Simple computer code for inviscid sloshing in a 2D rectangular basin with spectral-finite difference method

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Abstract

In this short paper, a small numerical code written in Julia language is presented for the classic and well-studied problem of 2D inviscid sloshing in a rectangular basin. The spectral method (Fourier series) was used for the spatial coordinate and finite difference was used in time. Comparison of results against two existing analytical formulas show relatively good agreement. The code can be used for moderately steep standing waves in preliminary calculations and as a classroom educational aid.

Keywords: Sloshing, Standing waves, Ideal Flow, Spectral method, Finite difference

Introduction

Standing waves in a stationary 2D rectangular basin for an homogeneous, inviscid and irrotational fluid is an old and well-documented problem which occurs in water and fuel tankers, lakes and reservoirs. There are many papers dealing with this topic. For example, Frandsen (2004) and Sobey (2009) gave analytical solutions to the problem. Also, numerous studies have tackled this problem using Fourier series in space (x coordinate) and finite difference in time, such as Forbes (2010), which presents a numerical method that works for both free and forced sloshing. As a result, this short paper cannot be called a new research, but rather it is a repetition of work done in the past by numerous authors. The merit of this paper is availability of the code and the possible use for basic calculations and in elementary wave hydrodynamics courses taught in the university.

A schematic of the problem is shown in Fig. 1. The variables are x and z= horizontal and vertical coordinates; u and w = horizontal and vertical velocity components; L = basin length, h = initial water height, H = wave height, η = wave elevation and ϕ = velocity potential; t = time, ρ = fluid density, P = pressure and g = 9.81 m/s^2 = gravity's acceleration.

A 2nd order analytical solution of this problem is given by Frandsen (2004), which is similar to the solution of Wu and Taylor (1994). The wave height is $H\approx\hat{a}/2$ in this formulation. This solution is given by:

$$\beta_1 = \frac{1}{8}\frac{3\omega_n^4 - g^2k_n^2}{\omega_n^4} - \frac{3}{2}\frac{\omega_n^4 - g^2k_n^2}{\omega_n^2\left(4\omega_n^2 - \omega_{2n}^2\right)}, \quad \beta_2 = \frac{1}{2}\frac{\omega_n^2\omega_{2n}^2 - \omega_n^4 - 3g^2k_n^2}{\omega_n^2\left(4\omega_n^2 - \omega_{2n}^2\right)}$$
$$\omega_n = \sqrt{gk_n \tanh k_n h}, \quad \omega_{2n} = \sqrt{g2k_n \tanh\left(2k_n h\right)}, \quad k_n = n\pi/L, \hat{\varepsilon} = \hat{a}\omega_n^2/g \tag{1}$$

A 5th order solution of the standing waves problem is given by Sobey (2009):

$$
\begin{aligned}
&\eta = b_1(t)\cos k_n x + b_2(t)\cos 2k_n x, \quad b_1(t) = \hat{a}\cos\omega_n t \\
&b_2(t) = \frac{\hat{a}^2\omega_n^2}{g}\left(\beta_0 + \beta_1\cos 2\omega_n t + \beta_2\cos\omega_{2n} t\right), \quad \beta_0 = \frac{1}{8}\frac{\omega_n^4 + g^2 k_n^2}{\omega_n^4} \\
&\phi(x,z,t) = \frac{gt}{k}\sum_{i=1}^{N}\varepsilon^i \tilde{D}_i + \left(\frac{g^{1/2}}{k^{3/2}}\right)\sum_{i=1}^{N}\varepsilon^i\sum_{j=0}^{i}\sum_{m=0}^{i}\tilde{A}_{ijm}\left[\cosh jkz + \tanh jq\sinh jkz\right]\cos jkx\sin m\omega t, \\
&\eta(x,t) = \frac{1}{k}\sum_{i=1}^{N}\varepsilon^i\sum_{j=0}^{i}\sum_{m=0}^{i}\tilde{b}_{ijm}\cos jkx\cos m\omega t, \omega = \sqrt{gk}\sum_{i=1}^{N}\varepsilon^{i-1}\tilde{C}_i, \omega = \frac{2\pi}{T} \; ,k = \frac{\pi}{L}, \varepsilon = \frac{kH}{2}
\end{aligned}
\tag{2}
$$

Wherein coefficients $\tilde{A}$, $\tilde{b}$, $\tilde{C}$ and $\tilde{D}$ are given by Sobey (2009), and T = wave period. The values of these coefficients are also available in https://github.com/DanialAmini/Standing-waves-5th-order which contains a proof of derivation as well.

Governing equations and solution method

For standing waves formed by a homogeneous, inviscid and irrotational (ideal) fluid in a 2D rectangular basin, the governing equations along with solid wall and free surface boundary conditions are the following:

$$
\begin{aligned}
&\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad 0 \le x \le L, -h \le z \le \eta; \quad \phi = \phi(x,z,t); \quad \eta = \eta(x,t); \\
&\frac{\partial\phi}{\partial x} = 0 \quad \text{at} \quad x = 0, x = L; \quad \frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z = -h; \\
&\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x}, \quad \frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + g\eta = 0 \quad \text{at} \quad z = \eta
\end{aligned}
\tag{3}
$$

The pressure is calculated from

$$
\frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + gz + \frac{P}{\rho} = 0 \tag{4}
$$

The Laplace equation is the continuity equation. The last two equations are kinematic and dynamic (Bernoulli) boundary conditions. Following the paper of Sobey (2009), the equations are rewritten in dimensionless form. Defining $k=\pi/L$, $k \doteq [1/\mathrm{L}]$, $\sqrt{gk} \doteq [1/\mathrm{T}]$ (L and T are units of length and time). As a result, the following transforms are needed

$$
\begin{aligned}
&kz = z^*, \quad k\eta = \eta^*, kx = x^*, kL = \pi, \quad kh = q, t\sqrt{gk} = t^*, \Delta t\sqrt{gk} = \Delta t^*, g^* = 1, \\
&\phi = \phi^*\frac{\sqrt{gk}}{k^2}, \frac{\partial\phi}{\partial z} = \sqrt{\frac{g}{k}}\frac{\partial\phi^*}{\partial z^*}, \quad \frac{\partial\phi}{\partial x} = \sqrt{\frac{g}{k}}\frac{\partial\phi^*}{\partial x^*}, \quad \frac{\partial\phi}{\partial t} = \frac{g}{k}\frac{\partial\phi^*}{\partial t^*} \\
&\frac{\partial\eta}{\partial x} = \frac{\partial\eta^*}{\partial x^*}, \frac{\partial\eta}{\partial z} = \frac{\partial\eta^*}{\partial z^*}, \frac{\partial\eta}{\partial t} = \sqrt{\frac{g}{k}}\frac{\partial\eta^*}{\partial t^*}
\end{aligned}
\tag{5}
$$

And the governing equations become

$$\frac{\partial^2\phi^*}{\partial x^{*2}}+\frac{\partial^2\phi^*}{\partial z^{*2}}=0 \text{ at } 0\le x^*\le\pi,-q\le z^*\le\eta^*;\phi^*=\phi^*\left(x^*,z^*,t^*\right);\eta^*=\eta^*\left(x^*,t^*\right);$$
$$\frac{\partial\phi^*}{\partial x^*}=0 \text{ at } x^*=0,\pi;\ \frac{\partial\phi^*}{\partial z^*}=0 \text{ at } z^*=-q;\frac{\partial\phi^*}{\partial z^*}=\frac{\partial\eta^*}{\partial t^*}+\frac{\partial\eta^*}{\partial x^*}\frac{\partial\phi^*}{\partial x^*} \text{ at } z^*=\eta^*; \tag{6}$$
$$\frac{\partial\phi^*}{\partial t^*}+\frac{1}{2}\left[\left(\frac{\partial\phi^*}{\partial x^*}\right)^2+\left(\frac{\partial\phi^*}{\partial z^*}\right)^2\right]+\eta^*=0 \text{ at } z^*=\hat{\eta}^*$$

The analytical solution to the Laplace equation with bottom and side boundary conditions is

$$\phi^{*n}=\sum_{j=0}^{N}a_j^n\cos jx^*\left[\cosh jz^*+\tanh jq\sinh jz^*\right],\ \ \eta^{*n}=\sum_{j=1}^{N}b_j^n\cos jx^* \tag{7}$$

Where a_j and b_j= coefficients of potential and free surface functions; j = counter for Fourier series terms; N = number of Fourier terms; n = time step, and η^* is assumed to be a cosine series. The series of η doesn't include j=0 because mean surface water is z=0. Differentiating with respect to x and z is done on the trigonometric functions to yield:

$$u^{*n}=\frac{\partial\phi^*}{\partial x^*}=-\sum_{j=0}^{N}a_j^n\sin jx^*\left[\cosh jz^*+\tanh jq\sinh jz^*\right]$$
$$w^{*n}=\frac{\partial\phi^*}{\partial z^*}=\sum_{j=0}^{N}a_j^n\cos jx^*\left[\sinh jz^*+\tanh jq\cosh jz^*\right],\eta_x^{*n}=-\sum_{j=1}^{N}b_j^n\sin jx^*, \tag{8}$$

Differentiation with respect to time is done by finite difference:

$$\eta_{t^*}^{*n}=\frac{\partial\eta^{*n}}{\partial t^*}=\sum_{j=1}^{N}\frac{b_j^{n+1}-b_j^n}{\Delta t^*}\cos jx^*,$$
$$\phi_{t^*}^{*n}=\frac{\partial\phi^{*n}}{\partial t^*}=\sum_{j=0}^{N}\frac{a_j^{n+1}-a_j^n}{\Delta t^*}\cos jx^*\left[\cosh jz^*+\tanh jq\sinh jz^*\right]=\sum_{j=0}^{N}\frac{a_j^{n+1}-a_j^n}{\Delta t^*}F_j\left(z^*\right) \tag{9}$$

Where F is the function appearing in ϕ function. In the free surface boundary conditions, the terms not containing time derivatives are grouped together for convenience

$$FK^n\left(x^*\right)=w^*\left(\left\{a^n\right\},\eta^{*n},x^*\right)-\eta_{x^*}^*\left(\left\{b^n\right\},x^*\right)u^*\left(\left\{a^n\right\},\eta^{*n},x^*\right)$$
$$FD^n\left(x^*\right)=-\frac{1}{2}\left[u^*\left(\left\{a^n\right\},\eta^{*n},x^*\right)\right]^2-\frac{1}{2}\left[w^*\left(\left\{a^n\right\},\eta^{*n},x^*\right)\right]^2-\eta^*\left(\left\{b^n\right\},x^*\right) \tag{10}$$

Wherein FK and FD are right hand side of kinematic and dynamic boundary conditions calculated at $z^*=\eta^*$; $\left\{a_j^n\right\}$ and $\left\{b_j^n\right\}$ are vectors of coefficients. This form of writing is used because the calculation of η^* and $\eta_{x^*}^*$ needs knowledge about $\left\{b_j^n\right\}$ and x^*, and calculation of ϕ, u and w at $z^*=\eta^*$ needs $\left\{a_j^n\right\}$, x^* and η^* at that point. Consequently,

$$\sum_{j=1}^{N}\frac{b_j^{n+1}-b_j^n}{\Delta t^*}\cos jx^*=FK^n\left(x^*\right),\sum_{j=0}^{N}\frac{a_j^{n+1}-a_j^n}{\Delta t^*}F_j\left(x^*,\eta^{*n}\right)=FD^n\left(x^*\right) \tag{11}$$

Notice that for the dynamic boundary condition the following expression (Eq. (12)) is wrong because time differentiation is performed before substituting $z^*=\eta^*$:

$$\sum_{j=0}^{N} \frac{a_j^{n+1} F_j\left(x^*, \eta^{*n+1}\right) - a_j^n F_j\left(x^*, \eta^{*n}\right)}{\Delta t^*} = FD^n\left(x^*\right) \tag{12}$$

Proceeding with Eq. (11), coefficients $\left\{b_j^{n+1}\right\}$ are directly found from orthogonality of Fourier series:

$$b_j^{n+1} = b_j^n + RL_j^n, RL_j^n = \frac{2\Delta t^*}{\pi} \int_0^{\pi} \cos jx^* FK^n\left(x^*\right) dx^*, 1 \le j \le N \tag{13}$$

Orthogonality cannot be used for dynamic boundary condition (even though cosh jz^* + tanh jq sinh $jz^* \approx 1$). As a result, a system of equations should be constructed. Multiplying by cosine terms:

$$\frac{2}{\pi} \sum_{j=0}^{N} \int_0^{\pi} a_j^{n+1} F_j\left(x^*, \eta^{*n}\right) \cos kx^* dx^* = \frac{2}{\pi} \int_0^{\pi} \left[\phi\left(\left\{a_j^n\right\}, x^*\right) + \Delta t^* FD^n\left(x^*\right)\right] \cos kx^* dx^* \tag{14}$$

for $0 \le k \le N$. Due to lack of orthogonality, terms with $j \ne k$ don't vanish. Also, $\phi\left(\left\{a_j^n\right\}, x^*, \eta^{*n}\right) = \sum_{j=0}^{N} a_j^{n+1} F_j\left(x^*, \eta^{*n}\right)$. Consequently:

$$\left[C_{kj}^n\right]_{(N+1)\times(N+1)} \left\{a_j^{n+1}\right\}_{(N+1)\times 1} = \left\{RD_k^n\right\}_{(N+1)\times 1} \tag{15}$$

Solving this equation gives the value of $\left\{a_j^{n+1}\right\}$ vector. It is not necessary to include k=0 in the coefficient matrix but this is only a slight added calculation. Eqs. (13) and (15) are the forward difference formula. The code uses 4th order Runge-Kutta method:

$$\begin{gathered} b_j^* = b_j^n + RL_j^n/2, \left[C_{kj}^n\right]\left\{a_j^*\right\} = \left\{RD_k^n\right\}/2; b_j^{**} = b_j^n + RL_j^*/2, \left[C_{kj}^n\right]\left\{a_j^{**}\right\} = \left\{RD_k^*\right\}/2; \\ b_j^{***} = b_j^n + RL_j^{**}/2, \left[C_{kj}^n\right]\left\{a_j^{***}\right\} = \left\{RD_k^{**}\right\}/2; b_j^{n+1} = b_j^n + RL_j^n/6 + RL_j^*/3 + RL_j^{**}/3 + RL_j^{***}/3 \\ \left[C_{kj}^n\right]\left\{a_j^{**}\right\} = \left\{RD_k^*\right\}/6 + \left\{RD_k^*\right\}/3 + \left\{RD_k^*\right\}/3 + \left\{RD_k^*\right\}/6; \end{gathered} \tag{16}$$

The integrals are evaluated with the trapezoidal rule with M+1 integration points:

$$\int_0^{\pi} f(x) dx = \frac{\pi}{M}\left[\frac{1}{2} f(0) + \frac{1}{2} f(\pi) + \sum_{i=1}^{M-1} f\left(\frac{i\pi}{M}\right)\right] \tag{17}$$

The calculations mentioned here were written as a small code in Julia language and available in https://github.com/DanialAmini/Sloshing-Finite-Difference. Julia is a free numerical programming language created by scientists in MIT (Bezanson et al. 2012).

Verification of results

In the first example, the numerical code is compared against the 5th order analytical formula of Sobey (2009), as shown in Fig. 2. Parameters were: h=100 m, L=74.614 m (transition/deep flow), T=10 s and H=10 m. For the numerical code, N=10 Fourier terms with M=100 integration points were used with a time step of dt^*=0.0064 or dt=0.01 s. The initial condition for the free surface were values of $\eta_0 = b_j \cos jx^*$ with $b_j \ne 0$ for $1 \le j \le 5$. The flow potential was zero at first (ϕ_0=0 or a_j=0). The results show good agreement between analytical and numerical results, which means that the numerical code is probably of $O[(0.5H/h)^5]$ accuracy order, similar to formulas of Sobey (2009). Other notable points are: in nonlinear waves with finite amplitude, trough is not deep but the peak is steeper compared with a regular half-cosine waveform.

In the second example, the numerical code is compared against the 2nd order analytical formula of Frandsen (2004), as shown in Fig. 3. Parameters were: h=0.5 m, L=1.5 m (shallow/transition flow), T_2=0.995 s (period of second mode) and $H_{initial}$=0.142 m. For the numerical code, N=10 Fourier terms with M=100 integration points were used as before, with a time step of dt^*=0.00907 or dt=0.002 s. The initial condition for the free surface was $\eta_0^* = 0.148\cos\left(2x^*\right)$. The flow potential was zero at first. The results show good agreement between analytical and numerical results at first, but the underpredictions of maximum and minimum of water height become apparent, and a small lag in time appears too. These deficiencies of the 2nd order model are also observed by Frandsen (2004). Consequently, the presented numerical method has an order of accuracy higher than 2.

It is interesting to mention that when starting from quiet flow (ϕ=0) and initial condition of only one cosine function (such as non-zero second mode of the second example), the wave is not periodic when wave amplitude is high, but it will stay periodic if wave amplitude is small (for $0.5H/h \leq 0.01$). The formula of Sobey (2009) is periodic in time because small values were derived for the coefficients other than the main (first) mode so that the wave stays periodic as time passes. But in the formula of Frandsen (2004), the interaction between modes does not allow the wave to stay perfectly periodic with the peaks not quite of the same height in each cycle. The wave profiles for a representative cycle is shown in Fig. 4.

Lastly, the code can be used to calculate evolution of standing waves starting from an arbitrary irregular waveform $\eta_0^*\left(x^*\right)$ by using $b_j = \int_0^{\pi} \eta_0^*\left(x\right)\cos jx^* dx^*$ to obtain initial values of b_j. The studied problems had quiet and still flow at the beginning, corresponding to initial value of a_j=0, but a similar integral can be used if the fluid is not still at first.

Conclusions

A numerical code implementation of standing inviscid waves in 2 dimensions was presented. The code is an implementation of the method explained in Forbes (2010) for the special case of free sloshing. The spatial coordinate (x) was expanded using Fourier series, and finite difference was used for time marching with Runge-Kutta method. The results were compared against two existing analytical formulas with finite wave heights, showing that the proposed method has an order of accuracy of around 5 in terms H/h (wave height to water depth ratio).

References

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Figures

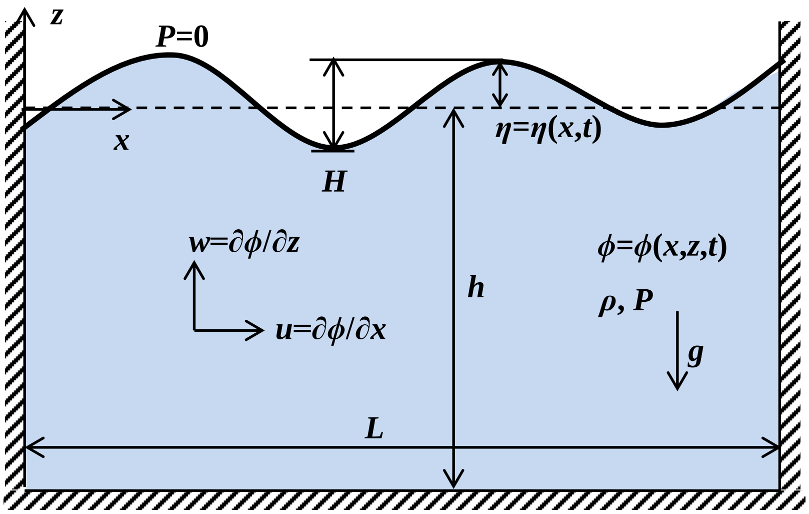

Fig. 1. Schematic of 2D rectangular basin with standing waves

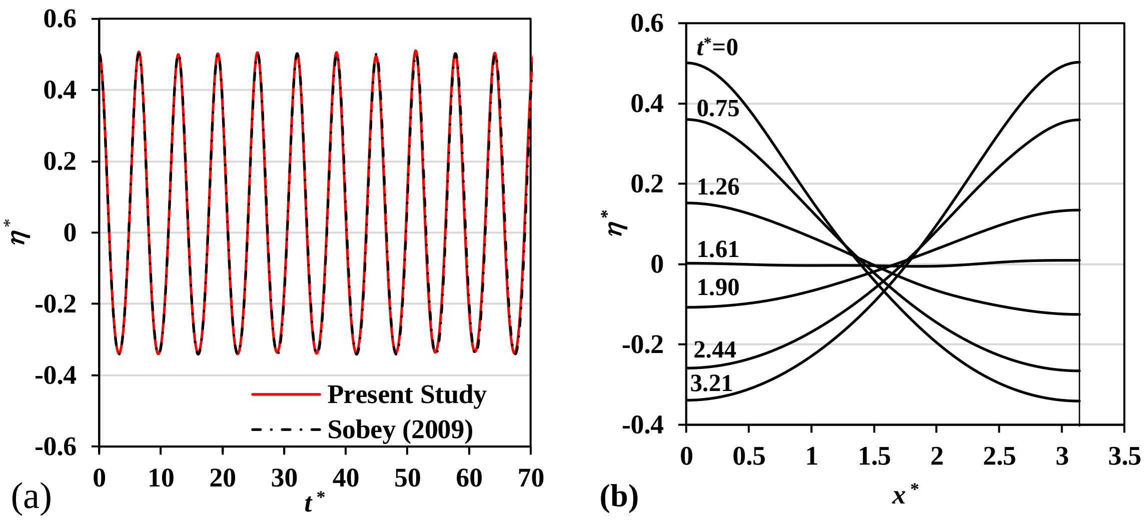

Fig. 2. First example: (a) comparison of results against analytical formula of Sobey (2009) at the left side of the free surface and (b) wave profile at different times

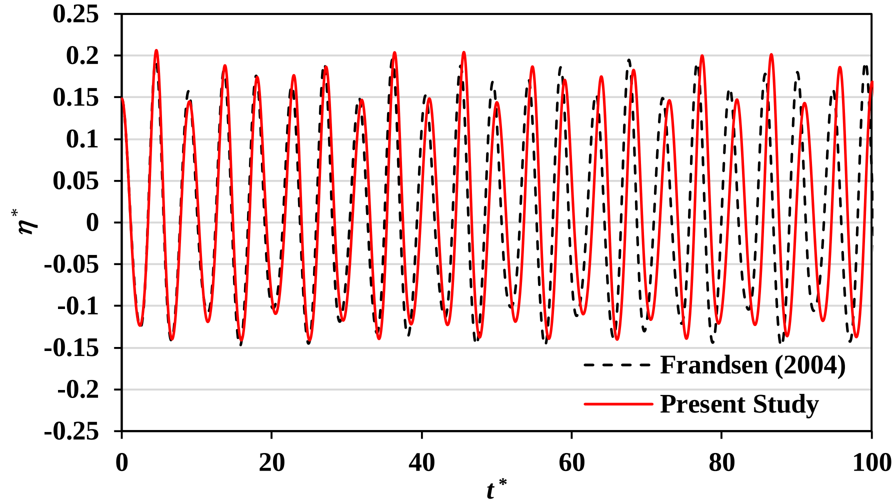

Fig. 3. Comparison of results for second sloshing mode against formula of Frandsen (2004)

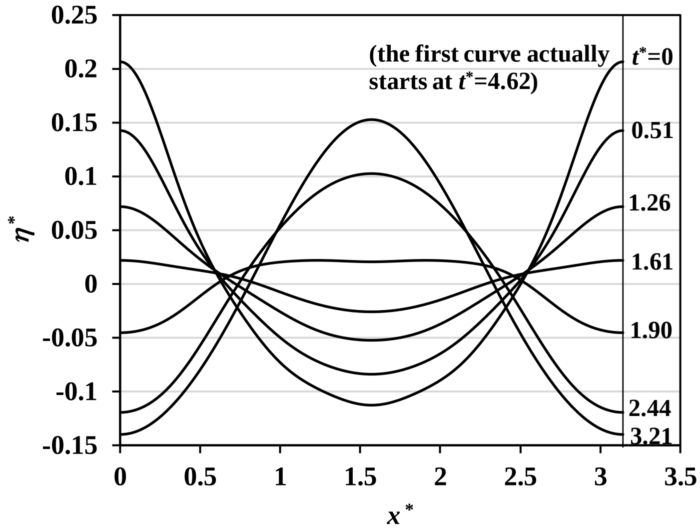

Fig. 4. Wave profiles for the second example