SSY285 - Assignment 1

Daniel Söderqvist Johannes Lundahl Marteinn Sigþórsson

November 17, 2023

Introduction

The dynamic model of the DC motor with flywheel can be divided into two sections to easily express equations related to either electrical or mechanical. There will be in total 6 equations to describe the total system, see equation 1-3 and 5-7 and the ..... The system inputs are considered to be the voltage, v_a, and the additional external torque, T_e that is applied to the system.

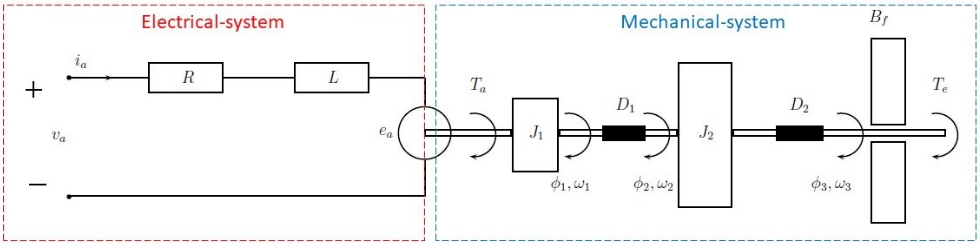

Figure 1: Dynamic model of a DC motor with flywheel

Using Kirchhoff's and Ohm's law on the electrical system we obtain:

$$v_a - i_a R - L\frac{di}{dt} - e_a = 0 \tag{1}$$

The mechanical system equations can be obtained using Newton's second law for rotation and equilibrium equations as follows:

$$T_a - D_1(\theta_1 - \theta_2) = \mathcal{J}_1\dot{\omega}_1 \tag{2}$$

$$D_1(\theta_1 - \theta_2) - D_2(\theta_2 - \theta_3) = \mathcal{J}_2\dot{\omega}_2 \tag{3}$$

$$D_2(\theta_2 - \theta_3) - B_f\omega_3 + T_e = 0 \tag{4}$$

To obtain the entire dynamics of the system, we also need to make use of the fact that the angular velocities can be expressed as the derivative of the angles. Equation 4 can now be substituted into equation 7 since they describe the same dynamics:

$$\omega_1 = \dot{\phi}_1 \tag{5}$$

$$\omega_2 = \dot{\phi}_2 \tag{6}$$

$$\omega_3 = \dot{\phi}_3 = \frac{T_e}{B_f} + \frac{D_2(\theta_2 - \theta_3)}{B_f} \tag{7}$$

Question b)

Since the inductance is very small, the voltage drop over the inductor will be close to zero, thus $L\frac{di}{dt} \approx 0$. Equation 1 will then lose one term and therefore one can obtain an expression for i_a and substitute it into equation2 to be able to reduce the state vector,x by eliminating i_a. The input vector, u, will contain the voltage and the torque as described before. These vectors will look as follows:

$$x = \begin{bmatrix} \phi_1 \\ \omega_1 \\ \phi_2 \\ \omega_2 \\ \phi_3 \end{bmatrix}, \qquad u = \begin{bmatrix} v_a \\ T_e \end{bmatrix} \tag{8}$$

The total state space model with the A and B matrices will become:

$$\underbrace{\begin{bmatrix} \dot{\phi}_1 \\ \dot{\omega}_1 \\ \dot{\phi}_2 \\ \dot{\omega}_2 \\ \dot{\phi}_3 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{D_1}{\mathcal{J}_1} & -\frac{K_e K_t}{\mathcal{J}_1 R} & \frac{D_1}{\mathcal{J}_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{D_1}{\mathcal{J}_2} & 0 & \frac{-(D_1+D_2)}{\mathcal{J}_2} & 0 & \frac{D_2}{\mathcal{J}_2} \\ 0 & 0 & \frac{D_2}{B_f} & 0 & -\frac{D_2}{B_f} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \phi_1 \\ \omega_1 \\ \phi_2 \\ \omega_2 \\ \phi_3 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{K_t}{\mathcal{J}_1 R} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B_f} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u} \tag{9}$$

Question c)

Case 1:
By using the state variables case 1 can be determined by the following C and D matrices:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \phi_1 \\ \omega_1 \\ \phi_2 \\ \omega_2 \\ \phi_3 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u} \tag{10}$$

Case 2:
By isolating for i_a in equation 1 and using equation 7, the output can be determined for case 2. The C and D matrices for case 2 are as follows:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 0 & -\frac{K_E}{R} & 0 & 0 & 0 \\ 0 & 0 & \frac{D_2}{B_f} & 0 & -\frac{D_2}{B_f} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \phi_1 \\ \omega_1 \\ \phi_2 \\ \omega_2 \\ \phi_3 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B_f} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u} \tag{11}$$

Question d)

The eigenvalues of the A-matrix describe the stability of the system. If all the poles lie in the left half plane, the system can be considered to be stable. The eigenvalues were calculated in Matlab as:

$$\begin{aligned} \lambda_1 &= -391.4 + 1.4791i \\ \lambda_2 &= -391.4 - 1.4791i \\ \lambda_3 &= -0.000 + 0.000i \\ \lambda_4 &= -270.8 + 0.000i \\ \lambda_5 &= -946.4 + 0.000i \end{aligned} \tag{12}$$

One can observe that all eigenvalues lie in the left half plane. By using the `pzplot()` command in Matlab, the following plots were obtained:

As you can see in figure 3 there is one pole and one zero located in the same spot, which was verified using the output from the `pzmap` function. Case 2 has a pole and a zero in $0.000 + 0.000i$ which means that pole-zero cancellation occurs and therefore the system becomes fully stable and not marginally stable anymore. This can be seen later in the simulation plots.

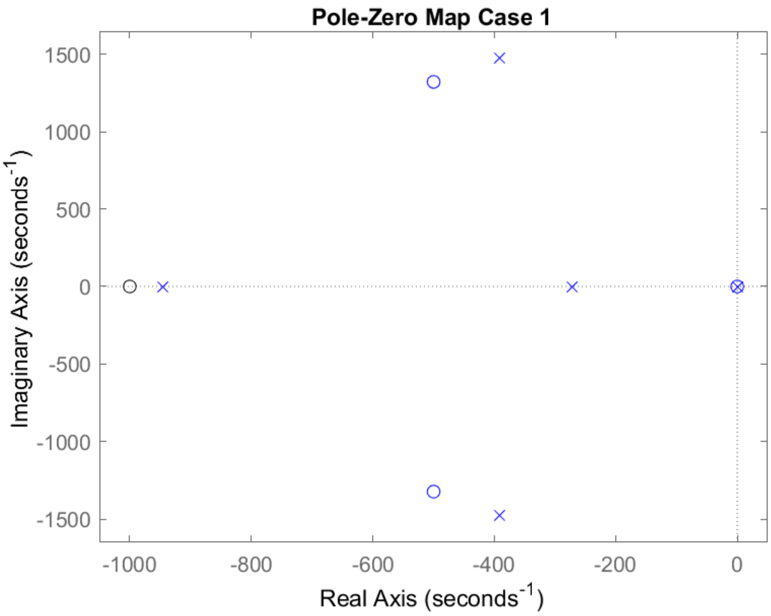

Figure 2: Poles for the Case 1 - system

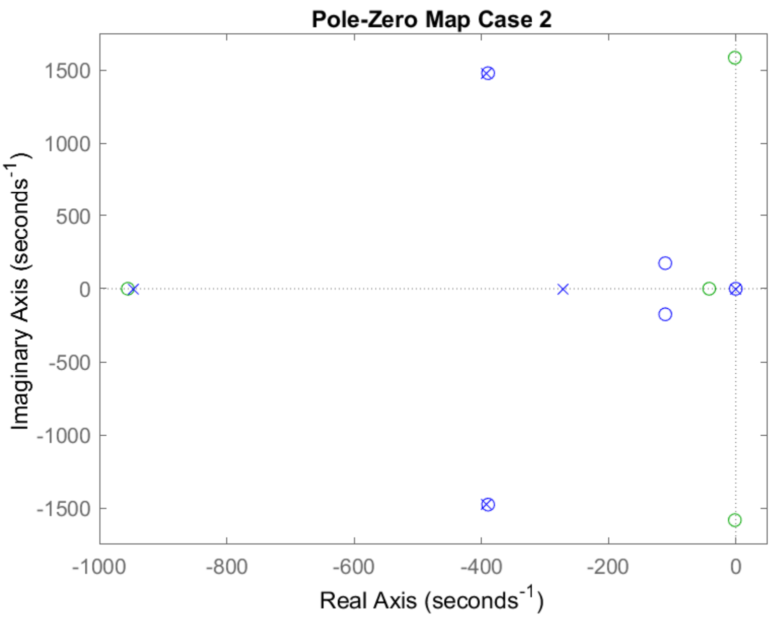

Figure 3: Poles and zeros for the Case 2 - system

Question e)

The transfer functions when the external torque is zero are:

```
From input to output...
     s^5 + 1000 s^4 + 2.55e06 s^3 + 2.5e09 s^2 + 1e11 s - 0.003096
 1:  -------------------------------------------------------------
     s^5 + 2000 s^4 + 3.55e06 s^3 + 3.05e09 s^2 + 6e11 s + 0.6388

                          5e12 s - 0.8039
 2:  ------------------------------------------------------------
     s^5 + 2000 s^4 + 3.55e06 s^3 + 3.05e09 s^2 + 6e11 s + 0.6388

Continuous-time transfer function.
```

Figure 4: Transfer function for the system without external torque T_e

Further, the poles and the zeros are presented below, where a pole-zero cancellation occurs once again. Since all poles and zeros are in the right half plane, the system is the minimum phase.

$$\begin{aligned} p_1 &= -391.4 + 1.4791i \\ p_2 &= -391.4 - 1.4791i \\ p_3 &= -0.000 + 0.000i \\ p_4 &= -270.8 + 0.000i \\ p_5 &= -946.4 + 0.000i \end{aligned} \tag{13}$$

Question f)

The aim of this task is to simulate the system over time given the inputs with an applied starting voltage and an applied external force after the system has stabilized.
It was decided to make new C and D matrices considering all angles, angular velocities, and the current as outputs to get a better overview of the results from the simulation.

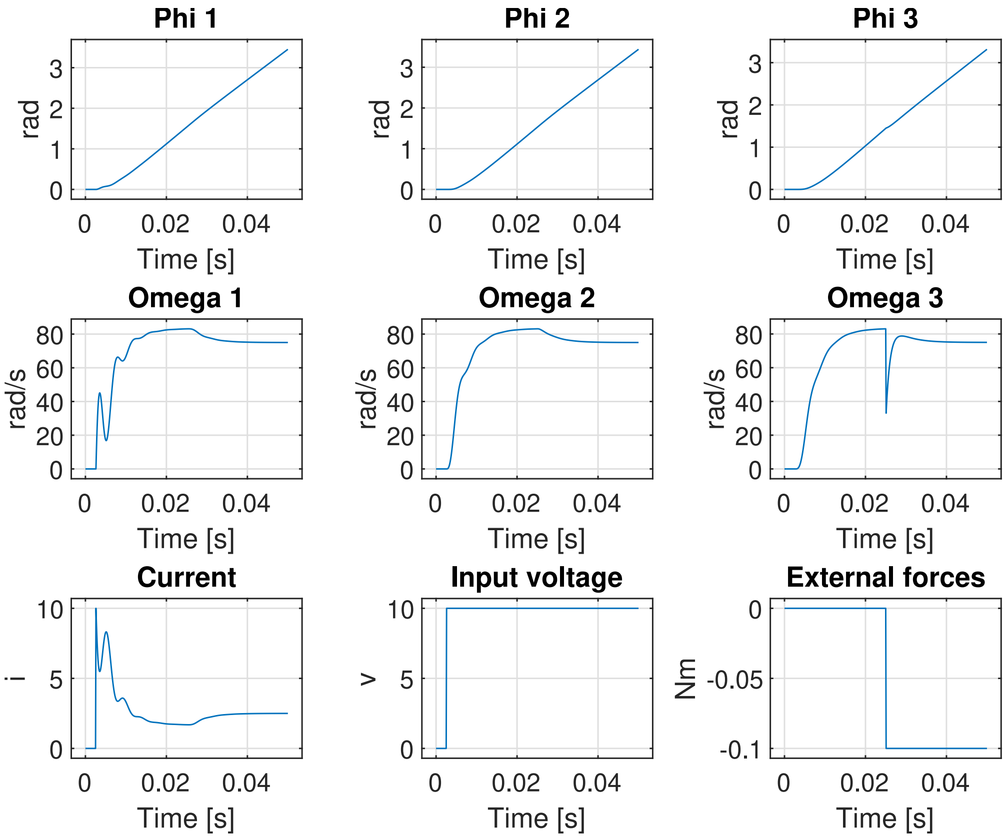

Figure 5: Output simulation for Case 1 and 2

By taking a look at the angles, one can observe, that it's reasonable that the angles increase in a somewhat linear way since the axis is continuously spinning after applying the 10 voltage of input and increasing the angles while the input is still being applied. The slope of the curve decreases by a small amount when the external force is applied which is reasonable since an external force is applied in the opposite direction.

If we look at the angular velocities we can see that all 3 stabilizes relatively fast which is a sign of stability in which we previously described and saw that we had. Looking at the system without any impacts in terms of inputs, the system will maintain constant, which is clearly a sign of marginally stability or stability. If the system was about to variate in any way even though no input is applied, the system would be unstable. This can also be visible when applying the external force since the system is restabilizing.

The result from the current plot is also reasonable. When first applying the voltage there's a bigger load for the system to start but when it's spinning after some time-steps then the load decreases as the system is less demanding when already spinning.