Line Integrals

We start with the vector field,

$$\overrightarrow{\mathbf{F}}(x, y, z) = \langle M(x, y, z), N(x, y, z), R(x, y, z)\rangle$$
$$= M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + R(x, y, z)\overrightarrow{k}\ .$$

and the three-dimensional, smooth curve C given by

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\overrightarrow{k}, \quad a \leqslant t \leqslant b\,.$$

The line integral of $\overrightarrow{\mathbf{F}}$ along C is:

$$\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r} = \int_a^b \overrightarrow{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$$

Dot Product

Also, $\overrightarrow{\mathbf{F}}(\vec{r}(t))$ is a shorthand for,

$$\overrightarrow{\mathbf{F}}(\vec{r}(t)) = \overrightarrow{\mathbf{F}}(x(t), y(t), z(t)).$$

Example 1 Evaluate $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r}$ where

$$\overrightarrow{\mathbf{F}}(x, y, z) = 8x^2yz\vec{i} + 5z\vec{j} - 4xy\vec{k}.$$

and C is the curve given by

$$\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}, \quad 0 \leqslant t \leqslant 1.$$

Solution. we first need the vector field evaluated along the curve.

$$\overrightarrow{\mathbf{F}}(\vec{r}(t)) = 8t^2(t^2)(t^3)\vec{i} + 5t^3\vec{j} - 4t(t^2)\vec{k} = 8t^7\vec{i} + 5t^3\vec{j} - 4t^3\vec{k}.$$

Next, we need the derivative of the parameterization:

$$\vec{r}\,'(t) = \vec{i} + 2t\vec{j} + 3t^2\vec{k}.$$

Finally, let's get the dot product taken care of:

$$\overrightarrow{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}\,'(t) = (8t^7z\vec{i} + 5t^3\vec{j} - 4t^3\vec{k}) \cdot (\vec{i} + 2t\vec{j} + 3t^2\vec{k})$$

$$= 8t^7 + 10t^4 - 12t^5$$

The line integral is then,

$$\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r} = \int_0^1 (8t^7 + 10t^4 - 12t^5)dt = (t^8 + 2t^5 - 2t^6)\Big|_0^1 = 1.$$

Example 2 Evaluate $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r}$ where

$$\overrightarrow{\mathbf{F}}(x, y, z) = xz\vec{i} - yz\vec{k}.$$

and C is the line segment from $(-1,2,0)$ to $(3,0,1)$.

Solution. The parameterization for the line:

$$\vec{r}(t) = (4t-1)\vec{i} + (2-2t)\vec{j} + t\vec{k}, \quad 0 \leqslant t \leqslant 1.$$

Let us get the vector field evaluated along the curve:

$$\overrightarrow{\mathbf{F}}(\vec{r}(t)) = (4t-1)t\vec{i} - (2-2t)(t)\vec{k} = (4t^2 - t)\vec{i} - (2t - 2t^2)\vec{k}.$$

Next, we need the derivative of the parameterization:

$$\vec{r}'(t) = 4\vec{i} - 2\vec{j} + \vec{k}.$$

The dot product is then:

$$\overrightarrow{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}'(t) = 4(4t^2 - t) - (2t - 2t^2) = 18t^2 - 6t.$$

The line integral becomes,

$$\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r} = \int_0^1 (18t^2 - 6t)dt = (6t^3 - 3t^2)\Big|_0^1 = 3.$$

For the vector field,

$$\vec{\mathbf{F}} = \langle M, N, R\rangle = M\vec{i} + N\vec{j} + R\vec{k}\ .$$

and the three-dimensional, smooth curve C given by

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \quad a \leqslant t \leqslant b,$$

the line integral of $\vec{\mathbf{F}}$ along C is:

$$\begin{aligned}
\int_C \vec{\mathbf{F}} \cdot d\vec{r} &= \int_a^b (M\vec{i} + \vec{j} + R\vec{k}) \cdot (x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k})\ dt \\
&= \int_a^b (Mx'(t) + Ny'(t) + Rz'(t))\ dt \\
&= \int_a^b Mx'(t)\ dt + \int_a^b Ny'(t)\ dt + \int_a^b Rz'(t))\ dt \\
&= \int_C M\ dx + \int_C N\ dy + \int_C R\ dz \\
&= \int_C M\ dx + N\ dy + R\ dz\,.
\end{aligned}$$

$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = = \int_C M\ dx + N\ dy + R\ dz\,.$$

The Fundamental Theorem of Calculus:

The Fundamental Theorem of Calculus says us how to evaluate definite integrals:

$$\int_a^b F'(x)dx = F(b) - F(a)\,.$$

There is a version of this for line integrals over certain kinds of vector fields. Here it is:

Theorem. *Suppose that C is a smooth curve given by $\vec{r}(t), a \leqslant t \leqslant b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then,*

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))\,.$$

Note that $\vec{r}(a)$ represents the initial point on C, while $\vec{r}(b)$ represents the final point on C.

Proof. Let us start by just computing the line integral:

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\ dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right)\ dt$$

Now, at this point we can use the *chain rule* to simplify the integrand as follows:

$$\left(\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}\right)=\frac{d}{dt}[f(\vec{r}(t)].$$

Now, we just need to use the *Fundamental Theorem of Calculus* for single integrals:

$$\int_C \nabla f\cdot d\vec{r}=\int_a^b \frac{d}{dt}[f(\vec{r}(t)]\,dt=f(\vec{r}(b))-f(\vec{r}(a)).$$

Example 3 Evaluate $\int_C \nabla f\cdot d\vec{r}$ where

$$f(x,y,z)=\cos(\pi x)+\sin(\pi y)-xyz,$$

and C is any path that starts at $(1,1/2,2)$ and ends at $(2,1,-1)$.

Solution. Let $\vec{r}(t), a\leqslant t\leqslant b$, be any path that starts at $(1,1/2,2)$ and ends at $(2,1,-1)$. Then, $\vec{r}(a)=(1,1/2,2)$ and $\vec{r}(b)=(2,1,-1)$.

The integral is then,

$$\begin{aligned}\int_C \nabla f\cdot d\vec{r}&=f(\vec{r}(b))-f(\vec{r}(a))=f(2,1,-1)-f(1,1/2,2)\\&=\cos(2\pi)+\sin(\pi)+2-\cos(\pi)-\sin(\frac{\pi}{2})+1=4.\end{aligned}$$

Example 4 Evaluate $\int_C \vec{\mathbf{F}}\cdot d\vec{r}$ where

$$\vec{\mathbf{F}}=(2x^3y^4+x)\vec{i}+(2x^4y^3+y)\vec{j},$$

and C is given by $\vec{r}(t)=(t\cos(\pi t)-1)\vec{i}+\sin\left(\frac{\pi t}{2}\right)\vec{j},\ 0\leqslant t\leqslant 1.$

Solution. This vector field is conservative and that a potential function for the vector field is,

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + C\,.$$

Using this we know that integral must be independent of path and so all we need to do is use the Fundamental Theorem to do the evaluation:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

where, $\vec{r}(1)) = \langle -2,1 \rangle$ and $\vec{r}(0)) = \langle -1,0 \rangle$. So, the integral is,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = f(-2,1) - f(-1,0) = \left(\frac{21}{2} + C\right) - \left(\frac{1}{2} + C\right) = 10.$$