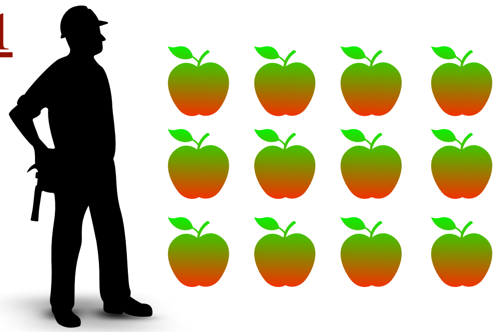
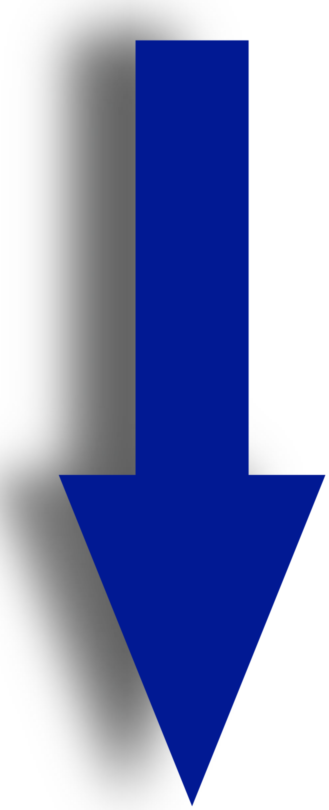
Generating Function (Ordinary)

1. Introductory Examples

E.1





Grace	Frank	Mary
$4 \leqslant$	$2 \leqslant$	$2 \leqslant \quad \leqslant 5$
4	3	5
4	4	4
4	5	3
4	6	2
5	2	5
5	3	4
5	4	3
5	5	2
6	2	4
6	2	4
6	3	3
7	2	3
7	3	2
8	2	2

$$f(x) = (\quad Grace \quad)(\quad Frank \quad)(\quad Mary \quad)$$

$$x^4 + x^5 + x^6 + x^7 + x^8$$

$$x^2 + x^3 + x^4 + x^5 + x^6$$

$$x^2 + x^3 + x^4 + x^5$$

$$f(x) = (x^4 + x^5 + \cdots + x^8)(x^2 + x^3 + \cdots + x^6)(x^2 + x^3 + x^4 + x^5)$$

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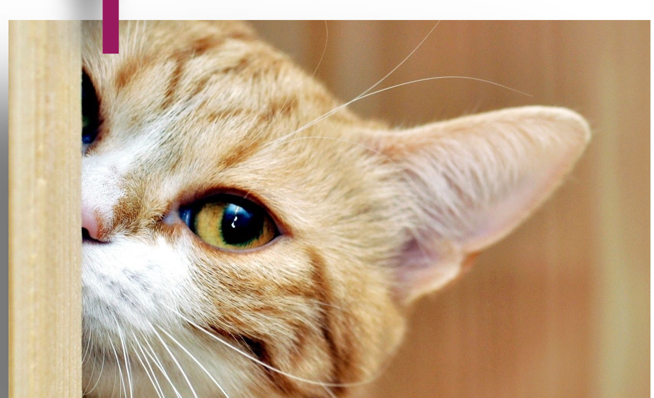
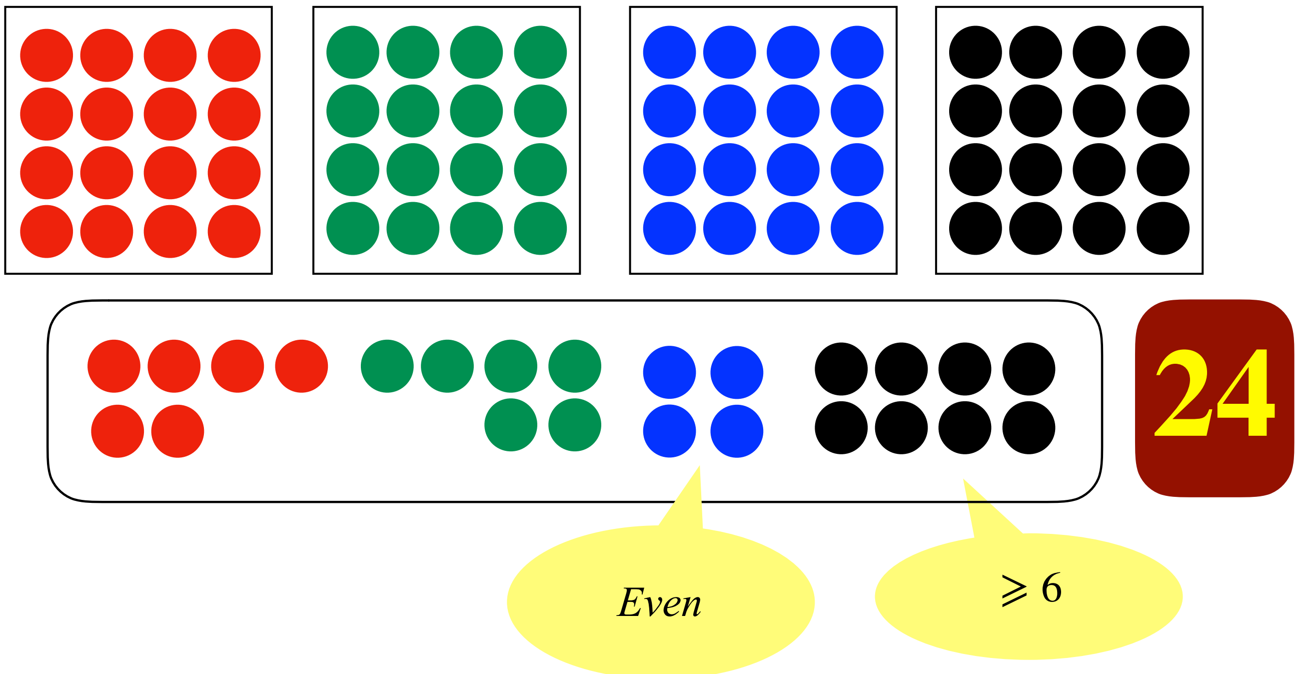
$$x^4x^3x^5, \quad x^5x^2x^5, \quad x^6x^4x^2, \quad x^4x^3x^5, \quad x^3x^4x^5, \quad x^8x^2x^2, \cdots$$

$$f(x) = x^8 + 3x^9 + 6x^{10} + 10x^{11} + \boxed{14x^{12}} + 16x^{13} + 16x^{14} + 14x^{15} + 10x^{16} + 6x^{17} + 3x^{18} + x^{19}$$



E.2



$$1 + x^1 + x^2 + x^3 + \cdots + x^{23} + x^{24}$$

$$1 + x^1 + x^2 + x^3 + \cdots + x^{23} + x^{24}$$

$$1 + x^2 + x^4 + x^6 + x^8 + \cdots + x^{24}$$

$$x^6 + x^7 + x^8 + \cdots + x^{23} + x^{24}$$

So the answer of the problem is the coefficient of x^{24} in the generating function:

$$f(x) = (1 + x^1 + x^2 + \cdots + x^{24})(1 + x^1 + x^2 + \cdots + x^{24})$$
$$(1 + x^2 + x^4 + \cdots + x^{24})(x^6 + x^7 + \cdots + x^{24})$$

$$x^9x^5x^4x^6, \quad x^5x^2x^8x^7, \quad x^{10}x^2x^{12}, \quad x^{10}x^{14}, \quad \cdots$$

$$f(x) = \cdots + \boxed{?}\, x^{24} + \cdots$$

E.3

How many integer solutions are there for the equation

$$x_1 + x_2 + x_3 + x_4 = 25, \quad x_i \geqslant 0, \text{ for all } 1 \leqslant i \leqslant 4?$$

In this example, the generating function is:

x_1 $\quad 1 + x^1 + x^2 + x^3 + \cdots + x^{24} + x^{25}$

x_2 $\quad 1 + x^1 + x^2 + x^3 + \cdots + x^{24} + x^{25}$

x_3 $\quad 1 + x^1 + x^2 + x^3 + \cdots + x^{24} + x^{25}$

x_4 $\quad 1 + x^1 + x^2 + x^3 + \cdots + x^{24} + x^{25}$

$$f(x) = (1 + x^1 + x^2 + x^3 + \cdots + x^{25})(1 + x^1 + x^2 + x^3 + \cdots + x^{25})$$
$$(1 + x^1 + x^2 + x^3 + \cdots + x^{25})(1 + x^1 + x^2 + x^3 + \cdots + x^{25})$$

$$\Rightarrow f(x) = (1 + x^1 + x^2 + x^3 + \cdots + x^{25})^4$$

$$\Rightarrow f(x) = \cdots + \boxed{?}\, x^{25} + \cdots$$

2. Definitions and Examples Calculation Techniques

Let $a_0, a_1, a_2, \ldots, a_n, \ldots,$ be a sequence of real numbers. The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

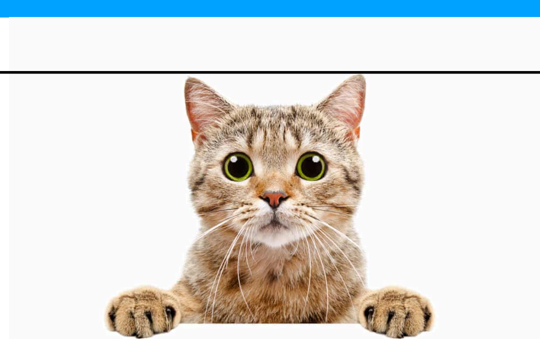
is called the *generating function* for the given sequence.

For any $n \in \mathbb{Z}^+$, we have

$$\underbrace{(1+x)^n}_{f(x)} = \underbrace{\binom{n}{0}}_{a_0} + \underbrace{\binom{n}{1}}_{a_1} x + \underbrace{\binom{n}{2}}_{a_2} x^2 + \cdots + \underbrace{\binom{n}{n}}_{a_n} x^n .$$

So $(1+x)^n$ is the generating function for the sequence:

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}, \ldots, \binom{n}{n}, 0, 0, 0, \ \ldots$$



For any $n \in \mathbb{Z}^+$, we have

$$(1 - x^{n+1}) = (1 - x)(1 + x + x^2 + x^3 + \cdots + x^n)$$

$$\boxed{\frac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + x^3 + \cdots + x^n,}$$

So $\dfrac{1 - x^{n+1}}{1 - x}$ is the generating function for the sequence:

$$1,\ 1,\ 1,\ 1, \ldots,\ 1,\ 0,\ 0,\ 0, \ldots,$$

Extending this idea, we find that

$$1 = (1 - x)(1 + x + x^2 + x^3 + \cdots + x^n + \cdots)$$

$$\boxed{\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,}$$

So $\dfrac{1}{1 - x}$ is the generating function for the sequence:

$$1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1, \ldots,$$

With

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

taking the derivative yields

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots + x^n + \cdots),$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots,$$

So $\frac{1}{(1-x)^2}$ is the generating function for the sequence:

1, 2, 3, 4, 5,…,

$\times\, x$

$$\frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n + \cdots,$$

So $\frac{x}{(1-x)^2}$ is the generating function for the sequence:

0, 1, 2, 3, 4, 5,…,

Continuing from last part

$$\frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}(0 + x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots),$$

$$\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1} + \cdots,$$

Hence,

$$\frac{x+1}{(1-x)^3}$$

generates

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2, \ldots,\ n^2,\ \ldots$$

and

$$\frac{x(x+1)}{(1-x)^3}$$

generates

$$0^2,\ 1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2, \ldots,\ n^2,\ \ldots$$

$$f_0(x) = \boxed{\frac{1}{1-x}} = \boxed{1 + x + x^2 + x^3 + \cdots + x^n + \cdots,}$$

$$f_1(x) = x\frac{d}{dx}f_0(x) = \boxed{\frac{x}{(1-x)^2}} = \boxed{0 + x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots,}$$

$$f_2(x) = x\frac{d}{dx}f_1(x) = \boxed{\frac{x^2+x}{(1-x)^3}} = \boxed{0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n + \cdots,}$$

$$f_3(x) = x\frac{d}{dx}f_2(x) = \boxed{\frac{x^3+4x^2+x}{(1-x)^4}} = \boxed{0^3 + 1^3x + 2^3x^2 + 3^3x^3 + \cdots + n^3x^n + \cdots,}$$

$$f_4(x) = x\frac{d}{dx}f_3(x) = \boxed{\frac{x^4+11x^3+11x^2+x}{(1-x)^5}} = \boxed{0^4 + 1^4x + 2^4x^2 + \cdots + n^4x^n + \cdots,}$$