Conservative Vector Fields

When is a vector field $\overrightarrow{\mathbf{F}}$ conservative?

Definitions.

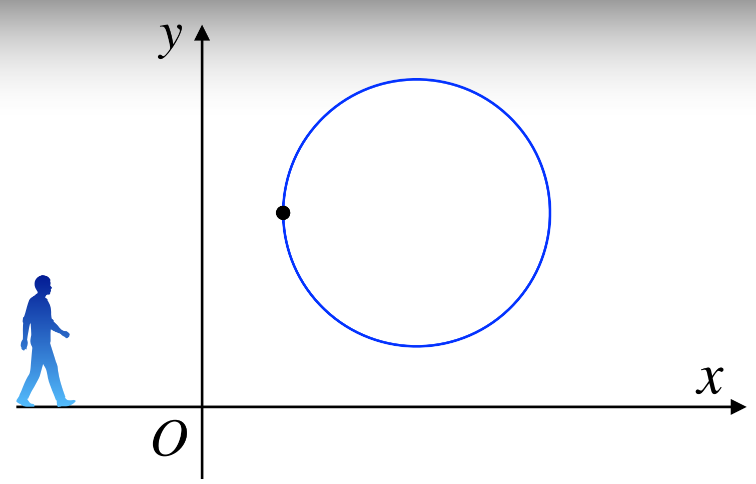
First suppose that $\overrightarrow{\mathbf{F}}$ is a *continuous* vector field in some domain D.

$\overrightarrow{\mathbf{F}}$ is a ***conservative*** vector field if there is a function f such that $\nabla f = \overrightarrow{\mathbf{F}}$.

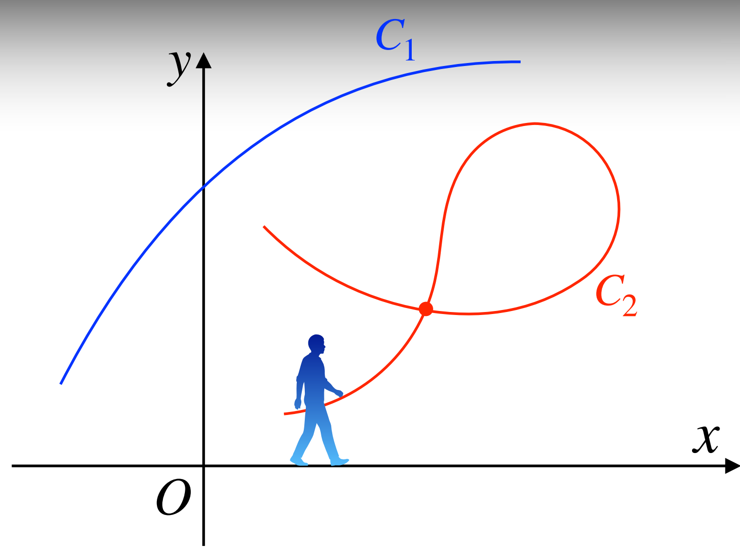
The function f is called a ***potential function*** for the vector field $\overrightarrow{\mathbf{F}}$.

A path C is called ***closed*** if its initial and final points are the same point.

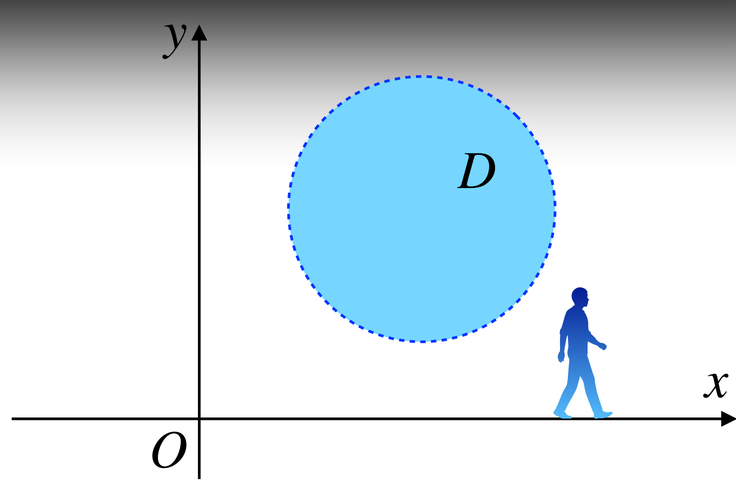
For example, a circle is a closed path.



A path C is **simple** if it does not cross itself.

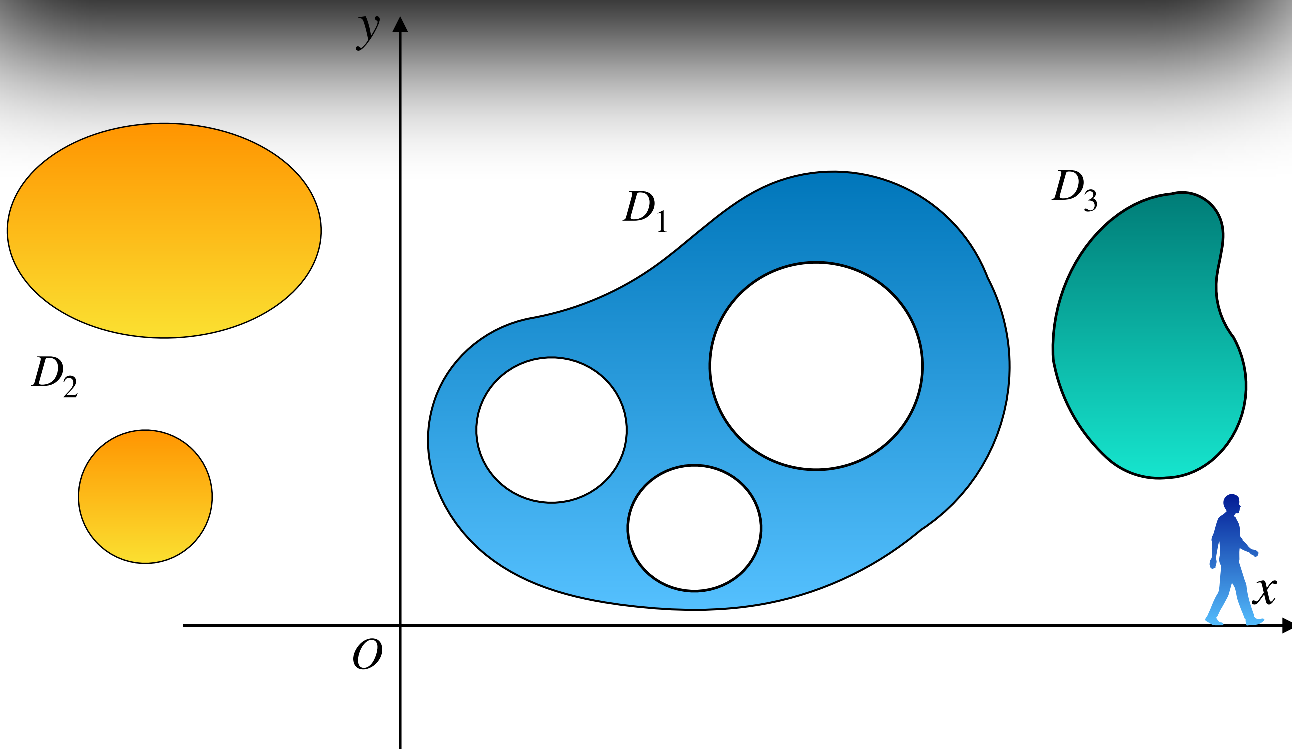


A region D is ***open*** if it does not contain any of its boundary points.



A region D is ***connected*** if we can connect any two points in the region with a path that lies completely in D.

A region D is if it is **simply-connected** and it contains no holes.



Let

$$\overrightarrow{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y)\rangle = M(x,y)\vec{i} + N(x,y)\vec{j}$$

be a vector field on an ***open*** and ***simply-connected*** region D. If M and N have continuous first order partial derivatives in D and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then the vector field $\overrightarrow{\mathbf{F}}$ is conservative.

Let

$$\overrightarrow{\mathbf{F}}(x,y,z) = \langle M(x,y,z), N(x,y,z), R(x,y,z)\rangle$$
$$= M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + R(x,y,z)\overrightarrow{k}\ .$$

be a vector field on an ***open*** and ***simply-connected*** region D. Then, if M, N and R have continuous first order partial derivatives in D and

$$\frac{\partial R}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial R}{\partial x} = \frac{\partial M}{\partial z}, \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

then the vector field $\overrightarrow{\mathbf{F}}$ is conservative.

Example 1 Determine if the following vector fields are conservative or not.

1. $\overrightarrow{\mathbf{F}}(x,y) = (x^2 - yx)\vec{i} + (y^2 - xy)\vec{j}\ .$

We have

$$M(x,y) = x^2 - yx \quad \Longrightarrow \quad \frac{\partial M}{\partial y} = -x,$$

$$N(x,y) = y^2 - xy \quad \Longrightarrow \quad \frac{\partial N}{\partial x} = -y,$$

So, since the two partial derivatives are not the same this vector field is *NOT conservative*.

2. $\overrightarrow{\mathbf{F}}(x,y) = (2xe^{xy} + x^2ye^{xy})\vec{i} + (x^3e^{xy} + 2y)\vec{j}$.

We have

$$M(x,y) = 2xe^{xy} + x^2ye^{xy}$$

and so

$$\frac{\partial M}{\partial y} = 2x^2e^{xy} + x^2e^{xy} + x^3ye^{xy} = 3x^2e^{xy} + x^3ye^{xy}.$$

Similarly, we obtain

$$N(x,y) = x^3e^{xy} + 2y \quad \Longrightarrow \quad \frac{\partial N}{\partial x} = 3x^2e^{xy} + x^3ye^{xy},$$

The two partial derivatives are equal and so this is a *conservative* vector field.

How to find a potential function:

Let us assume that the vector field

$$\overrightarrow{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y)\rangle = M(x,y)\vec{i} + N(x,y)\vec{j}$$

is *conservative*, and so we know that a *potential function*,

$$f(x,y)$$

exists, such that

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = M\vec{i} + N\vec{j} = \overrightarrow{\mathbf{F}},$$

or by setting components equal we have,

$$\begin{cases} \frac{\partial f}{\partial x} = M \\ \frac{\partial f}{\partial y} = N \end{cases}$$

$$\frac{\partial f}{\partial x} = M \quad \Longrightarrow \quad f(x, y) = \int M(x, y)dx$$

or

$$\frac{\partial f}{\partial y} = N \quad \Longrightarrow \quad f(x, y) = \int N(x, y)dy$$

Example 2 Determine if the following vector fields are conservative and find a potential function for the vector field if it is conservative.

1. $\overrightarrow{\mathbf{F}}(x, y) = (2x^3y^4 + x)\vec{i} + (2x^4y^3 + y)\vec{j}$.

First, we identify M, N and then check that the vector field is conservative:

$$M(x, y) = 2x^3y^4 + x \quad \Longrightarrow \quad \frac{\partial M}{\partial y} = 8x^3y^3,$$
$$N(x, y) = 2x^4y^3 + y \quad \Longrightarrow \quad \frac{\partial N}{\partial x} = 8x^3y^3,$$

So, the vector field is *conservative*.

Now let us find the *potential function* $f(x, y)$. We have

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = (2x^3y^4 + x)\vec{i} + (2x^4y^3 + y)\vec{j} = \overrightarrow{\mathbf{F}},$$

or by setting components equal we have,

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x^3y^4 + x \\ \dfrac{\partial f}{\partial y} = 2x^4y^3 + y \end{cases}$$

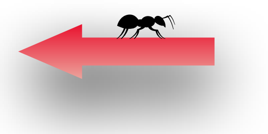
Here is the first integral.

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x \Longrightarrow f(x,y) = \int (2x^3y^4 + x)dx = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + g(y)$$

where $g(y)$ is the *constant* of integration.

Now, let us differentiate $f(x,y)$ (including the $g(y)$) with respect to y and set it equal to N since that is what the derivative is supposed to be:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x^3y^4 + x \\ \dfrac{\partial f}{\partial y} = 2x^4y^3 + y \end{cases}$$



$$\frac{\partial f}{\partial y} = 2x^4y^3 + y \Longrightarrow 2x^4y^3 + g'(y) = 2x^4y^3 + y \Longrightarrow g'(y) = y$$

$$\Longrightarrow g(y) = \frac{1}{2}y^2 + C.$$

So, putting this all together we can see that a potential function for the vector field is,

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + C.$$

Note that we can always check our work by verifying that $\nabla f = \overrightarrow{\mathbf{F}}$.

2. $\overrightarrow{\mathbf{F}}(x,y) = (2xe^{xy} + x^2ye^{xy})\vec{i} + (x^3e^{xy} + 2y)\vec{j}$.

We have already verified that this vector field is *conservative*.

Finally, we find the *potential function* $f(x, y)$. We have

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = (2xe^{xy} + x^2ye^{xy})\vec{i} + (x^3e^{xy} + 2y)\vec{j} = \overrightarrow{\mathbf{F}},$$

or by setting components equal we have,

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xe^{xy} + x^2ye^{xy} \\ \dfrac{\partial f}{\partial y} = x^3e^{xy} + 2y \end{cases}$$

Here is the first integral.

$$\frac{\partial f}{\partial y} = x^3e^{xy} + 2y \Longrightarrow f(x, y) = \int (x^3e^{xy} + 2y)dy = x^2e^{xy} + y^2 + g(x)$$

where $g(x)$ is the *constant* of integration.

If we differentiate this with respect to x and set equal to M we get:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xe^{xy} + x^2ye^{xy} \\ \dfrac{\partial f}{\partial y} = x^3e^{xy} + 2y \end{cases}$$

$$\frac{\partial f}{\partial x} = e^{xy}(2x + x^2) \implies 2xe^{xy} + x^2ye^{xy} + g'(x) = 2xe^{xy} + x^2ye^{xy}$$

$$\implies g'(x) = 0 \implies g(x) = C.$$

Here is the potential function for this vector field.

$$f(x, y) = x^2e^{xy} + y^2 + C.$$

Assume that the vector field

$$\begin{aligned}\overrightarrow{\mathbf{F}}(x,y,z) &= \langle M(x,y,z), N(x,y,z), R(x,y,z)\rangle \\ &= M(x,y,z)\vec{i}+N(x,y,z)\vec{j}+R(x,y,z)\overrightarrow{k}\ .\end{aligned}$$

is *conservative*, and so we know that a *potential function*,

$$f(x,y,z)$$

exists, such that

$$\nabla f = \frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\overrightarrow{k} = M\vec{i}+N\vec{j}+R\overrightarrow{k} = \overrightarrow{\mathbf{F}},$$

or by setting components equal we have,

$$\begin{cases} \dfrac{\partial f}{\partial x} = M \\ \dfrac{\partial f}{\partial y} = N \\ \dfrac{\partial f}{\partial z} = R \end{cases}$$

$$\frac{\partial f}{\partial x} = M \implies f(x,y,z) = \int M(x,y,z)\ dx$$

or

$$\frac{\partial f}{\partial y} = N \implies f(x,y,z) = \int N(x,y,z)\ dy$$

or

$$\frac{\partial f}{\partial z} = R \implies f(x,y,z) = \int N(x,y,z)\ dz$$

Example 3 Find a potential function for the vector field,

$$\vec{\mathbf{F}}(x,y,z) = 2xy^3z^4\vec{i} + 3x^2y^2z^4\vec{j} + 4x^2y^3z^3\vec{k}\ .$$

We want to find the *potential function* $f(x,y,z)$. We have

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = (2xy^3z^4)\vec{i} + (3x^2y^2z^4)\vec{j} + 4x^2y^3z^3\vec{k} = \vec{\mathbf{F}}$$

or by setting components equal we have,

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xy^3z^4 \\ \dfrac{\partial f}{\partial y} = 3x^2y^2z^4 \\ \dfrac{\partial f}{\partial z} = 4x^2y^3z^3 \end{cases}$$

Here is the first integral.

$$\frac{\partial f}{\partial x} = 2xy^3z^4 \Longrightarrow f(x,y,z) = \int (2xy^3z^4)dx = x^2y^3z^4 + g(y,z)$$

where $g(y,z)$ is the *constant* of integration.

Now, let us differentiate $f(x,y,z)$ (including the $g(y,z)$) with respect to y and set it equal to N:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xy^3z^4 \\ \dfrac{\partial f}{\partial y} = 3x^2y^2z^4 \\ \dfrac{\partial f}{\partial z} = 4x^2y^3z^3 \end{cases}$$

$$\frac{\partial f}{\partial y} = 3x^2y^2z^4 + g_y(y,z) = 3x^2y^2z^4 \Longrightarrow g_y(y,z) = 0 \Longrightarrow g(y,z) = h(z)\,.$$

$$f(x,y,z) = x^2y^3z^4 + h(z)$$

Finally, we differentiate $f(x,y,z)$ with respect to z and set it equal to R:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xy^3z^4 \\ \dfrac{\partial f}{\partial y} = 3x^2y^2z^4 \\ \dfrac{\partial f}{\partial z} = 4x^2y^3z^3 \end{cases}$$

$$\frac{\partial f}{\partial z} = 4x^2y^3z^3 + h'(z) = 4x^2y^3z^3 \Longrightarrow h'(z) = 0 \Longrightarrow h(z) = C\,.$$

The potential function for this vector field is then,

$$f(x,y,z) = x^2y^3z^4 + C\,.$$

Example 4 Find a potential function for the vector field,

$$\overrightarrow{\mathbf{F}}(x,y,z) = (2x\cos(y) - 2z^3)\vec{i} + (3 + 2ye^z - x^2\sin(y))\vec{j} + (y^2e^z - 6xz^2)\vec{k}\,.$$

We want to find the *potential function* $f(x,y,z)$. We have

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = M\vec{i} + N\vec{j} + R\vec{k} = \vec{\mathbf{F}}$$

or by setting components equal we have,

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x\cos(y) - 2z^3 \\ \dfrac{\partial f}{\partial y} = 3 + 2ye^z - x^2\sin(y) \\ \dfrac{\partial f}{\partial z} = y^2e^z - 6xz^2 \end{cases}$$

For this example, we integrate the first one with respect to x, the second one with respect to y and the third one with respect to z:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x\cos(y) - 2z^3 \implies x^2\cos(y) - 2xz^3 \\ \dfrac{\partial f}{\partial y} = 3 + 2ye^z - x^2\sin(y) \implies 3y + y^2e^z + x^2\cos(y) \\ \dfrac{\partial f}{\partial z} = y^2e^z - 6xz^2 \implies y^2e^z - 2xz^3 \end{cases}$$

The potential function for this vector field is then,

$$f(x, y, z) = x^2\cos(y) - 2xz^3 + 3y + y^2e^z + C.$$