Analysis I für IGI
Blatt 13
Lösungsvorschlag

Bemerkung: A4 waere fuer eine Klausur zu schwer

Aufgabe 1

$$\int \frac{x^3+x^2-4x-4}{x^2-4x+3}\,dx$$

Zählergrad > Nennergrad -> erst mal Polynomdivision.

$$\begin{array}{l} (x^3+x^2-4x-4) : (x^2-4x+3) = x+5 \\ \underline{-(x^3-4x^2+3x)} \\ \quad 5x^2-7x-4 \\ \quad \underline{-(5x^2-20x+15)} \\ \qquad 13x-19 \end{array}$$

Also $\frac{x^3+x^2-4x-4}{x^2-4x+3} = x+5+\frac{13x-19}{x^2-4x+3}$

Mit MNF: $(x^2-4x+3) = (x-1)(x-3)$. Damit folgt

$$\frac{13x-19}{x^2-4x+3} = \frac{13x-19}{(x-1)(x-3)} = \frac{c_1}{x-1} + \frac{c_2}{x-3}$$

Zuhaltemethode: $c_2 = \frac{13x-19}{x-1}\Big|_{x=3} = 10$, $c_1 = \frac{13x-19}{x-3}\Big|_{x=1} = 3$

Damit folgt

$$\int \frac{x^3+x^2-4x-4}{x^2-4x+3}\,dx = \int x+5+\frac{3}{x-1}+\frac{10}{x-3}\,dx = \frac{1}{2}x^2+5x+3\ln|x-1|+10\ln|x-3|.$$

Aufgabe 2

uneigentliche Integrale.

a) $\int_2^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \left[2\sqrt{x}\right]_2^b = \lim_{b\to\infty} 2\sqrt{b} - 2\sqrt{2} = \infty$

(div.)

b) $\int_0^1 \frac{\arccos(x)}{\sqrt{1-x^2}}\,dx = \lim_{a\to 0} \int_a^1 \frac{\arccos(x)}{\sqrt{1-x^2}}\,dx = \lim_{a\to 0} \left[\frac{-1}{2}\arccos^2(x)\right]_a^1 = 0 + \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{\pi^2}{8}$

c) $\int_0^\infty \frac{k}{\mu} \cdot \left(\frac{x}{\mu}\right)^{k-1} e^{-\left(\frac{x}{\mu}\right)^k}\,dx = \lim_{b\to\infty} \left[-e^{-\left(\frac{x}{\mu}\right)^k}\right]_0^b = 0-(-1) = 1$

d) $\int_0^1 \ln(x)\,dx = \lim_{a\to 0} \int_a^1 \ln(x)\,dx = \lim_{a\to 0} \left[x\cdot\ln(x) - x\right]_a^1 = \lim_{a\to 0}\left(-1 - a\ln(a) + a\right) = -1$

denn $\lim_{a\to 0} a\ln(a) = \lim_{a\to 0} \frac{\ln(a)}{\frac{1}{a}} \overset{l'H, \frac{\infty}{\infty}}{=} \lim_{a\to 0} \frac{\frac{1}{a}}{-\frac{1}{a^2}} = \lim_{a\to 0} -a = 0$

e) $\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{x}{1+x^2}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\cdot\ln(1+x^2)\right]_0^b = \lim_{b\to\infty} \frac{1}{2}\cdot\ln(1+b^2) = \infty$

f) $\int_0^\infty e^{-\mu x}\,dx = \lim_{b\to\infty} \int_0^b e^{-\mu x}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\mu} e^{-\mu x}\right]_0^b = \lim_{b\to\infty}\left(-\frac{1}{\mu} e^{-\mu b} + \frac{1}{\mu}\right)$

$= \begin{cases} \frac{1}{\mu}, & \text{falls } \mu > 0 \\ \infty, & \text{falls } \mu < 0 \end{cases}$

Aufgabe 3

$f: [-1,1] \to \mathbb{R}$, $f(x) = -3(x^2-1)$. Volumen Rotationskörper?

$$V = \pi \int_{-1}^{1} f(x)^2 dx = 9\pi \int_{-1}^{1} (x^2-1)^2 dx = 9\pi \cdot \int_{-1}^{1} x^4 - 2x^2 + 1 \, dx$$

$$= 9\pi \left[x^5/5 - \tfrac{2}{3}x^3 + x \right]_{-1}^{1} = \frac{48}{5}\pi$$

Aufgabe 4

$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$; $\int_{-\infty}^{\infty} \phi(x) dx = 1$

a) $\int_{-\infty}^{\infty} x \cdot \phi(x) dx$?

Möglichkeit 1 $x \cdot \phi(x)$ ist punktsymmetrisch, deswegen ist $\int_{-\infty}^{\infty} x \cdot \phi(x) dx = 0$

Möglichkeit 2: $\int_{-\infty}^{\infty} x \cdot \phi(x) = \lim_{c \to \infty} \int_{-c}^{c} x \, \phi(x) dx = \lim_{c \to \infty} \int_{-c}^{c} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

$$= \lim_{c \to \infty} \left[-\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right]_{-c}^{c} = \underline{\underline{0}}$$

$$\int_{-\infty}^{\infty} x^2 \phi(x) = 2 \cdot \int_{0}^{\infty} x^2 \phi(x) dx = 2 \cdot \left(\lim_{c \to \infty} \frac{1}{\sqrt{2\pi}} \int_{0}^{c} \underset{u}{x} \; \underbrace{x e^{-x^2/2}}_{v'} dx \right) \overset{\text{p.I.}}{=} \lim_{c \to \infty} \left(\frac{2}{\sqrt{2\pi}} \left(\left[-x e^{-x^2/2} \right]_0^c + \int_0^c e^{-\frac{x^2}{2}} dx \right) \right)$$

$$= \lim_{c \to \infty} \left(\frac{2}{\sqrt{2\pi}} \cdot \underbrace{(-c \cdot e^{-c^2/2})}_{\to 0} \right) + \frac{2}{\sqrt{2\pi}} \cdot \underbrace{\int_0^{\infty} e^{-x^2/2} dx}_{\sqrt{2\pi}/2} = \underline{\underline{1}}$$

Hier haben wir benutzt, dass

$\int_0^{\infty} \phi(x) dx = 1/2$ gilt. Das stimmt, da $\phi(-x) = \phi(x)$

b) z.z. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\Gamma(\tfrac{1}{2}) = \int_0^{\infty} x^{-\frac{1}{2}} e^{-x} dx = \lim_{b \to \infty} \int_0^{b} x^{-\frac{1}{2}} e^{-x} dx \overset{\substack{x = u^2/2 \\ dx = u}}{=} \lim_{b \to \infty} \int_0^{\sqrt{2b}} \frac{\sqrt{2}}{u} e^{-u^2/2} \, u \, du$$

$$= \sqrt{2} \cdot \int_0^{\infty} e^{-u^2/2} du = \sqrt{2} \cdot \frac{\sqrt{2\pi}}{2} = \underline{\underline{\sqrt{\pi}}}$$

mit der gleichen Begründung wie oben

Aufgabe 5

$f: [1,\infty) \to \mathbb{R}, \quad f(x) = \frac{1}{x}$

a) Volumen zw $[1,b)$, $b>1$?

$$V(b) = \pi \int_1^b f(x)^2 dx = \pi \int_1^b \frac{1}{x^2} dx = \pi \cdot \left[-\frac{1}{x}\right]_1^b = \underline{\underline{\pi - \frac{\pi}{b}}}$$

b) uneigtl. Vol?

$$V_\infty = \lim_{b \to \infty} V(b) = \lim_{b \to \infty} \pi - \frac{\pi}{b} = \underline{\underline{\pi}}$$

c) uneigtl. Mantelfläche?

$$M(b) = 2\pi \int_1^b f(x) \sqrt{1 + f'(x)^2} dx = 2\pi \int_1^b \frac{1}{x} \cdot \sqrt{1 + \frac{1}{x^4}} dx \geq 2\pi \int_1^b \frac{1}{x} dx = 2\pi \ln(b)$$

Da $\lim_{b \to \infty} \ln(b) = \infty$ konvergiert $\lim_{b \to \infty} M(b)$ also nicht.