7.10.14

8| a) z.z. $\max_{1\le j} |x_j - x_{j-1}| \to 0$ für $n \to \infty$

$$x_j - x_{j-1} = a \cdot \left(\frac{b}{a}\right)^{j/n} - a \cdot \left(\frac{b}{a}\right)^{j-1/n}$$

$$= a\left(\frac{b}{a}\right)^{\frac{j-1}{n}} \Big[\underbrace{\left(\frac{b}{a}\right)^{1/n}}_{\to 1} - 1\Big] \to 0 \quad \text{für } n\to\infty$$

zws

b) $$\sigma(\Pi_n\ f, \xi_j) = \sum_{j=1}^{n} f(x_j)\ |x_j - x_{j-1}|$$

$$= \sum_{j=1}^{n} \frac{x_j - x_{j-1}}{x_j}$$

$$= \sum_{j=1}^{n} \left| 1 - \frac{a\left(\frac{b}{a}\right)^{\frac{j-1}{n}}}{a\left(\frac{b}{a}\right)^{\frac{j}{n}}} \right|$$

$$= \sum_{j=1}^{n} \left| 1 - \left(\frac{a}{b}\right)^{1/n} \right|$$

$$= n\left(1 - \left(\frac{a}{b}\right)^{1/n}\right) = -\ln\left(\frac{a}{b}\right)$$

Freitag

iii) $f(x) = \begin{cases} \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Sei $x_k = \frac{1}{2\pi k}$ $\lim_{k \to \infty} f(x_k) = \lim_{k \to \infty} \underbrace{\cos(2\pi k)}_{=1} = 1$

$\boxed{\begin{matrix} x_k > 0 \\ y_k > 0 \end{matrix}}$

$y_k = \frac{1}{2\pi k + \frac{1}{2}\pi}$ $\lim f(y_k) = \lim_{k \to \infty} \underbrace{\cos(2\pi k + \frac{\pi}{2})}_{=0} = 0$

Prinzipiell:

$\tilde{x}_k = -\frac{1}{2\pi k}$ $\tilde{y} = \frac{1}{-\pi k - \frac{\pi}{2}}$

7 ii) $y' = \underbrace{\frac{3x^2}{x^3-1}}_{f(x)} y + \underbrace{x+1}_{g(x)}$ $\quad y(2)=1 \quad y(\xi)=\eta$

$$y_0(x) = \exp\left(\int_\xi^x \frac{3t^2}{t^3-1}\,dt\right) \overset{u=t^3-1}{=} \exp\left(\int_{\xi_0}^{x^3-1} \frac{1}{u}\,du\right)$$

$$= \exp(\ln|x^3-1|)$$

$$= |x^3-1|$$

$$y(x) = \left(1 + \int_2^x \frac{t+1}{t^3-1}\,dt\right)|x^3-1|$$

$$= \left(1 + \int_2^x \frac{t+1}{(t-1)(t^2+t+1)}\,dt\right)|x^3-1|$$

$$= \left(1 + \frac{2}{3}\int_2^x \frac{1}{t-1}\,dt + \frac{1}{3}\int_2^x \frac{-2t-1}{t^2+t+1}\,dt\right)|x^3-1|$$

$$= \left[1 + \frac{2}{3}\left[\ln(t-1)\right]_2^x - \frac{1}{3}\int_2^x \frac{2t+1}{t^2+t+1}\,dt\right]|x^3-1|$$

$u = t^2+t+1$

$u' = 2t+1$

$$= \left(1 + \frac{2}{3}\ln(x-1) - \frac{1}{3}\int_7^{x^2+x+1} \frac{1}{u}\,du\right)|x^3-1|$$

$$= \left(1 + \frac{2}{3}\ln|x-1| - \frac{1}{3}\left(\ln|t^2+t+1|\right]_2^x\right)|x^3-1|$$

$$= \left(1 + \frac{2}{3}\ln|x-1| - \frac{1}{3}\ln|x^2+x+1| - \frac{1}{3}\ln|7|\right)(x^3-1)$$

NR:

$(t^3-1):(t+1) = t^2+t+1$

$-t^3+t^2$

$+t^2$

$-t^2+t$

$+t-1$

$$\frac{t+1}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1}$$

$$= \frac{At^2+At+A+Bt^2+Ct-Bt-C}{Q(t)}$$

$$= \frac{t^2(A+B)+t(A+C-B)+A-C}{Q(t)}$$

$\Rightarrow A+B=0 \Rightarrow A=-B$

$A-C=1$

$A+C-B=1 \quad 2A+C=1$

$C=1-2A$

$A-C = A-1+2A = 1$

$3A=2$

$A=2/3$

$B=-\frac{2}{3} \quad C=-$

Steffy Aufgaben Freitag

2 i) $e^x - x = \frac{3}{2}$ $\qquad [0,\infty)$

$I = [0,1]$ kompakt $f(x) = e^x - x$

$f(0)\ e^0 - 0 = \cancel{1}$ $\qquad f(1) = \underbrace{e^1}_{>2{,}7} - 1 -> 1{,}7$

$\frac{3}{2} \in (f(0), f(1))$

nach ZWS $\exists x_0 : f(x_0) = \frac{3}{2}$

2 ii) $f: [-4; 2] \to \mathbb{R}$ $\quad x \mapsto -x^3 - 2x^2 + x - 4$

$f'(x) = -3x^2 - 4x + 1 \stackrel{!}{=} 0$ $\qquad x_{1/2} = \frac{4 \pm \sqrt{16+12}}{-6}$

$f''(x) = -6x - 4$

$x_1 = \frac{4+\sqrt{28}}{-6}$ $\qquad x_2 = \frac{4-\sqrt{28}}{-6}$

$f''(x_1) = 4 + \sqrt{28} - 4 > 0 \Rightarrow$ TP bei x_1 $\qquad x_1 \approx \frac{4+5}{-6} \approx -\frac{3}{2}$

$f''(x_2) = 4 - \sqrt{28} - 4 < 0 \Rightarrow$ HP bei x_2 $\qquad x_2 \approx \frac{4-5}{-6} \approx +\frac{1}{6}$

$f(-4) = +64 - 32 - 4 - 4 = 24$

$f(2) = -8 - 8 + 2 - 4 = -18 \Rightarrow I = [-18, 24]$

2023

2) a) $R(x) = \frac{1}{(x^2-a^2)^2} = \frac{1}{(x+a)^2(x-a)^2} =$

$$= \frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \frac{A_3}{(x-a)} + \frac{A_4}{(x-a)^2}$$

$$= \frac{A_1(x+a)(x^2-2ax+a^2) + A_2(x^2-2ax+a^2) + A_3(x-a)(x^2+2ax+a^2) + A_4(x^2+2ax+a^2)}{Q(x)}$$

$$= \frac{A_1(x^3-2ax^2+a^2x+ax^2-2a^2x+a^3) + \cancel{A_2(\ldots)} (x^3+2ax^2+a^2x-ax^2-2a^2x-a^3) + A_4(\ldots)}{Q(x)} + A_2\ldots + A_3$$

$$= \frac{x^3(A_1+A_3) + x^2(A_1-aA_1+A_2+aA_3+A_4) + x(-a^2A_1-2aA_2-a^2A_3+2aA_4)}{Q(x)}$$

F

$$+ \frac{a^3A_1 + a^2A_2 - a^3A_3 + a^2A_4}{Q(x)}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ -a & 1 & a & 1 & 0 \\ -a^2 & -2a & -a^2 & +2a & 0 \\ a^3 & a^2 & -a^3 & a^2 & 1 \end{array}\right)$$

b) rg $A = m$ universell lösbar, rg $A = n$ ~~eindeutig~~ maximal eine Lösung „eindeutig lösbar" oder nicht lösbar

c) $w \in \ker A$ $v+w$ ist Lösung

$Av = b$: $A(v+w) = Av + Aw = Av + 0 = b$ da

d) Satz über Partialbruchzerlegung

1. Klausur

a) F ist Linearform wenn F: lineare abb.
$F(\lambda x) = \lambda f(x)$
$F(x+y) = f(x+y)$
und $\mathbb{R}^n \to \mathbb{R}$

b) $\|x\| = \sqrt{\langle x,x\rangle}$

c) $\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$= \cancel{|\langle x,x\rangle| + |\langle x,y\rangle + \langle y,x\rangle| + |\langle y,y\rangle|^2}$

$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$

$+ \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$

$= 2(\|x\|^2 + \|y\|^2$

d) wohldefiniert:

$\langle f_1 + f_2, g\rangle = \langle f_1 g, + f_2 g\rangle$ und $\langle f, g_1 + g_2\rangle = \langle f, g_1\rangle + \langle f, g_2\rangle$

$\langle \lambda f, g\rangle = \lambda\langle f, g\rangle = \langle f, \lambda g\rangle \Rightarrow$ Bilinearform

$\langle f, g\rangle = \langle g, f\rangle \Rightarrow$ symmetrisch

$\langle f, f\rangle = \int \underbrace{f^2(x)}_{\geq 0} dx \geq 0$ und $\langle f, f\rangle = 0$ für $f \equiv 0$

da für $f \in C(I)$ und $f(x_0) \neq 0$
gibt $\exists\delta: U_\delta(x_0)$ mit $f(x) \neq 0$
$\forall x \in U_\delta(x_0)$
$\Rightarrow \int^b f^2(x)dx \neq 0$

e) (M,d) $d(x,y) \geq 0$ $d(x,y) = 0 \Leftrightarrow x = y$
$d(x,y) = d(y,x)$
$d(x,z) \leq d(x,y) + d(y,z)$ $M \neq 0$ (M,d) heißt
[illegible]

8.10.16

S1 $\frac{d}{dx} x^{-2} = \lim_{x \to x_0} \frac{\frac{1}{x^2} - \frac{1}{x_0^2}}{x - x_0} = \lim_{x \to x_0} \frac{x_0^2 - x^2}{(x - x_0) x^2 x_0^2} = \lim_{x \to x_0} \frac{(x_0 - x)(x + x_0)}{(x - x_0) x^2 x_0^2}$

$= \lim_{x \to x_0} - \frac{x + x_0}{x^2 x_0^2} = - \frac{2}{x_0^3}$

05.10.23

positive definit, symmetrische Bilinearformen

$\langle x, x \rangle \geq 0 \qquad \langle x, y \rangle = \langle y, x \rangle$

b) $\langle x, y \rangle = \sum_{i,j=1}^{n} a_{ij} x_i \bar{y}_j = x^T A \bar{y}$

Nein $\quad A > 0$

c) A muss positiv definit und symmetrisch sein

d) $\langle x, y \rangle = \langle y, x \rangle \qquad x^T A \bar{y} = y^T A \bar{x} \qquad (x^T A \bar{y})^{+} = \bar{y}^T \bar{A}^T \bar{x}$

wenn gilt $\quad \bar{A}^T = A^T$

$x^T A \bar{x} \geq 0$ Definition eines positiv definiten Matrix

$\forall x \neq 0$

d) (Met) $d(x,y) \geq 0 \qquad d(x,y) = 0 \Leftrightarrow x = y$

$d(x,y) = d(y,x)$

$d(x,z) \leq d(x,y) + d(y,z)$

e) 1-Norm

$\left(\sum_{i=1} |x_i|^p \right)^{1/p}$

5) $\frac{x-y}{x} < \ln\frac{x}{y} < \frac{x-y}{y}$ MWS: $f(x) = \ln(x)$ $f'(x) = \frac{1}{x}$

$a = \cancel{x}$

$b = \cancel{x}$

$\exists z \in [x,y]$ für das gilt

$$f'(z) = \boxed{\frac{\ln(\cancel{x}) - \ln(y)}{x-y} = \frac{1}{z}}$$

$\cancel{y} < z < x$

$$\ln\left(\frac{x}{y}\right) = \frac{x-y}{z}$$

$z >$

$\Rightarrow \frac{1}{y} > \frac{1}{z} > \frac{1}{x}$

$\boxed{\Leftrightarrow \frac{x-y}{y} > \frac{x-y}{z} > \frac{x-y}{x}}$

$\Rightarrow$ ln

$$\Rightarrow \frac{x-y}{\cancel{x}} < \ln\left(\frac{x}{y}\right) < \frac{x-y}{\cancel{y}}$$

08.10.16

8) $f(t) = \log(1+t) - \frac{t}{\sqrt{1+t}}$ $f'(t) = \frac{1}{1+t} - \frac{\sqrt{1+t} - t\frac{1}{2\sqrt{1+t}}}{1+t}$

$f(0) = 0$

$f(x) \dots$

$$= \frac{1 + \sqrt{1+t} - \frac{t}{2}\frac{1}{\sqrt{1+t}}}{1+t}$$

$\exists z \in [0,x)$

$$\frac{1 \mp \sqrt{1+z} + \frac{z}{2}\frac{1}{\sqrt{1+z}}}{1+z} = \log(1+\cancel{x}) - \frac{x}{\sqrt{1+x}}$$

$$= \frac{1+\sqrt{1+z}}{1+z} - \frac{z}{2\sqrt{1+z}} = \log(1+x) - \frac{x}{\sqrt{1+x}}$$

$$\leq \frac{1 \mp 1 \mp z/2 + \frac{z}{2}\frac{1}{\sqrt{1+z}}}{1+z} = \frac{z/2 \cdot \left(-1 + \frac{1}{\sqrt{1+z}}\right)}{1+z} < 0$$

$$\Rightarrow \log(1+x) - \frac{x}{\sqrt{1+x}} < 0 \Rightarrow \text{Ungl.}$$

$f(x)=\sinh(x) \quad f^{-1}(x) = \mathrm{Arsinh}(x)$

4) c) $\cancel{f'}\ \frac{d}{dx}\,\mathrm{Arsinh}(x)$

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\frac{e^x-e^{-x}}{2} = \frac{e^x+e^{-x}}{2} = \cosh(x) = \sqrt{1+\sinh^2(x)}$$

$$f'(f^{-1}(x)) = \sqrt{1+\sinh^2(\mathrm{Arsinh}(x))}$$

$$= \sqrt{1+x^2}$$

$$\Rightarrow \frac{d}{dx} f^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

$\boxed{8.10.16}$ 7) Z: $\boxed{f^{(n)}(x) = (-1)^n\, n!\, \frac{4(x-n)}{(x+1)^{n+2}}}$

Induktion: ~~$f^{(1)}(x) = \frac{d}{dx}\frac{4x}{(1+x^2)^2} = \frac{(1+x^2)4-4x\cdot 2x}{(1+x^2)^2}$~~

Anfang

~~$= \frac{4+4x^2-8x^2}{(1+x^2)^2}$~~

~~$= \frac{4(1-x^2)}{(1+x^2)^2}$~~

$(x+n)(n+2)$

$xn - n^2 + 2x - \cancel{2n}$

$= 2x - n^2$

$$f(x) = \frac{4x}{(1+x)^2}$$

Anfang $n=1$: $f'(x) = \frac{(1+x)^2 4 - 4x\, 2(1+x)}{(1+x)^4}$

$$= \frac{(1+x)[4+4x-8x-\cancel{8x^2}]}{(1+x)^4}$$

$$= \frac{4(1-x)\cancel{2x^2}}{(1+x)^3}$$

$$= (-1)^1 1!\, \frac{4(x-1)}{(1+x)^3}$$

IH:

IS: $f^{(n+1)} = \frac{d}{dx} f^{(n)} = \frac{d}{dx}(-1)^n n!\, \frac{4(x-n)}{(x+1)^{n+2}}$

$$= (-1)^n n!\, \frac{(x+1)^{n+2}4 - 4(x-n)(n+2)(x+1)^{n+1}}{(x+1)^{2n+4}}$$

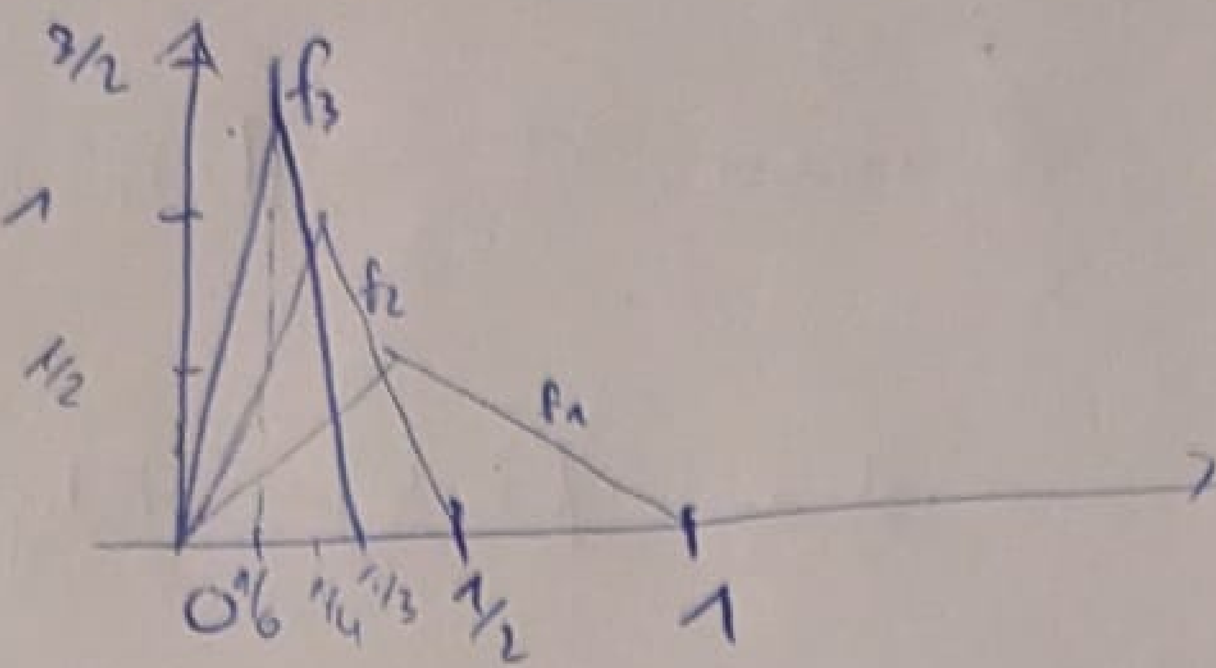
$$= (-1)^n n!\, \frac{(x+1)^{n+1}[4(x+1) - 4(x-n)(n+2)]}{x^{2n+4}}$$

$$= (-1)^n n!\, \frac{4[(x+1)-(x-n)(n+2)]}{}$$

2023

6a)

(1)



$f_1(x) = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 1-x & \frac{1}{2} \le x < 1 \\ 0 & \end{cases}$

$f_2(x) = \begin{cases} 4x & 0 \le x < \frac{1}{4} \\ 4(1/2 - x) & \frac{1}{4} \le x < 1/2 \\ 0 & x \in [1/2, 1] \end{cases}$

$f_3(x) = \begin{cases} 9x & 0 \le x < 1/6 \\ 9(\frac{1}{3} - x) & 1/6 \le x < 1/3 \\ 0 & 1/3 \le x \le 1 \end{cases}$

$\int_0^1 f_k(x)dx = \int_0^{1/2k} k^2 x + \int_{1/2k}^{1/k} k^2(\frac{1}{k} - x) +$

(3) $= \left[\frac{k^2}{2} x^2\right]_0^{1/2k} + \left[kx - \frac{k^2}{2}x^2\right]_{1/2k}^{1/k} = \frac{k^2}{2}\frac{1}{4k^2} + \frac{k}{k} - \frac{k^2}{2}\frac{1}{k^2} - \frac{k}{2k} + \frac{k^2}{2}\frac{1}{4k^2}$

$= \frac{1}{8} + 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{8} = \frac{1}{4}$

(2) ~~$\forall x \; \exists k \in$~~ $\forall x \in [0,1] \; \exists k \in \mathbb{N}$ sodass $\frac{1}{k} < x \Rightarrow x \in \frac{1}{k} \le x \le 1$

Nach Def ist $f_k(x) = 0 \Rightarrow \lim_{k \to \infty} f_k(x) = 0$

27.07.2023

5 b) $f: [a,b] \to \mathbb{R}$ Riemann-int. $\Rightarrow |f(t)| \leq M > 0$ auf $[a,b]$

$F(x) = \int_a^x f(t)\,dt$ ist auf $[a,b]$ stetig

$$|F(x) - F(x_0)| = \left| \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt \right|$$

$$= \left| \int_a^x f(t)\,dt - \left(\int_a^x f(t)\,dt + \int_x^{x_0} f(t)\,dt \right) \right|$$

$$= \left| \int_x^{x_0} f(t)\,dt \right| \leq \left| \int_x^{x_0} M\,dt \right| = M|x - x_0| < M\delta < \varepsilon$$

Wähle $\delta < \frac{\varepsilon}{M}$

5a) Zeige $\frac{d}{dx}\left(-\frac{x}{a}\cot(ax) + \frac{1}{a^2}\ln\sin(ax)\right) = \frac{x}{\sin^2(ax)}$

NR: $\frac{d}{dx}\cot(ax) = \frac{d}{dx}\frac{\cos(ax)}{\sin(ax)} = \frac{-a\sin^2(ax) - a\cos^2(ax)}{\sin^2(ax)} = \frac{-a}{\sin^2(ax)}$

$$\frac{d}{dx}\left(\frac{-x}{a}\cot(ax) + \frac{1}{a^2}\ln(\sin(ax))\right) = \cancel{\frac{-1}{a}\cot(ax)} - \frac{x}{a}\frac{-a}{\sin^2(ax)} + \cancel{\frac{1}{a^2}\frac{\cos(ax)}{\sin(ax)}}$$

$$= \frac{x}{\sin^2(ax)}$$

weitere Stammfkt: $+c$

Stetig Freitag

4| a) $\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in U_\delta(x_0) \Rightarrow |f(x) - f(x_0)| < \varepsilon$

b) $\lim_{x \to a} f(x) = f(a)$ und $\tilde{f} : [a,b] \to \mathbb{R}$ ist stetig

c) $f(x) = \begin{cases} x & x \leq 2 \\ 0 & x > 2 \end{cases}$

d) $f(x) = \begin{cases} x-1 & x > 1 \\ \sin(\pi x) & x \leq 1 \end{cases}$

Stetig:

$f(x_0) = \sin(\pi) = 0$

$\lim_{x \to x_0^+} f(x) = \lim_{x \to 1^+} x - 1 = 0$

$\lim_{x \to x_0^-} f(x) = \lim_{x \to 1^-} \sin(\pi x) = 0 \Rightarrow$ stetig

f' $c_r = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{1+h-1-0}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$

$c_l = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{\sin(\pi(1+h))}{h}$

$\overset{\text{L'Hop.}}{=} \lim_{h \to 0^-} \frac{\overbrace{\cos(\pi(1+h))}^{\to \cos(\pi) = -1} \cdot \pi}{1}$

$= \lim_{h \to 0^-} \frac{-\pi}{1} = -\pi \neq 1$

$\Rightarrow f$ nicht diff'bar bei $x_0 = 1$

e) $\sin(\pi x)$ und $x-1$ stetig diff'bar als Komposition stetig diff'barer Funktionen für $x_0 \neq 1$

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6)

$$\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt \overset{PI}{=} \left[\frac{(x-t)^n}{n!} f^{(n)}(t)dt\right]_{x_0}^{x} + \int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt$$

$$\overset{PI}{=} -\frac{(x-x_0)^n}{n!} f^{(n)}(x_0)\,dt \mp \left[\frac{(x-t)^{n-1}}{(n-1)!} f^{(n-1)}(t)\right]_{x_0}^{x} \pm \int \frac{(x-t)^{n-2}}{(n-2)!} f^{(n-1)}(t)\,dt$$

$$\vdots$$

$$= \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)\,dt \mp \frac{(x-x_0)^{n-1}}{(n-1)!} f^{(n-1)}(x_0)\,dt + \dots \int \frac{x-t}{1!} f^{(n+1)}(t)\,dt$$

$$= -\sum_{k=2}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)\,dt + \left[(x-t)f'(t)\right]_{x_0}^{x} + \int_{x_0}^{x} f'(t)\,dt$$

$$= -\sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + f(x) - f(x_0)$$

$$\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt = -\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + f(x)$$

$| + \sum \dots$