1. Fibonacci Sequence:

Fibonacci numbers

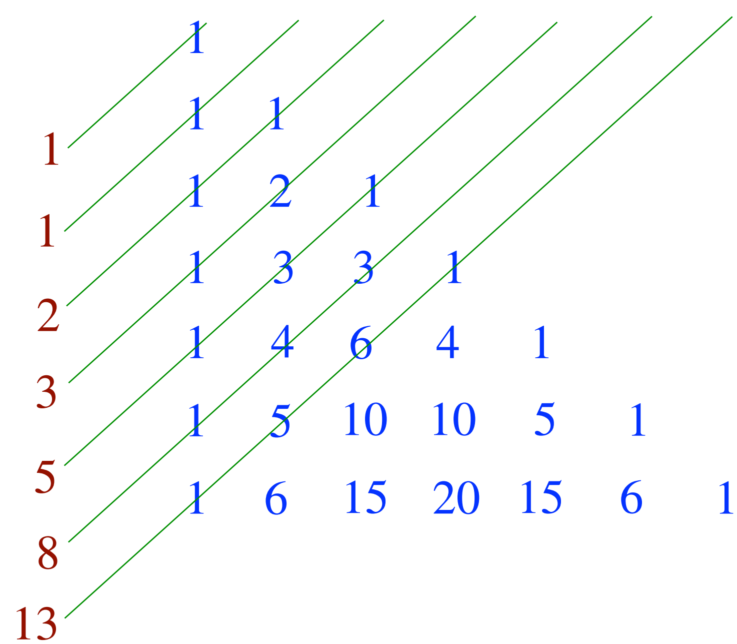
$\{F_n\}_{n=0}^{\infty} \longrightarrow F_0,\ F_1,\ F_2,\ F_3,\ F_4,\ F_5,\ \ldots,\ F_n,\ \cdots$ **A000045**

$$\begin{cases} F_0 = 0, \quad F_1 = 1 \\ F_n = F_{n-1} + F_{n-2}, \quad n \geqslant 2 \end{cases}$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, …

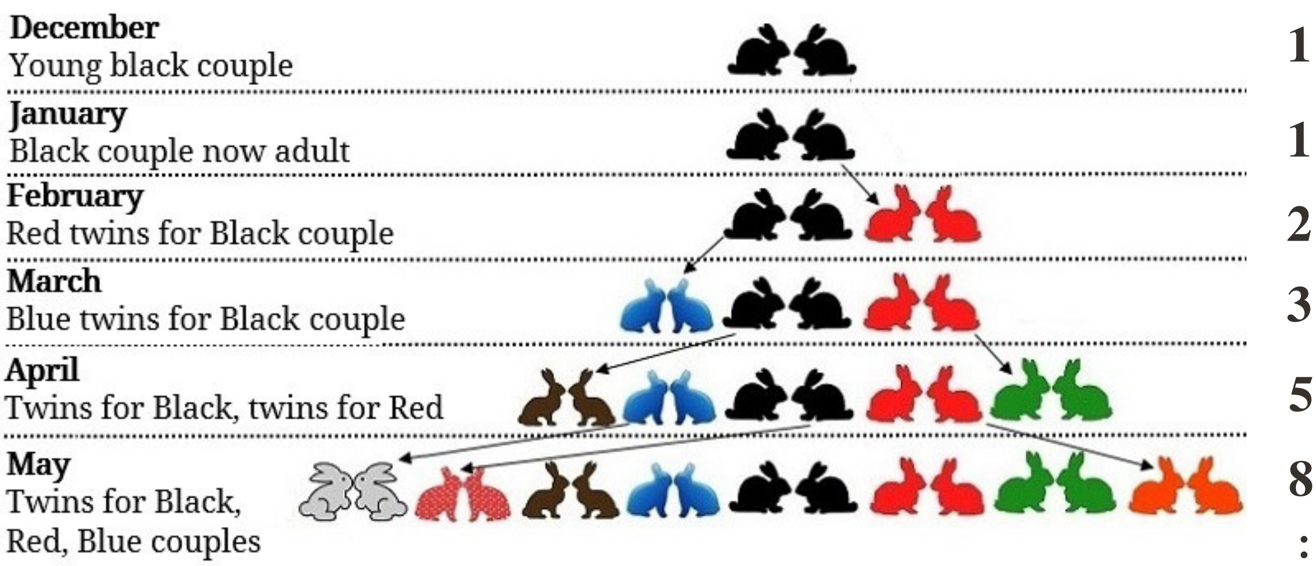
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

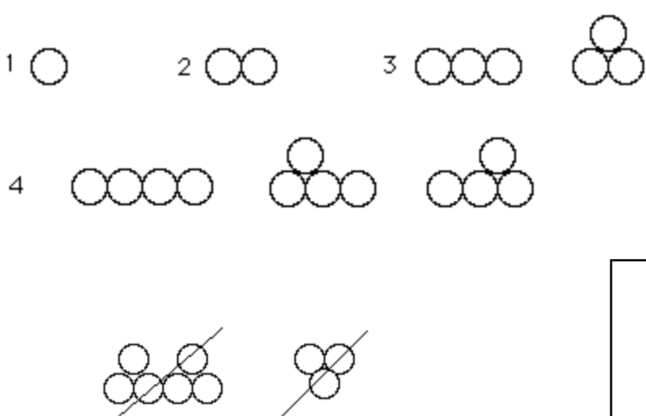
(1) The Fibonacci numbers occur in the sums of diagonals in *Pascal's triangle*:

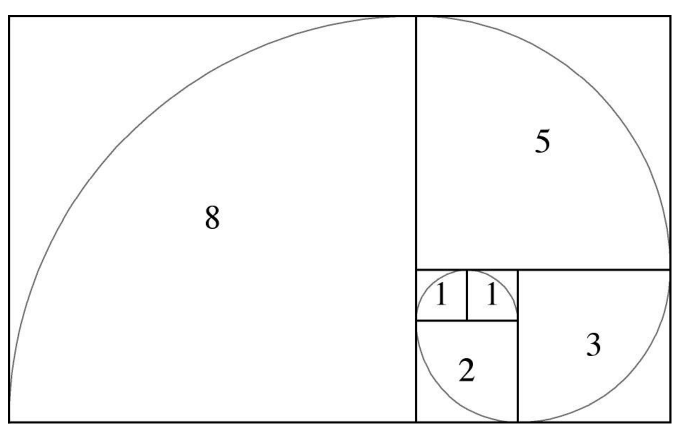


(2) Fibonacci discovered the sequence by posing the following question:

If a pair of rabbits is placed in an enclosed area, how many rabbits will be born there if we assume that every month a pair of rabbits produces another pair and that rabbits begin to bear young two months after their birth?







Sum of the Fibonacci Numbers

1. *The sum of the first n Fibonacci numbers can be expressed as*

$$F_1 + F_2 + \cdots + F_{n-1} + F_n = F_{n+2} - 1.$$

Solution. From the definition of the Fibonacci sequence, we know

$$\begin{aligned} F_1 &= F_3 - F_2, \\ F_2 &= F_4 - F_3, \\ F_3 &= F_5 - F_4, \\ &\vdots \\ F_{n-1} &= F_{n+1} - F_n, \\ F_n &= F_{n+2} - F_{n+1}, \end{aligned}$$

We now add these equations to find

$$F_1 + F_2 + \cdots + F_{n-1} + F_n = F_{n+2} - F_2\,.$$

Recalling that $F_2 = 1$, we see this equation is the result. □

Sum of Odd Terms

2. *The sum of the odd terms of the Fibonacci sequence*

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}\,.$$

Solution. Again looking at individual terms, we see from the definition of the sequence that

$$\begin{aligned} F_1 &= F_2 - F_0, \\ F_3 &= F_4 - F_2, \\ F_5 &= F_6 - F_4, \\ &\vdots \\ F_{2n-1} &= F_{2n} - F_{2n-2}, \end{aligned}$$

If we now add these equations term by term, we get

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n} - F_0 = F_{2n} . \square$$

Sum of Even Terms

3. *The sum of the even terms of the Fibonacci sequence*

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1.$$

Solution. From Problem **1**, we have

$$F_1 + F_2 + \cdots + F_{2n-1} + F_{2n} = F_{2n+2} - 1.$$

Subtracting our equation for the *sum of odd terms*, we obtain

$$\begin{aligned} F_2 + F_4 + F_6 + \cdots + F_{2n} &= F_{2n+2} - 1 - (F_1 + F_3 + F_5 + \cdots + F_{2n-1}) . \\ &= F_{2n+2} - 1 - F_{2n} \\ &= F_{2n+1} - 1, \end{aligned}$$

as we desired. $\square$

Sum of Fibonacci Numbers with Alternating Signs

4. *The sum of the Fibonacci numbers with alternating signs*

$$F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{n+1}F_n = (-1)^{n+1}F_{n-1} + 1.$$

Solution. Building further from our progress with sums, we can subtract our even sum equation from our odd sum equation to find

$$F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n-1} - F_{2n} = F_{2n} - F_{2n+1} + 1 = -F_{2n-1} + 1.$$

Now, adding F_{2n+1} to both sides of this equation, we obtain

$$F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n-1} - F_{2n} + F_{2n+1} = F_{2n+1} - F_{2n-1} + 1 = F_{2n} + 1.$$

Combining above equations, we arrive at the sum of Fibonacci numbers with alternating signs:

$$F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{n+1}F_n = (-1)^{n+1}F_{n-1} + 1. \ \square$$

Sum of Squares

5. *The sum of the squares of the first n Fibonacci numbers*

$$F_1^2 + F_2^2 + \cdots + F_{n-1}^2 + F_n^2 = F_nF_{n+1}.$$

Solution. Note that

$$F_k^2 = F_kF_k = F_k(F_{k+1} - F_{k-1}) = F_kF_{k+1} - F_{k-1}F_k.$$

If we add the equations

$$\begin{aligned} F_1^2 &= F_1F_2 - F_0F_1, \\ F_2^2 &= F_2F_3 - F_1F_2, \\ F_3^2 &= F_3F_4 - F_2F_3, \\ &\vdots \\ F_n^2 &= F_nF_{n+1} - F_{n-1}F_n, \end{aligned}$$

term by term, we arrive

$$F_1^2 + F_2^2 + \cdots + F_{n-1}^2 + F_n^2 = F_nF_{n+1} - F_0F_1 = F_nF_{n+1}. \ \square$$

Another Important Formula

6. *We have*

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}\,.$$

Solution. We will now begin this proof by induction on m.

For $m = 1$: $F_{n-1}F_1 + F_nF_2 = F_{n-1} + F_n = F_{n+1}$,

For $m = 2$: $F_{n-1}F_2 + F_nF_3 = F_{n-1} + 2F_n = F_{n-1} + F_n + F_n = F_{n+1} + F_n = F_{n+2}\,.$

Thus, we have now proved the basis of our induction. Now suppose our formula to be true for $m = 1,2,\ldots,k+1$. We shall prove that it also holds for $m = k+2$. So, by induction, assume

$$F_{n+k} = F_{n-1}F_k + F_nF_{k+1},$$

and

$$F_{n+k+1} = F_{n-1}F_{k+1} + F_nF_{k+2}\,.$$

If we add these two equations term by term, we obtain

$$F_{n+k+2} = F_{n+k} + F_{n+k+1} = F_{n-1}(F_k + F_{k+1}) + F_n(F_{k+1} + F_{k+2}) = F_{n-1}F_{k+2} + F_nF_{k+3}$$

which was the required result. So, by induction we have proven our initial formula holds true for $m = k+2$, and thus for all values of m. $\square$

Difference of Squares of Fibonacci Numbers

7. *We have*

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2\,.$$

Solution. Continuing from the previous formula (Problem **6**) in this problem, let $m = n$. We obtain

$$F_{2n} = F_{n-1}F_n + F_nF_{n+1} = F_n(F_{n-1} + F_{n+1})\,.$$

Since

$$F_n = F_{n+1} - F_{n-1},$$

we can now rewrite the formula as follows:

$$F_{2n} = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1})$$

or

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2\,. \ \square$$

The Golden Ratio.

8. *We have*

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}\,.$$

Solution. Since $F_{n+1} = F_n + F_{n-1}$, by definition, it follows that

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n},$$

Now, let

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = L\,.$$

We then see that

$$\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \lim_{n\to\infty} \frac{1}{\frac{F_n}{F_{n-1}}} = \frac{1}{L}\,.$$

We now have the statement

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n},$$

which is equivalent to the the equation

$$L = 1 + \frac{1}{L} \quad \Longrightarrow \quad L^2 - L - 1 = 0 \quad \Longrightarrow \quad L = \frac{1 \pm \sqrt{5}}{2}\,.$$

Thus, we arrive at our desired result of

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}\,. \;\square$$