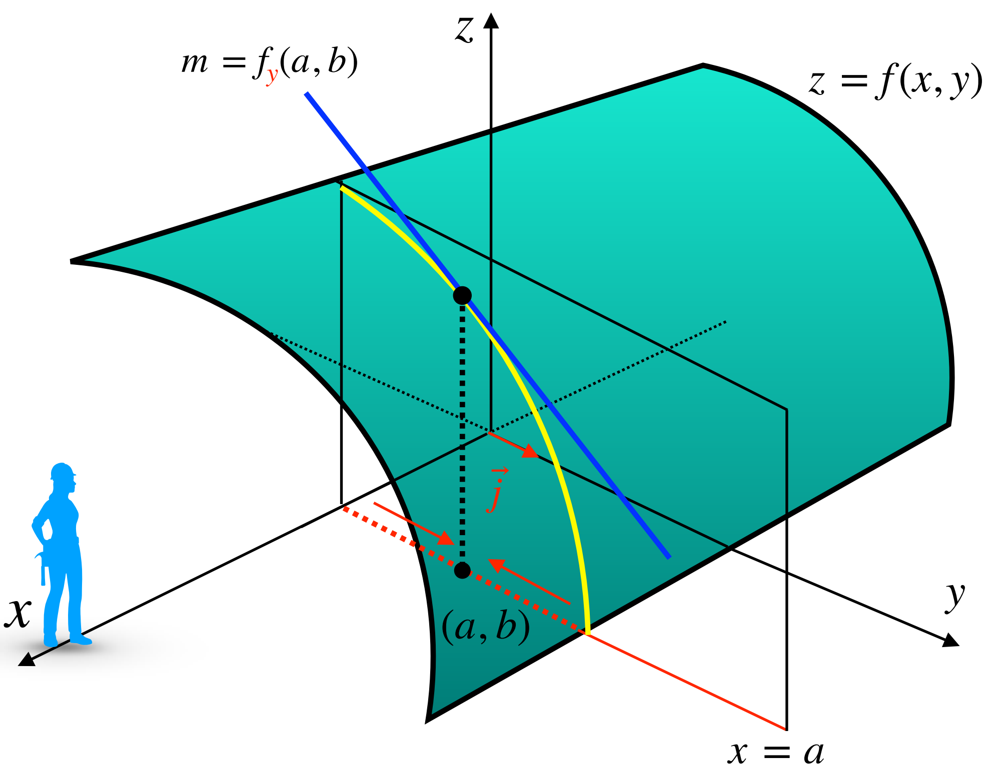
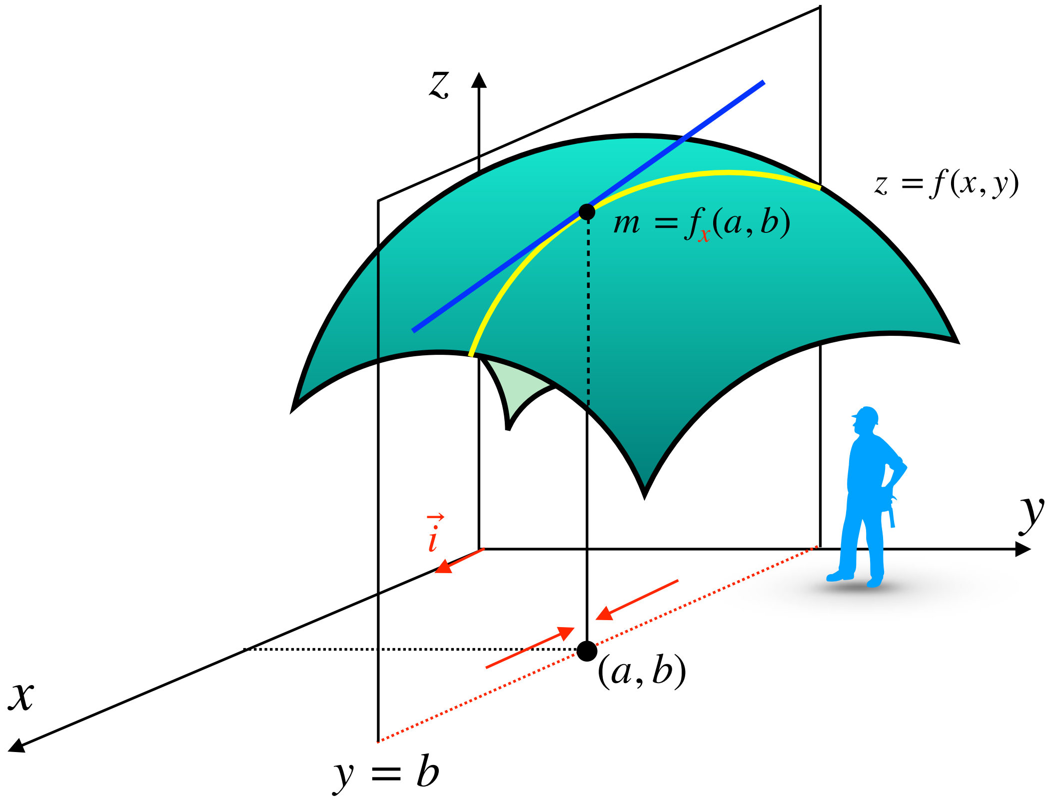
8. Directional Derivatives

Partial Derivatives

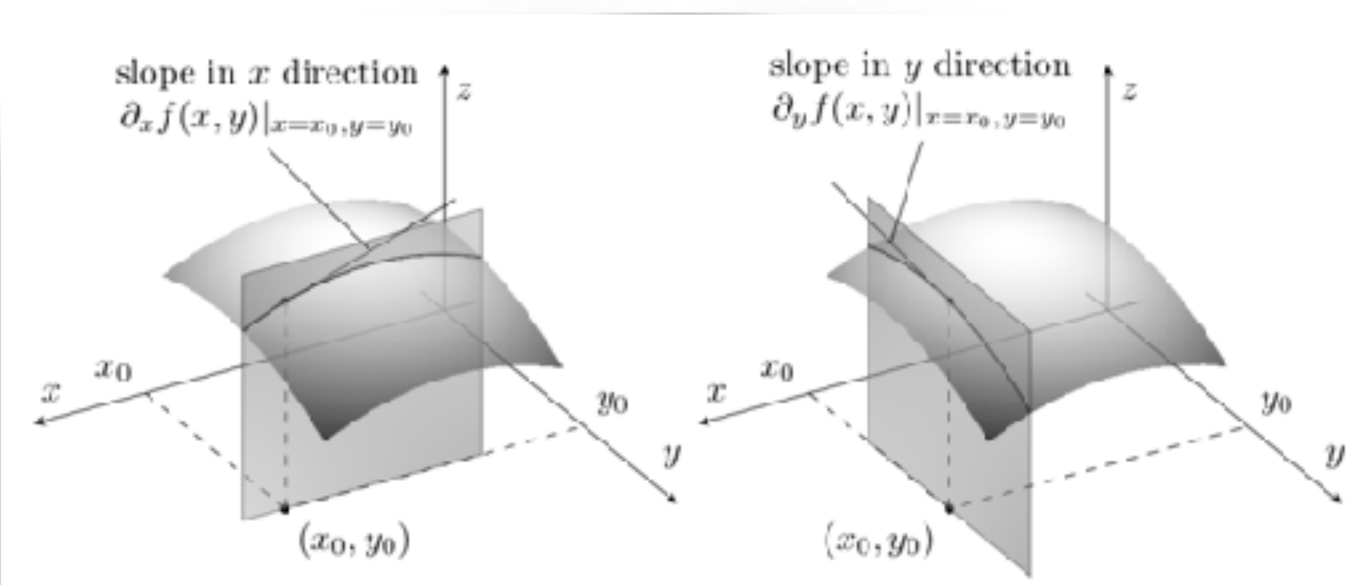
The formal definitions of the two partial derivatives. Given the function $z = f(x, y)$:

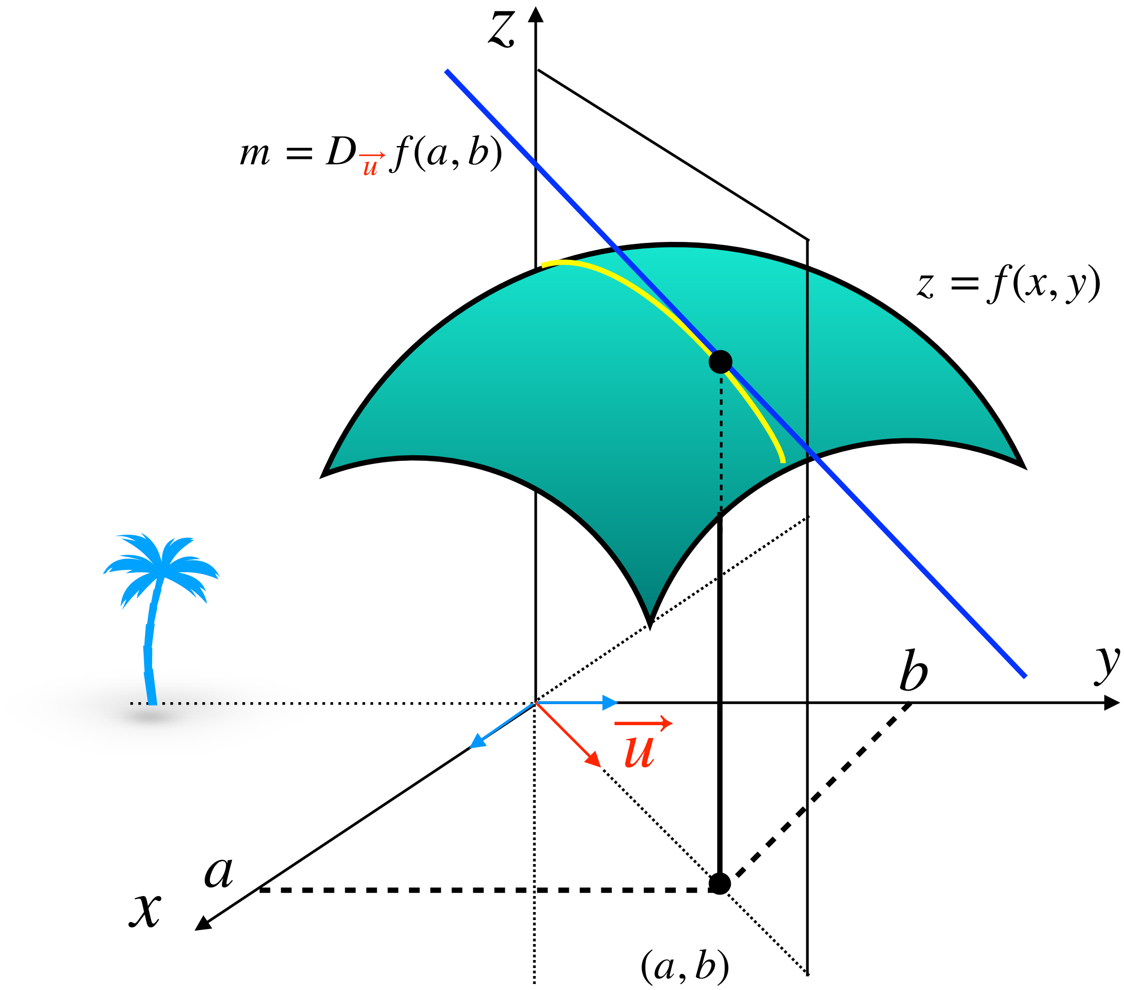
$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} = D_{\vec{j}} f(x, y)\,.$$





$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = D_{\vec{i}} f(x, y) .$$





Directional Derivative

Directional Derivative of $f(x,y)$ at (a,b) in the Direction of a unit vector $\vec{u}$:

If $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, we define the direction derivative $D_{\vec{u}} f$ at the point (a,b) by

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

provided that the limit exists.

If $\overrightarrow{u} = \vec{i}$, then $u_1 = 1$ and $u_2 = 0$, and we have

$$D_{\vec{i}} f(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b).$$

If $\overrightarrow{u} = \vec{j}$, then $u_1 = 0$ and $u_2 = 1$, and we have

$$D_{\vec{j}} f(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h} = f_y(a,b).$$

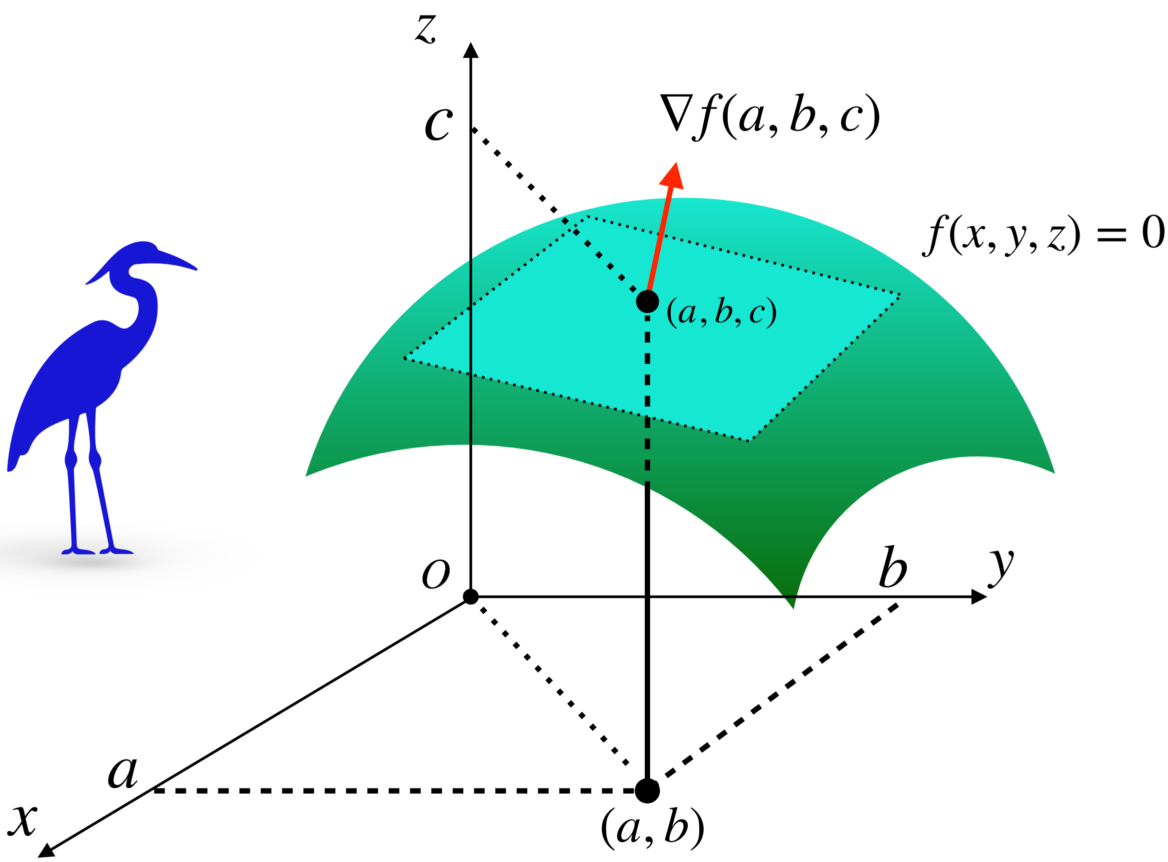
$$D_{\overrightarrow{u}} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \overrightarrow{u}$$

$$D_{\overrightarrow{u}} f(a,b,c) = \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle \cdot \overrightarrow{u}$$

The **gradient of** f or **gradient vector of** f

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

$$\nabla f(a,b,c) = \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle$$



Example 1 Find each of the directional derivatives.

1. $D_{\vec{u}} f(2,0)$ where $f(x,y) = xe^{xy} + y$ and $\vec{u}$ is the unit vector in the direction of $\theta = \dfrac{2\pi}{3}$.

Solution. The unit vector giving the direction is,

$\vec{u} = \langle \cos\theta, \sin\theta \rangle = \langle \cos\left(\dfrac{2\pi}{3}\right), \sin\left(\dfrac{2\pi}{3}\right) \rangle = \langle -\dfrac{1}{2}, \dfrac{\sqrt{3}}{2} \rangle$.

Moreover, we have

$$\nabla f(x,y) = \left\langle e^{xy} + xye^{xy}, x^2e^{xy} + 1 \right\rangle.$$

So, the directional derivative is, $D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$:

$$D_{\vec{u}} f(x, y) = \left(-\frac{1}{2}\right)(e^{xy} + xye^{xy}) + \left(\frac{\sqrt{3}}{2}\right)(x^2e^{xy} + 1)\,.$$

Now, plugging in the point in question gives,

$$D_{\vec{u}} f(2,0) = \left(-\frac{1}{2}\right)(1) + \left(\frac{\sqrt{3}}{2}\right)(5) = \frac{5\sqrt{3} - 1}{2}\,. \square$$

2. $D_{\vec{u}} f(x, y, z)$ where $f(x, y, z) = x^2z + y^3z^2 - xyz$ in the direction of $\vec{v} = \langle -1,0,3 \rangle$.

Solution. Since $\|\vec{v}\| = \sqrt{1 + 0 + 9} = \sqrt{10}$, we obtain

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{10}}\langle -1,0,3 \rangle = \langle -\frac{1}{\sqrt{10}},0,\frac{3}{\sqrt{10}} \rangle\,.$$

Moreover , we have

$$\nabla f(x, y, z) = \langle 2xz - yz, 3y^2z^2 - xz, x^2 + 2y^3z - xy \rangle\,.$$

The directional derivative is then,

$$D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u} =$$
$$\left(-\frac{1}{\sqrt{10}}\right)(2xz - yz) + 0(3y^2z^2 - xz) + \left(\frac{3}{\sqrt{10}}\right)(x^2 + 2y^3z - xy) =$$

$$\frac{1}{\sqrt{10}}(3x^2 + 6y^3z - 3xy - 2xz + yz) . \square$$

3. $D_{\overrightarrow{u}} f(x, y)$ for $f(x, y) = x \cos y$ in the direction of $\overrightarrow{v} = \langle 2,1 \rangle$.

Solution. Since $\|\overrightarrow{v}\| = \sqrt{4+1} = \sqrt{5}$, we obtain

$$\overrightarrow{u} = \frac{\overrightarrow{v}}{\|\overrightarrow{v}\|} = \frac{1}{\sqrt{5}}\langle 2,1 \rangle = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle .$$

Moreover , we have

$$\nabla f(x, y) = \langle \cos y, -x \sin y \rangle .$$

The directional derivative is then,

$$D_{\overrightarrow{u}} f(x, y) = \nabla f(x, y) \cdot \overrightarrow{u} = \langle \cos y, -x \sin y \rangle \cdot \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$$

$$= \frac{1}{\sqrt{5}}(2 \cos y - x \sin y) . \square$$

4. $D_{\overrightarrow{u}} f(x, y, z)$ for $f(x, y, z) = \sin(yz) + \ln(x^2)$ at $(1,1,\pi)$ in the direction of $\overrightarrow{v} = \langle 1,1,-1 \rangle$.

Solution. Since $\|\overrightarrow{v}\| = \sqrt{1+1+1} = \sqrt{3}$, we obtain

$$\overrightarrow{u} = \frac{\overrightarrow{v}}{\|\overrightarrow{v}\|} = \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\rangle .$$

Moreover , we have

$$\nabla f(x,y,z) = \langle \frac{2}{x}, z\cos(yz), y\cos(yz)\rangle,$$

and so

$$\nabla f(1,1,\pi) = \langle \frac{2}{1}, \pi\cos(\pi), \cos(\pi)\rangle = \langle 2, -\pi, -1\rangle .$$

The directional derivative is then, $D_{\overrightarrow{u}} f(1,1,\pi) = \nabla f(1,1,\pi)\cdot \overrightarrow{u}$:

$$D_{\overrightarrow{u}} f(1,1,\pi) = \langle 2,-\pi,-1\rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\rangle$$

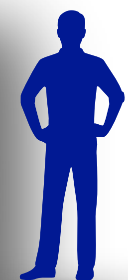
$$= \frac{1}{\sqrt{3}}(2-\pi+1) = \frac{3-\pi}{\sqrt{3}} . \ \square$$

Notation 1.

$$\nabla f = \langle f_x, f_y \rangle, \quad \text{and} \quad \nabla f = \langle f_x, f_y, f_z \rangle .$$

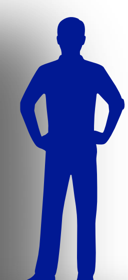
$$\nabla f = f_x \vec{i} + f_y \, \vec{j}, \qquad \nabla f = f_x \vec{i} + f_y \, \vec{j} + f_z \, \vec{k},$$

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$$



$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \, \vec{j},$$

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$



$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k},$$

Notation 2.

Scalar field: A scalar field associates a scalar value to every point in a space.

Vector field: A vector field is an assignment of a vector to each point in a subset of space.

Scalar field:

- $f : \mathbb{R} \to \mathbb{R}, \qquad y = f(x),$
- $f : \mathbb{R}^2 \to \mathbb{R}, \qquad z = f(x, y)$
- $f : \mathbb{R}^3 \to \mathbb{R}, \qquad v = f(x, y, z)$
- $f : \mathbb{R}^n \to \mathbb{R}, \qquad v = f(x_1, x_2, \ldots, x_n)$

Vector field:

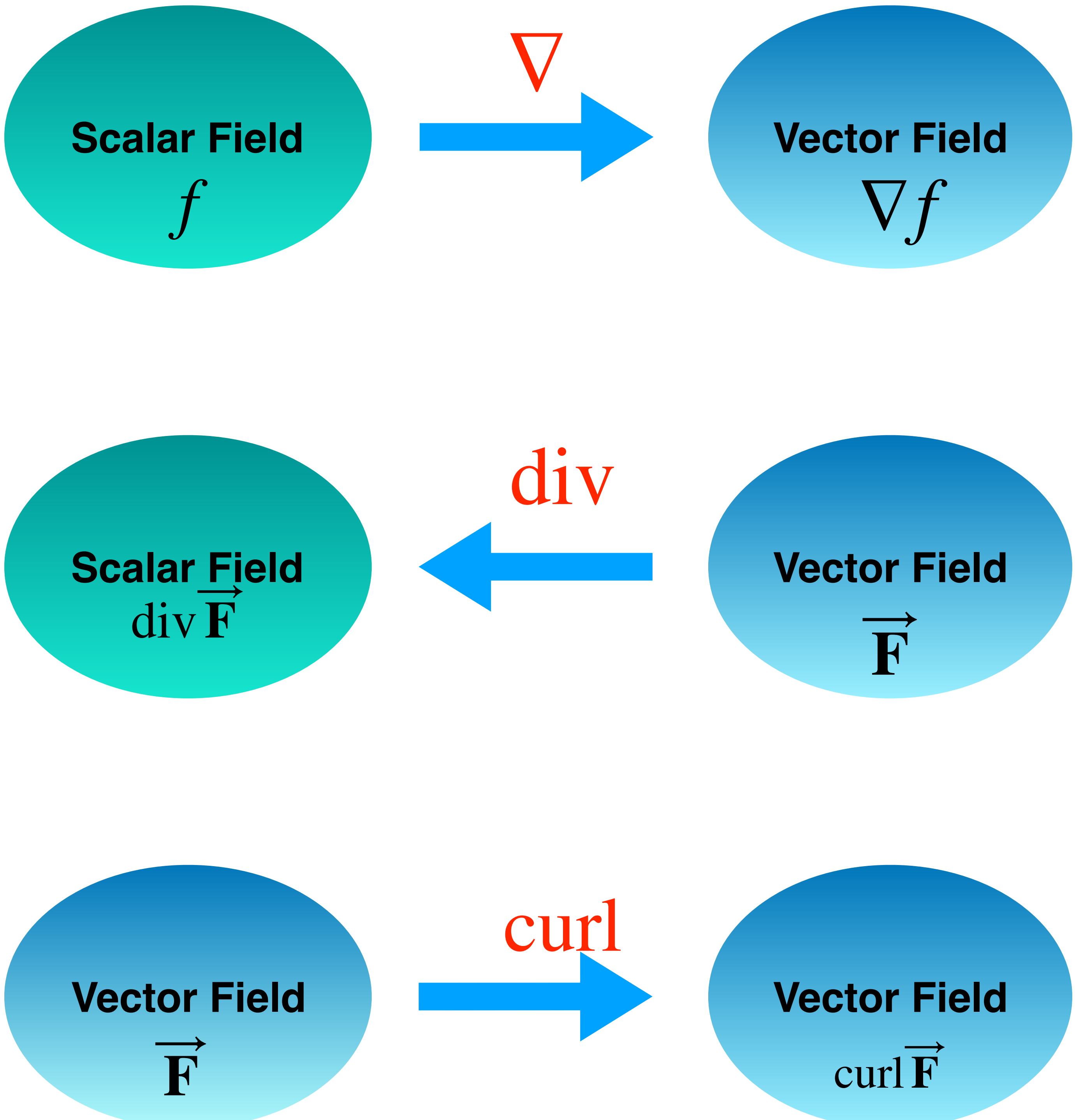
- $\vec{r} : \mathbb{R} \to \mathbb{R}^2, \qquad \vec{r}(t) = \langle f(t), g(t) \rangle$

- $\overrightarrow{\mathbf{F}} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\overrightarrow{\mathbf{F}}(x, y) = \langle M(x, y), N(x, y) \rangle = M(x, y)\vec{i} + N(x, y)\vec{j}\ .$$

- $f : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\begin{aligned}\overrightarrow{\mathbf{F}}(x, y, z) &= \langle M(x, y, z), N(x, y, z), R(x, y, z) \rangle\ . \\ &= M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + R(x, y, z)\,\vec{k}\end{aligned}$$


Scalar Field
f
∇
Vector Field
∇f
Scalar Field
$\operatorname{div} \overrightarrow{\mathbf{F}}$
div
Vector Field
$\overrightarrow{\mathbf{F}}$
Vector Field
$\overrightarrow{\mathbf{F}}$
curl
Vector Field
$\operatorname{curl} \overrightarrow{\mathbf{F}}$

$$\operatorname{div}\vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}}, \qquad \operatorname{curl}\vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}}.$$

$$\vec{\mathbf{F}}(x, y, z) = = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

$$\operatorname{div}\vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle M, N, R \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial R}{\partial z}$$

$$\operatorname{curl}\vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial M}{\partial z}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

Example 2 Compute $\operatorname{div}\vec{\mathbf{F}}$ for the vector field

$$\vec{\mathbf{F}}(x, y, z) = x^2y\vec{i} + xyz\vec{j} - x^2y^2\vec{k}.$$

We have

$$\operatorname{div} \overrightarrow{\mathbf{F}} = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(xyz)}{\partial y} + \frac{\partial(-x^2y^2)}{\partial z} = 2xy + xz\,.$$

$$\operatorname{div}\ (\operatorname{curl} \overrightarrow{\mathbf{F}}) = 0.$$

Proof. We have $\operatorname{div}\ (\operatorname{curl} \overrightarrow{\mathbf{F}}) = \nabla \cdot (\nabla \times \overrightarrow{\mathbf{F}}) = 0.$ $\square$

Example 3 Compute $\operatorname{curl} \overrightarrow{\mathbf{F}}$ for the vector field

$$\overrightarrow{\mathbf{F}}(x, y, z) = yz^2 \vec{i} + xy \vec{j} + yz \vec{k}\,.$$

Solution. We have

$$\operatorname{curl} \overrightarrow{\mathbf{F}} = \nabla \times \overrightarrow{\mathbf{F}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xy & yz \end{vmatrix} = z\vec{i} + 2yz\vec{j} + (y - z^2)\vec{k}\,.$$

Laplace operator

The next topic that we want to briefly mention is the Laplace operator:

$$\operatorname{div}\,(\nabla f) = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}\,.$$

The **Laplace operator** is then defined as,

$$\nabla^2 = \nabla \cdot \nabla$$

Practice Problems

For problems 1 & 2 compute $\operatorname{div}\overrightarrow{\mathbf{F}}$ and $\operatorname{curl}\overrightarrow{\mathbf{F}}$:

1. $\overrightarrow{\mathbf{F}}(x, y, z) = x^2y\,\vec{i} - (z^3 - 3x)\,\vec{j} + 4y^2\,\vec{k}\,.$
2. $\overrightarrow{\mathbf{F}}(x, y, z) = (3x + 2z^2)\,\vec{i} + \dfrac{x^3y^2}{z}\,\vec{j} - (z - 7x)z\,\vec{k}\,.$

Conservative Vector Fields

The vector field $\overrightarrow{\mathbf{F}}$ is *conservative* if and only if there exists a *potential function* f such that $\nabla f = \overrightarrow{\mathbf{F}}$.

The vector field $\overrightarrow{\mathbf{F}}$ is *conservative* if and only if $\operatorname{curl} \overrightarrow{\mathbf{F}} = 0$.

The vector field

$$\overrightarrow{\mathbf{F}}(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}$$

is *conservative* if and only if $\operatorname{curl} \overrightarrow{\mathbf{F}} = 0$, that is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The vector field

$$\overrightarrow{\mathbf{F}}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + R(x, y, z)\vec{k},$$

is *conservative* if and only if $\operatorname{curl} \overrightarrow{\mathbf{F}} = 0$, that is

$$\frac{\partial R}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial R}{\partial x} = \frac{\partial M}{\partial z}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Example 4 Determine if the following vector fields are conservative or not.

1. $\overrightarrow{\mathbf{F}}(x, y) = (x^2 - yx)\vec{i} + (y^2 - xy)\vec{j}$.
2. $\overrightarrow{\mathbf{F}}(x, y) = (2xe^{xy} + x^2ye^{xy})\vec{i} + (x^3e^{xy} + 2y)\vec{j}$.
3. $\overrightarrow{\mathbf{F}}(x, y) = (2x^3y^4 + x)\vec{i} + (2x^4y^3 + y)\vec{j}$.
4. $\overrightarrow{\mathbf{F}}(x, y, z) = 2xy^3z^4\vec{i} + 3x^2y^2z^4\vec{j} + 4x^2y^3z^3\vec{k}$.
5. $\overrightarrow{\mathbf{F}}(x, y, z) = (2x\cos y - 2z^3)\vec{i} + (3 + 2ye^z - x^2\sin y)\vec{j} + (y^2e^z - 6xz^2)\vec{k}$.