Lösungsvorschlag Blatt 3

1) a) $\sum_{k=1}^{n} \log(k) = \log(1) + \log(2) + \ldots + \log(n) = \log(1 \cdot 2 \cdot \ldots \cdot n) = \log(n!)$

b) $\sum_{k=0}^{n} e^{kx} = \sum_{k=0}^{n} (e^x)^k = \frac{(e^x)^{n+1} - 1}{e^x - 1} = \frac{e^{nx+x} - 1}{e^x - 1}$

c) $\prod_{k=0}^{n} e^{kx} = \cancel{e^{0+1x+2x+\ldots+nx}}\ e^{0+x+2x+\ldots+nx} = e^{x \sum_{k=0}^{n} k}$

$= e^{x \frac{n(n+1)}{2}}$

2) a) $2^x = 5 \iff x \ln(2) = \ln(5) \iff x = \frac{\ln(5)}{\ln(2)}$

b) $\ln(x) + \ln(x-1) = 0 \iff \ln(x(x-1)) = 0 \iff x(x-1) = 1$

$\iff x^2 - x - 1 = 0 \implies x_{1/2} = \frac{1 \pm \sqrt{5}}{2}$, aber da $\ln(x)$ für

$x \leq 0$ nicht definiert ist darf nur $x = \frac{1+\sqrt{5}}{2}$ eingesetzt werden.

c) $2^x \cdot 3^{x+1} = 5^x \iff \ln(2^x 3^{x+1}) = \ln(5^x) \iff x \ln(2) + (x+1)\ln(3) = x \ln(5)$

$\iff x = \frac{-\ln(3)}{\ln(2) + \ln(3) - \ln(5)}$

d) $\ln(x) - \ln(x^2) + 1 = 0 \iff \ln\left(\frac{x}{x^2}\right) = -1 \iff \frac{1}{x} = e^{-1} \iff x = e$

e) $e^x + e^{-x} = 2$ Setze $y = e^x$ $(\iff x = \ln(y))$

$\implies y + \frac{1}{y} = 2 \iff y^2 + 1 = 2y \iff y^2 - 2y + 1 = 0 \iff (y-1)^2 = 0$

$\iff y = 1 \implies x = \ln(1) = 0$

f) $4^x - 2^{x+1} = 3 \iff 2^x \cdot 2^x - 2 \cdot 2^x = 3 \iff 2^x(2^x - 2) = 3$

Setze $y = 2^x$ $\left(x = \frac{\ln(y)}{\ln(2)}\right)$ $\implies y(y-2) = 3 \iff y^2 - 2y - 3 = 0$

$\implies y_{1/2} = \frac{2 \pm 4}{2}$, $y = -1$ darf nicht eingesetzt werden $(2^x = -1$ ↯$)$

$\implies y = 3 \implies x = \frac{\ln(3)}{\ln(2)}$

3) a): $A(n) \Leftrightarrow \sum_{k=1}^{n-1} k \log\left(\frac{k+1}{k}\right) = n\log(n) - \log(n!)$

IS: $A(2)$: $\sum_{k=1}^{n} k \log\left(\frac{k+1}{k}\right) = \log(2) = 2\cdot\log(2) - \log(2!)$ ✓

IH: $A(n)$ gelte für ein festes aber beliebiges $n \in \mathbb{N}$

IS: $A(n) \Rightarrow A(n+1) = \sum_{k=1}^{n} k \log\left(\frac{k+1}{k}\right) = n \log\left(\frac{n+1}{n}\right) + \sum_{k=1}^{n-1} k \log\left(\frac{k+1}{k}\right)$

$\overset{IH}{=} n\left(\log(n+1) - \log(n)\right) + n\log(n) - \log(n!)$

$= n\left(\log(n+1)\right) - \log(n!) = n\log(n+1) + \log(n+1) - \left(\log(n+1) + \log(n!)\right)$

$= (n+1)\log(n+1) - \log((n+1)!)$ □

b) $\sum_{k=1}^{n-1} k\log\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n-1} k\left(\log(k+1) - \log(k)\right) = \sum_{k=1}^{n-1} k\log(k+1) - \underbrace{\sum_{k=1}^{n-1} k\log(k)}$ 0 für $k=1$

$= \sum_{k=2}^{n}(k-1)\log k - \sum_{k=2}^{n-1} k\log(k) = \sum_{k=2}^{n} k\log(k) - \sum_{k=2}^{n}\log(k) - \sum_{k=2}^{n-1} k\log(k)$

$= n\log(n) - \sum_{k=1}^{n}\log k \overset{1a}{=} n\log(n) - \log(n!)$ □

4) a) z.z.: $\sin(x+y) - \sin(x-y) = 2\cos(x)\sin(y)$

Additionstheoreme: ①: $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

②: $\sin(x-y) = \sin(x+(-y)) = \sin(x)\cos(-y) + \cos(x)\sin(-y)$
$= \sin(x)\cos(y) - \cos(x)\sin(y)$

①-② liefert: $\sin(x+y) - \sin(x-y) = \cos(x)\sin(y) + \cos(x)\sin(y) = 2\cos(x)\sin(y)$ □

b) z.z.: $\frac{1}{2} + \sum_{k=1}^{n}\cos(kx) = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)}$

Es gilt: $2\sin\left(\frac{x}{2}\right)\sum_{k=1}^{n}\cos(kx) = \sum_{k=1}^{n} 2\cos(kx)\sin\left(\frac{x}{2}\right) \overset{a)}{=} \sum_{k=1}^{n}\left(\sin\left(kx+\frac{x}{2}\right) - \sin\left(kx-\frac{x}{2}\right)\right)$

Ind.-shift 2. Summe $\rightarrow$ $= \sum_{k=1}^{n}\sin\left(kx+\frac{x}{2}\right) - \sum_{k=0}^{n-1}\sin\left(kx+\frac{x}{2}\right) = \sin\left(nx+\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) = \sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\left(\frac{x}{2}\right)$

$\Rightarrow \frac{1}{2} + \sum_{k=1}^{n}\cos kx = \frac{1}{2} + \frac{2\sin\left(\frac{x}{2}\right)\sum_{k=1}^{n}\cos(kx)}{2\sin\left(\frac{x}{2}\right)} = \frac{1}{2} + \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)}$ □

c) $\cos x = b$ ist für $b > 1$ oder $b < -1$ unlösbar, wenn $x \in \mathbb{R}$

Für $-1 \le b \le 1$ hat die Gleichung unendlich viele Lösungen. Genau eine davon fällt in das Intervall $[0, \pi]$, hier ist die Gleichung also eindeutig lösbar. Diese eindeutige Lösung nennen wir Arcuscosinus von b. (Falls $x_1 \in [0,\pi]$ Lösung $\Rightarrow \exists x_2 \in [-\pi, 0]$ mit $x_1 = -x_2$, s.d. x_2 ebenfalls Lösung. Alle weiteren: $x_1 + 2k\pi$ oder $x_2 + 2k\pi$, $k \in \mathbb{Z}$ (Periodizität))