ESS101 Mod & Sim
Assignment 1 - Lagrange Modelling
Due date: 2023-09-15

In this assignment we will investigate the modelling of complex mechanical systems based on the Lagrange approach. **Please note that the page limit for the report on this assignment is 6 pages.**

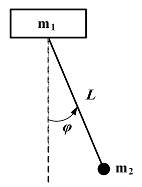
Instructions

The assignments comprise an important part of the examination in this course. Hence, it is important to comply with the following rules and instructions:

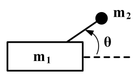
- The assignment is pursued in groups of two students. For group formation procedures, see Canvas.
- The findings from each assignment are described in a short report, written by each group independently.
- The report should provide clear answers to the questions, including your motivations, explanations, observations from simulations etc. Figures included in the report should have legends, and axes should be labelled.
- Since the assignments are part of the examination in the course, plagiarism is of course not allowed. If we observe that this happens anyway, it will be reported.
- The report should be uploaded to Canvas *before the deadline*. Matlab code is uploaded as separate files. If the deadline is not met, we cannot guarantee that you will be able to complete the assignment during the course.
- The written report is graded PASS, REVISE or RESUBMIT. With a decision REVISE, you will get a new due date to provide a revised report with changes highlighted. With a decision RESUBMIT, you will need to submit a completely rewritten report.

1. **Hovering mass** We consider a helicopter hovering a mass. We will describe the system as two masses coupled by a rigid link. The 3D positions of the masses will be described by $\mathrm{p}_1 \in \mathbb{R}^3$ and $\mathrm{p}_2 \in \mathbb{R}^3$. We will consider the force $\mathrm{u} \in \mathbb{R}^3$ applied to mass 1 (helicopter) as the external force.

   Hint: to make your life easy: note the positions of the masses using vectors $\mathrm{p}_{1,\dots,3} \in \mathbb{R}^3$ *and avoid detailing them as e.g.* $x_k, y_k, z_k \in \mathbb{R}$



(a) Schematic of a helicopter hovering a mass



(b) Top view of the system

   (a) We will first model this system using “minimal coordinates”, where the position of the helicopter is described by its cartesian coordinates $\mathrm{p}_1 \in \mathbb{R}^3$ and the position of the handing mass by two angles θ, ϕ describing the orientation of the cable. A force $\mathrm{u} \in \mathbb{R}^3$ is acting on the helicopter. Your generalized coordinates will have the form:

$$q = \begin{bmatrix} \mathrm{p}_1 \\ \theta \\ \phi \end{bmatrix} \in \mathbb{R}^5 \tag{1}$$

Hint: here it will be crucial for your own sanity to use the Matlab symbolic toolbox!!

To derive your Lagrange function, we advise you to first write the position of the hanging mass p_2 as a function of q, and use the Matlab symbolic toolbox to compute $\dot{\mathrm{p}}_2(\mathrm{q},\dot{\mathrm{q}}) = \frac{\partial \mathrm{p}_2}{\partial \mathrm{q}}\dot{\mathrm{q}}$. The kinetic and potential energy associated to the hanging mass will then read as:

$$T_2 = \frac{1}{2}m_2\dot{\mathrm{p}}_2(\mathrm{q},\dot{\mathrm{q}})^\top \dot{\mathrm{p}}_2(\mathrm{q},\dot{\mathrm{q}}), \qquad V_2 = m_2 g \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathrm{p}_2(\mathrm{q}) \tag{2}$$

Your model, fully assembled, should have the form

$$\dot{\mathrm{q}} = \mathrm{v} \tag{3a}$$
$$M(\mathrm{q})\dot{\mathrm{v}} = \mathrm{b}(\mathrm{q},\dot{\mathrm{q}},\mathrm{u}), \tag{3b}$$

where you need to specify the functions M and b. (Please note that M and b are just arbitrary names given to the expressions that you must identify.)

(b) Write down the model equations using the constrained Lagrange approach. Your generalized coordinates will then be:

$$\mathrm{q} = \begin{bmatrix} \mathrm{p}_1 \\ \mathrm{p}_2 \end{bmatrix} \in \mathbb{R}^6 \tag{4}$$

and your (scalar) constraint will be of the form:

$$C = \frac{1}{2}\left(\mathrm{e}^\top \mathrm{e} - L^2\right), \quad \text{where} \quad \mathrm{e} = \mathrm{p}_1 - \mathrm{p}_2 \tag{5}$$

The dynamics are then given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathrm{q}}} - \frac{\partial \mathscr{L}}{\partial \mathrm{q}} = \mathrm{Q} \tag{6a}$$
$$\mathrm{C}(\mathrm{q}) = 0 \tag{6b}$$

And should yield in this case a model of the form:

$$\dot{\mathrm{q}} = v \tag{7a}$$
$$M\dot{\mathrm{v}} = \mathrm{b}(\mathrm{q},\mathrm{z},\mathrm{u}) \tag{7b}$$
$$0 = \mathrm{C}(\mathrm{q}) \tag{7c}$$

(Here $M(\mathrm{q})$ and $\mathrm{b}(\mathrm{q},\dot{\mathrm{q}},\mathrm{u})$ are not the same as in the previous question!).

Hint: it is fairly easy to do these computations on pen-and-paper

Compare the complexity of your two models (i.e. the complexity of the symbolic expressions for M and b). What do you conclude?

2. **Explicit vs. Implicit model**

(a) Put your hovering mass model of question 1b) in the form:

$$\begin{bmatrix} M & \mathrm{a}(\mathrm{q}) \\ \mathrm{a}(\mathrm{q})^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathrm{q}} \\ \mathrm{z} \end{bmatrix} = \mathrm{c}\left(\mathrm{q},\dot{\mathrm{q}},\mathrm{u}\right) \tag{28}$$

specify what the functions a and c are.

(b) Compare the model in the form (28) to its explicit counterpart (make sure to do that using the Matlab symbolic toolbox):

$$\begin{bmatrix} \ddot{\mathrm{q}} \\ \mathrm{z} \end{bmatrix} = \begin{bmatrix} M & \mathrm{a}(\mathrm{q}) \\ \mathrm{a}(\mathrm{q})^\top & 0 \end{bmatrix}^{-1} \mathrm{c}\left(\mathrm{q},\dot{\mathrm{q}},\mathrm{u}\right) \tag{29}$$

What do you observe?
Note that it is not necessary to include the model equations in the report – the Matlab code to produce the expressions is enough!