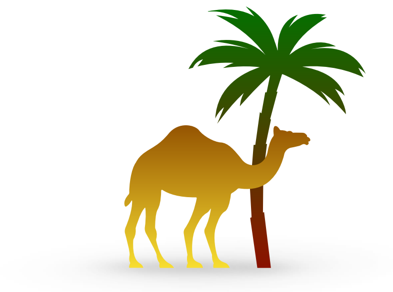

Problems and Solutions:

1. *How many functions* $f : \{1,2,3,4,5\} \to \{a, b, c, d\}$ *are there?*

Solution. We have

$$f : \begin{array}{l} 1 \to \boxed{} \\ 2 \to \boxed{} \\ 3 \to \boxed{} \\ 4 \to \boxed{} \\ 5 \to \boxed{} \end{array} \Longrightarrow f : \begin{array}{l} 1 \to \boxed{4} \\ 2 \to \boxed{4} \\ 3 \to \boxed{4} \\ 4 \to \boxed{4} \\ 5 \to \boxed{4} \end{array} \Longrightarrow 4 \times 4 \times 4 \times 4 \times 4 = 4^5$$

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2. *How many functions* $f : \{1,2,3\} \to \{1,2,3,4,5,6,7,8\}$ *are injective?*

Solution. We have

$$f : \begin{array}{l} 1 \to \boxed{} \\ 2 \to \boxed{} \\ 3 \to \boxed{} \end{array} \Longrightarrow f : \begin{array}{l} 1 \to \boxed{8} \\ 2 \to \boxed{7} \\ 3 \to \boxed{6} \end{array} \Longrightarrow 8 \times 7 \times 6 = P(8,3)$$

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3. *How many integer solutions are there to the equation*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13.$$

*(An **integer solution** to an equation is a solution in which the unknown must have an integer value.)*

1. where $x_i \geq 0$ for each x_i?

2. where $x_i > 0$ for each x_i?

3. where $x_i \geq 2$ for each x_i?

Solution. We have

1. $$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \xrightarrow{x_i \geq 0}$$

$$\boxed{} + \boxed{} + \boxed{} + \boxed{} + \boxed{} = 1,1,1,1,1,1,1,1,1,1,1,1,1$$

$$\longrightarrow \quad | \quad | \quad | \quad | \quad = 1,1,1,1,1,1,1,1,1,1,1,1,1$$

$$\longrightarrow 1,1 \,|\, 1,1,1 \,|\, 1,1,1,1,1 \,|\, 1 \,|\, 1,1$$

$$\longrightarrow \binom{13+5-1}{5-1} = \binom{17}{4}$$

2. $$x_1 + x_2 + x_3 + x_4 + x_5 = 13 \xrightarrow{x_i \geq 1}$$

$$\boxed{1} + \boxed{1} + \boxed{1} + \boxed{1} + \boxed{1} = 1,1,1,1,1,1,1,1$$

$$\longrightarrow 1 \,|\, 1 \,|\, 1 \,|\, 1 \,|\, 1 = 1,1,1,1,1,1,1,1$$

$$\longrightarrow 1\ 1,1 \,|\, 1 \,|\, 1\ 1,1,1 \,|\, 1 \,|\, 1\ 1,1,1 \longrightarrow \binom{8+5-1}{5-1} = \binom{12}{4}$$

3. $x_1 + x_2 + x_3 + x_4 + x_5 = 13 \xrightarrow{x_i \geq 2}$

$$\boxed{1,1} + \boxed{1,1} + \boxed{1,1} + \boxed{1,1} + \boxed{1,1} = 1,1,1$$

$$\Rightarrow 1,1 \mid 1,1 \mid 1,1 \mid 1,1\ 1 \mid 1,1\ 1,1 \Rightarrow \binom{3+5-1}{5-1} = \binom{7}{4}$$

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4. Determine the number of integer solutions for

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40,$$

where

A. $x_i \geqslant 0,\ 1 \leqslant i \leqslant 5,$

B. $x_i \geqslant -3,\ 1 \leqslant i \leqslant 5,$

Solution. A. We have

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40, \xrightarrow{x_i \geq 0,\ y \geq 1} x_1 + x_2 + x_3 + x_4 + x_5 + y = 40,$$

$$\boxed{} + \boxed{} + \boxed{} + \boxed{} + \boxed{} + \boxed{1} = 1,1,\ldots,1,1.$$

$$\Rightarrow \mid\ \mid\ \mid\ \mid\ \mid\ 1 = 1,1,1,\ldots,1,1$$

$$\Rightarrow \binom{39+6-1}{6-1} = \binom{44}{5}$$

$$8,\ 7,\ 3,\ 5,\ 15, \Longleftrightarrow 8,\ 7,\ 3,\ 5,\ 15,\ 2$$

B. We have $x_i \geqslant -3,\ 1 \leqslant i \leqslant 5,$

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40, \xrightarrow{y_i = x_i + 3 \geq 0,} y_1 + y_2 + y_3 + y_4 + y_5 < 55,$$

$$\xrightarrow{y_i \geq 0,\ y \geq 1} y_1 + y_2 + y_3 + y_4 + y_5 + y = 55,$$

$$\boxed{} + \boxed{} + \boxed{} + \boxed{} + \boxed{} + \boxed{1} = 1,1,\ldots,1,1.$$

$$\longrightarrow \quad |\ \ |\ \ |\ \ |\ \ |\ \ 1 \quad = 1,1,1,\ldots,1,1$$

$$\longrightarrow \binom{54+6-1}{6-1} = \binom{59}{5}$$

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5. Determine the number of integer solutions for

$$x_1 + x_2 + x_3 + x_4 = 32,$$

where

A. $x_i \geqslant 0,\ 1 \leqslant i \leqslant 4,$

B. $x_i > 0,\ 1 \leqslant i \leqslant 4,$

C. $x_1, x_2 \geqslant 5,\ \ x_3, x_4 \geqslant 4,$

D. $x_i \geqslant 8,\ 1 \leqslant i \leqslant 4,$

E. $x_i \geqslant -2,\ 1 \leqslant i \leqslant 4,$

F. $x_1, x_2, x_3 > 0,\ 0 < x_4 \leqslant 25,$

Solution. We have

A. $\binom{32+4-1}{4-1} = \binom{35}{3}$

B. $\binom{28+4-1}{4-1} = \binom{31}{3}$

C. $\binom{14+4-1}{4-1} = \binom{17}{3}$

D. $\binom{0+4-1}{4-1} = \binom{3}{3} = 1$

E. $\binom{40+4-1}{4-1} = \binom{43}{3}$ $\quad y_i = x_i + 2 \geq 0 \implies y_1 + y_2 + y_3 + y_4 = 40,$

F. $\binom{28+4-1}{4-1} - \binom{3+4-1}{4-1} = \binom{31}{3} - \binom{6}{3}$

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6. Find the following sums:

A. $\sum_{i=1}^{n} i\binom{n}{i} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$

Solution 1. We put $S = \sum_{i=0}^{n} i\binom{n}{i}$. Then we have

$$S = 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$$

Since $\binom{n}{i} = \binom{n}{n-i}$ we obtain

$$S = n\binom{n}{n} + (n-1)\binom{n}{n-1} + (n-2)\binom{n}{n-2} + \cdots + 0\binom{n}{0}$$
$$= n\binom{n}{0} + (n-1)\binom{n}{1} + (n-2)\binom{n}{2} + \cdots + 0\binom{n}{n}.$$

It follows from

$$S = 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$$
$$S = n\binom{n}{0} + (n-1)\binom{n}{1} + (n-2)\binom{n}{2} + \cdots + 0\binom{n}{n},$$

that

$$S + S = n\binom{n}{0} + n\binom{n}{1} + n\binom{n}{2} + n\binom{n}{3} + \cdots + n\binom{n}{n}$$
$$= n\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}\right] = n2^n,$$

and so $S = n2^{n-1}$. $\square$

Solution 2. We define

$$f(x) = (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

Then

$$f'(x) = n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}.$$

Putting $x = 1$, we get

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + 4\binom{n}{4} + \cdots + n\binom{n}{n},$$

as desired. $\square$

B. $$\sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i} = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n}$$

Solution. We define

$$f(x) = (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

Then

$$\int f(x)\, dx = \int (1+x)^n\, dx = \int \left[\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n\right] dx,$$

and so

$$\frac{(x+1)^{n+1}}{n+1} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \cdots + \binom{n}{n}\frac{x^{n+1}}{n+1} + C.$$

If $x = 0$, then $C = \dfrac{1}{n+1}$, an hence we have

$$\frac{(x+1)^{n+1}}{n+1} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \cdots + \binom{n}{n}\frac{x^{n+1}}{n+1} + \frac{1}{n+1}.$$

or equivalently

$$\frac{(x+1)^{n+1}-1}{n+1} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \cdots + \binom{n}{n}\frac{x^{n+1}}{n+1}.$$

Putting $x = 1$, we get

$$\frac{2^{n+1}-1}{n+1} = \binom{n}{0} + \binom{n}{1}\frac{1}{2} + \binom{n}{2}\frac{1}{3} + \cdots + \binom{n}{n}\frac{1}{n+1}. \ \square$$

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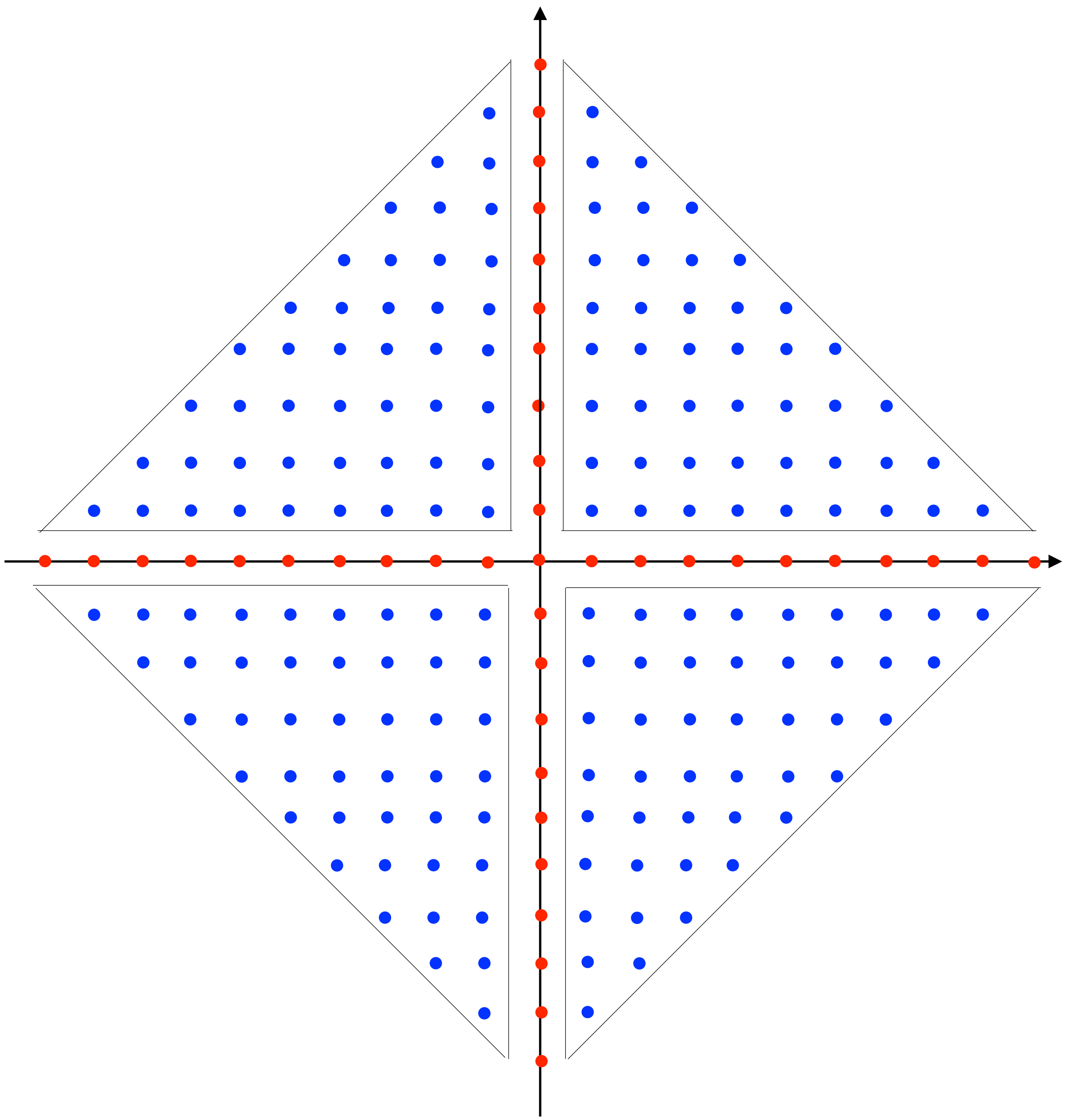
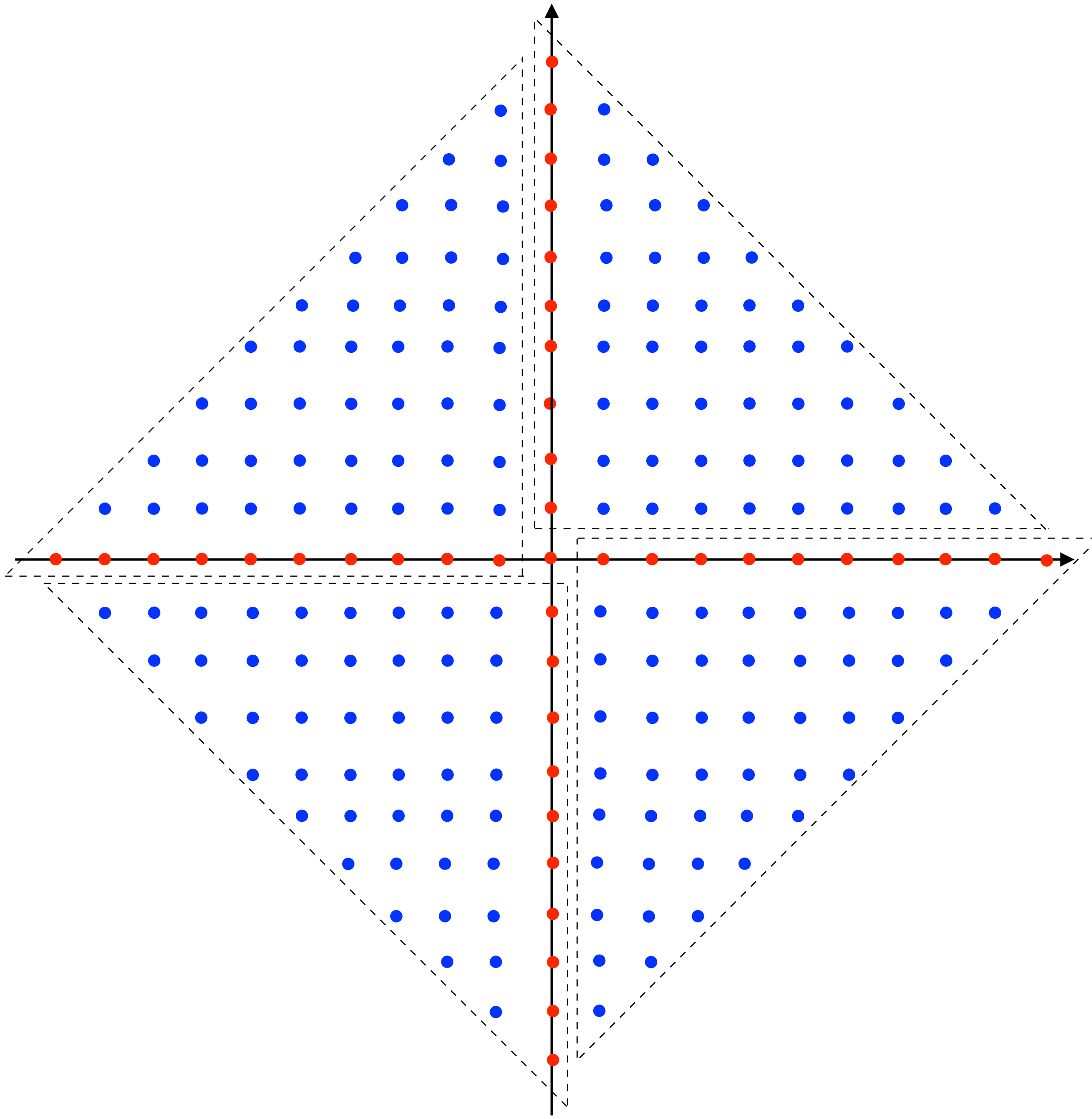
7. Find the number of integer solutions for the inequality $|x| + |y| \leqslant 10$.

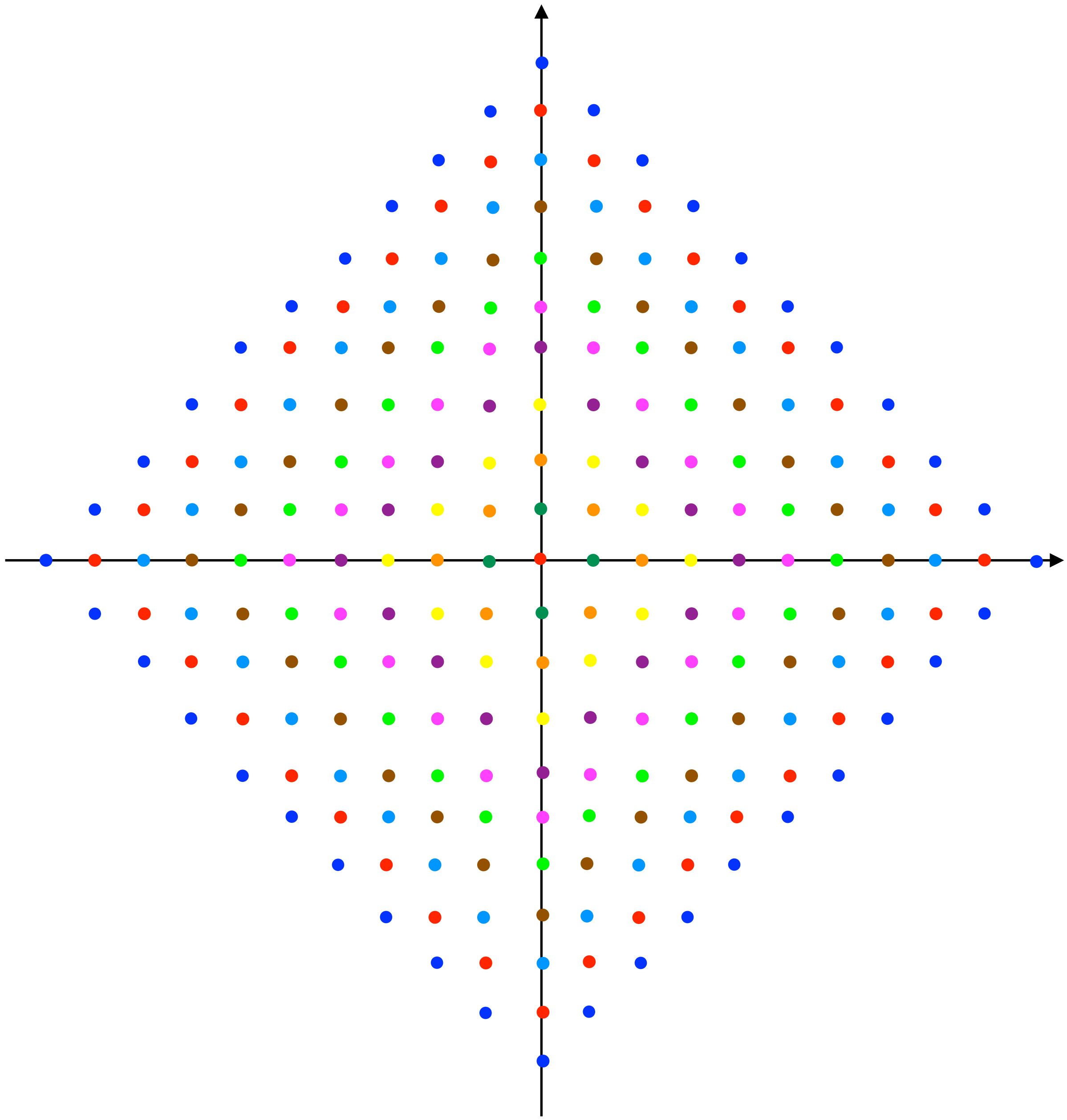
Solution. 221. $\square$



$$4(1+2+3+\cdots+9)+4(10)+1=4\frac{9(9+1)}{2}+41=221$$



$$4(1+2+\cdots+10)+1 = 4\frac{10(10+1)}{2}+1 = 221.$$



$$1+4+8+12+\cdots+40 = 1+4(1+2+3+\cdots+10) = 1+4\frac{10(10+1)}{2} = 221.$$