Problem 1. *Prove that* $0^2 + 1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Solution. We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

$$\Longrightarrow \quad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots,$$

$$\xrightarrow{\times x} \quad \frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots,$$

$$\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}(0 + x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots),$$

$$\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1} + \cdots,$$

$$\xrightarrow{\times x} \quad \frac{x(x+1)}{(1-x)^3} = 0 + 1x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n + \cdots,$$

$$\Longrightarrow \quad f(x) = 0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n + \cdots,$$

$$g(x) = \frac{f(x)}{1-x} = \frac{1}{1-x} \cdot \frac{x(x+1)}{(1-x)^3} = \frac{x(x+1)}{(1-x)^4}$$

$$\Longrightarrow \quad 0^2,\ \ 0^2 + 1^2,\ \ 0^2 + 1^2 + 2^2,\ \ 0^2 + 1^2 + 2^2 + 3^2,\ \ \ldots,$$

The coefficient of x^n in

$$g(x) = \frac{x(x+1)}{(1-x)^4}$$

is

$$0^2 + 1^2 + 2^2 + 3^2 + \cdots + n^2 .$$

$$g(x) = \frac{x(x+1)}{(1-x)^4} = \underbrace{(x+x^2)}_{a_i}\underbrace{\left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x^2) + \cdots\right]}_{b_i}$$

$$\begin{aligned}
c_n &= a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_{n-1}b_1 + a_nb_0 \\
&= a_1b_{n-1} + a_2b_{n-2} \\
&= 1\binom{-4}{n-1}(-1)^{n-1} + 1\binom{-4}{n-2}(-1)^{n-2} \\
&= (-1)^{n-1}\binom{4+(n-1)-1}{n-1}(-1)^{n-1} + (-1)^{n-2}\binom{4+(n-2)-1}{n-2}(-1)^{n-2} \\
&= \binom{n+2}{n-1} + \binom{n+1}{n-2} = \frac{(n+2)!}{(n-1)!3!} + \frac{(n+1)!}{(n-2)!3!}
\end{aligned}$$

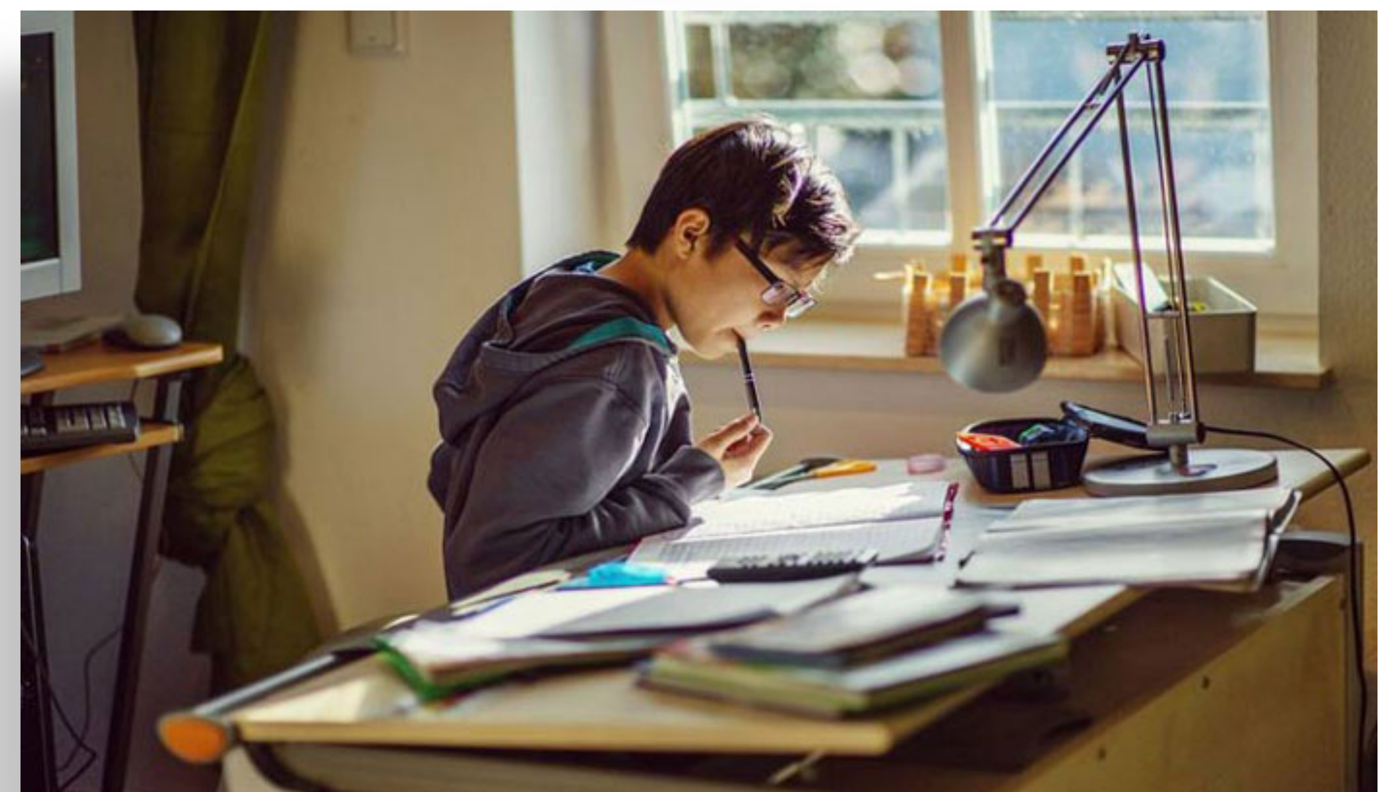
$$= \frac{(n+2)(n+1)n}{3!} + \frac{(n+1)n(n-1)}{3!} = \frac{n(n+1)}{6}\Big[n+2+n-1\Big]$$

$$= \frac{n(n+1)(2n+1)}{6}. \square$$

Problem 2. *Prove that* $0^3 + 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$

Solution. Similar to **Problem 1**. $\square$

$$0^3 + 1^3 + 2^3 + 3^3 + \cdots + n^3 = (0+1+2+3+\cdots+n)^2.$$



Problem 3. *Show that the number of partitions of a positive integer n where no summand appears more than twice equals the number of partitions of n where no summand is divisible by* 3.

Solution. On the one hand, we have

$$
\begin{array}{lcl}
1 & \Rightarrow & 1+x+x^2 \\
2 & \Rightarrow & 1+x^2+x^4 \\
3 & \Rightarrow & 1+x^3+x^6 \\
\vdots & \vdots & \vdots \\
n & \Rightarrow & 1+x^n+x^{2n} \\
\vdots & \vdots & \vdots
\end{array}
$$

$$f(x) = (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\cdots(1+x^n+x^{2n})\cdots$$

On the other hand, we have

$$
\begin{array}{lcl}
1 & \Rightarrow & 1+x+x^2+x^3+x^4+x^5+\cdots = \left(\dfrac{1}{1-x}\right) \\
2 & \Rightarrow & 1+x^2+x^4+x^6+x^8+x^{10}+\cdots = \left(\dfrac{1}{1-x^2}\right) \\
4 & \Rightarrow & 1+x^4+x^8+x^{12}+x^{16}+x^{20}+\cdots = \left(\dfrac{1}{1-x^4}\right) \\
5 & \Rightarrow & 1+x^5+x^{10}+x^{15}+x^{20}+x^{25}+\cdots = \left(\dfrac{1}{1-x^5}\right) \\
7 & \Rightarrow & 1+x^7+x^{14}+x^{21}+x^{28}+x^{35}+\cdots = \left(\dfrac{1}{1-x^7}\right) \\
\vdots & \vdots & \vdots
\end{array}
$$

$$g(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^7}\right)\cdots$$

Since

$$1+x+x^2 = \frac{1-x^3}{1-x}, \quad 1+x^2+x^4 = \frac{1-x^6}{1-x^2}, \quad 1+x^3+x^6 = \frac{1-x^9}{1-x^3}, \quad \cdots$$

we obtain

$$\begin{aligned} f(x) &= (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\cdots(1+x^n+x^{2n})\cdots \\ &= \left(\frac{\cancel{1-x^3}}{1-x}\right)\left(\frac{\cancel{1-x^6}}{1-x^2}\right)\left(\frac{\cancel{1-x^9}}{\cancel{1-x^3}}\right)\left(\frac{\cancel{1-x^{12}}}{1-x^4}\right)\left(\frac{\cancel{1-x^{15}}}{1-x^5}\right)\cdots \\ &= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-x^7}\right)\cdots = g(x) \end{aligned}$$

The coefficient of x^n in $f(x)$ = The coefficient of x^n in $g(x)$

the number of partitions of a positive integer n where no summand appears more than twice = the number of partitions of n where no summand is divisible by 3. $\square$

Problem 4. *Show that the number of partitions of a positive integer n where no summand is divisible by 4 equals the number of partitions of n where no even summand is repeated.*