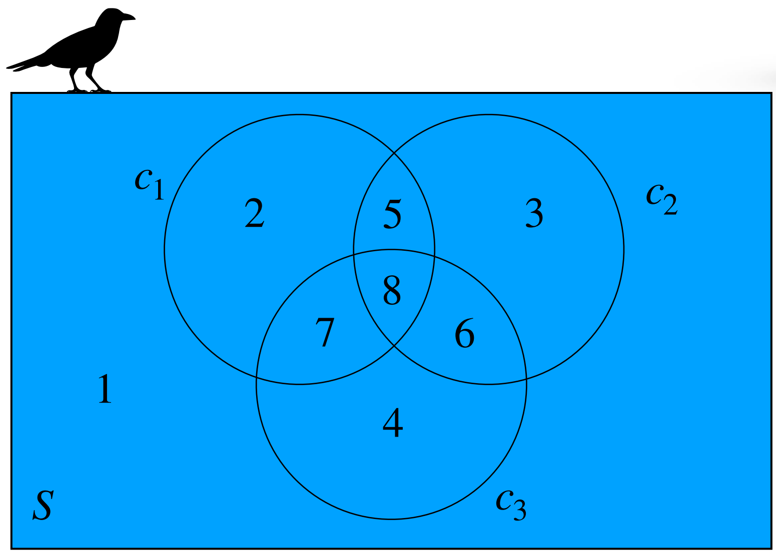
A Generalization of the Principle of Inclusion and Exclusion



$$E_1 = 2 \cup 3 \cup 4$$

$$E_1 = N(c_1) + N(c_2) + N(c_3) - 2[N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] + 3N(c_1c_2c_3)$$

$$E_1 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3$$

$$E_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - 3N(c_1c_2c_3)$$

$$E_2 = S_2 - \binom{3}{1} S_3$$

Theorem 2. *(Generalization of the Principle of Inclusion and Exclusion)* *Consider a set* S, *with* $|S| = N$, *and conditions* c_i, $1 \leqslant i \leqslant t$, *each of which may be satisfied by some of the elements of* S. *The number of elements of* S *that satisfy* *exactly* m *of the conditions* c_i, $1 \leqslant i \leqslant t$, *is given by*

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{t-m} \binom{t}{t-m} S_t .$$

If $m = 0$ *we obtain Theorem* 1.

A combinatorial proof. $x \in S$

- x satisfies *fewer* than m conditions.

$$0 = 0$$

- x satisfies *exactly* m conditions.

$$1 = 1$$

- x satisfies r of the conditions, where $m < r \leqslant t$.

$$0 = ?$$

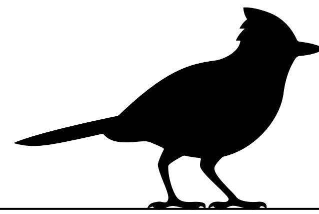
x is counted:

$$\binom{r}{m} \text{ times in } S_m,$$

$$\binom{r}{m+1} \text{ times in } S_{m+1},$$

$$\vdots$$

$$\binom{r}{r} \text{ times in } S_r,$$



So on the right-hand side, x is counted:

$$\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{t}{r-m}\binom{r}{r}.$$

But, for $0 \leqslant k \leqslant r-m$, we have

$$\begin{aligned}
\binom{m+k}{k}\binom{r}{m+k} &= \frac{(m+k)!}{k!m!}\frac{r!}{(m+k)!(r-m-k)!} \\
&= \frac{\cancel{(m+k)!}}{k!m!}\frac{r!}{\cancel{(m+k)!}(r-m-k)!} \\
&= \frac{1}{k!m!}\frac{r!}{(r-m-k)!} \\
&= \frac{r!}{m!(r-m)!}\frac{(r-m)!}{k!(r-m-k)!} \\
&= \binom{r}{m}\binom{r-m}{k}
\end{aligned}$$

$$\binom{r}{m} - \binom{m+1}{1}\binom{r}{m+1} + \binom{m+2}{2}\binom{r}{m+2} - \cdots + (-1)^{r-m}\binom{t}{r-m}\binom{r}{r}$$

$$= \binom{r}{m}\binom{r-m}{0} + \binom{r}{m}\binom{r-m}{1} + \binom{r}{m}\binom{r-m}{2} + \cdots + (-1)^{r-m}\binom{r}{m}\binom{r-m}{r-m}$$

$$= \binom{r}{m}\left[\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} - \cdots + (-1)^{r-m}\binom{r-m}{r-m}\right]$$

$$= \binom{r}{m}[1-1]^{r-m} = \binom{r}{m} \cdot 0 = 0.$$

$$\boxed{0} = \boxed{0}$$

and the formula is verified. □

$$\boxed{L_m = S_m - \binom{m}{m-1}S_{m+1} + \binom{m+1}{m-1}S_{m+2} - \cdots + (-1)^{t-m}\binom{t-1}{m-1}S_t.}$$

When $m = 1$, we have

$$L_1 = S_1 - \binom{1}{0}S_2 + \binom{2}{0}S_3 - \cdots + (-1)^{t-1}\binom{t-1}{0}S_t.$$

$$= S_1 - S_2 + S_3 - \cdots + (-1)^{t-1}S_t.$$

$$= S_0 - \left(S_0 - S_1 + S_2 - S_3 - \cdots + (-1)^t S_t\right) = |S| - \overline{N}.$$

Derangements: Nothing Is in Its Right Place

We want to arrange the numbers $1, 2, 3, \ldots, n$ so that:

- 1 is not in first place,
- 2 is not in second place,

  $\vdots$
- n is not in nth place,

These arrangements are called the *derangements* of $1, 2, 3, \ldots, n$.

- Condition c_1: *1 is in first place,*
- Condition c_2: *2 is in second place,*
- Condition c_3: *3 is in third place,*

  $\vdots$
- Condition c_n: n is in nth place.

$$d_n = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n$$

$$S_0 = n!$$

$$S_1 = \binom{n}{1}(n-1)! = \frac{n!}{1!\cancel{(n-1)!}}\cancel{(n-1)!} = \frac{n!}{1!},$$

$$S_2 = \binom{n}{2}(n-2)! = \frac{n!}{2!\cancel{(n-2)!}}\cancel{(n-2)!} = \frac{n!}{2!},$$

$$S_3 = \binom{n}{3}(n-3)! = \frac{n!}{3!\cancel{(n-3)!}}\cancel{(n-3)!} = \frac{n!}{3!},$$

$$\vdots$$

$$S_n = \binom{n}{n}(n-n)! = \frac{n!}{n!\cancel{(n-n)!}}\cancel{(n-n)!} = \frac{n!}{n!},$$

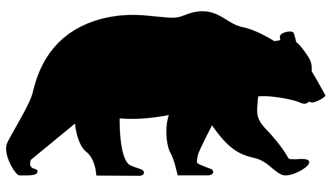
$$d_n = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \cdots + (-1)^n \frac{n!}{n!}$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}\right]. \square$$

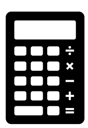
Example 1. The number of derangements of 1,2,3,4 is

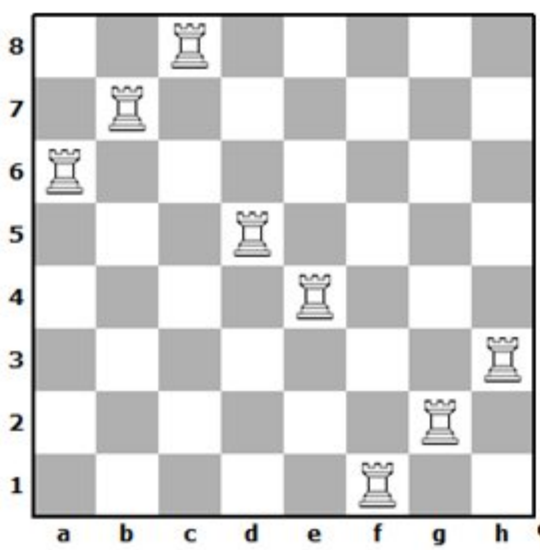
$$d_4 = 4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = (4)(3) - 4 + 1 = 9.$$

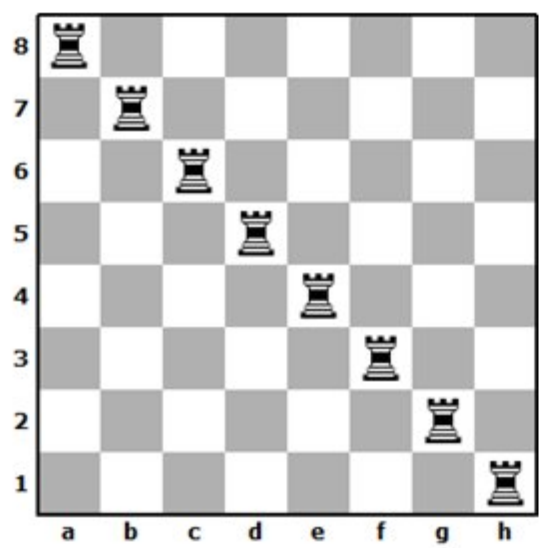


2143	3142	4123
2341	3412	4312
2413	3421	4321

Rook Polynomial





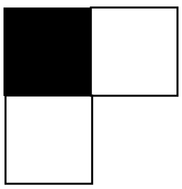


C

$$r(C, x) = 1 + x$$

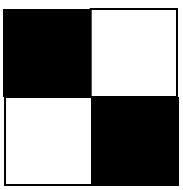
C

$$r(C, x) = 1 + 2x$$



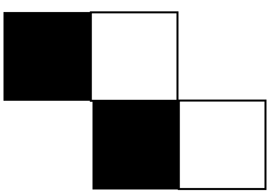
C

$$r(C, x) = 1 + 3x + x^2$$



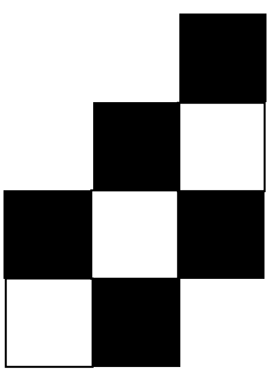
C

$$r(C, x) = 1 + 4x + 2x^2$$



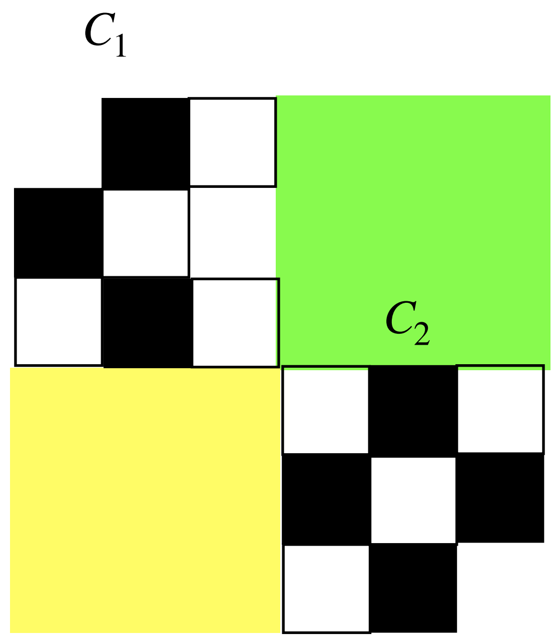
C

$$r(C, x) = 1 + 4x + 3x^2$$



$$r(C, x) = 1 + 8x + 16x^2 + 7x^3$$

C



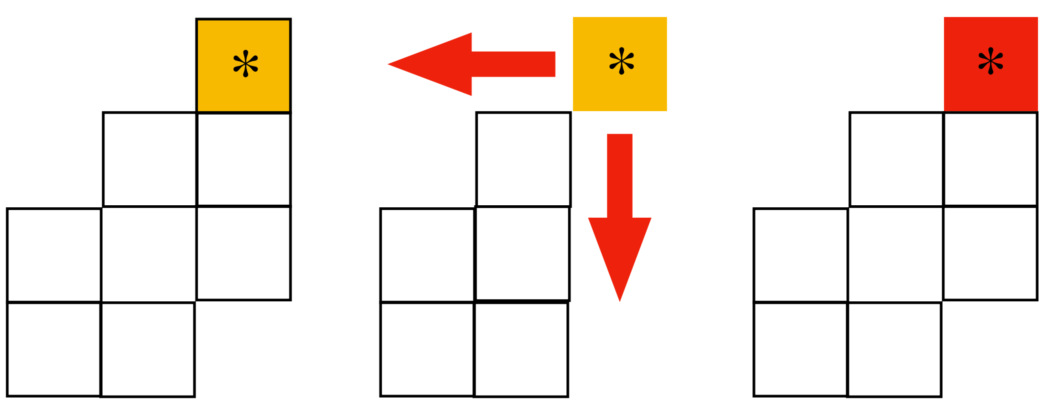
C

$$C = C_1 \cup C_2$$

$$r(C, x) = r(C_1, x) \cdot r(C_2, x)$$

$$C = C_1 \cup C_2 \cup \cdots \cup C_n$$
$$r(C, x) = r(C_1, x) \cdot r(C_2, x) \ \cdots \ r(C_n, x)$$

$$r(C, x) = r_0(C) + r_1(C)x + r_2(C)x^2 + \ \cdots \ + r_n(C)x^n$$



C $\qquad$ C_s $\qquad$ C_e

$$r_k(C) = r_{k-1}(C_s) + r_k(C_e)$$

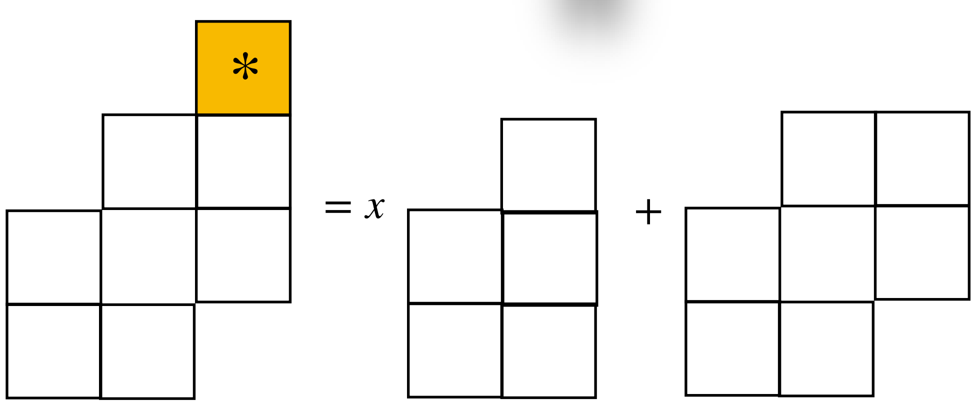
$$r_k(C)x^k = r_{k-1}(C_s)x^k + r_k(C_e)x^k$$

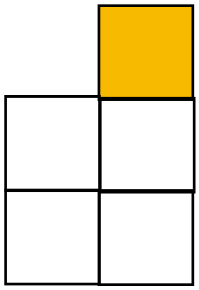
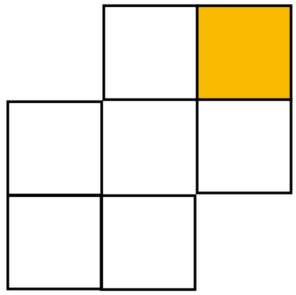
$$\sum_{k=1}^{n} r_k(C)x^k = \sum_{k=1}^{n} r_{k-1}(C_s)x^k + \sum_{k=1}^{n} r_k(C_e)x^k$$

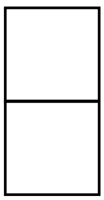
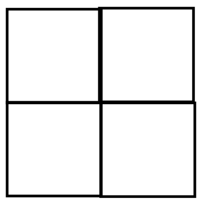
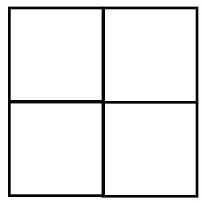
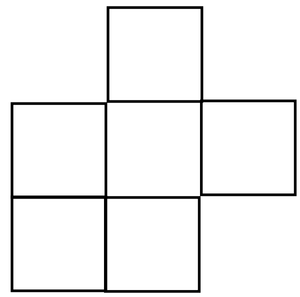
$$\sum_{k=1}^{n} r_k(C)x^k = x\sum_{k=1}^{n} r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^{n} r_k(C_e)x^k$$

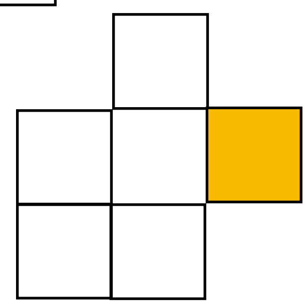
$$\sum_{k=0}^{n} r_k(C)x^k = x\sum_{k=1}^{n} r_{k-1}(C_s)x^{k-1} + \sum_{k=0}^{n} r_k(C_e)x^k$$

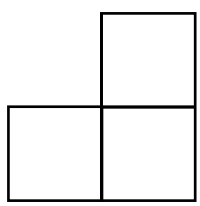
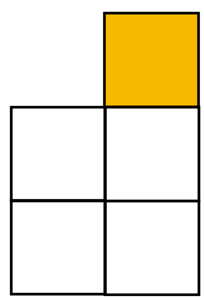
$$\boxed{r(C,x) = x\ r(C_s,x) + r(C_e,x)}$$

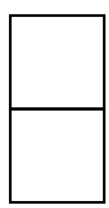
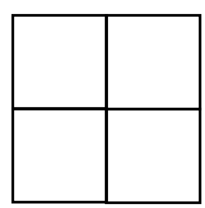


$= x($  $) + ($  $)$

$= x(x$  $+$  $) + (x$  $+$  $)$

$= x[x(1 + 2x) + (1 + 4x + 2x^2)] + x(1 + 4x + 2x^2) + ($  $)$

$= 2x + 9x^2 + 6x^3 + (x$  $+$  $)$

$= 2x + 9x^2 + 6x^3 + x(1 + 3x + x^2) + (x$  $+$  $)$

$$= 3x + 12x^2 + 7x^3 + [x(1 + 2x) + (1 + 4x + 2x^2)]$$

$$= 1 + 8x + 16x^2 + 7x^3 . \square$$