1. Partitions of integers.

In Number Theory, the **partition function** $p(n)$ represents the number of possible partitions of a non-negative integer n.

For instance, $p(4) = 5$, because the integer 4 has the five partitions:

$$1+1+1+1,\ 1+1+2,\ 1+3,\ 2+2,\ \text{and}\ 4.$$

By convention $p(0) = 1$, as there is one way (the empty sum) of representing zero as a sum of positive integers.

The first few values of the partition function, starting with $p(0) = 1$, are:

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, …

(sequence A000041 in the OEIS).

On-Line Encyclopedia of Integer Sequences

Without regard to order

$$\begin{aligned}
p(1) = 1 &: \quad 1\\
p(2) = 2 &: \quad 2 = 1+1\\
p(3) = 3 &: \quad 3 = 2+1 = 1+1+1\\
p(4) = 5 &: \quad 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1\\
p(5) = 7 &: \quad 5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1
\end{aligned}$$

$$p(100) = 190{,}569{,}292$$
$$p(1000) = 24{,}061{,}467{,}864{,}032{,}622{,}473{,}692{,}149{,}727{,}991 \approx 2.40615 \times 10^{31}$$
$$p(10000) = 36{,}167{,}251{,}325{,}\ldots{,}906{,}916{,}435{,}144 \approx 3.61673 \times 10^{106}$$

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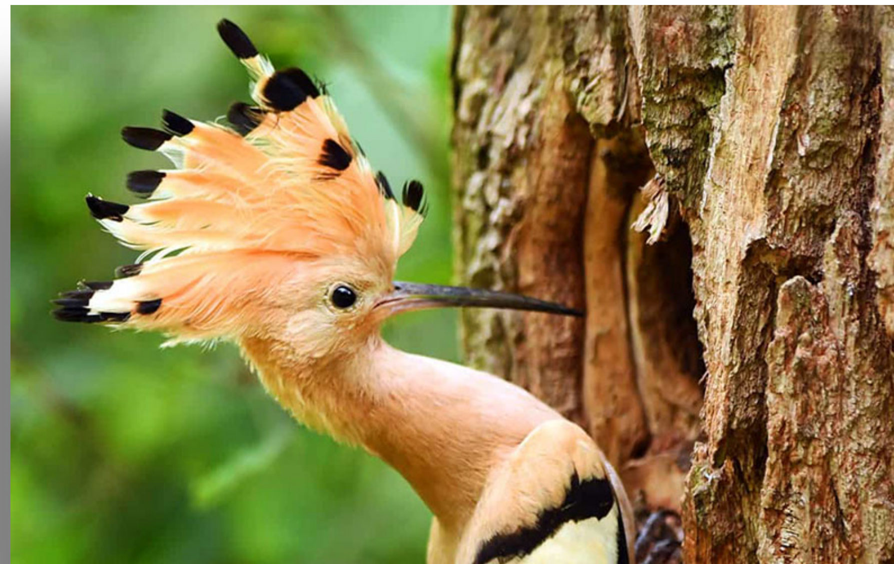

1 $\longrightarrow$ $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10}$

2 $\longrightarrow$ $1 + x^2 + x^4 + x^6 + x^8 + x^{10}$

3 $\longrightarrow$ $1 + x^3 + x^6 + x^9$

4 $\longrightarrow$ $1 + x^4 + x^8$

5 $\longrightarrow$ $1 + x^5 + x^{10}$

6 $\longrightarrow$ $1 + x^6$

7 $\longrightarrow$ $1 + x^7$

8 $\longrightarrow$ $1 + x^8$

9 $\longrightarrow$ $1 + x^9$

10 $\longrightarrow$ $1 + x^{10}$

$$f(x) = (1 + x + \cdots + x^{10})(1 + x^3 + x^6 + x^9)(1 + x^4 + x^8)(1 + x^5 + x^{10})(1 + x^6)$$
$$(1 + x^7)(1 + x^8)(1 + x^9)(1 + x^{10})$$

$$f(x) = \cdots + \boxed{42}\, x^{10} + \cdots$$

$$f(x) = p(0) + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + p(5)x^5 + p(6)x^6$$
$$+ p(7)x^7 + p(8)x^8 + p(9)x^9 + p(10)x^{10} + a x^{11} + b x^{12} + c x^{13} + \cdots$$

1 $\quad 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + \cdots$

2 $\quad 1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12} + x^{14} + x^{16} + x^{18} + \cdots$

3 $\quad 1 + x^3 + x^6 + x^9 + x^{12} + x^{15} + x^{18} + x^{21} + x^{24} + x^{27} + \cdots$

4 $\quad 1 + x^4 + x^8 + x^{12} + x^{16} + x^{20} + x^{24} + x^{28} + x^{32} + x^{36} + \cdots$

5 $\quad 1 + x^5 + x^{10} + x^{15} + x^{20} + x^{25} + x^{30} + x^{35} + x^{40} + x^{45} + \cdots$

6 $\quad 1 + x^6 + x^{12} + x^{18} + x^{24} + x^{30} + x^{36} + x^{42} + x^{48} + x^{54} + \cdots$

7 $\quad 1 + x^7 + x^{14} + x^{21} + x^{28} + x^{35} + x^{42} + x^{49} + x^{56} + x^{63} + \cdots$

8 $\quad 1 + x^8 + x^{16} + x^{24} + x^{32} + x^{40} + x^{48} + x^{54} + x^{64} + x^{72} + \cdots$

9 $\quad 1 + x^9 + x^{18} + x^{27} + x^{36} + x^{45} + x^{54} + x^{63} + x^{72} + x^{81} + \cdots$

10 $\quad 1 + x^{10} + x^{20} + x^{30} + x^{40} + x^{50} + x^{60} + x^{70} + x^{80} + x^{90} + \cdots$

1 → $\dfrac{1}{1-x}$

2 → $\dfrac{1}{1-x^2}$

3 → $\dfrac{1}{1-x^3}$

4 → $\dfrac{1}{1-x^4}$

5 → $\dfrac{1}{1-x^5}$

6 → $\dfrac{1}{1-x^6}$

7 → $\dfrac{1}{1-x^7}$

8 → $\dfrac{1}{1-x^8}$

9 → $\dfrac{1}{1-x^9}$

10 → $\dfrac{1}{1-x^{10}}$

$$f(x) = \prod_{i=1}^{10} \frac{1}{1-x^i} = p(0) + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + p(5)x^5 + p(6)x^6$$
$$+ p(7)x^7 + p(8)x^8 + p(9)x^9 + p(10)x^{10} + a\,x^{11} + bx^{12} + cx^{13} + \cdots$$

$$\boxed{f(x) = \prod_{i=1}^{r} \frac{1}{1-x^i} = p(0) + p(1)x + p(2)x^2 + \cdots + p(r)x^r + a\,x^{r+1} + bx^{r+2} + \cdots}$$

Example 1. Find the generating function for $p_d(n)$, the number of partitions of positive integer n into distinct summands.

Solution. Before we start, let us consider the 11 partition of 6 :

1) $1+1+1+1+1+1$ **2)** $1+1+1+1+2$
3) $1+1+1+3$ **4)** $1+1+4$
5) $1+1+2+2$ **6)** $1+5$
7) $1+2+3$ **8)** $2+2+2$
9) $2+4$ **10)** $3+3$
11) 6

$\Rightarrow$ $p_d(6) = 4.$

$1 \Rightarrow 1+x$

$2 \Rightarrow 1+x^2$

$3 \Rightarrow 1+x^3$

$4 \Rightarrow 1+x^4$

$5 \Rightarrow 1+x^5$

$\vdots \quad \vdots \quad \vdots$

$n \Rightarrow 1+x^n$

$$p_d(x) = (1+x)(1+x^2)\cdots(1+x^n)\cdots = \prod_{i=1}^{n}(1+x^i)$$
$$= \cdots + p_d(n)x^n + \cdots$$

Example 2. Find the generating function for $p_o(n)$, the number of partitions of positive integer n into odd summands.

Solution. Again, let us consider the 11 partition of 6 :

1) $1+1+1+1+1+1$ **2)** $1+1+1+1+2$
3) $1+1+1+3$ **4)** $1+1+4$
5) $1+1+2+2$ **6)** $1+5$
7) $1+2+3$ **8)** $2+2+2$
9) $2+4$ **10)** $3+3$
11) 6

$\Rightarrow$ $p_o(6)=4.$

$1 \Rightarrow 1+x+x^2+x^3+x^4+x^5+\cdots$

$3 \Rightarrow 1+x^3+x^6+x^9+x^{12}+x^{15}+\cdots$

$5 \Rightarrow 1+x^5+x^{10}+x^{15}+x^{20}+x^{25}+\cdots$

$\vdots \quad \vdots \quad \vdots$

$2i+1 \Rightarrow 1+x^{2i+1}+x^{4i+2}+x^{6i+3}+\cdots$

$$p_o(x)=(1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots$$
$$=\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^7}\right)\cdots$$
$$=\cdots+\ p_o(n)x^n+\ \cdots$$

Note that

$$1+x=\frac{1-x^2}{1-x}, \quad 1+x^2=\frac{1-x^4}{1-x^2}, \quad 1+x^3=\frac{1-x^6}{1-x^3}, \quad \cdots$$

Now, we have

$$\begin{aligned} p_d(x) &= (1+x)(1+x^2)\cdots(1+x^n)\cdots \\ &= \left(\frac{\cancel{1-x^2}}{1-x}\right)\left(\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\right)\left(\frac{\cancel{1-x^6}}{1-x^3}\right)\left(\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\right)\cdots \\ &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^7}\right)\cdots = p_o(x) \end{aligned}$$

$$p_d(x)=p_o(x) \quad \Longrightarrow \quad p_d(n)=p_o(n)$$

2. The Exponential Generating Function

Let $a_0, a_1, a_2, \ldots, a_n, \ldots,$ be a sequence of real numbers. The function

$$f(x) = a_0 + a_1\frac{x^1}{1!} + a_2\frac{x^2}{2!} + \cdots + a_n\frac{x^n}{n!} + \cdots = \sum_{i=0}^{\infty} a_i\frac{x^i}{i!}$$

is called the *exponential generating function* for the given sequence.

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

$$f(x) \qquad a_0 \qquad a_1 \qquad a_2 \qquad a_n$$

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n}, 0, 0, 0, \ldots$$

$$(1+x)^n = \binom{n}{0} + 1!\binom{n}{1}\frac{x^1}{1!} + 2!\binom{n}{2}\frac{x^2}{2!} + \cdots + n!\binom{n}{n}\frac{x^n}{n!}$$

$$= P(n,0) + P(n,1)\frac{x^1}{1!} + P(n,2)\frac{x^2}{2!} + \cdots + P(n,n)\frac{x^n}{n!}.$$

$$(1+x)^n = P(n,0) + P(n,1)\frac{x^1}{1!} + P(n,2)\frac{x^2}{2!} + \cdots + P(n,n)\frac{x^n}{n!}.$$

Example 3. Examining the Maclurian series expansion for e^x, we find

$$\boxed{e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}}$$

So e^x is the exponential generating function for the sequence:

$1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots,$

Example 4. In how many ways can four of the letters in **ENGINE** be arranged?

Solution. In the following we list the possible sections of size 4 from the letters **E, N, G, I, N, E** along with the number of arrangements those four letters determine.

E	**E**	**N**	**N**	$\frac{4!}{2!2!} = 6$
E	**E**	**G**	**N**	$\frac{4!}{2!} = 12$
E	**E**	**I**	**N**	$\frac{4!}{2!} = 12$
E	**E**	**G**	**I**	$\frac{4!}{2!} = 12$
E	**G**	**N**	**N**	$\frac{4!}{2!} = 12$
E	**I**	**N**	**N**	$\frac{4!}{2!} = 12$
G	**I**	**N**	**N**	$\frac{4!}{2!} = 12$
E	**I**	**G**	**N**	$4! = 24$

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E $\quad 1 + x + \dfrac{x^2}{2!}$

N $\quad 1 + x + \dfrac{x^2}{2!}$

G $\quad 1 + x$

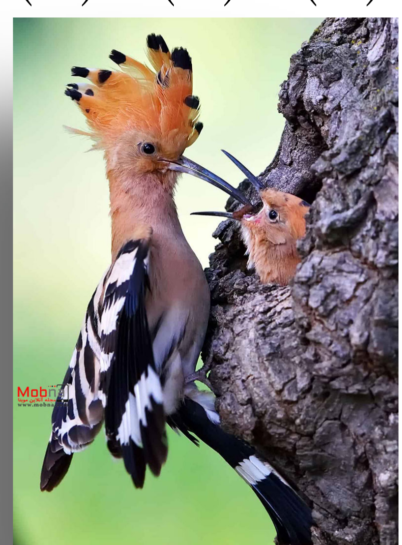
I $\quad 1 + x$

$$f(x) = \left(1 + x + \frac{x^2}{2!}\right)^2 (1+x)^2 = \cdots + \boxed{?}\ \frac{x^4}{4!} + \cdots$$

$$f(x) = \cdots + \left[\left(\frac{x^4}{2!2!}\right) + \left(\frac{x^4}{2!}\right) + \left(\frac{x^4}{2!}\right) + \left(\frac{x^4}{2!}\right) + \left(\frac{x^4}{2!}\right) + \left(\frac{x^4}{2!}\right) + \left(\frac{x^4}{2!}\right) + x^4\right] + \cdots$$

$$= \cdots + \left[\left(\frac{4!}{2!2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + 4!\right] \frac{x^4}{4!} + \cdots$$

$$= \cdots + \boxed{102}\ \frac{x^4}{4!} + \cdots \quad \square$$



3. The Summation Operator

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots = \sum_{i=0}^{\infty} a_ix^i$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

$$\Longrightarrow \quad g(x) = \frac{1}{1-x} \cdot f(x)$$

$$g(x) = \frac{f(x)}{1-x} = (1 + x + x^2 + \cdots + x^n + \cdots)(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots + (a_0 + a_1 + a_2 + \cdots + a_n)x^n + \cdots$$

The coefficient x^n in $g(x) = a_0 + a_1 + a_2 + \cdots + a_n$.

Example 5. Prove that $0 + 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

Solution. We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

$$\Longrightarrow \quad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots,$$

$\times\, x$

$$\Rightarrow \quad f(x) = \frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots,$$

$$\Rightarrow \quad g(x) = \frac{f(x)}{1-x} = \frac{1}{1-x} \cdot \frac{x}{(1-x)^2} = \frac{x}{(1-x)^3}$$

$$\frac{x}{(1-x)^3} = x(1-x)^{-3} = x\sum_{r=0}^{\infty}\binom{3+r-1}{r}x^r = x\sum_{r=0}^{\infty}\binom{r+2}{r}x^r$$

$$\Rightarrow \quad \binom{n-1+2}{n-1} = \binom{n+1}{n-1} = \binom{n+1}{2} = \frac{(n+1)n}{2}.$$

$$\Rightarrow \quad \boxed{0 + 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}.}$$