Problems and Solutions:

8. *How many nonnegative integers less than a million contain all the digits 1,2,3,4? How many numbers consist of these four digits alone?*

Solution. Let $S = \{0,1,2,3,\ldots,10^6 - 1\}$. We consider the following conditions:

- condition c_1: $k \in S$, *and k does not contain the digit* 1.
- condition c_2: $k \in S$, *and k does not contain the digit* 2.
- condition c_3: $k \in S$, *and k does not contain the digit* 3.
- condition c_4: $k \in S$, *and k does not contain the digit* 4.

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4$$

$$S_0 = |S| = 10^6,$$

$$S_1 = \sum_{1 \leqslant i \leqslant 4} N(i) = \binom{4}{1} \times 9^6 = 4 \times 9^6$$

$$S_2 = \sum_{1 \leqslant i < j \leqslant 4} N(c_i c_j) = \binom{4}{2} \times 8^6 = 6 \times 8^6$$

$$S_3 = \sum_{1 \leqslant i < j < k \leqslant 4} N(c_i c_j c_k) = \binom{4}{3} \times 7^6 = 4 \times 7^6$$

$$S_4 = N(c_1 c_2 c_3 c_4) = \binom{4}{4} \times 6^6 = 1 \times 6^6$$

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4 = 10^6 - 4 \times 9^6 + 6 \times 8^6 - 4 \times 7^6 + 1 \times 6^6 = 23{,}160.$$

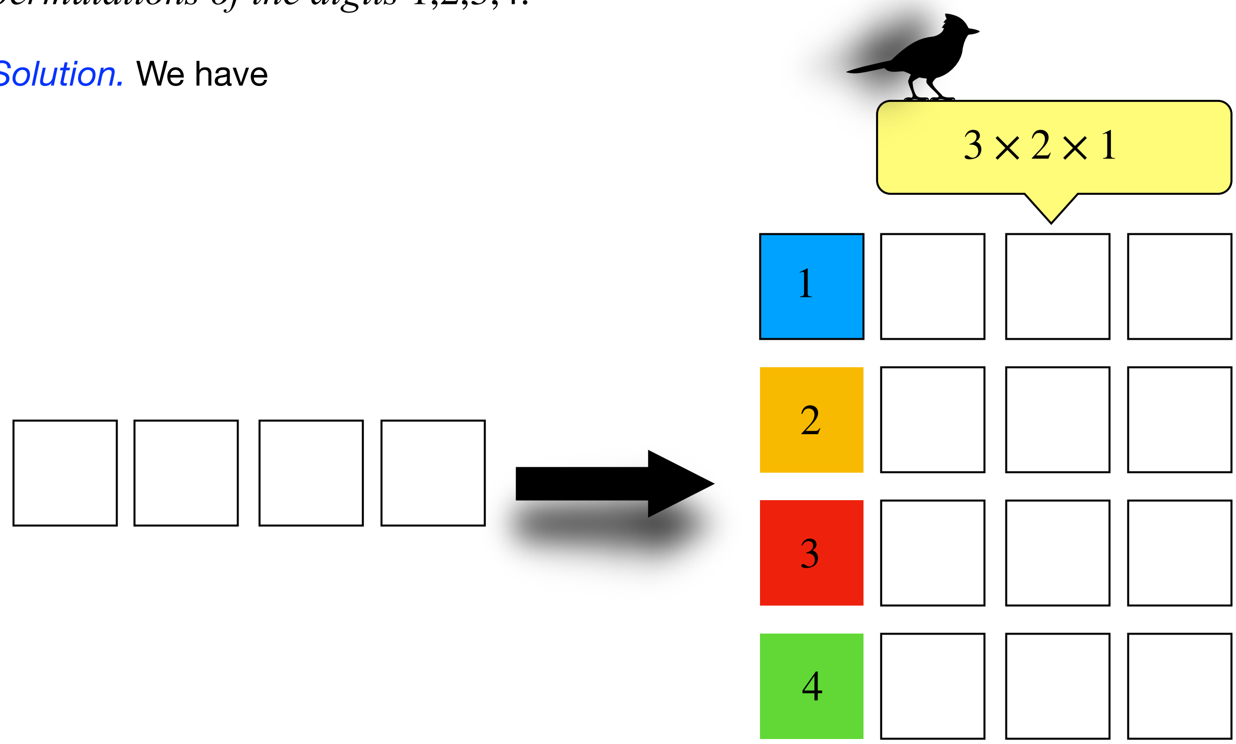
The digits 1,2,3,4 alone generate

$$4 + 4^2 + 4^3 + 4^4 + 4^5 + 4^6 = \frac{4^7 - 1}{4 - 1} = 5{,}460$$

numbers. □

9. *Find the sum of the four-digit numbers obtained in all possible permutations of the digits* 1,2,3,4.

Solution. We have



Each digit appears in each order (digit place) 6 times. Now, we obtain the sum

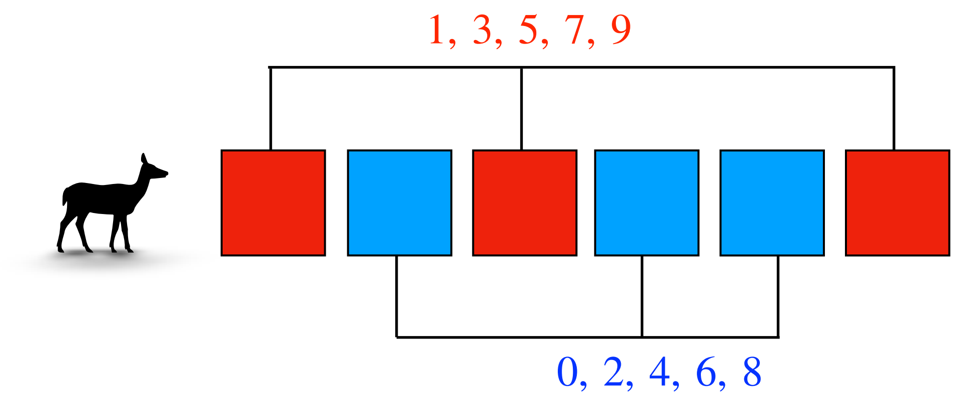
$$6(1+2+3+4) = 60,$$

for each place. Therefore, the total sum is

$$\underbrace{60}_{\text{1's place}} + 60(\underbrace{10}_{\text{10's place}}) + 60(\underbrace{100}_{\text{100's place}}) + 60(\underbrace{1000}_{\text{1000's place}}) = 66{,}660.$$

10. *How many six-digit numbers are there in which three digits are even and three are odd?*

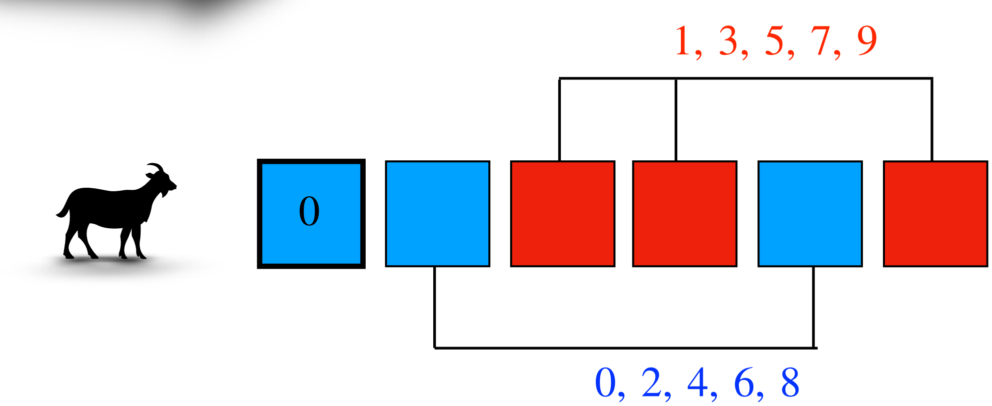
Solution. We have



The positions for odd digits can be chosen in $\binom{6}{3} = 20$ ways.

Each position can occupied by one of 5 digits (either even or odd).

$$\binom{6}{3} \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 20 \times 5^6 .$$

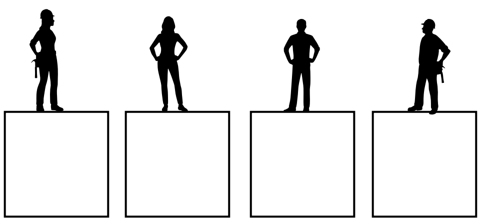


$$\binom{5}{2} \times 5 \times 5 \times 5 \times 5 \times 5 = 10 \times 5^5 .$$

$$20 \times 5^6 - 10 \times 5^5 = 281{,}250$$

11. *How many four-digit numbers can be formed from the digits of the number* 123,153.

Solution. We have



Four distinct digits: $1,\ 2,\ 3,\ 5$

Two identical and two distinct digits

$1,\ 1,\ 2,\ 3$ $1,\ 1,\ 2,\ 5$ $1,\ 1,\ 3,\ 5$

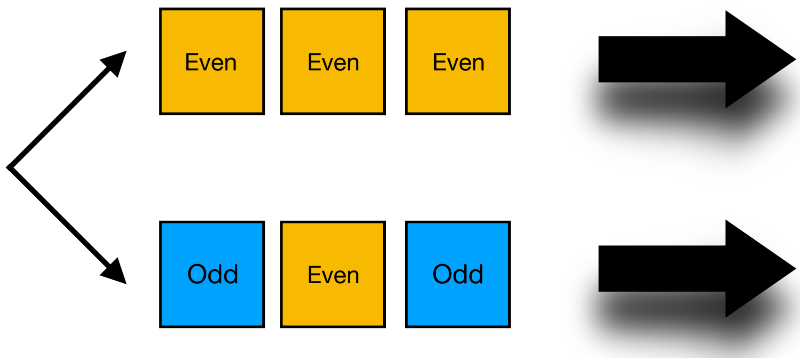
$1,\ 2,\ 3,\ 3$ $1,\ 3,\ 3,\ 5$ $2,\ 3,\ 3,\ 5$

Two pairs of the same digits: $1,\ 1,\ 3,\ 3$

$$4! + 6 \times \frac{4!}{2!1!1!} + \frac{4!}{2!2!} = 24 + 6 \times 12 + 6 = 102.$$

12. *In how many ways can three numbers be chosen from the natural numbers* 1 *to* 30 *so that their sum is even?*

Solution. We have



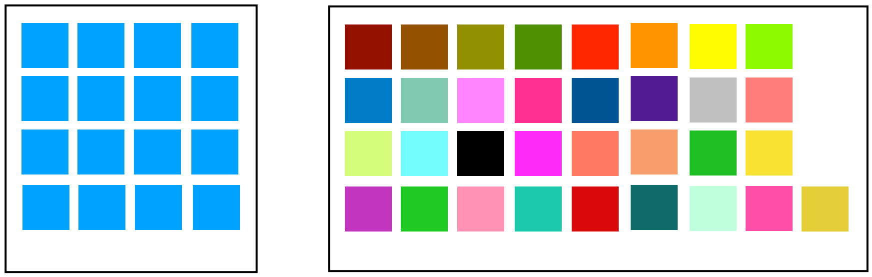
$$\binom{15}{3}$$

$$\binom{15}{2}\binom{15}{1}$$

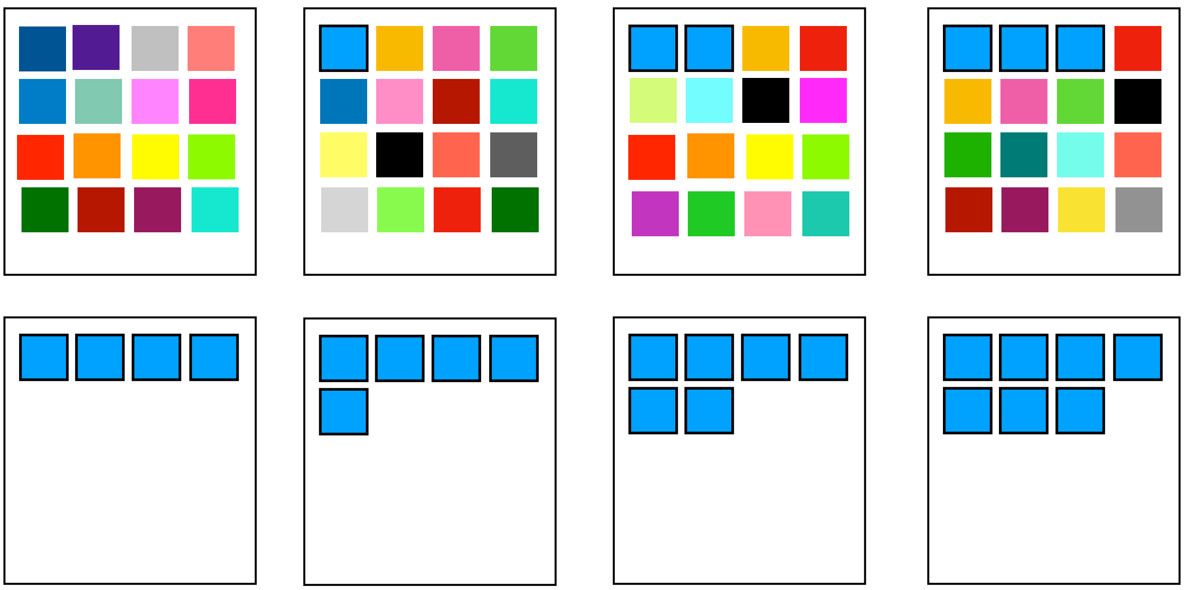
$$\binom{15}{3} + \binom{15}{2}\binom{15}{1} = 2{,}030.$$

13. *There are* $3n+1$ *objects of which n are identical and the remaining are distinct. Prove that n objects can be selected from them in* 2^{2n} *ways.*

Solution. We have



n identical objects $\qquad$ $2n+1$ *distinct objects*



$$\binom{2n+1}{n}+\binom{2n+1}{n-1}+\binom{2n+1}{n-2}+\binom{2n+1}{n-3}+\cdots+\binom{2n+1}{0}=$$

$$\frac{1}{2}\left[\binom{2n+1}{2n+1}+\binom{2n+1}{2n}+\cdots+\binom{2n+1}{n+1}+\binom{2n+1}{n}+\binom{2n+1}{n-1}+\cdots+\binom{2n+1}{0}\right]$$

$$\frac{1}{2}2^{2n+1}=2^{2n}.$$

14. *How many pairs of integers x, y lie between 1 and 1000 such that $x^2 + y^2$ is divisible by 49?*

Solution. We have

$x \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 7$ ⟶ $x^2 \equiv 0, 1, 4, 2, 2, 4, 1 \pmod 7$

⟶ $x^2 + y^2 \equiv 0 \pmod 7$ only when $x \equiv 0 \pmod 7$ and $y \equiv 0 \pmod 7$

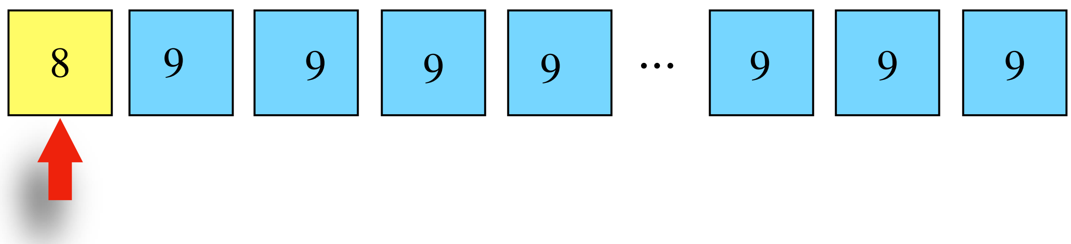
⟶ $x^2 + y^2 \equiv 0 \pmod{49}$

⟶ $\left[\frac{1000}{7}\right]^2 = 142^2 = 20{,}164$ with regard for order

⟶ $\binom{142+2-1}{2} = \binom{143}{2} = 10{,}153.$ if we disregard order

15. *Of the number from* 1 *to* 10,000,000*, which are more numerous, those that contain a unit or those that do not?*

Solution. The number of m-digit numbers without any ones is $8 \times 9^{m-1}$.



Hence, between 1 to 10,000,000, there are

$$8(1+9+9^2+9^3+9^4+9^5+9^6) = 8 \times \frac{9^7-1}{9-1} = 4{,}782{,}968,$$

numbers whose representations do not contain 1. This is *less than half* of 10^7 .

16. *The numbers from* 1 *to* 100,000,000 *are written down in succession so that we have the sequence of digits*

$$12345678910111213141516171 8\ldots 100{,}000{,}000.$$

Prove that the number of all the digits of this sequence is equal to the number of zeros in the sequence

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots\ ,\ 1{,}000{,}000{,}000.$$

Solution. The given sequence contains

$$(1 \times 9) + (2 \times 90) + (3 \times 900) + \cdots + (8 \times 90{,}000{,}000) + 9$$

digits.

Now we compute the number of zeros in the sequence

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots\ ,\ 1{,}000{,}000{,}000.$$

000000001	000000010	000000100	⋯	
000000002	000000011	000000101	⋯	
000000003	000000012	000000102	⋯	
⋮	⋮	⋮		
000000009	000000019	000000119	⋯	
	000000020	000000120	⋯	1000000000
	⋮	⋮		
	000000099	000000199	⋯	
		000000200	⋯	
		⋮		
		000000999	⋯	000000000

10	10	10	10	10	10	10	10	10

The number of digits: 9×10^9

Each digit appearing as many times as any other one. We thus have

$\boxed{9 \times 10^8}$ zeros.

But these zeros include the zeros we appended:

$$8 \times 9, \quad 7 \times 90, \quad 6 \times 900, \quad 5 \times 9{,}000, \quad 4 \times 90{,}000, \ldots$$

Now, we obtain

$$\begin{aligned}
& 9 \times 10^8 - (8 \times 9 + 7 \times 90 + \cdots + 1 \times 90{,}000{,}000) \\
&= 9^2 \times (1 + 10 + 10^2 + 10^3 + \cdots + 10^7) + 9 - (8 \times 9 + 7 \times 90 + \cdots + 1 \times 90{,}000{,}000) \\
&= 9 \times (9 + 90 + 900 + 9{,}000 + \cdots + 9 \times 10^7) + 9 - (8 \times 9 + 7 \times 90 + \cdots + 1 \times 90{,}000{,}000) \\
&= [9 \times 9 - 8 \times 9] + [9 \times 90 - 7 \times 90] + [9 \times 900 - 6 \times 900] + \cdots + [9 \times 9 \times 10^7 - 1 \times 9 \times 10^7] + 9 \\
&= (1 \times 9) + (2 \times 90) + (3 \times 900) + \cdots + (8 \times 90{,}000{,}000) + 9. \ \square
\end{aligned}$$