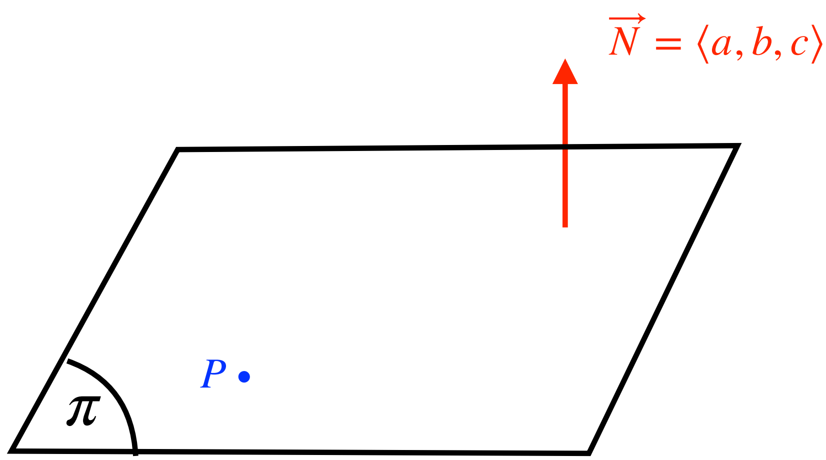
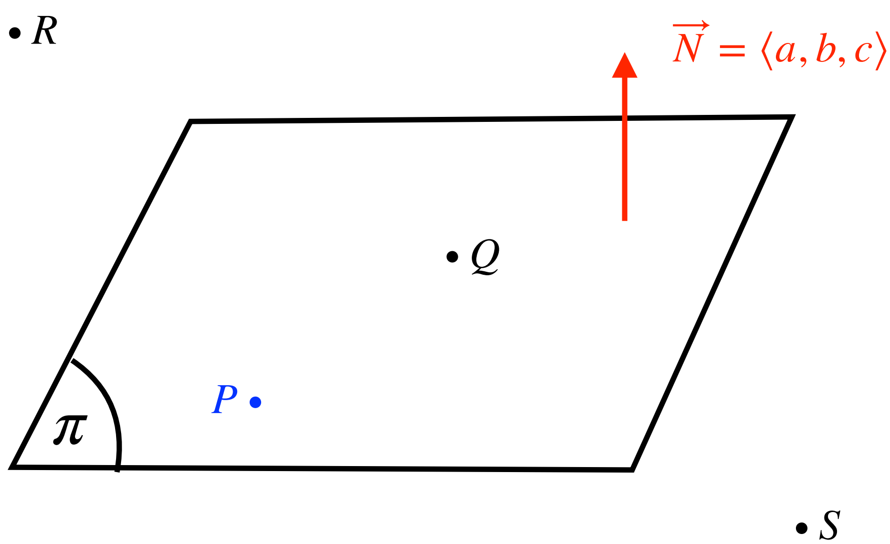
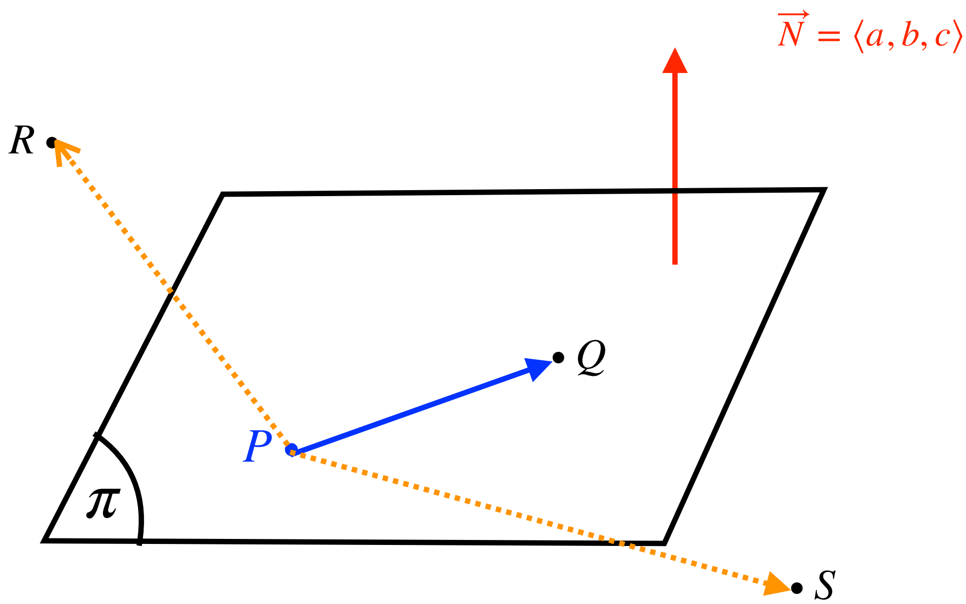
# 3. Plane and Line

## Section 1-1: Equations of planes

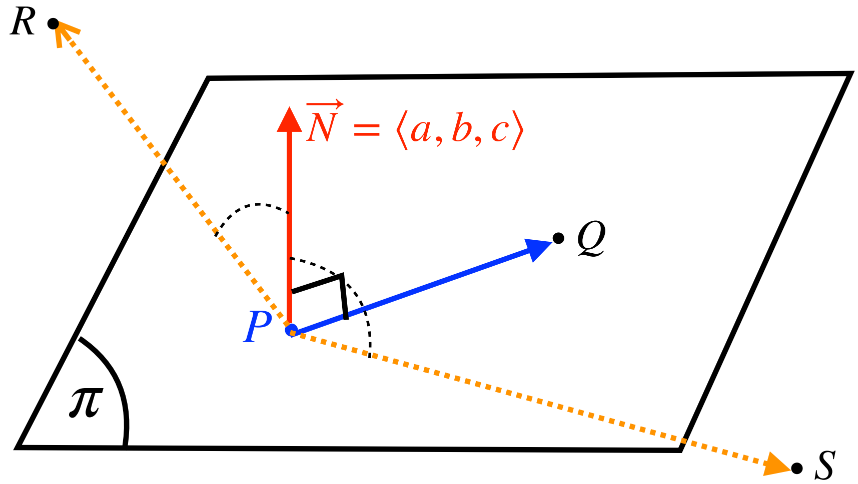
$$\pi \left| \begin{array}{l} \vec{N} = \langle a, b, c \rangle \quad \text{Normal vector} \\ P = (x_0, y_0, z_0) \end{array} \right.$$



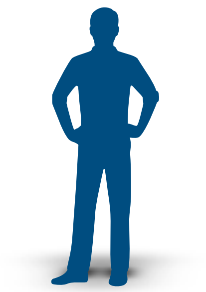






$\overrightarrow{PQ} \perp \overrightarrow{N}$



$$Q = (x, y, z) \in \pi \left| \begin{array}{l} \overrightarrow{N} = \langle a, b, c \rangle \\ P = (x_0, y_0, z_0) \end{array} \right. \iff \overrightarrow{PQ} \perp N$$



$$\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$\overrightarrow{N} = \langle a, b, c \rangle$$

$$\Longrightarrow \overrightarrow{N} \cdot \overrightarrow{PQ} = 0 \Longrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This is called the *scalar equation of plane*. Often this will be written as,

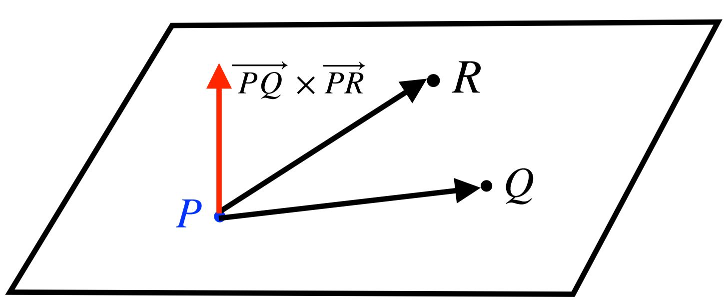
$$ax + by + cz + d = 0,$$

where $d = -(ax_0 + by_0 + cz_0)$.

**Example 1** Determine the equation of the plane that contains the points:

$$P = (1, -2, 0), \quad Q = (3, 1, 4), \quad R = (0, -1, 2).$$

We have $\overrightarrow{PQ} = \langle 2, 3, 4 \rangle$, $\overrightarrow{PR} = \langle -1, 1, 2 \rangle$, and $\overrightarrow{N} = \overrightarrow{PQ} \times \overrightarrow{PR}$.



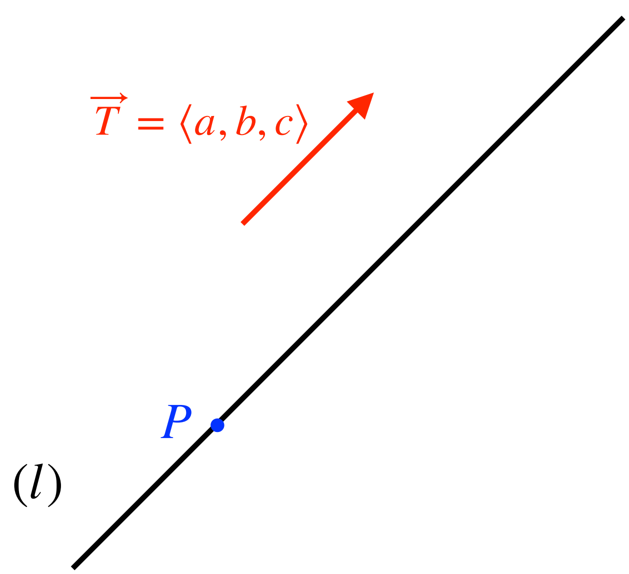
$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = 2\vec{i} - 8\vec{j} + 5\vec{k}\ .$$

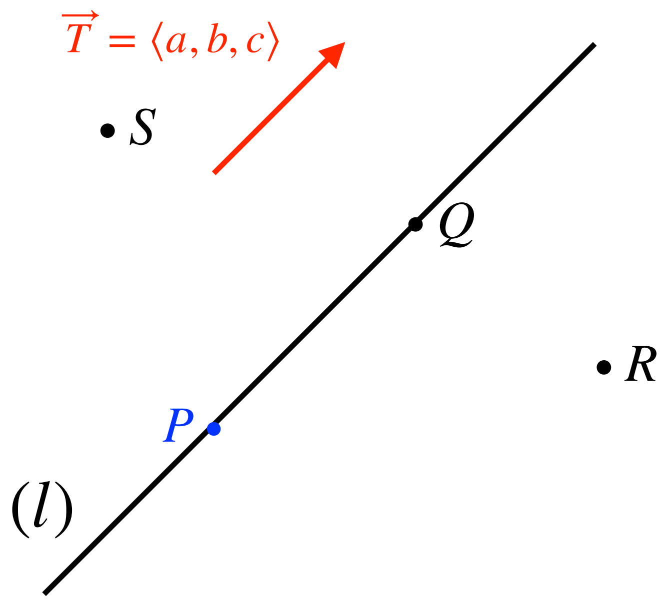
$$\pi \left| \begin{array}{l} \vec{N} = \langle 2, -8, 5 \rangle \\ P = (1, -2, 0) \end{array} \right. \implies 2(x-1) - 8(y+2) + 5(z-0) = 0.$$

$$2x - 8y + 5z - 18 = 0.$$

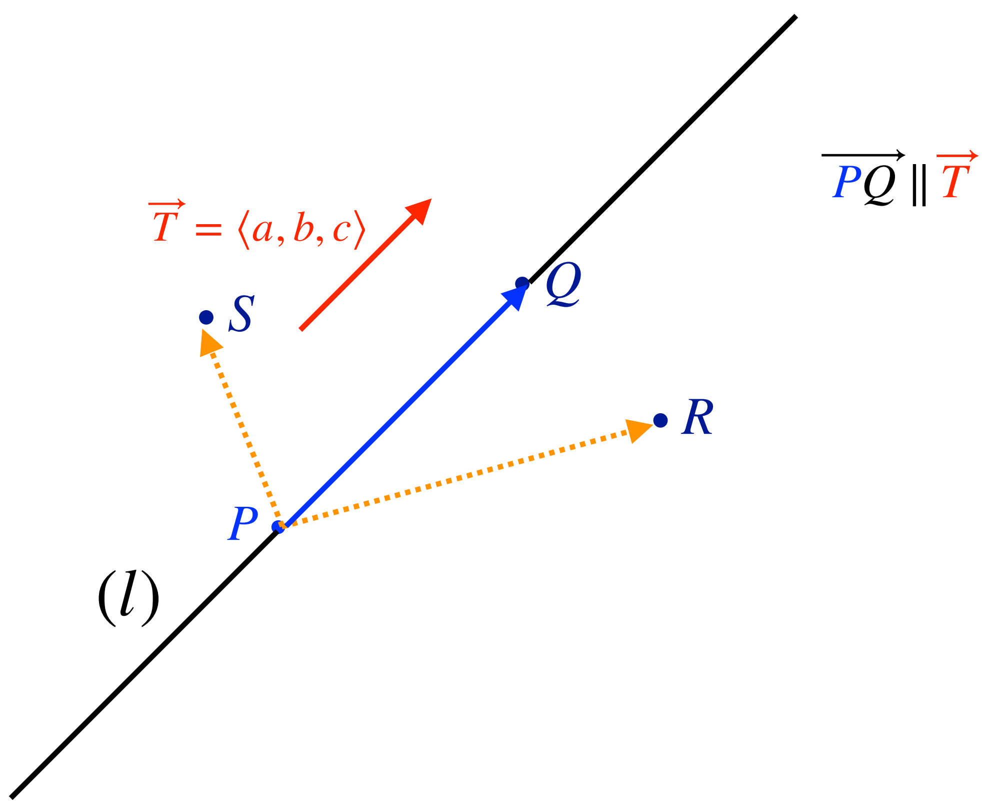
## Section 1-2 : Equations of lines

$$(l) \left| \begin{array}{l} \vec{T} = \langle a, b, c \rangle \\ P = (x_0, y_0, z_0) \end{array} \right.$$



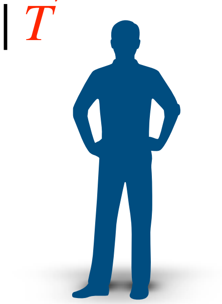


$\overrightarrow{T} = \langle a, b, c \rangle$
$S$
$Q$
$R$
$P$
$(l)$

$\overrightarrow{PQ} \parallel \overrightarrow{T}$
$\overrightarrow{T} = \langle a, b, c \rangle$
$S$
$Q$
$R$
$P$
$(l)$

$$Q=(x,y,z)\in l \left| \begin{array}{l} \overrightarrow{T}=\langle a,b,c\rangle \\ P=(x_0,y_0,z_0) \end{array} \right. \iff \overrightarrow{PQ}\parallel\overrightarrow{T}$$



$$\overrightarrow{PQ}=\langle x-x_0, y-y_0, z-z_0\rangle$$
$$\overrightarrow{T}=\langle a,b,c\rangle$$

$$\Longrightarrow \exists t \quad \overrightarrow{PQ}=t\overrightarrow{T} \Longrightarrow \quad (x-x_0)=ta, \ (y-y_0)=tb, \ (z-z_0)=tc\,.$$

- Hence, we have

$$x=x_0+ta, \quad y=y_0+tb, \quad z=z_0+tc\,.$$

This is called the *parametric equations of the line.*

- Moreover, if $a\neq 0$, $b\neq 0$, $c\neq 0$, then

$$\frac{(x-x_0)}{a}=\frac{(y-y_0)}{b}=\frac{(z-z_0)}{c}=t\,.$$

This is called the *symmetric equations of the line*.

- Finally, the *vector form of the line* is

$$\vec{r}(t)=\langle x_0,y_0,z_0\rangle+t\langle a,b,c\rangle\,.$$

**Example 2** Write down the equation of the line that passes through the points $A=(1,4,-3), B=(2,-1,3)$.

We have $\overrightarrow{T}=\overrightarrow{AB}=\langle 1,-5,6\rangle$ hence the symmetric form of this line is

$$\frac{x-1}{1}=\frac{y-4}{-5}=\frac{z+3}{6}\,.$$

## Practical Problems

For problems 1 – 3 write down the equation of the plane.

1. The plane containing the points $(4, -3, 1)$, $(-3, -1, 1)$ and $(4, -2, 8)$.
2. The plane containing the point $(3, 0, -4)$ and orthogonal to the line given by $\vec{r}(t) = \langle 12 - t, 1 + 8t, 4 + 6t \rangle$.
3. The plane containing the point $(-8, 3, 7)$ and parallel to the plane given by $4x + 8y - 2z = 45$.

For problems 4 & 5 determine if the two planes are parallel, orthogonal or neither.

4. The plane given by $4x - 9y - z = 2$ and the plane given by $x + 2y - 14z = -6$.
5. The plane given by $-3x + 2y + 7z = 9$ and the plane containing the points $(-2, 6, 1)$, $(-2, 5, 0)$ and $(-1, 4, -3)$.

For problems 6 & 7 determine where the line intersects the plane or show that it does not intersect the plane.

6. The line given by $\vec{r}(t) = \langle -2t, 2 + 7t, -1 - 4t \rangle$ and the plane given by $4x + 9y - 2z = -8$.
7. The line given by $\vec{r}(t) = \langle 4 + t, -1 + 8t, 3 + 2t \rangle$ and the plane given by $2x - y + 3z = 15$.
8. Find the line of intersection of the plane given by $3x + 6y - 5z = -3$ and the plane given by $-2x + 7y - z = 24$.
9. Determine if the line given by $x = 8 - 15t, y = 9t, z = 5 + 18t$ and the plane by $10x - 6y - 12z = 7$ are parallel, orthogonal or neither.

For problems $10$ & $11$ give the equation of the line in vector form, parametric form and symmetric form.

10. The line through the points $(2,-4,1), (0,4,-10)$.
11. The line through the point $(-7,2,4)$ and parallel to the line given by $x=5-8t, y=6+t, z=-12t$.
12. Is the line through the points $(2,0,9)$ and $(-4,1,-5)$ parallel, orthogonal or neither to the line given by $\vec{r}(t)=\langle 5,1-9t,-8-4t\rangle$?

For problems $13$ & $14$ determine the intersection point of the two lines or show that they do not intersect.

13. The line given by $x=8+t, y=5+6t, z=4-2t$ and the line given by $\vec{r}(t)=\langle -7+12t,3-t,14+8t\rangle$.
14. The line passing through the points $(1,-2,13)$ and $(2,0,-5)$ and the line given by $\vec{r}(t)=\langle 2+4t,-1-t,3\rangle$.
15. Does the line given by $x=9+21t, y=-7, z=12-11t$ intersect the $xy$-plane? If so, give the point.
16. Does the line given by $x=9+21t, y=-7, z=12-11t$ intersect the $xz$-plane? If so, give the point.