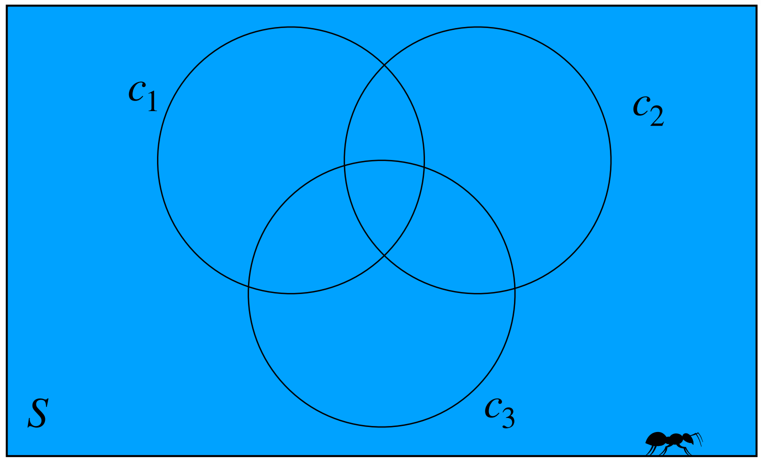
The Principal of Inclusion and Exclusion

Notation:



$S, \quad |S| = N,$

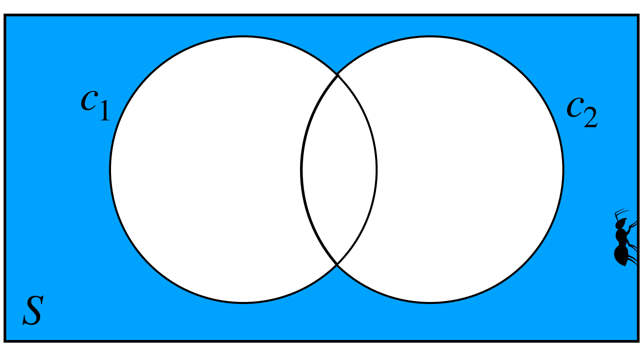
$c_1, c_2, \ldots, c_t \quad \Longrightarrow \quad N(c_1), N(c_2), \ldots, N(c_t)$

$\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_t \quad \Longrightarrow \quad N(\bar{c}_1) = N - N(c_1), \ \ldots \ , N(\bar{c}_t) = N - N(c_t)$

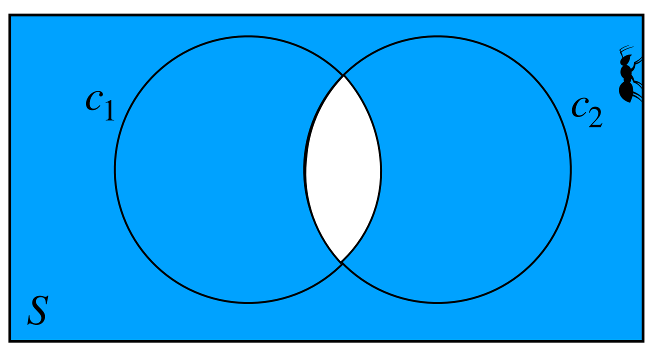
$N(c_1 c_2 \cdots c_k)$

$N(\overline{c_1 c_2 \cdots c_k}) = N - N(c_1 c_2 \ldots c_k)$

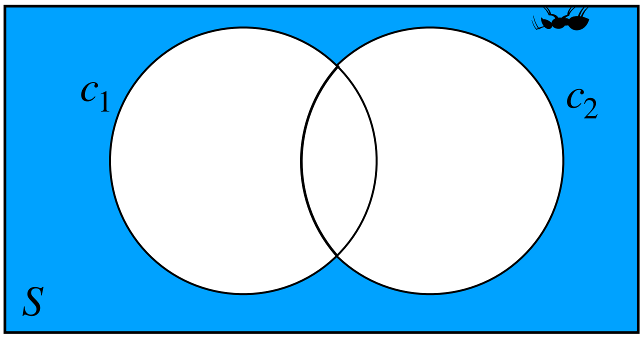
$N(\bar{c}_1 \bar{c}_2 \cdots \bar{c}_k)$



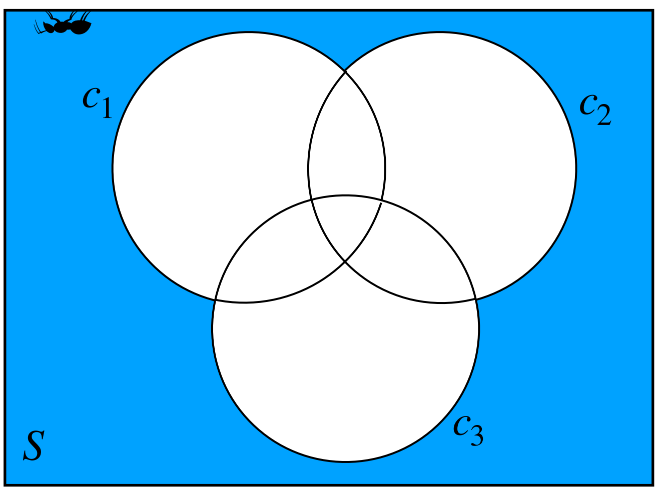
$N(\bar{c}_1 \bar{c}_2)$



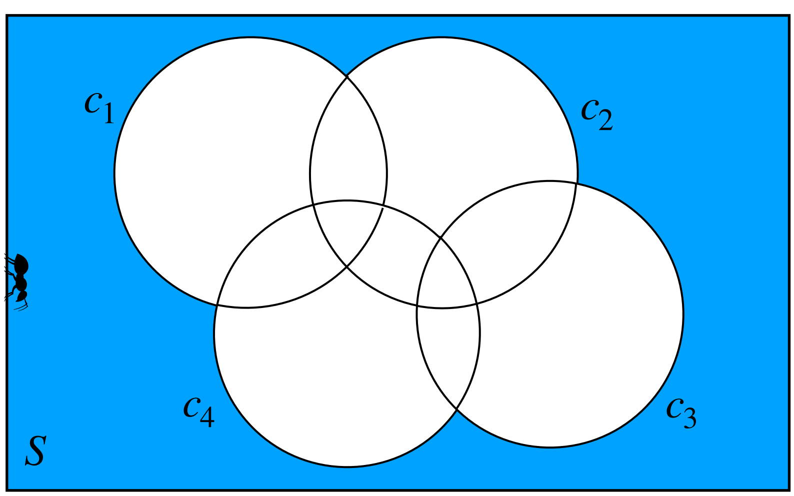
$N(\overline{c_1 c_2})$



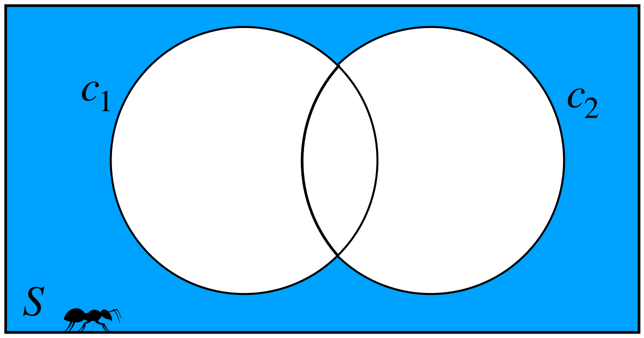
$N(\bar{c}_1\bar{c}_2)$



$N(\bar{c}_1\bar{c}_2\bar{c}_3)$

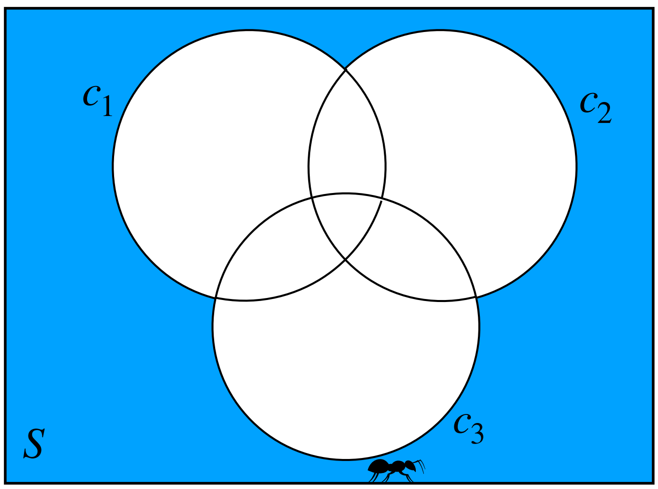


$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4)$



$N(\bar{c}_1\bar{c}_2)$

$$N(\bar{c}_1\bar{c}_2) = N - (N(c_1) + N(c_2)) + N(c_1c_2)$$



$N(\bar{c}_1\bar{c}_2\bar{c}_3)$

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = N - (N(c_1) + N(c_2) + N(c_3)) + (N(c_1c_2) + N(c_1c_3) + N(c_2c_3)) - N(c_1c_2c_3)$$

$$\begin{aligned} N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N & \\ & -(N(c_1) + N(c_2) + N(c_3) + N(c_4)) \\ & +(N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)) \\ & -(N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)) \\ & +N(c_1c_2c_3c_4) \end{aligned}$$

Theorem 1. *(The Principle of Inclusion and Exclusion) Consider a set S, with $|S| = N$, and conditions c_i, $1 \leqslant i \leqslant t$, each of which may be satisfied by some of the elements of S. The number of elements of S that satisfy none of the conditions c_i, $1 \leqslant i \leqslant t$, is denoted by $\overline{N} = N(\bar{c}_1\bar{c}_2\bar{c}_3\cdots\bar{c}_t)$ where*

$$
\begin{aligned}
\overline{N} = N \\
&-(N(c_1) + N(c_2) + N(c_3) + \cdots + N(c_t)) \\
&+(N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)) \\
&-(N(c_1c_2c_3) + N(c_1c_2c_4) + \cdots + N(c_1c_2c_t) + \cdots + N(c_{t-2}c_{t-1}c_t)) \\
&\vdots \\
&+(-1)^t N(c_1c_2c_3\cdots c_t)
\end{aligned}
$$

or

$$
\begin{aligned}
\overline{N} &= N - \sum_{1\leqslant i\leqslant t} N(c_i) + \sum_{1\leqslant i<j\leqslant t} N(c_ic_j) - \sum_{1\leqslant i<j<k\leqslant t} N(c_ic_jc_k) + \cdots + (-1)^t N(c_1c_2c_3\cdots c_t) \\
&= S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t
\end{aligned}
$$

A combinatorial proof. $x \in S$

$$ \boxed{} = \boxed{} $$

- x satisfies *none* of the conditions:

$$ \boxed{1} = \boxed{1} $$

- x satisfies *exactly* r of the conditions $1 \leqslant r \leqslant t$:

$$0 = 1 - \binom{r}{1} + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r \binom{r}{r} = [1-1]^r = 0 \quad \square$$

Example 1. Determine the number of positive integers n where $1 \leqslant n \leqslant 100$ and n is not divisible by 2,3 or 5.

Solution.

- Condition c_1: *m is divisible by* 2.
- Condition c_2: *m is divisible by* 3.
- Condition c_3: *m is divisible by* 5.

$$\overline{N} = N - (N(c_1) + N(c_2) + N(c_3)) + (N(c_1c_2) + N(c_1c_3) + N(c_2c_3)) - N(c_1c_2c_3)$$

$$= S_0 - S_1 + S_2 - S_3$$

$$\boxed{S_0 = N = 100}$$

$$N(c_1) = \left[\frac{100}{2}\right] = 50, \quad N(c_2) = \left[\frac{100}{3}\right] = 33, \quad N(c_3) = \left[\frac{100}{5}\right] = 20,$$

$$\boxed{S_1 = 50 + 33 + 20 = 103}$$

$$N(c_1c_2) = \left[\frac{100}{6}\right] = 16, \quad N(c_1c_3) = \left[\frac{100}{10}\right] = 10, \quad N(c_2c_3) = \left[\frac{100}{15}\right] = 6,$$

$$\boxed{S_2 = 16 + 10 + 6 = 32}$$

$$N(c_1c_2c_3) = \left[\frac{100}{30}\right] = 3,$$

$$\boxed{S_3 = 3.}$$

$\overline{N} = S_0 - S_1 + S_2 - S_3 = 100 - 103 + 32 - 3 = 26.$ $\square$

Example 2. Find the number of nonnegative integers solution to the equation:

$$x_1 + x_2 + x_3 + x_4 = 18,$$

with the restriction that $x_i \leqslant 7,\ 1 \leqslant i \leqslant 4$.

Solution.

- Condition c_1: $x_1 \geqslant 8$.
- Condition c_2: $x_2 \geqslant 8$.
- Condition c_3: $x_3 \geqslant 8$.
- Condition c_4: $x_4 \geqslant 8$.

$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4$

$$\boxed{S_0 = \binom{18+4-1}{4-1} = \binom{21}{3}}$$

$$\boxed{S_1 = \binom{4}{1}\binom{10+4-1}{4-1} = \binom{4}{1}\binom{13}{3}}$$

$$\boxed{S_2 = \binom{4}{2}\binom{2+4-1}{4-1} = \binom{4}{2}\binom{5}{3}}$$

$$\boxed{S_3 = 0}$$

$$\boxed{S_4 = 0}$$

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4 = \binom{21}{3} - \binom{4}{2}\binom{5}{3} + \binom{4}{1}\binom{13}{3} - 0 + 0 = 246. \ \square$$

Example 3. For $n \in \mathbb{Z}^+$, let $\phi(n)$ be the number of positive integers m, where $1 \leqslant m < n$ and $\gcd(m, n) = 1$, that is m, n are relatively prime. This function is known as *Euler's phi function*. If we write

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t},$$

where $p_1, p_2, \ldots, p_t$ are distinct primes and $a_i \geqslant 1$, for all $1 \leqslant i \leqslant t$, then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_t}\right)$$

Solution. (Special Case!) Let $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$, and $S = \{1,2,\ldots,n\}$.

- Condition c_1: *$k \in S$, and k is divisible by p_1,*
- Condition c_2: *$k \in S$, and k is divisible by p_2,*
- Condition c_3: *$k \in S$, and k is divisible by p_3,*
- Condition c_4: *$k \in S$, and k is divisible by p_4,*

$1 \leqslant k < n, \gcd(k, n) = 1 \iff$ *k is not divisible by any of the primes* p_i, $1 \leqslant i \leqslant 4$

$$\phi(n) = \overline{N} = S_0 - S_1 + S_2 - S_3 + S_4$$

$$S_0 = |S| = n,$$

$$S_1 = \left[\frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3} + \frac{n}{p_4}\right]$$

$$S_2 = \left[\frac{n}{p_1p_2} + \frac{n}{p_1p_3} + \frac{n}{p_1p_4} + \frac{n}{p_2p_3} + \frac{n}{p_2p_4} + \frac{n}{p_3p_4}\right]$$

$$S_3 = \left[\frac{n}{p_1p_2p_3} + \frac{n}{p_1p_2p_4} + \frac{n}{p_1p_3p_4} + \frac{n}{p_2p_3p_4}\right]$$

$$S_4 = \frac{n}{p_1p_2p_3p_4}$$

$$\phi(n) = \overline{N} = S_0 - S_1 + S_2 - S_3 + S_4 =$$

$$n - \left[\frac{n}{p_1} + \frac{n}{p_2} + \frac{n}{p_3} + \frac{n}{p_4}\right] + \left[\frac{n}{p_1p_2} + \frac{n}{p_1p_3} + \frac{n}{p_1p_4} + \frac{n}{p_2p_3} + \frac{n}{p_2p_4} + \frac{n}{p_3p_4}\right]$$

$$- \left[\frac{n}{p_1p_2p_3} + \frac{n}{p_1p_2p_4} + \frac{n}{p_1p_3p_4} + \frac{n}{p_2p_3p_4}\right] + \frac{n}{p_1p_2p_3p_4} =$$

$$\frac{n}{p_1p_2p_3p_4}\ [p_1p_2p_3p_4 - (p_2p_3p_4 + p_1p_3p_4 + p_1p_2p_4 + p_1p_2p_3) + (p_3p_4 + p_2p_4 +$$

$$+ p_2p_3 + p_1p_4 + p_1p_3 + p_1p_2) - (p_1 + p_2 + p_3 + p_4) + 1] =$$

$$\frac{n}{p_1p_2p_3p_4}(p_1 - 1)(p_2 - 1)(p_3 - 1)(p_4 - 1) =$$

$$n\left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 1}{p_2}\right)\left(\frac{p_3 - 1}{p_3}\right)\left(\frac{p_4 - 1}{p_4}\right)$$

$$n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)\left(1 - \frac{1}{p_4}\right). \square$$

$$\phi(n) = p_1^{a_1-1} p_2^{a_2-1} \cdots p_t^{a_t-1} \left(p_1 - 1\right)\left(p_2 - 1\right) \cdots \left(p_t - 1\right)$$