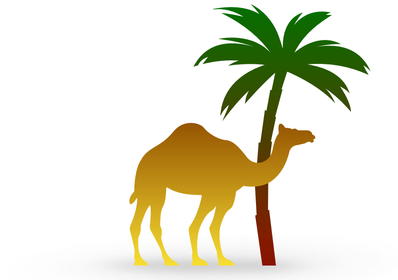
The Rules of Sum and Product



The Rule of Sum:

$A: \quad (A_1, w_1),\ (A_2, w_2),\ \ldots,\ (A_n, w_n),$

The number of ways to do $A = w_1 + w_2 + w_3 + \cdots + w_n$

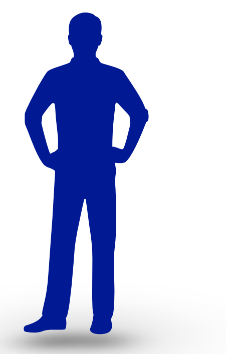
The Rule of Product:

$A: \quad (S_1, w_1),\ (S_2, w_2),\ \ldots,\ (S_n, w_n),$

The number of ways to do $A = w_1 \times w_2 \times w_3 \times \cdots \times w_n$

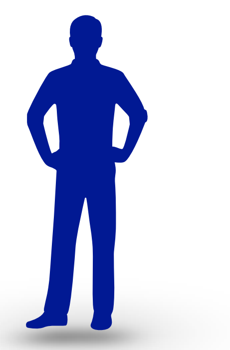
Permutations

$$P(n,n) = n(n-1)(n-2)(n-3)\cdots(3)(2)(1) = n!$$



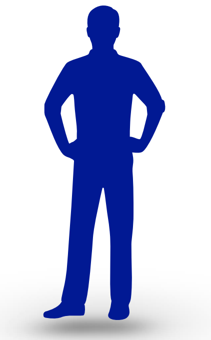
r-Permutations

$$P(n,r) = \frac{n!}{(n-r)!}$$



$$P(n,r) = n(n-1)(n-2)(n-3)\cdots(n-r+1)$$

k- Combinations

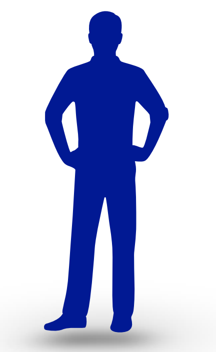


$$C(n,k) = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!}$$

(Binomial Theorem) For all $n \geqslant 0$, we have

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Arrangements with repetition

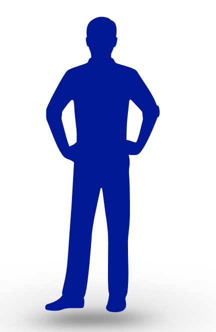


$$P(n; n_1, n_2, \ldots, n_k) = \binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

(Multinomial Theorem) For all $n \geqslant 0$, and $k \geqslant 1$, we have

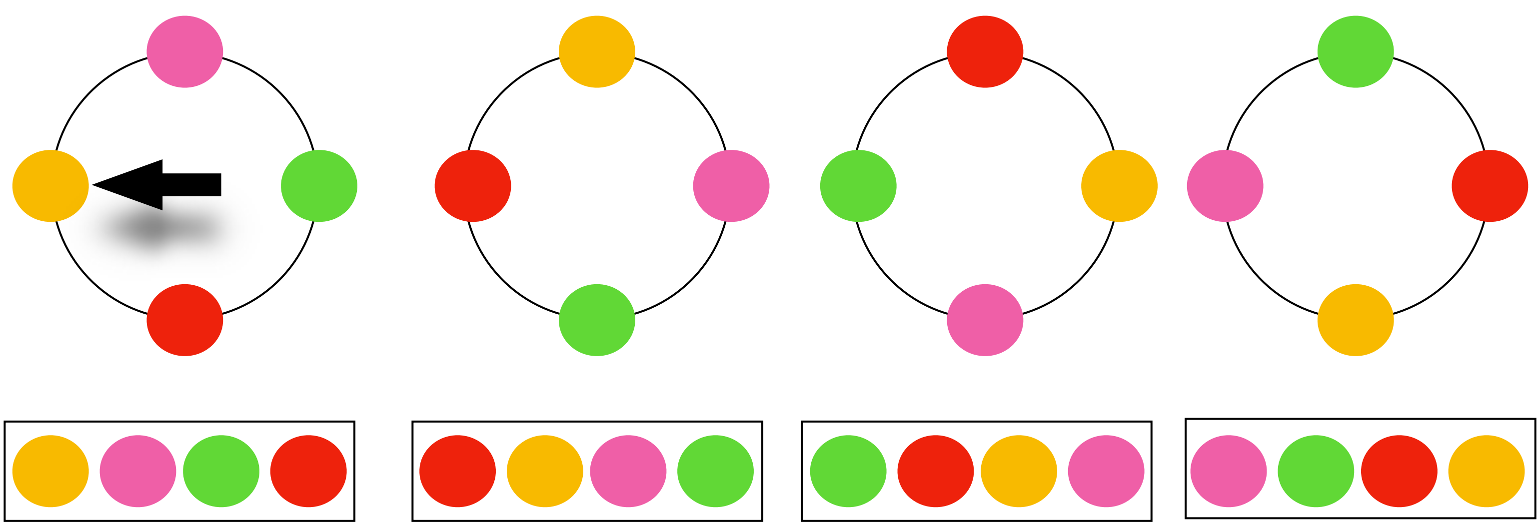
$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

Combination With Repetition



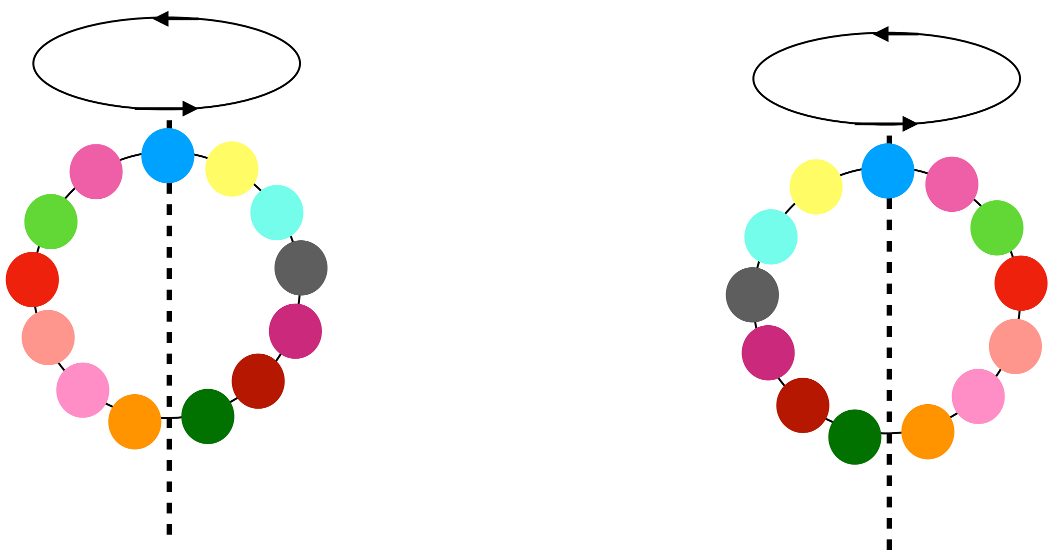
$$C^*(n,r) = \binom{r+n-1}{n-1} = \binom{n+r-1}{r}$$

Circular Arrangement

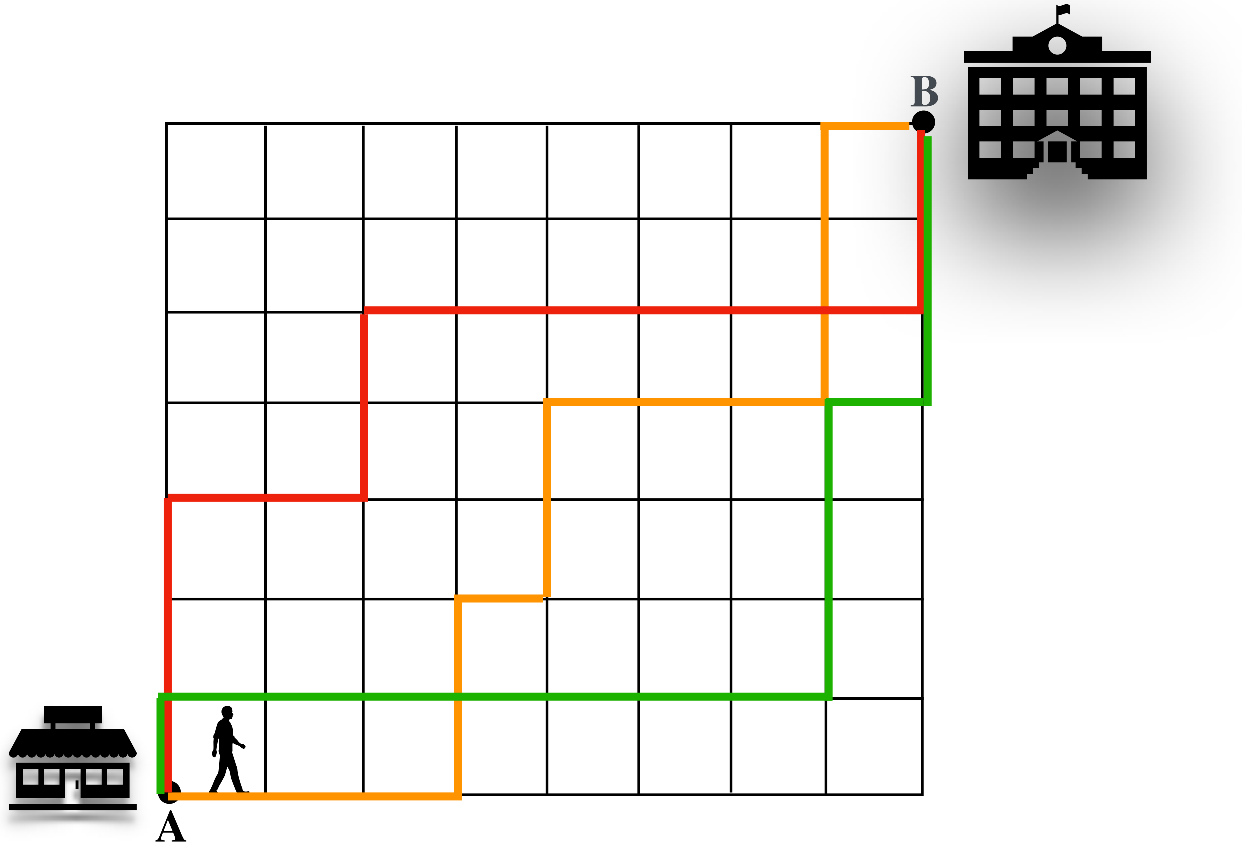


$$|S| = n \implies \mathrm{Cir}(n) = \frac{n!}{n} = (n-1)!$$

Necklace Arrangements:



$$|S| = n \implies \mathrm{Nec}(n) = \frac{(n-1)!}{2}$$



Path 1: RRRUURUURRRUUUR

Path 2: URRRRRRRUUURUUU

Path 3: UUURRUUURRRRRRUU

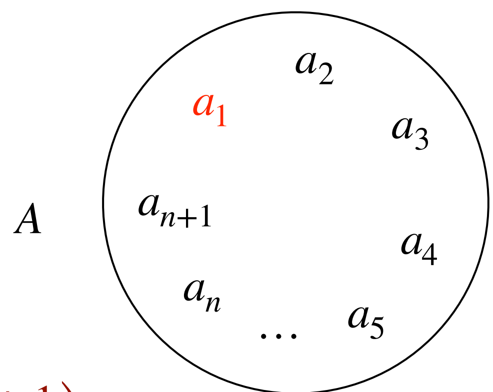
$$\binom{8+7}{8,7} = \frac{15!}{8!7!}$$

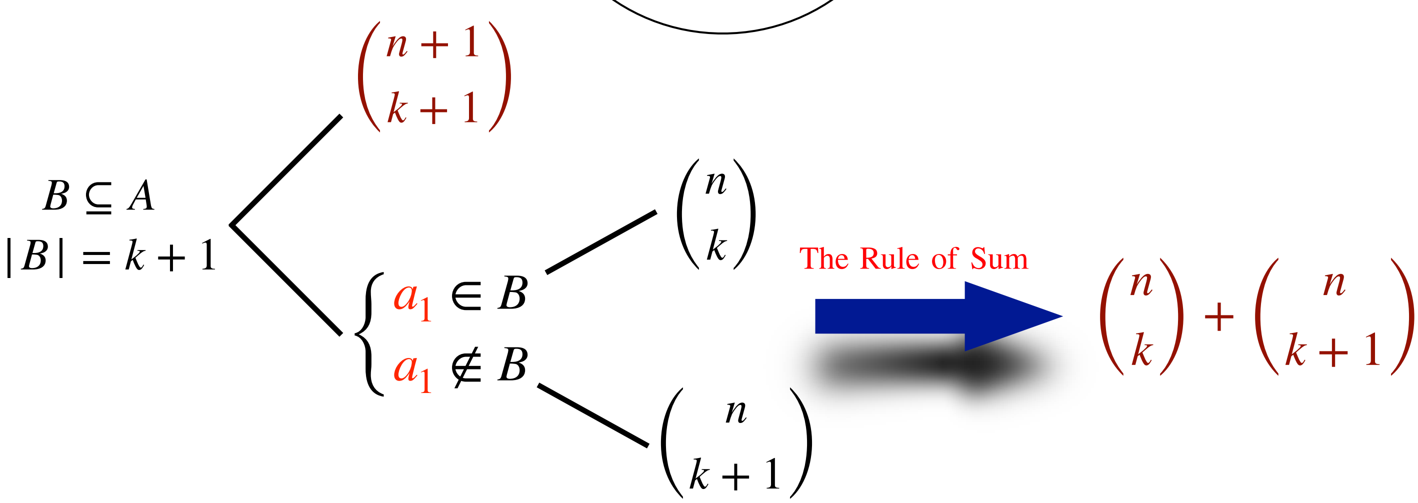
1. Prove that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

Algebraic Proof.

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}$$
$$= \frac{(k+1)n! + (n-k)n!}{(k+1)!(n-k)!}$$
$$= \frac{(n+1)n!}{(k+1)!(n-k)!} = \frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1}. \quad \square$$

Combinatorial proof.





$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}. \quad \square$$

2. *(Vandermonde's Identity)* Prove that

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Combinatorial proof.

- The **LHS** counts the number of ways to choose a committee of r people from a group of m men and n women.

- The **RHS** counts the same thing according to cases depending on the number of men on the committee, which can range from 0 to r. If there are t men, then there must be $r-t$ women. Since in such a case there are

$$\binom{m}{t}$$

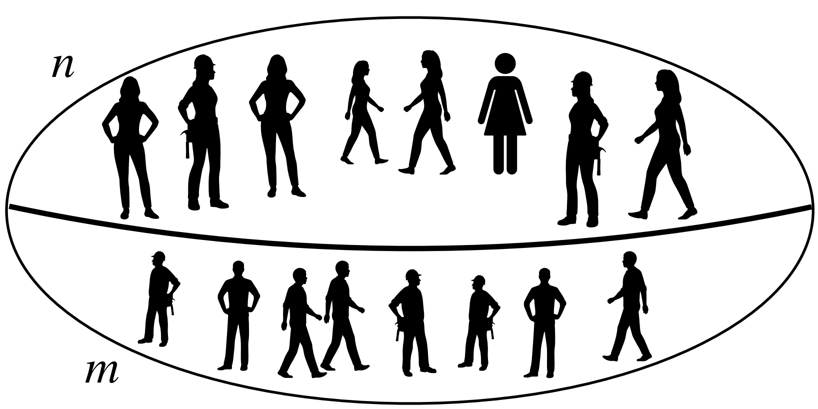
ways to select the men, and

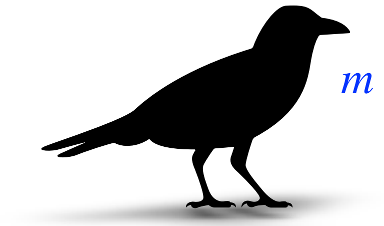
$$\binom{n}{r-t}$$

ways to select the women, the number of such committees is (the Rule of Product)

$$\binom{m}{t}\binom{n}{r-t}.$$

The result now follows from the Rule of Sum. $\square$





$m = n = r \implies$ $\boxed{\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2}$

3. Prove that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Proof. We have
$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Taking $a = -1$ and $b = 1$, we get

$$0 = (-1+1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} + \cdots + (-1)^n \binom{n}{n}$$

which implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots \quad \square$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$

4. Prove that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}.$$

Proof. Let

$$f(x) := (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Then, we obtain

$$f'(x) := n(1+x)^{n-1} = \sum_{i=1}^{n} \binom{n}{i} i x^{i-1}.$$

Taking $x = 1$, we get

$$n2^{n-1} = \sum_{i=1}^{n} \binom{n}{i} i,$$

as required. $\square$