Lösungsvorschlag Blatt 12

1) a) (i) $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1} = -\frac{1}{x}$

(ii) $\int \sqrt{x\sqrt{x}}\,dx = \int (x\,x^{\frac{1}{2}})^{\frac{1}{2}} = \int x^{\frac{1}{2}} x^{\frac{1}{4}}\,dx = \int x^{\frac{3}{4}}\,dx = \frac{4}{7} x^{\frac{7}{4}}$

(iii) $\int \frac{x+2}{\sqrt{x}}\,dx = \int \sqrt{x}\,dx + \int \frac{2}{\sqrt{x}}\,dx = \int x^{\frac{1}{2}}\,dx + 2\int x^{-\frac{1}{2}}\,dx = \frac{2}{3} x^{\frac{3}{2}} + 4x^{\frac{1}{2}}$

(iv) $\int \underset{g}{x}\,\underset{f'}{e^{3x}}\,dx \overset{PI}{=} \frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x}\,dx = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x}$

(v) $\int \underset{f'}{x^{\frac{1}{3}}}\,\underset{g}{\ln(x)}\,dx = \frac{3}{4} x^{\frac{4}{3}} \ln(x) - \int \frac{3}{4} x^{\frac{4}{3}} \frac{1}{x}\,dx = \frac{3}{4} x^{\frac{4}{3}} \ln(x) - \frac{3}{4} \cdot \frac{3}{4} x^{\frac{4}{3}}$

(vi) $\int \ln(x)\,dx = \int \underset{f'}{1} \cdot \underset{g}{\ln|x|}\,dx \overset{PI}{=} x\ln(x) - \int x\,\frac{1}{x}\,dx = x\ln(x) - x$

(vii) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\int \frac{1}{2\sqrt{x}} \sin(\sqrt{x})\,dx = -2\cos\sqrt{x}$

(viii) $\int \frac{1}{x\ln(x)}\,dx = \int \frac{\frac{1}{x}}{\ln(x)}\,dx \overset{\int \frac{f'}{f} = \ln(f)}{=} \ln(\ln(x))$

(ix) $\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int 3x^2\sqrt{x^3+1} = \frac{1}{3} \cdot \frac{2}{3}(x^3+1)^{\frac{3}{2}}$

(x) $\int \frac{1}{(x-1)^2}\,dx = \int (x-1)^{-2}\,dx = -(x-1)^{-1} = \frac{-1}{x-1}$

(xi) $\int \frac{1}{x^2-4}\,dx$

PBZ: $\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{c_1}{x-2} + \frac{c_2}{x+2}$

Die Zuhaltemethode liefert $c_1 = \frac{1}{4}$, $c_2 = -\frac{1}{4}$

$\Rightarrow \int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{x-2}\,dx - \frac{1}{4}\int \frac{1}{x+2}\,dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2|$

(xii) $\int \frac{-x^2+x+4}{(1-x)^2(3x+1)}\,dx$

PBZ: $\cancel{\frac{-x^2+x+4}{(1-x)^2}}$ hat ~~zwei NS~~: zweifache NS:

$$\frac{-x^2+x+4}{(1-x)^2(3x+1)} = \frac{c_1}{1-x} + \frac{c_2}{(1-x)^2} + \frac{c_3}{3x+1}$$

Zuhaltemethode: $c_2 = 1 \quad c_3 = 2$

c_1 durch Einsetzen (z.B. $x=0$) $\Rightarrow \frac{4}{1\cdot 1} = c_1 + 1 + 2 \Rightarrow c_1 = 1$

$$\Rightarrow \int \frac{-x^2+x+4}{(1-x)^2(3x+1)}\,dx = \int \frac{1}{1-x}\,dx + \int \frac{1}{(1-x)^2}\,dx + 2\int \frac{1}{3x+1}\,dx = -\ln|1-x| + \frac{1}{1-x} + \frac{2}{3}\ln|3x+1|$$

b) $$\int \cos^3(2x)\,dx = \int \cos(2x)(1-\sin^2(2x))\,dx = \int \cos(2x) - \cos(2x)\sin^2(2x)\,dx$$

$$= \int \cos(2x)\,dx - \int \cos(2x)\sin^2(2x)\,dx = \frac{1}{2}\sin(2x) - \int \cos(2x)\sin^2(2x)\,dx$$

$$\overset{u=\sin(2x)}{\underset{\frac{du}{dx}=2\cos(2x)}{=\!=}} \frac{1}{2}\sin(2x) - \int \cos(2x)\,u^2\,\frac{1}{2\cos(2x)}\,du = \frac{1}{2}\sin(2x) - \int \frac{1}{2}u^2\,du = \frac{1}{2}\sin(2x) - \frac{1}{6}u^3$$

$$= \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x)$$

2) a) $g: [c,d] \to \mathbb{R}$ stetig, $f: [a,b] \to [c,d]$ stetig, diffbar $\quad$ z.z.: $\int_{f(a)}^{f(b)} g(x)\,dx = \int_a^b g(f(t))\,f'(t)\,dt$

Da g stetig gibt es eine Stammfunktion $G(x) = \int g(x)\,dx$ mit $G'(x) = g(x)\ \forall x \in [c,d]$

Nach dem HS folgt $\int_{f(a)}^{f(b)} g(x)\,dx = G(f(a)) - G(f(b)) = (G\circ f)(b) - (G\circ f)(a)$

Da $G\circ f$ stetig diffbar folgt: $(G\circ f)'(t) = (G(f(t)))' = G'(f(t))\cdot f'(t)$

$$\overset{HS}{\Rightarrow} G(f(b)) - G(f(a)) = \int_a^b G'(f(t))\,f'(t)\,dt \qquad \square$$

b) $$\int_{e^2}^{e^3} \frac{1}{x\ln(x)\ln(\ln(x))}\,dx \overset{a)}{=} \int_2^3 \frac{1}{e^t\cdot t\cdot \ln(t)}\cdot e^t\,dt = \int_2^3 \frac{1}{t\ln(t)}\,dt$$

$$= \int_2^3 \frac{\frac{1}{t}}{\ln(t)} \overset{\frac{f'}{f}}{=} [\ln(\ln(t))]_2^3 = \ln(\ln(3)) - \ln(\ln(2))$$