2 products **A and B**
3 periods

Production cost per unit:

	t=1	t=2	t=3
A	3	6	9
B	4	8	12

Inventory holding cost per unit:

	t=1	t=2	t=3
A	2	2	2
B	5	5	5

Demand :

	t=1	t=2	t=3
A	7	7	8
B	2	4	5

Joint production capacity: **20 units per period**

Assume initital and ending inventories = 0

Set-based AMPL model for production planning with multiple products

```
set PROD;
param T;
var X{PROD, 1..T} >= 0;
var I{PROD, 1..T} >= 0;

param ic{PROD, 1..T};
param pc{PROD, 1..T};
param demand{PROD, 1..T};
param cap{1..T};
param initinv{PROD};
param endinv{PROD};

minimize totalcosts:
  sum{p in PROD, t in 1..T} pc[p,t] * X[p,t]
+ sum{p in PROD, t in 1..T} ic[p,t] * I[p,t];
s.t. Con1 {t in 1..T}:            sum{p in PROD} X[p,t] <= cap[t];
s.t. Con2 {p in PROD,t in 2..T}:  I[p,t] = I[p,t-1] + X[p,t] - demand[p,t];
s.t. Con3 {p in PROD}:            I[p,1] = initinv[p] + X[p,1] - demand[p,1];
s.t. Con4 {p in PROD}:            I[p,T] >= endinv[p];
```

```
data;
set PROD := A B;
param T := 3;
param pc :      1  2  3 :=
           A    3  6  9
           B    4  8  12 ;

param ic :      1  2  3 :=
           A    2  2  2
           B    5  5  5 ;

param demand :  1  2  3 :=
           A    7  7  8
           B    2  4  5 ;

param cap :=        1 20    2 20    3 20;
param initinv :=    A 0     B 0;
param endinv :=     A 0     B 0;
```

A product allocation problem (single period)

– See also Transportation model in ch. 3

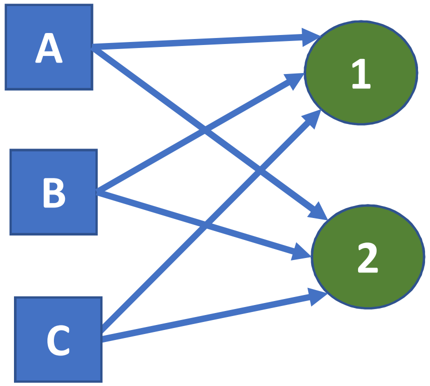
A company has three factories (A, B and C), which supply products to two markets; Market 1 and Market 2

- Demand is **72** for Market 1 and **50** for Market 2
- Capacity for the three factories:

A	35
B	60
C	50

- Variable cost per unit supplied from factory to market:

	market 1	market 2
A	45	41
B	47	49
C	48	46



The company wants to decide the quantity supplied from each factory to each market, so that variable costs are minimized

Simple mathematical formulation

$Min\ 45\ X_{A1} + 41\ X_{A2} + 47\ X_{B1} + 49\ X_{B2} + 48\ X_{C1} + 46\ X_{C2}$

such that

$X_{A1} + X_{A2} \le 35$

$X_{B1} + X_{B2} \le 60$

$X_{C1} + X_{C2} \le 50$

$X_{A1} + X_{B1} + X_{C1} = 72$

$X_{A2} + X_{B2} + X_{C2} = 50$

where $X_{i,j}$ is the quantity supplied from factory i to market j

Set-based mathematical formulation

$$Min \sum_{\substack{i=A,B,C \\ j=1,2}} c_{i,j} X_{i,j}$$

such that

$$\sum_{j=1,2} X_{i,j} \le K_i \quad \forall\, i$$

«for each factory i»

$$\sum_{i=A,B,C} X_{i,j} = D_j \quad \forall\, j$$

«for each market j»

$$X_{i,j} \ge 0\ \forall\, i,j$$

where
$X_{i,j}$ is the quantity supplied from factory i to market j
$c_{i,j}$ is the variable cost
K_i is the factory capacity
D_j is the market demand

AMPL models of the product allocation problem

Simple:

```
var XA1 >= 0;
var XA2 >= 0;
var XB1 >= 0;
var XB2 >= 0;
var XC1 >= 0;
var XC2 >= 0;

minimize TotalCosts:
    45*XA1 + 41*XA2 + 47*XB1 + 49*XB2 + 48*XC1 + 46*XC2 ;

subject to Constraint1: XA1 + XA2  <=  35;
subject to Constraint2: XB1 + XB2  <=  60;
subject to Constraint3: XC1 + XC2  <=  50;
subject to Constraint4: XA1 + XB1 + XC1  =  72;
subject to Constraint5: XA2 + XB2 + XC2  =  50;
```

Set based:

```
set FACTORY;
set MARKET;
param capacity {i in FACTORY};
param demand {j in MARKET};
param unitcost {i in FACTORY, j in MARKET};
var Shipped {i in FACTORY, j in MARKET} >= 0;

minimize TotalCost:
  sum {i in FACTORY, j in MARKET} unitcost[i,j] * Shipped[i,j];
subject to Constraint1 {i in FACTORY}:
  sum {j in MARKET} Shipped[i,j] <= capacity[i];
subject to Constraint2 {j in MARKET}:
  sum {i in FACTORY} Shipped[i,j] = demand[j];
data;
set FACTORY := A B C;
set MARKET := M1 M2;
param capacity :=   A 35    B 60    C 50 ;
param demand :=     M1 72   M2 50 ;
param unitcost :
       M1      M2   :=
    A  45      41
    B  47      49
    C  48      46   ;
```

Set-based model expanded

```
ampl: expand;
minimize TotalCost:
        45*Shipped['A','M1'] + 41*Shipped['A','M2'] + 47*Shipped['B','M1'] +
        49*Shipped['B','M2'] + 48*Shipped['C','M1'] + 46*Shipped['C','M2'];

subject to Constraint1['A']:
        Shipped['A','M1'] + Shipped['A','M2'] <= 35;

subject to Constraint1['B']:
        Shipped['B','M1'] + Shipped['B','M2'] <= 60;

subject to Constraint1['C']:
        Shipped['C','M1'] + Shipped['C','M2'] <= 50;

subject to Constraint2['M1']:
        Shipped['A','M1'] + Shipped['B','M1'] + Shipped['C','M1'] = 72;

subject to Constraint2['M2']:
        Shipped['A','M2'] + Shipped['B','M2'] + Shipped['C','M2'] = 50;
```

Mathematical model vs AMPL model (simple)

Simple mathematical formulation:

$$\mathit{Min}\ 45\,X_{A1} + 41\,X_{A2} + 47\,X_{B1} + 49\,X_{B2} + 48\,X_{C1} + 46\,X_{C2}$$

such that

$$X_{A1} + X_{A2} \leq 35$$

$$X_{B1} + X_{B2} \leq 60$$

$$X_{C1} + X_{C2} \leq 50$$

$$X_{A1} + X_{B1} + X_{C1} = 72$$

$$X_{A2} + X_{B2} + X_{C2} = 50$$

Simple AMPL model

```
var XA1 >= 0;
var XA2 >= 0;
var XB1 >= 0;
var XB2 >= 0;
var XC1 >= 0;
var XC2 >= 0;

minimize TotalCosts:
    45*XA1 + 41*XA2 + 47*XB1 + 49*XB2 + 48*XC1 + 46*XC2 ;

subject to Constraint1: XA1 + XA2  <=  35;
subject to Constraint2: XB1 + XB2  <=  60;
subject to Constraint3: XC1 + XC2  <=  50;
subject to Constraint4: XA1 + XB1 + XC1  =  72;
subject to Constraint5: XA2 + XB2 + XC2  =  50;
```

Mathematical model vs AMPL model (set-based)

Set-based mathematical formulation:

$$Min \sum_{\substack{i=A,B,C \\ j=1,2}} c_{i,j} X_{i,j}$$

such that

$$\sum_{j=1,2} X_{i,j} \leq K_i \quad \forall\, i$$

$$\sum_{i=A,B,C} X_{i,j} = D_j \qquad \forall\, j$$

$$X_{i,j} \geq 0 \;\; \forall\, i,j$$

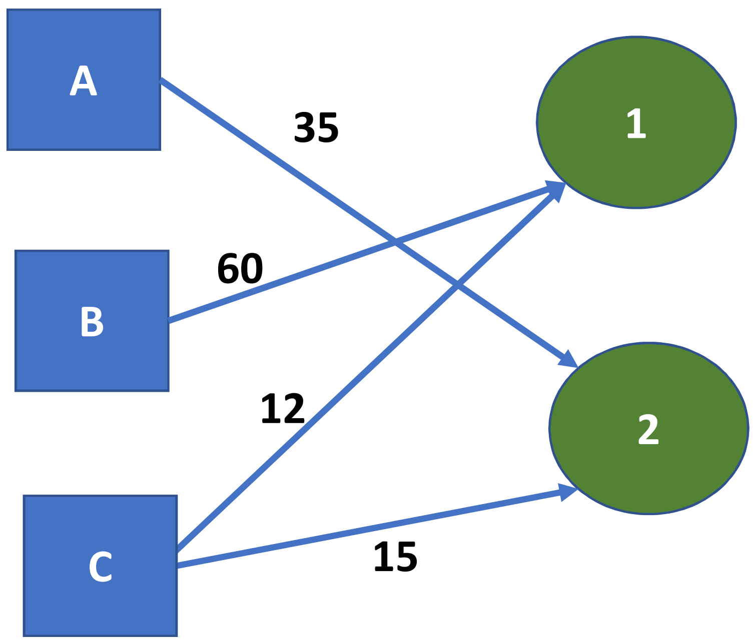
Set-based AMPL model

```
set FACTORY;
set MARKET;
param capacity {i in FACTORY};
param demand {j in MARKET};
param unitcost {i in FACTORY, j in MARKET};
var Shipped {i in FACTORY, j in MARKET} >= 0;

minimize TotalCost:
  sum {i in FACTORY, j in MARKET} unitcost[i,j] * Shipped[i,j];
subject to Constraint1 {i in FACTORY}:
  sum {j in MARKET} Shipped[i,j] <= capacity[i];
subject to Constraint2 {j in MARKET}:
  sum {i in FACTORY} Shipped[i,j] = demand[j];
data;
set FACTORY := A B C;
set MARKET := M1 M2;
param capacity :=   A 35    B 60    C 50 ;
param demand :=     M1 72   M2 50 ;
param unitcost :
        M1      M2   :=
    A   45      41
    B   47      49
    C   48      46   ;
```

Optimal solution

```
ampl: solve;
optimal solution; objective 5521
3 simplex iterations
ampl: display Shipped;
Shipped [*,*]
:    M1    M2     :=
A     0    35
B    60     0
C    12    15
;
```



Extension (try to solve until next meeting): Product allocation problem with time dimension

The following extensions:

- Four periods
- The product can be shipped in a later period than it was produced (using inventories at factories)
- Variable cost:

	market 1	market 2
A	45	41
B	47	49
C	48	46

Capacities as before:

A	35
B	60
C	50

- Demand for each market is time dependent:

demand	market 1	market 2
t = 1	46	18
t = 2	61	17
t = 3	72	50
t = 4	83	36

- Holding cost = 3 per unit on inventory per period
- Initial inventory at factory A = 12, factory B = 0, factory C = 5.
- The company wants to decide the quantity to be produced at each factory in each period and the quantity supplied from each factory to each market in each period, so that variable costs are minimized