6. Vector Functions (II)

Section 6-1 :Arc Length With Vector Functions

First of all, we recall the arc length for parametric equations is given by:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\,.$$

A natural extension:

$$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \Longrightarrow \quad L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt,$$

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad \Longrightarrow \quad L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt,$$

on the interval $a \leqslant t \leqslant b$. There is a nice simplification that we can make for these:

$$L = \int_a^b \|\vec{r}\,'(t)\|\, dt\,.$$

Example 1 Determine the length of the curve

$$\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle,$$

on the interval $0 \leqslant t \leqslant 2\pi$.

Solution. We will first need the tangent vector and its magnitude:

$$\vec{r}\,'(t) = \langle 2, 6\cos(2t), -6\sin(2t)\rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{4 + 36\cos^2 t + 36\sin^2 t} = \sqrt{40} = 2\sqrt{10}\,.$$

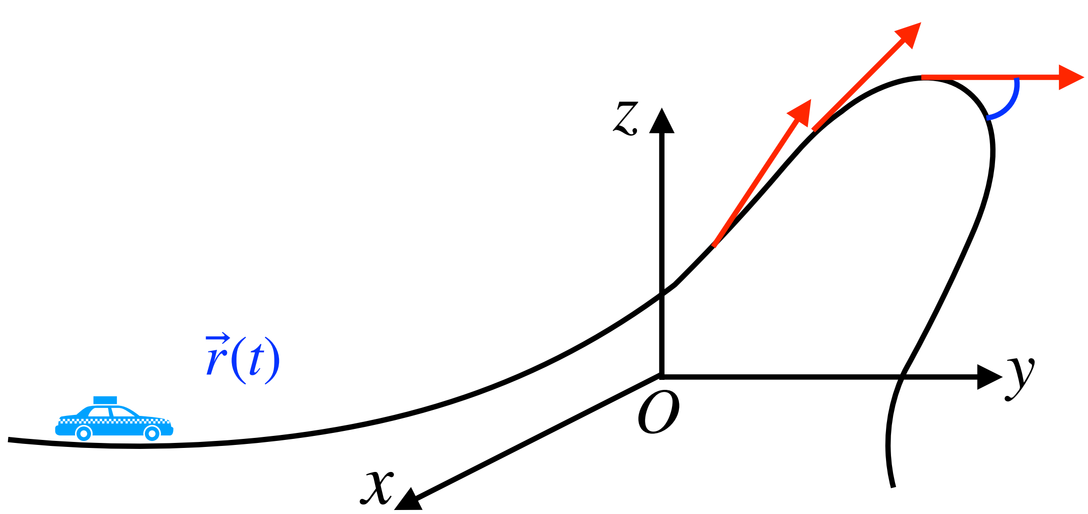
The length is then

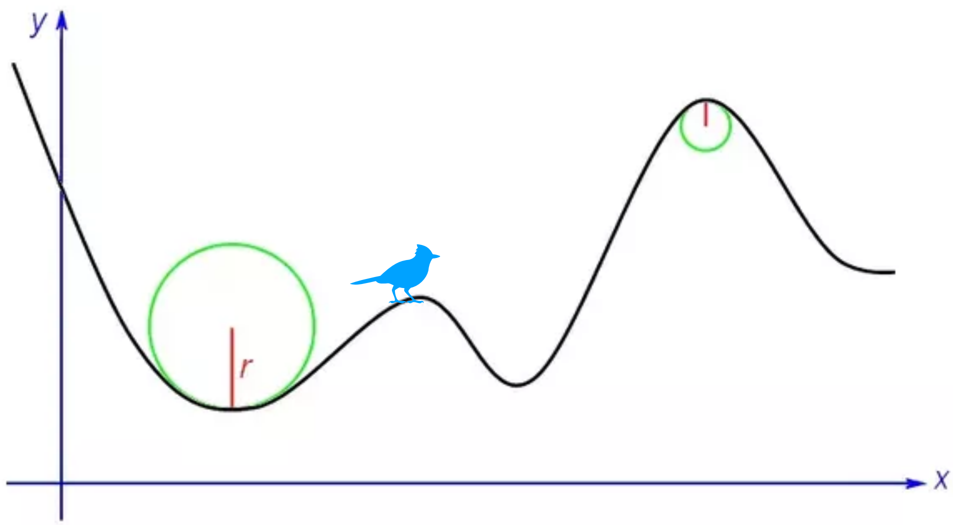
$$L = \int_0^{2\pi} \|\vec{r}\,'(t)\| dt = \int_0^{2\pi} 2\sqrt{10} dt = 4\sqrt{10}\pi\,.$$

Section 6-2 : Curvature

There are several formulas for determining the *curvature* for a smooth curve $\vec{r}(t)$:

$$\kappa = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} \qquad\qquad \kappa = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}\,.$$





Example 2 Determine the curvature for $\vec{r}(t) = \langle t, 3\sin t, 3\cos t\rangle$.

Solution. First, we computed the tangent and unit tangent vectors for this function:

$\vec{r}\,'(t) = \langle 1, 3\cos t, -3\sin t\rangle \quad \Longrightarrow \quad \|\vec{r}\,'(t)\| = \sqrt{1+9} = \sqrt{10},$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{\sqrt{10}}\langle 1, 3\cos t, -3\sin t\rangle\,.$$

The derivative of the unit tangent is,

$$\vec{T}\,'(t) = \frac{1}{\sqrt{10}}\langle 0, -3\sin t, -3\cos t\rangle,$$

and

$$\|\vec{T}\,'(t)\| = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}\,.$$

The curvature is then, $\kappa = \dfrac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \dfrac{3/\sqrt{10}}{\sqrt{10}} = \dfrac{3}{10}$.

Example 2 Determine the curvature of $\vec{r}(t) = \langle t^2, 0, t \rangle$.

Solution. We use the second form of the curvature:

$$\vec{r}'(t) = \langle 2t, 0, 1 \rangle, \qquad \vec{r}'(t) = \langle 2, 0, 0 \rangle\,.$$

Now, we obtain

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 2\vec{j}\,.$$

The magnitudes are,

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = 2, \qquad \|\vec{r}'(t)\| = \sqrt{4t^2 + 1}\,.$$

The curvature at any value of t is then,

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{2}{(4t^2 + 1)^{3/2}}\,.$$

Suppose that we have a curve given by $y = f(x)$ and we want to find its curvature. In this case we have

$$\vec{r}(x) = x\vec{i} + f(x)\,\vec{j} - 0\,\vec{k}\,.$$

If we then use the second formula for the curvature we will arrive at the following formula for the curvature:

$$\kappa = \frac{|\,f''(x)\,|}{\left(1 + [f'(x)]^2\right)^{3/2}}\,.$$

Section 6-3 : Torsion of a curve

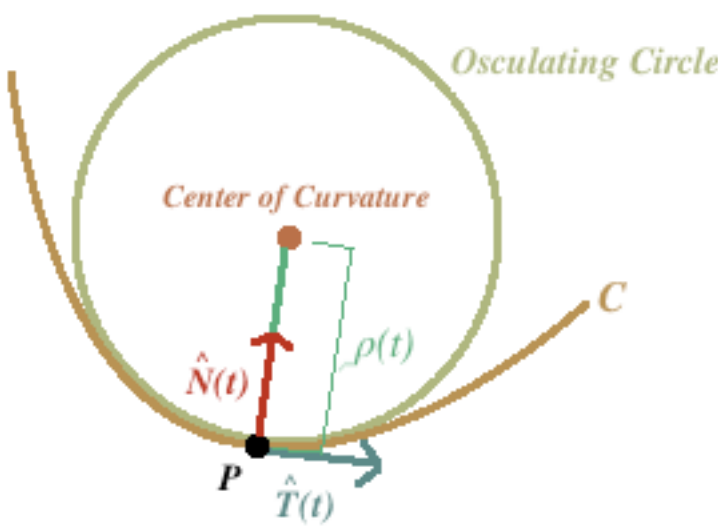
The torsion function of a smooth curve is

$$\tau = \frac{\vec{r}'(t) \cdot \left(\vec{r}''(t) \times \vec{r}'''(t)\right)}{\|\vec{r}'(t) \times \vec{r}''(t)\|^2}.$$

Section 6-4 : Osculating circle

The radius of the osculating circle : $\rho(t) = \dfrac{1}{\kappa(t)}$

The center of the osculating circle: $\vec{r}(t) + \dfrac{1}{\kappa(t)}\overrightarrow{N}(t)$



Practical Problems

For problems 1 & 2 determine the length of the vector function on the given interval.

1. $\vec{r}(t) = \langle 3 - 4t, \frac{1}{6t}, -9 - 2t \rangle$, on $[-6,8]$.
2. $\vec{r}(t) = \langle \frac{1}{3}t^3, 4t, \sqrt{2}t^2 \rangle$, on $[0,2]$.

Find the curvature for each the following vector functions.

3. $\vec{r}(t) = \langle \cos 2t, -\sin 2t, 4t \rangle$.
4. $\vec{r}(t) = \langle 4t, -t^2, 2t^3 \rangle$