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Faculty of Mathematics

Problems 3 - Calculus II

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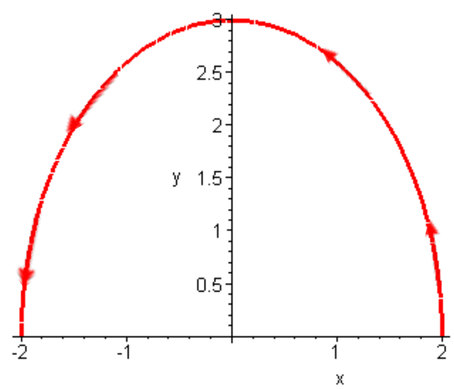
1. Find the work done by the force $\overrightarrow{\mathbf{F}}(x, y) = (2x + e^{-y})\ \vec{\mathbf{i}} + (4y - xe^{-y})\ \vec{\mathbf{j}}$ along the indicated curve:



Answer: -4.

2. Find a potential function for the given vector field

$$\overrightarrow{\mathbf{F}}(x, y, z) = (2xy^2 + 3xz^2)\ \vec{\mathbf{i}} + (2x^2y + 2y)\ \vec{\mathbf{j}} + (3x^2z - 2z)\ \overrightarrow{\mathbf{k}}.$$

Answer: $f(x, y, z) = x^2y^2 + \dfrac{3}{2}x^2z^2 + y^2 - z^2 + C.$

3. Find the line integral $\displaystyle\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r}$ of the vector field

$$\overrightarrow{\mathbf{F}}(x, y, z) = 3x^2z\ \vec{\mathbf{i}} + z^2\ \vec{\mathbf{j}} + (x^3 + 2yz)\ \overrightarrow{\mathbf{k}},$$

along the curve C parametrized by $\vec{r}'(t) = \left\langle \dfrac{\ln t}{\ln 2}, t^{\frac{3}{2}}, t\cos(\pi t) \right\rangle,\ 1 \leqslant t \leqslant 4.$

Answer: 159.

4. Let $\overrightarrow{\mathbf{F}}(x, y, z) = -y\ \vec{\mathbf{i}} + x\ \vec{\mathbf{j}} + \overrightarrow{\mathbf{k}}$ and let C be the portion of the helix given by $\vec{r}(t) = \langle \cos t, \sin t, \frac{t}{2\pi} \rangle$ on $[0,2\pi]$. Evaluate $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r}$.

Answer: $2\pi + 1$.

5. Evaluate $\int_C \overrightarrow{\mathbf{F}} \cdot d\vec{r}$, where $\overrightarrow{\mathbf{F}}(x, y, z) = \langle xy, yz, zx \rangle$ and $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $t \in [0,1]$.

Answer: $\frac{27}{28}$.

6. Compute $\operatorname{div} \overrightarrow{\mathbf{F}}$ for $\overrightarrow{\mathbf{F}}(x, y, z) = \langle x^2y, xyz, -x^2y^2 \rangle$.
Answer: $\operatorname{div} \overrightarrow{\mathbf{F}} = 2xy + xz$.

7. Find the Laplacian of $f(x, y, z) = x^2y^2z + 2xz$.

Answer: $2(x^2 + y^2)z$.

8. If $f(x, y, z) = 2xz - y^2z$, find $\nabla \times \nabla f$.
Answer: 0.

9. Find the line of intersection of two planes $x + y + z = 1$ and $x + 2y + 2z = 1$.
Answer: $x = 1, y = -t, z = t$.

10. Find the tangent line to the following curve:

$$C : \begin{cases} x^2 - 3xy + z^2 = 1, \\ 2x \tan^{-1}(xz) + 2y^2 - z = 1, \end{cases}$$

at point $(0,1,1)$.

Answer: $\dfrac{x}{8} = \dfrac{y-1}{3} = \dfrac{z-1}{12}$.

11. Find an equation of the line perpendicular to two vectors $\vec{u} = \langle 1,1,4 \rangle$ and $\vec{v} = \langle 0,-1,2 \rangle$ passing through the point $(0,1,3)$.

Answer: $x = 6t,\ y = -2t+1,\ z = -t+3$.

12. Find the area of the triangle with vertices at $A = (1,0,2), B = (3,1,0), C = (0,0,2)$.

Answer: $\dfrac{1}{2}\sqrt{5}$.

13. Consider the function $f(x,y) = x^3 + 3xy + y^3$. Find the critical points of the function.

Answer: $(0,0), (-1,-1)$.

14. Find the arc length of the curve parametrized by $\vec{r}(t) = \langle e^t, e^{-t}, \sqrt{2}\ t \rangle$, on the range $0 \leqslant t \leqslant \ln 2$.

Answer: $\dfrac{3}{2}$.

15. Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{2x^4 + y^2}$.

Answer: There does not exist.

16. Find an equation of the plane tangent to the surface $xyz - \ln z = 0$ at point $(0,1,1)$.

Answer: $x - z + 1 = 0$.

17. Find and classify all critical points of the function $f(x,y) = 2x^2 + y^4 - 4xy$ on the entire plane.

Answer: $(0,0), (1,1), (-1,-1)$, saddle, min, min, respectively.

18. Given two vectors $\overrightarrow{u} = \langle 1, -1, 0 \rangle$ and $\overrightarrow{v} = \langle 1, 0, 1 \rangle$ find the angle between them. Compute the area of the parallelogram by the vectors.

Answer: $\frac{\pi}{3}, \sqrt{3}$.

19. Find and classify all critical points of the function $f(x, y) = x^2y - 2xy - 5x^2 + 10x$ and classify them using the Second Derivative Test.

Answer: $(0,5), (2,5)$ saddle points.

20. Consider the function $f(x, y) = 3x^2 + 4y^2 - 2$. Find the directional derivative of $f(x, y)$ at the point $(1,1)$ in the direction of the vector $\overrightarrow{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$.

Answer: $3 + 4\sqrt{3}$.