Nonlinear pendulum

$$\ddot{\theta}+\omega_0^2\sin\theta=0, \qquad \omega_0=\sqrt{\frac{g}{L}}$$

Consider ω_0=1. The total energy is

$$E=(1-\cos\theta)+\frac{\dot{\theta}^2}{2}$$

First, the differential equation with second order time derivative is converted to a system of two first order systems

$$\begin{cases}\dfrac{d\dot{\theta}}{dt}=-\sin\theta\\ \dfrac{d\theta}{dt}=\dot{\theta}\end{cases}$$

Discretization. Forward Euler:

$$\begin{cases}\dfrac{\dot{\theta}^{n+1}-\dot{\theta}^n}{\Delta t}=-\sin\theta^n\\ \dfrac{\theta^{n+1}-\theta^n}{\Delta t}=\dot{\theta}^n\end{cases}\rightarrow\begin{cases}\dot{\theta}^{n+1}=\dot{\theta}^n-\Delta t\sin\theta^n\\ \theta^{n+1}=\theta^n+\Delta t\dot{\theta}^n\end{cases}$$

Corrector-Predictor:

$$\begin{cases}\dot{\theta}^*=\dot{\theta}^n-\Delta t\sin\theta^n\\ \theta^*=\theta^n+\Delta t\dot{\theta}^n\end{cases}, \quad \begin{cases}\dot{\theta}^{n+1}=\dot{\theta}^n-\Delta t\dfrac{\sin\theta^n+\sin\theta^*}{2}\\ \theta^{n+1}=\theta^n+\Delta t\dfrac{\dot{\theta}^n+\dot{\theta}^*}{2}\end{cases}$$

Crank-Nicolson

$$\begin{cases}\dfrac{\dot{\theta}^{n+1}-\dot{\theta}^n}{\Delta t}=-\dfrac{\sin\theta^n+\sin\theta^{n+1}}{2}\\ \dfrac{\theta^{n+1}-\theta^n}{\Delta t}=\dfrac{\dot{\theta}^n+\dot{\theta}^{n+1}}{2}\end{cases}$$

Because of the nonlinear term sin θ^{n+1}, Taylor series is used

$$\sin\theta^{n+1}=\sin\theta^*+\left(\theta^{n+1}-\theta^*\right)\cos\theta^*$$

Substituting in the equation

$$\begin{cases}\dfrac{\dot{\theta}^{n+1}-\dot{\theta}^n}{\Delta t}=-\dfrac{\sin\theta^n+\sin\theta^*+\left(\theta^{n+1}-\theta^*\right)\cos\theta^*}{2}=-\dfrac{\sin\theta^n+\sin\theta^*-\theta^*\cos\theta^*}{2}-\dfrac{\cos\theta^*}{2}\theta^{n+1}\\ \dfrac{\theta^{n+1}-\theta^n}{\Delta t}=\dfrac{\dot{\theta}^n+\dot{\theta}^{n+1}}{2}=\dfrac{\dot{\theta}^n}{2}+\dfrac{\dot{\theta}^{n+1}}{2}\end{cases}$$

After further simplification

$$\begin{cases}\dot{\theta}^{n+1}=\dot{\theta}^n-\dfrac{\Delta t\left(\sin\theta^n+\sin\theta^*-\theta^*\cos\theta^*\right)}{2}-\dfrac{\Delta t\cos\theta^*}{2}\theta^{n+1}\\ \theta^{n+1}=\theta^n+\Delta t\dfrac{\dot{\theta}^n}{2}+\Delta t\dfrac{\dot{\theta}^{n+1}}{2}\end{cases}$$

Rearranging the equation

$$\begin{cases} \dot{\theta}^{n+1} + a\theta^{n+1} = b \\ \theta^{n+1} + c\dot{\theta}^{n+1} = d \end{cases}$$

$$a = \frac{\Delta t \cos\theta^*}{2}, b = \dot{\theta}^n - \frac{\Delta t\left(\sin\theta^n + \sin\theta^* - \theta^*\cos\theta^*\right)}{2}$$

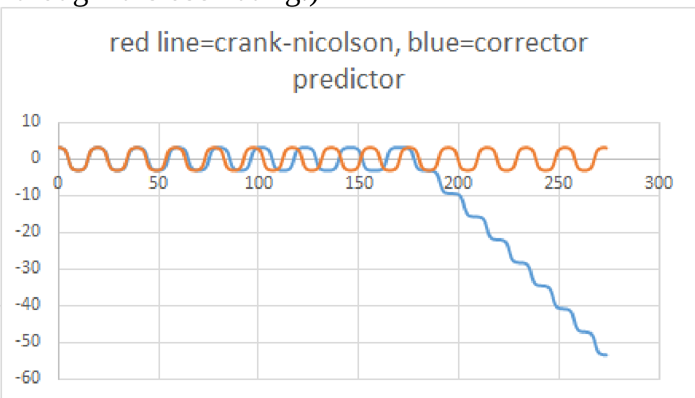
$$c = -\frac{\Delta t}{2}, d = \theta^n + \Delta t\frac{\dot{\theta}^n}{2}$$

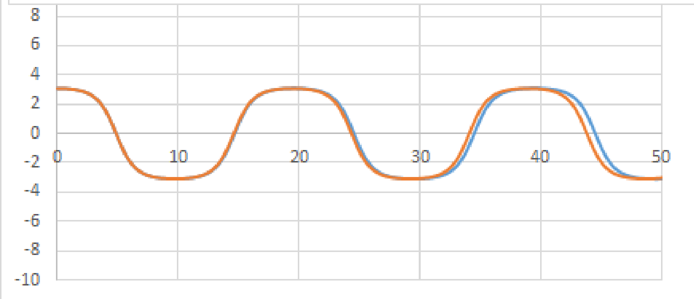
The above system of equations is solved for θ^{n+1} and $\dot{\theta}^{n+1}$, yielding

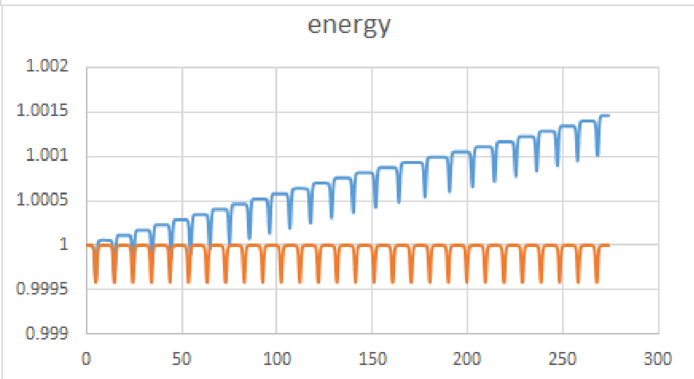
$$\theta^{n+1} = \frac{bc-d}{ac-1}, \dot{\theta}^{n+1} = \frac{ad-b}{ac-1}$$

The starting value of $\theta^*=\theta^n$ can be used, and after each iteration, we set $\theta^*=\theta^{n+1}$. Ten iterations are sufficient.

Example 1. $\theta(0) = \frac{\pi}{1.02}, \dot{\theta}(0) = 0, dt = 0.05$ (notice the flattened curves of oscillation. The corrector-predictor is gaining energy, and after some time, the pendulum swing-over takes place. Crank-nicolson has better energy conservation as the mean value is relatively constant although it is oscillating.)







Example 2. $\theta(0)=\frac{\pi}{2}, \dot{\theta}(0)=0, dt=0.05$ (good agreement)