# Mathematische Logik: Aufgabenbaltt 1 - Gruppe 12

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# Aufgabe 2

## a)

### Syntax

| | |
|---|---|
| W | Windows auf dem Computer |
| M | MacOS auf dem Computer |
| L | Linux auf dem Computer |
| G | Grafiktreiber installiert |
| D | Druckertreiber installiert |
| K | Konsole |
| A | Systemabsturz |
| H | Hausaufgaben gemacht |
| S | Computerspiele spielen |

$\tau = \{W, M, L, G, D, K, A, H, S\}$

### Semantik

$\forall x \in \tau, sei\ \mathfrak{I} : x \to \{0, 1\}\ sodass\ \mathfrak{I}(x) = 1, wenn\ x\ zutrifft.$

## b)

$\Psi_1 \coloneqq (W \wedge \neg M \wedge \neg L) \vee (\neg W \wedge M \wedge \neg L) \vee (\neg W \wedge \neg M \wedge L)$
$\Psi_2 \coloneqq (\neg G \wedge S) \to A$
$\Psi_3 \coloneqq H \to (D \wedge K)$
$\Psi_4 \coloneqq (L \wedge G \wedge \neg D) \vee (L \wedge \neg G \wedge D) \vee \neg L$
$\Psi_5 \coloneqq (W \wedge \neg K) \vee \neg W$
$\Psi_6 \coloneqq (H \wedge S) \vee (\neg H \wedge \neg S)$
$\Psi_7 \coloneqq \neg H \to A$
$\Psi_8 \coloneqq \neg A$

$[\![\Psi_1]\!]^{\mathfrak{I}} \coloneqq \max(\min([\![W]\!]^{\mathfrak{I}}, \neg[\![M]\!]^{\mathfrak{I}}, \neg[\![L]\!]^{\mathfrak{I}}), \min(\neg[\![W]\!]^{\mathfrak{I}}, [\![M]\!]^{\mathfrak{I}}, \neg[\![L]\!]^{\mathfrak{I}}), \min(\neg[\![W]\!]^{\mathfrak{I}}, \neg[\![M]\!]^{\mathfrak{I}}, \neg[\![L]\!]^{\mathfrak{I}}))$
$[\![\Psi_2]\!]^{\mathfrak{I}} \coloneqq \max(\neg \min([\![\neg G]\!]^{\mathfrak{I}}, [\![S]\!]^{\mathfrak{I}}), [\![A]\!]^{\mathfrak{I}})$
$[\![\Psi_3]\!]^{\mathfrak{I}} \coloneqq \max([\![A]\!]^{\mathfrak{I}}, \neg \min([\![D]\!]^{\mathfrak{I}}, [\![K]\!]^{\mathfrak{I}}))$
$[\![\Psi_4]\!]^{\mathfrak{I}} \coloneqq \max(\min([\![L]\!]^{\mathfrak{I}}, [\![G]\!]^{\mathfrak{I}}, [\![\neg D]\!]^{\mathfrak{I}}), \min([\![L]\!]^{\mathfrak{I}}, [\![\neg G]\!]^{\mathfrak{I}}, [\![D]\!]^{\mathfrak{I}}), [\![\neg L]\!]^{\mathfrak{I}})$
$[\![\Psi_5]\!]^{\mathfrak{I}} \coloneqq \max(\min([\![W]\!]^{\mathfrak{I}}, [\![\neg K]\!]^{\mathfrak{I}}), [\![\neg W]\!]^{\mathfrak{I}})$
$[\![\Psi_6]\!]^{\mathfrak{I}} \coloneqq \max(\min([\![H]\!]^{\mathfrak{I}}, [\![S]\!]^{\mathfrak{I}}), \min([\![\neg H]\!]^{\mathfrak{I}}, [\![\neg S]\!]^{\mathfrak{I}}))$
$[\![\Psi_7]\!]^{\mathfrak{I}} \coloneqq \max([\![H]\!]^{\mathfrak{I}}, [\![A]\!]^{\mathfrak{I}})$
$[\![\Psi_8]\!]^{\mathfrak{I}} \coloneqq [\![\neg A]\!]^{\mathfrak{I}}$

## c)

1. Aus $[\![\Psi_8]\!]^{\mathfrak{I}}$ folgt, dass $[\![A]\!]^{\mathfrak{I}} = 0$ gilt.
2. $[\![A]\!]^{\mathfrak{I}} = 0$ Eingesetzt in $[\![\Psi_2]\!]^{\mathfrak{I}}$ und $[\![\Psi_7]\!]^{\mathfrak{I}}$
   (a) $[\![\Psi_2]\!]^{\mathfrak{I}} \coloneqq \neg \min[\![\neg G]\!]^{\mathfrak{I}}, [\![S]\!]^{\mathfrak{I}})$
   (b) $[\![\Psi_7]\!]^{\mathfrak{I}} \coloneqq [\![H]\!]^{\mathfrak{I}}$

# Aufgabe 3

Give an appropriate positive constant $c$ such that $f(n) \leq c \cdot g(n)$ for all $n > 1$.

1. $f(n) = n^2 + n + 1$, $g(n) = 2n^3$
2. $f(n) = n\sqrt{n} + n^2$, $g(n) = n^2$
3. $f(n) = n^2 - n + 1$, $g(n) = n^2/2$

**Lösung**
We solve each solution algebraically to determine a possible constant $c$.

**Part One**

$$\begin{aligned} n^2 + n + 1 &= \\ &\leq n^2 + n^2 + n^2 \\ &= 3n^2 \\ &\leq c \cdot 2n^3 \end{aligned}$$

Thus a valid $c$ could be when $c = 2$.

**Part Two**

$$\begin{aligned} n^2 + n\sqrt{n} &= \\ &= n^2 + n^{3/2} \\ &\leq n^2 + n^{4/2} \\ &= n^2 + n^2 \\ &= 2n^2 \\ &\leq c \cdot n^2 \end{aligned}$$

Thus a valid $c$ is $c = 2$.

**Part Three**

$$\begin{aligned} n^2 - n + 1 &= \\ &\leq n^2 \\ &\leq c \cdot n^2/2 \end{aligned}$$

Thus a valid $c$ is $c = 2$.

# Aufgabe 4

Let $\Sigma = \{0, 1\}$. Construct a DFA $A$ that recognizes the language that consists of all binary numbers that can be divided by 5.

Let the state $q_k$ indicate the remainder of $k$ divided by 5. For example, the remainder of 2 would correlate to state $q_2$ because $7 \mod 5 = 2$.

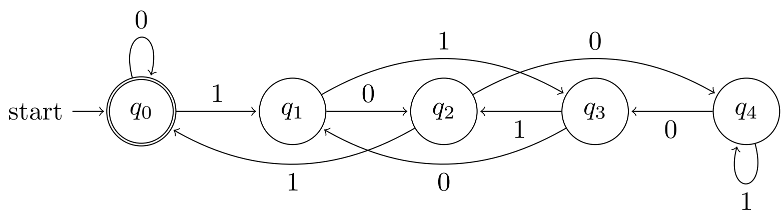


Figure 1: DFA, $A$, this is really beautiful, ya know?

**Justification**

Take a given binary number, $x$. Since there are only two inputs to our state machine, $x$ can either become $x0$ or $x1$. When a 0 comes into the state machine, it is the same as taking the binary number and multiplying it by two. When a 1 comes into the machine, it is the same as multipying by two and adding one.

Using this knowledge, we can construct a transition table that tell us where to go:

| | $x \mod 5 = 0$ | $x \mod 5 = 1$ | $x \mod 5 = 2$ | $x \mod 5 = 3$ | $x \mod 5 = 4$ |
|---|---|---|---|---|---|
| $x0$ | 0 | 2 | 4 | 1 | 3 |
| $x1$ | 1 | 3 | 0 | 2 | 4 |

Therefore on state $q_0$ or ($x \mod 5 = 0$), a transition line should go to state $q_0$ for the input 0 and a line should go to state $q_1$ for input 1. Continuing this gives us the Figure 1.

# Aufgabe 5

Write part of **Quick-Sort(**$list, start, end$**)**

**function** QUICK-SORT($list, start, end$)
    **if** $start \geq end$ **then**
        **return**
    **end if**
    $mid \leftarrow$ PARTITION($list, start, end$)
    QUICK-SORT($list, start, mid - 1$)
    QUICK-SORT($list, mid + 1, end$)
**end function**

Algorithm 1: Start of QuickSort

# Aufgabe 6

Suppose we would like to fit a straight line through the origin, i.e., $Y_i = \beta_1 x_i + e_i$ with $i = 1, \ldots, n$, $\mathrm{E}[e_i] = 0$, and $\mathrm{Var}[e_i] = \sigma_e^2$ and $\mathrm{Cov}[e_i, e_j] = 0, \forall i \neq j$.

## Part A

Find the least squares esimator for $\hat{\beta}_1$ for the slope $\beta_1$.

## Lösung

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2 \\ &= \sum_{i=1}^{n} \left(Y_i - \hat{\beta}_1 x_i\right)^2 \end{aligned}$$

By taking the partial derivative in respect to $\hat{\beta}_1$, we get:

$$\frac{\partial}{\partial \hat{\beta}_1}(RSS) = -2 \sum_{i=1}^{n} x_i (Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^{n} x_i (Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^{n} x_i Y_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^{n} x_i Y_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 \end{aligned}$$

Solving for $\hat{\beta}_1$ gives the final estimator for $\beta_1$:

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

## Part B

Calculate the bias and the variance for the estimated slope $\hat{\beta}_1$.

## Lösung

For the bias, we need to calculate the expected value $\mathrm{E}[\hat{\beta}_1]$:

$$\begin{aligned}
\mathrm{E}[\hat{\beta}_1] &= \mathrm{E}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i \mathrm{E}[Y_i]}{\sum x_i^2} \\
&= \frac{\sum x_i (\beta_1 x_i)}{\sum x_i^2} \\
&= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1
\end{aligned}$$

Thus since our estimator's expected value is $\beta_1$, we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned}
\mathrm{Var}[\hat{\beta}_1] &= \mathrm{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \sigma^2 \\
&= \frac{\sigma^2}{\sum x_i^2}
\end{aligned}$$

# Aufgabe 7

Prove a polynomial of degree $k$, $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$ is a member of $\Theta(n^k)$ where $a_k \ldots a_0$ are nonnegative constants.

*Proof.* To prove that $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$, we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0,\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are, $n^k \leq a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$ even if $c_1 = 1$ and $n_0 = 1$. This is because $n^k \leq c_1 \cdot a_k n^k$ for any nonnegative constant, $c_1$ and $a_k$.

Taking the second inequality, we prove it in the following way. By summation, $\sum_{i=0}^{k} a_i$ will give us a new constant, $A$. By taking this value of $A$, we can then do the following:

$$\begin{aligned} a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0 = & \\ & \leq (a_k + a_{k-1} \ldots a_1 + a_0) \cdot n^k \\ & = A \cdot n^k \\ & \leq c_2 \cdot n^k \end{aligned}$$

where $n_0 = 1$ and $c_2 = A$. $c_2$ is just a constant. Thus the proof is complete. □

# Aufgabe 18

Evaluate $\sum_{k=1}^{5} k^2$ and $\sum_{k=1}^{5}(k-1)^2$.

# Aufgabe 19

Find the derivative of $f(x) = x^4 + 3x^2 - 2$

# Aufgabe 6

Evaluate the integrals $\int_0^1 (1 - x^2)\mathrm{d}x$ and $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$.