Solving KdV equation with Fourier & Runge-Kutta methods (KdV-Fourier-RK4)

The governing equation is

$$u_t + 6uu_x + u_{xxx} = 0$$

According to Hany and Hassan (2013), the problem can be solved for u(x,0)=2sech^2(x−20) for 0≤x≤40. The analytical solution is u(x,t)=2 sech^2(x−20−4t).

If we assume zero boundary conditions at right and left boundaries or u(0,t)=u(40,t)=0, then it is possible to use the following sine Fourier series for u:

$$u(x,t) = \sum_{j=1}^{N} a_j(t) \sin jkx \quad , k = \frac{\pi}{L}$$

Alternatively, we can represent

$$u_i^n = \sum_{j=1}^{N} a_j^n \sin(jkx_i)$$

Here, i = counter for x (0≤i≤M, M=L/Δx), x_i=iΔx and n is related to the time step as t^n=(n-1)Δt.

We can analytically calculate u_x and u_{xxx}:

$$u_{x,i}^n = \sum_{j=1}^{N} a_j^n jk \cos(jkx_i), \; u_{xxx,i}^n = -\sum_{j=1}^{N} a_j^n j^3 k^3 \cos(jkx_i)$$

Also, u_t can be calculated with finite difference method:

$$u_{t,i}^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\sum_{j=1}^{N} a_j^{n+1} \sin jkx_i - \sum_{j=1}^{N} a_j^n \sin jkx_i}{\Delta t} = \sum_{j=1}^{N} \frac{a_j^{n+1} - a_j^n}{\Delta t} \sin jkx_i = \sum_{j=1}^{N} \frac{da_j^n}{dt} \sin jkx_i$$

But we have u_t=-6uu_x-u_{xxx}, as a result,

$$\sum_{j=1}^{N} \frac{da_j^n}{dt} \sin jkx_i = 6\left(\sum_{j'=1}^{N} a_{j'}^n \sin(j'kx_i)\right)\left(\sum_{j'=1}^{N} a_{j'}^n j'k \cos(j'kx_i)\right) + \sum_{j'=1}^{N} a_{j'}^n j'^3 k^3 \cos(j'kx_i) = f\left(\left\{a_{j'}^n\right\}, x_i\right)$$

Wherein j' is used to distinguish it from j index. Using the orthogonality of Fourier series:

$$\frac{da_j^n}{dt} = \int_0^L \sin(jkx)\; f\left(\left\{a_{j'}^n\right\}, x\right) dx, \quad 1 \le j \le N$$

The integral can be calculated with the trapezoidal rule:

$$\frac{da_j^n}{dt} = \sum_{i=0}^{M} \beta_i \sin(jkx_i)\; f\left(\left\{a_{j'}^n\right\}, x_i\right) \Delta x, \quad \alpha_i = \frac{1}{2} \text{ for } i = 0, M, \text{ otherwise } \beta_i = 1$$

In other words

$$\frac{da_j^n}{dt} = g\left(j,\left\{a_{j'}^n\right\}\right) \quad \text{for } 1 \le j \le N, \quad g\left(j,\left\{a_{j'}^n\right\}\right) = \sum_{i=0}^{M} \beta_i \sin\left(jkx_i\right)\ f\left(\left\{a_{j'}^n\right\}, x_i\right)\Delta x$$

This problem can be solved with RK4 (Runge-Kutta order 4) time marching method to achieve better accuracy

$$a_j^* = a_j^n, k_j^* = g\left(j,\left\{a_{j'}^*\right\}\right)$$

$$a_j^{**} = a_j^n + k_j^* \frac{\Delta t}{2}, k_j^{**} = g\left(j,\left\{a_{j'}^{**}\right\}\right)$$

$$a_j^{***} = a_j^n + k_j^{**} \frac{\Delta t}{2}, k_j^{***} = g\left(j,\left\{a_{j'}^{***}\right\}\right)$$

$$a_j^{****} = a_j^n + k_j^{***} \Delta t, k_j^{****} = f\left(j,\left\{a_{j'}^{****}\right\}\right)$$

$$a_j^{n+1} = a_j^n + \Delta t\left(\frac{k_j^*}{6} + \frac{k_j^{**}}{3} + \frac{k_j^{***}}{3} + \frac{k_j^{****}}{6}\right)$$

For the first step, Fourier coefficients are found from the initial value

$$\sum_{j=1}^{N} a_j^0 \sin jkx = u\left(x,0\right) \Rightarrow a_j^0 = \frac{2}{\pi}\int_0^L \sin\left(jkx\right) u\left(x,0\right) dx = \frac{2}{\pi}\sum_{i=0}^{M} \beta_i \sin\left(jkx_i\right) u\left(x_i, i\right)\Delta x$$

References:

Hany N. HASSAN, Hassan K. SALEH (2013), Fourier spectral methods for solving some nonlinear partial differential equations, International Journal of Open Problems in Computer Science and Mathematics, Vol. 6, No. 2. Link: http://www.i-csrs.org/Volumes/ijopcm/vol.6/6.2.14.pdf