Benchmark problem:
Prothero-Robinson equation

$$\frac{dy}{dt}=\lambda\left(y-\phi(t)\right)+\dot{\phi}(t),\quad y_0=\phi(t_0),$$ Analytical solution: y=ϕ(t)

$$\phi(t)=\tan^{-1}2t,\ \lambda=-20,\ t_0=-1.9,\ \Delta t=3.4,\ \dot{\phi}(t)=\frac{2}{1+4t^2},\ \tan^{-1}\left(2\times(-1.9)\right)=-1.31347$$

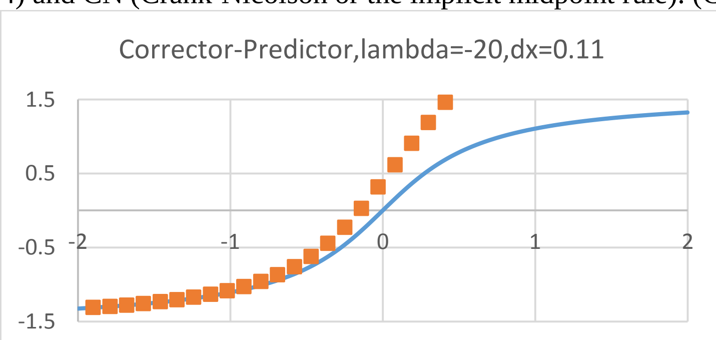
Implicit midpoint rule: $\dfrac{y^{n+1}-y^n}{\Delta t}=\dfrac{1}{2}\left[f\left(t^{n+1},y^{n+1}\right)+f\left(t^n,y^n\right)\right]$

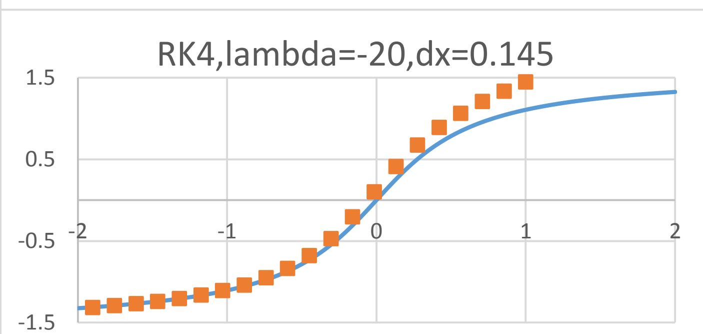
The equation is linear with respect to y, therefore the above equation can be easily simplified

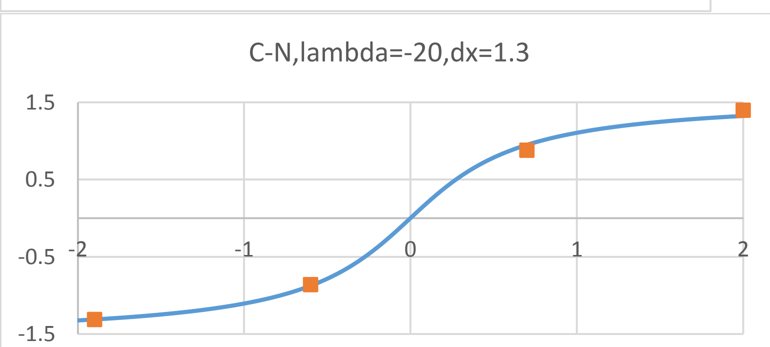
$$\frac{y^{n+1}-y^n}{\Delta t}=\frac{1}{2}\left[\lambda\left(y^{n+1}-\phi\left(t^{n+1}\right)\right)+\dot{\phi}\left(t^{n+1}\right)+f\left(t^n,y^n\right)\right]$$

$$\rightarrow y^{n+1}=\frac{\dfrac{y^n}{\Delta t}+\dfrac{1}{2}\left[-\lambda\phi\left(t^{n+1}\right)+\dot{\phi}\left(t^{n+1}\right)+f\left(t^n,y^n\right)\right]}{\left(\dfrac{1}{\Delta t}-\dfrac{\lambda}{2}\right)}$$

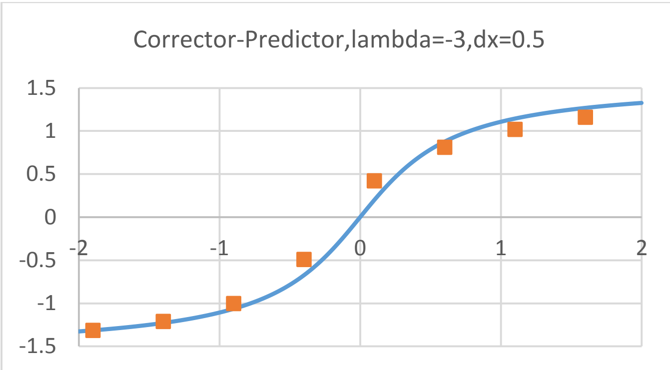
Comparison of results: three methods are considered. CP (corrector-predictor), RK4 (Runge Kutta order 4) and CN (Crank-Nicolson or the implicit midpoint rule). (Coding done with VBA)

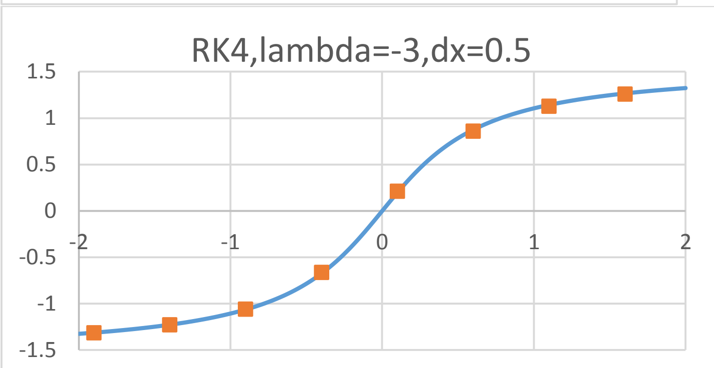


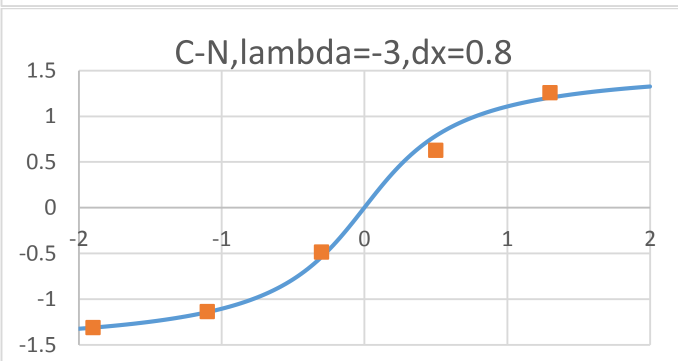




It is obvious that even RK4 has stability issues (forwards methods always have stability issues unless the time step is small). For smaller λ values, stability issues are significantly reduced. For example, $\lambda = -3$ results in better results:







Reference:

Hairer, Ernst, and Gerhard Wanner. "Radau methods." (6-page pdf is in researchgate)
Paper can be accessed from:
https://www.researchgate.net/publication/302467919_Radau_Methods