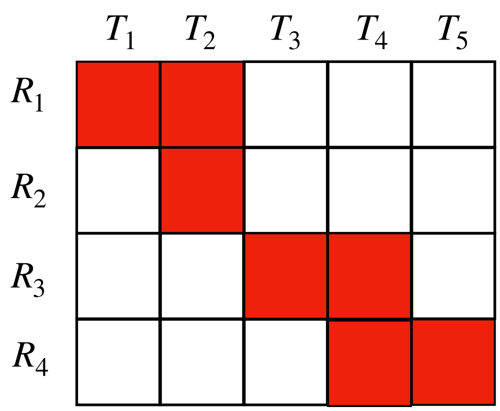
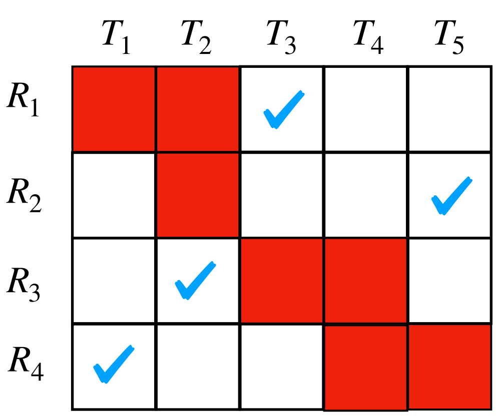
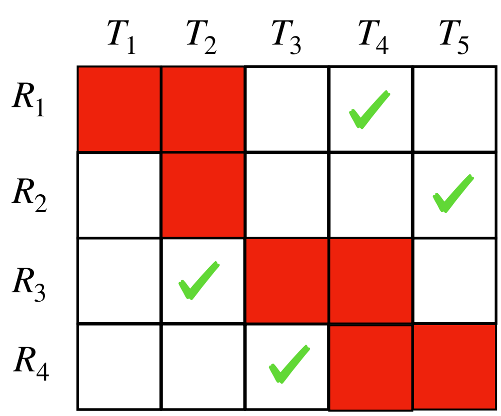
Arrangements With Forbidden Positions

Problem 1. In making seating arrangements, the shaded square of the figure means relative R_i will not sit at table T_j. Determine the number of ways that we can seat these four relatives at five different tables.

Solution. We have







- Condition c_1: R_1 is seated in a forbidden position but at different tables.
- Condition c_2: R_2 is seated in a forbidden position but at different tables.
- Condition c_3: R_3 is seated in a forbidden position but at different tables.
- Condition c_4: R_4 is seated in a forbidden position but at different tables.

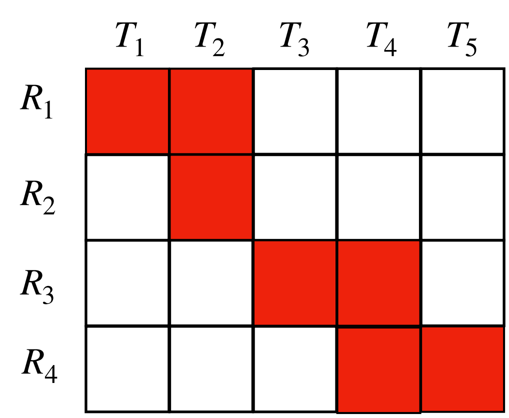
$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = \overline{N} = S_0 - S_1 + S_2 - S_3 + S_4$$

The total number of ways we can place the four relatives:

$$S_0 = |N| = 5! = 1(5!)$$

R_1	R_2	R_3	R_4
5	4	3	2

$$S_1 = N(c_1) + N(c_2) + N(c_3) + N(c_4)$$



$N(c_1) = 4! + 4!$ $(R_1 \to T_1$ or $R_1 \to T_2)$
$N(c_2) = 4!$ $(R_2 \to T_2)$
$N(c_3) = 4! + 4!$ $(R_3 \to T_3$ or $R_3 \to T_4)$
$N(c_4) = 4! + 4!$ $(R_4 \to T_4$ or $R_4 \to T_5)$

$$S_1 = (4! + 4!) + 4! + (4! + 4!) + (4! + 4!) = 7(4!)$$

$$S_2 = N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)$$

$N(c_1c_2) = 3!$ $(R_1 \to T_1$ and $R_2 \to T_2)$

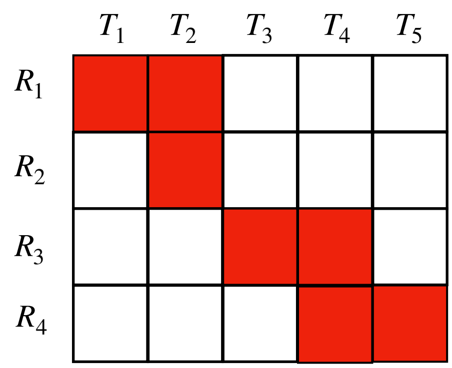
$N(c_1c_3) = 4(3!)$ $[(R_1, R_3) \to (T_1, T_3); (R_1, R_3) \to (T_2, T_3);$
$(R_1, R_3) \to (T_1, T_4); (R_1, R_3) \to (T_2, T_4)]$

$N(c_1c_4) = 4(3!)$ $[(R_1, R_4) \to (T_1, T_4); (R_1, R_4) \to (T_2, T_4);$
$(R_1, R_4) \to (T_1, T_5); (R_1, R_4) \to (T_2, T_5)]$

$N(c_2c_3) = 2(3!)$ $[(R_2, R_3) \to (T_2, T_3); (R_2, R_3) \to (T_2, T_4)]$

$N(c_2c_4) = 2(3!)$ $[(R_2, R_4) \to (T_2, T_4); (R_2, R_4) \to (T_2, T_5)]$

$N(c_3c_4) = 3(3!)$ $[(R_3, R_4) \to (T_3, T_4); (R_3, R_4) \to (T_3, T_5);$
$(R_3, R_4) \to (T_2, T_5)]$



$$S_2 = 3! + 4(3!) + 4(3!) + 2(3!) + 2(3!) + 3(3!) = 16(3!).$$

$$S_3 = N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)$$

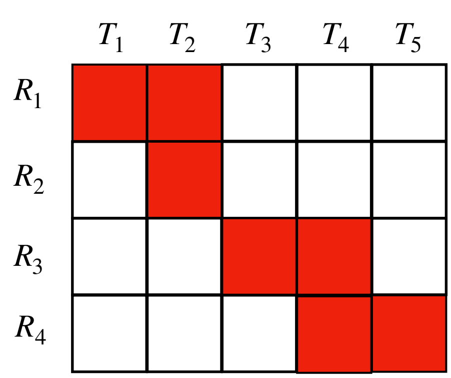
$N(c_1c_2c_3) = 2(2!)$ $[(R_1, R_2, R_3) \to (T_1, T_2, T_3); (R_1, R_2, R_3) \to (T_1, T_2, T_4)]$

$N(c_1c_2c_4) = 2(2!)$ $[(R_1, R_2, R_4) \to (T_1, T_2, T_4); (R_1, R_2, R_4) \to (T_1, T_2, T_5)]$

$N(c_1c_3c_4) = 6(2!)$ $[(R_1, R_3, R_4) \to (T_1, T_3, T_4); (R_1, R_3, R_4) \to (T_1, T_3, T_5);$
$(R_1, R_3, R_4) \to (T_1, T_4, T_5); (R_1, R_3, R_4) \to (T_2, T_3, T_4);$
$(R_1, R_3, R_4) \to (T_2, T_3, T_5); (R_1, R_3, R_4) \to (T_2, T_4, T_5)]$

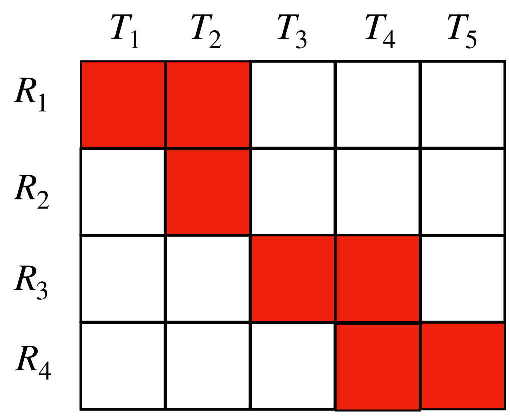
$N(c_2c_3c_4) = 3(2!)$ $[(R_2, R_3, R_4) \to (T_2, T_3, T_4); (R_2, R_3, R_4) \to (T_2, T_4, T_5)]$
$(R_2, R_3, R_4) \to (T_2, T_4, T_5)]$

$$S_3 = 2(2!) + 2(2!) + 6(2!) + 3(2!) = 13(2!)$$

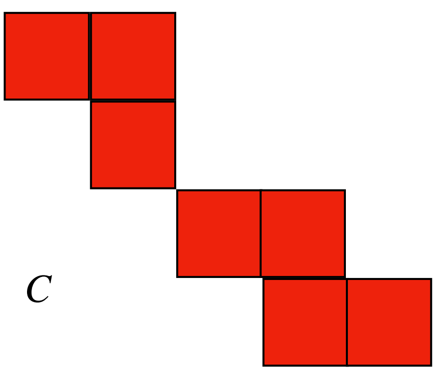


$S_4 = N(c_1c_2c_3c_4) = 3(1!)$

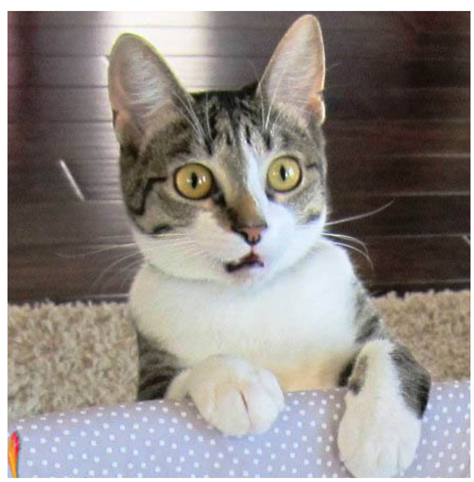
$(R_1, R_2, R_3, R_4) \to (T_1, T_2, T_3, T_4)]$
$(R_1, R_2, R_3, R_4) \to (T_1, T_2, T_3, T_5)]$
$(R_1, R_2, R_3, R_4) \to (T_1, T_2, T_4, T_4)]$



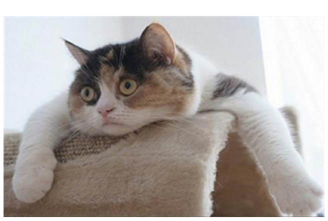
$$\begin{aligned} N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = \overline{N} &= S_0 - S_1 + S_2 - S_3 + S_4 \\ &= 1(5!) - 7(4!) + 16(3!) - 13(2!) + 3(1!) = 25 \end{aligned}$$



C



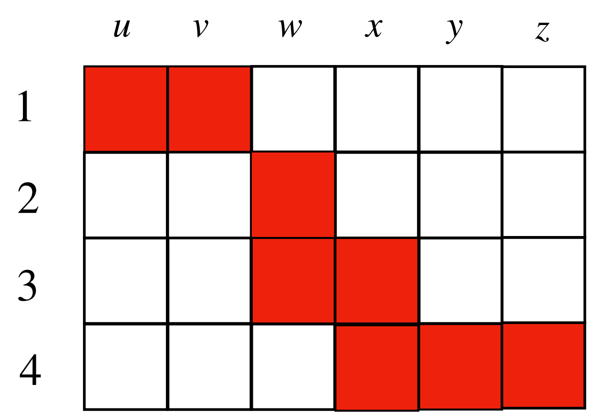
$$r(C, x) = (1 + 3x + x^2)(1 + 4x + 3x^2) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$$



Problem 2. Let $A = \{1,2,3,4\}$ and $B = \{u, v, w, x, y, z\}$. How many one-to-one functions $f : A \to B$ satisfy none of the following conditions?

$$\begin{aligned} f(1) &= u \text{ or } v, \\ f(2) &= w, \\ f(3) &= w \text{ or } x, \\ f(4) &= x, y, \text{ or } z\,. \end{aligned}$$

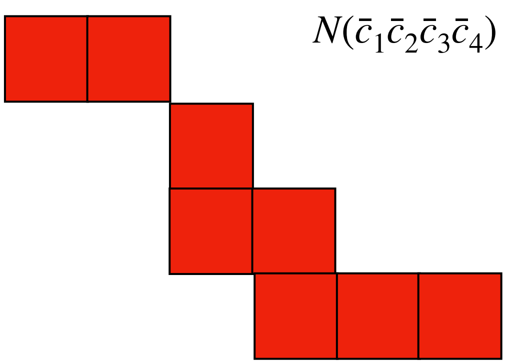
Solution. We have



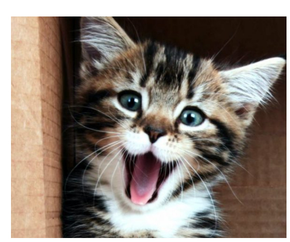
- Condition c_1: $f(1) = u \text{ or } v,$
- Condition c_2: $f(2) = w,$
- Condition c_3: $f(3) = w \text{ or } x,$
- Condition c_4: $f(4) = x, y, \text{ or } z\,.$

$$\begin{aligned} N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = \overline{N} &= S_0 - S_1 + S_2 - S_3 + S_4 \\ &= \square\frac{6!}{2!} - \square\frac{5!}{2!} + \square\frac{4!}{2!} - \square\frac{3!}{2!} + \square\frac{2!}{2!} \end{aligned}$$

$$r(C, x) = (1 + 2x)(1 + 6x + 9x^2 + 2x^3) = 1 + 8x + 21x^2 + 20x^3 + 4x^4$$

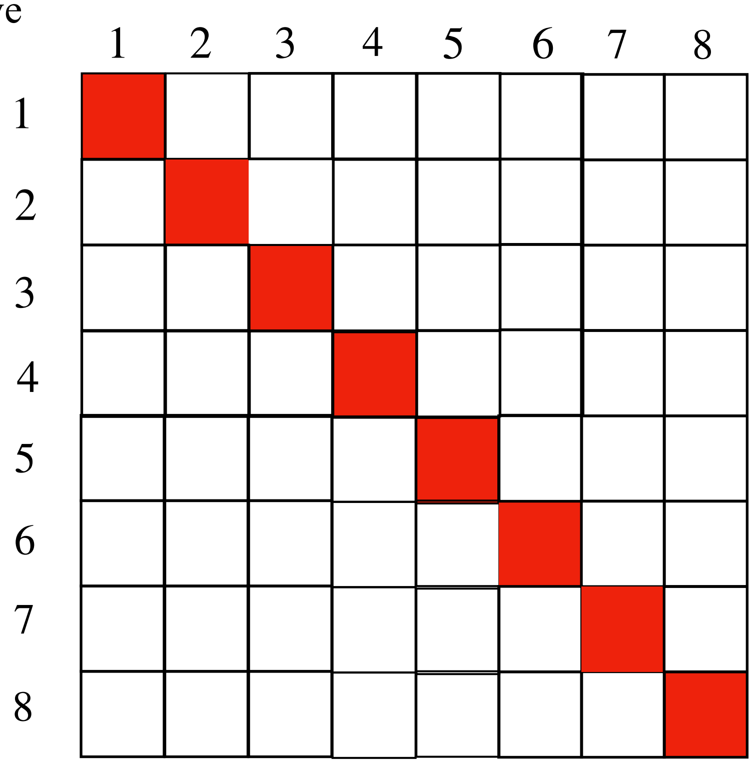


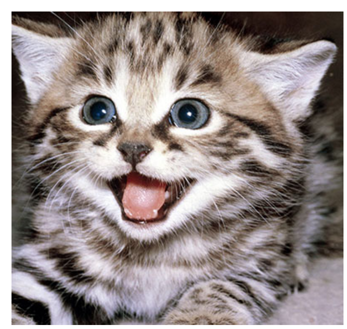
$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = \overline{N} = \frac{6!}{2!} - 8\frac{5!}{2!} + 21\frac{4!}{2!} - 20\frac{3!}{2!} + 4\frac{2!}{2!} = 76.$$



Problem 3. Let $A = \{1,2,3,\ldots,8\}$. How many one-to-one functions $f : A \to A$ satisfy $f(i) \neq i$ for all $i \in A$?

Solution. We have





- Condition c_i: $f(i) = i$, for $i = 1,2,\ldots,8$.

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\cdots\bar{c}_8) = \sum_{i=0}^{8}(-1)^i S_i = \sum_{i=0}^{8}(-1)^i r_i (8-i)!$$

$$r(C,x) = (1+x)^8 = \sum_{k=0}^{8}\binom{8}{k}x^k$$

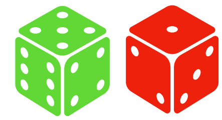
The number of such one-to-one function is:

$$\binom{8}{0}8! - \binom{8}{1}7! + \binom{8}{2}6! - \binom{8}{3}5! + \binom{8}{4}4! - \binom{8}{5}3! + \binom{8}{6}2! - \binom{8}{7}1! + \binom{8}{8}0!$$

$$= 8!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!}\right] = d_8 .$$

Problem 4. We roll two dice six times, where one is red die and the other green die. We know the following pairs did not occur:

$$(1,2), (2,1), (2,5), (3,4), (4,1), (4,5) \text{ and } (6,6).$$

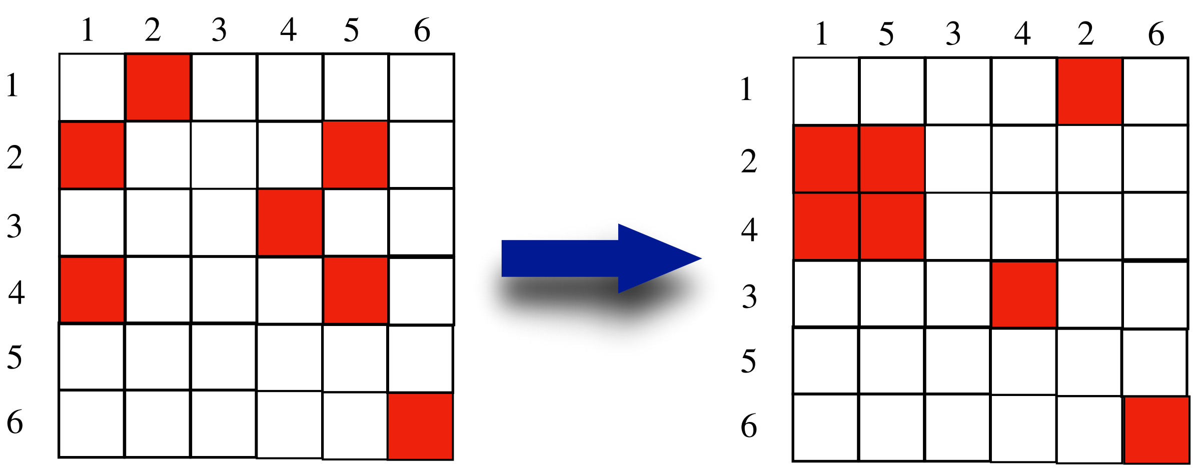


What is the probability that we obtain all six values both on red die and green die?

Solution. One of the solutions is like:

$$(1,1), (2,3), (4,4), (3,2), (5,6), (6,5).$$

In the following, chessboard C is depicted with seven shaded squares:



$$r(C,x) = (1+4x+2x^2)(1+x)^3 = 1+7x+17x^2+19x^3+10x^4+2x^5$$

- Condition c_i: all six values occur on both the red and green dies, but i on the red die is paired with one of the forbidden numbers on the green die.

$$(6!)N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4\bar{c}_5\bar{c}_6) = (6!)\sum_{i=0}^{6}(-1)^i S_i = (6!)\sum_{i=0}^{6}(-1)^i r_i (6-i)!$$

$$= 6![6! - 7(5!) + 17(4!) - 19(3!) + 10(2!) - 2(1!) + 0(0!)] = 6![192] = 138{,}240.$$

The probability of this even is:

$$\frac{138{,}240}{(29)^6} = 0.00023.$$