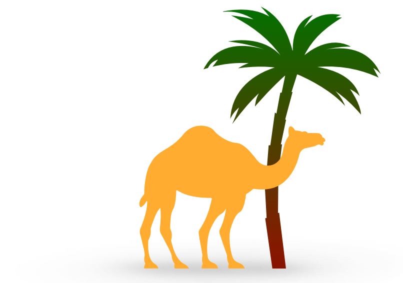
The Rules of Sum and Product



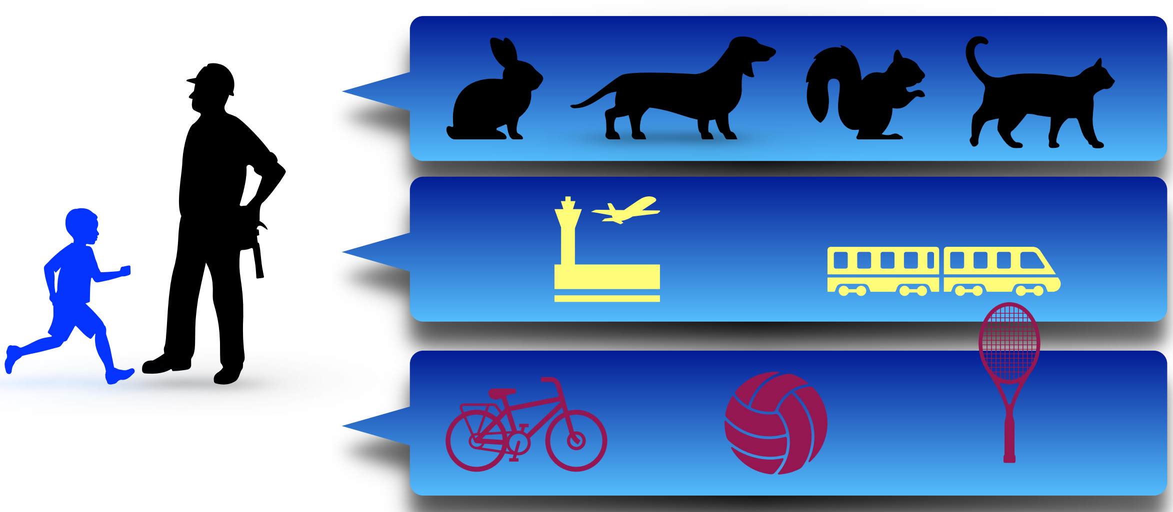
The Rule of Sum:

$$\mathbf{A}\ :\ \begin{cases} A_1 & \to & w_1 \\ A_2 & \to & w_2 \\ A_3 & \to & w_3 \\ & \vdots & \\ A_n & \to & w_n \end{cases} \implies \text{the number of ways to do } A = w_1 + w_2 + w_3 + \cdots + w_n$$

The Rule of Product:

$$\mathbf{A}\ :\ \begin{cases} S_1 & \to & w_1 \\ S_2 & \to & w_2 \\ S_3 & \to & w_3 \\ & \vdots & \\ S_n & \to & w_n \end{cases} \implies \text{the number of ways to do } A = w_1 \times w_2 \times w_3 \times \cdots \times w_n$$

Example 1. (Award)



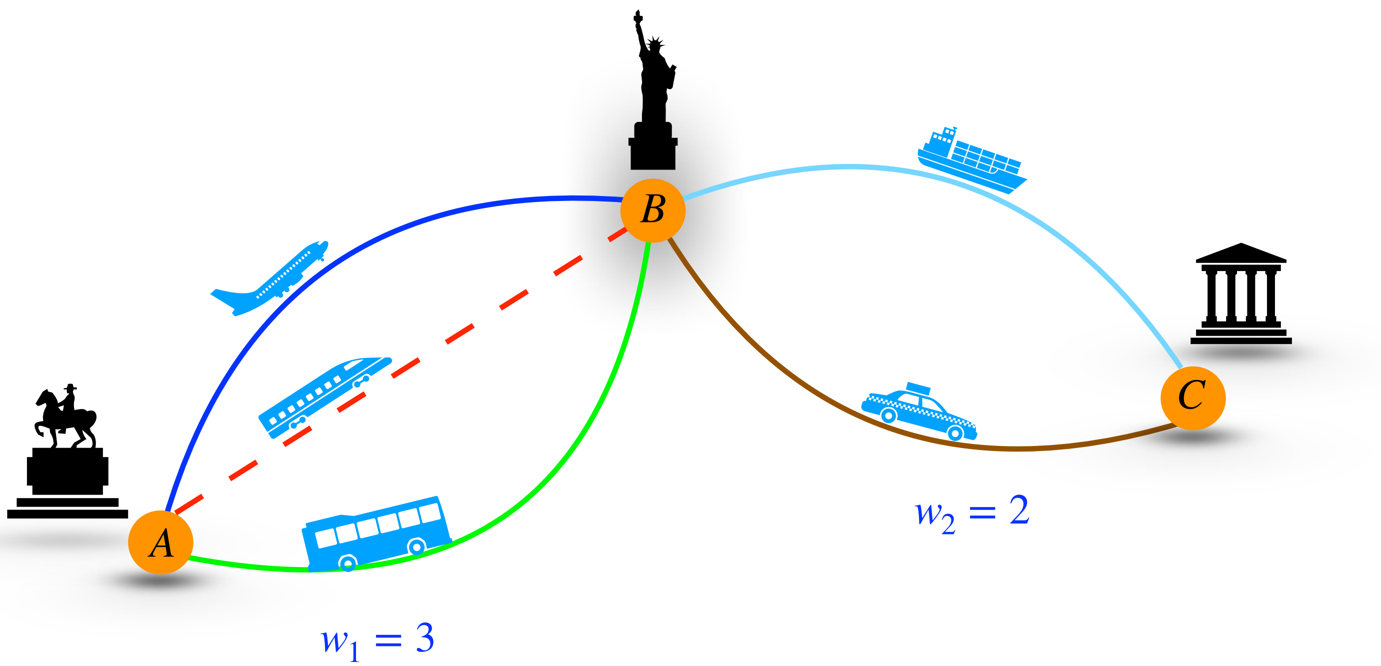
$w_1 = 4$

$w_2 = 2$

$w_3 = 3$

$4 + 2 + 3 = 9$

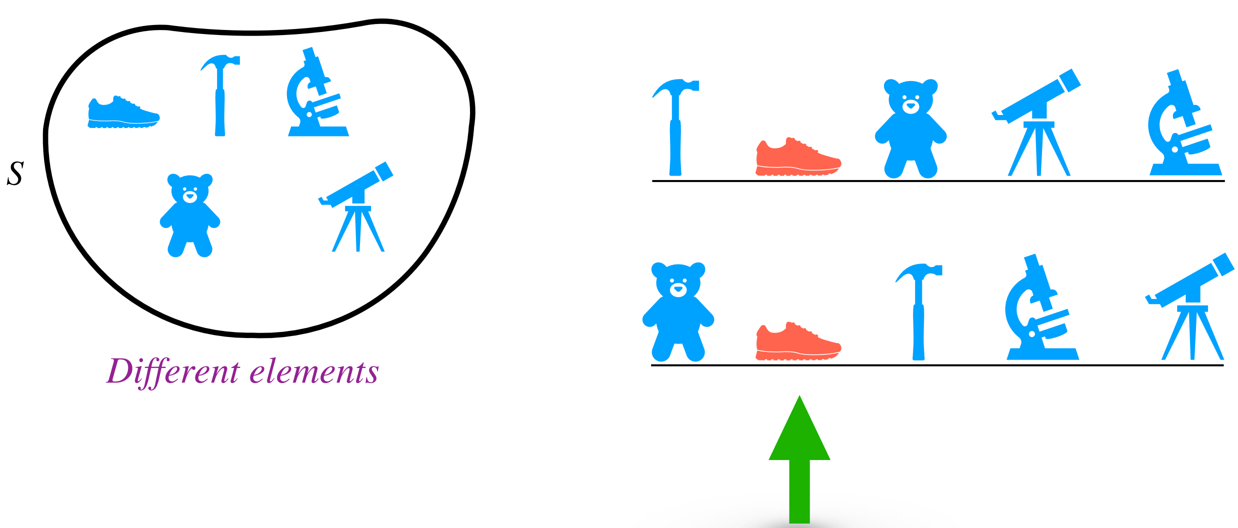
Example 2.



$$3 \times 2 = 6$$

Permutations

A *permutation* is an arrangement of some elements in which order matters. In other words, a permutation is an ordered combination of elements.

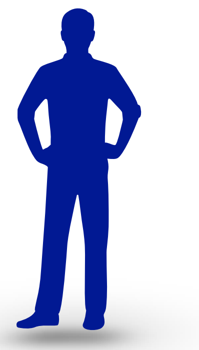


Assume we have a set S with n different elements: $S = \{a_1, a_2, \ldots, a_n\}$.

Question: *How many different permutations are there?*

$$\begin{array}{rcccccccc} & S_1 & S_2 & S_3 & S_4 & & S_{n-1} & S_n \\ P(n,n) = & \boxed{n} & \boxed{n-1} & \boxed{n-2} & \boxed{n-3} & \cdots & \boxed{2} & \boxed{1} \end{array}$$

$$P(n,n) = n(n-1)(n-2)(n-3)\cdots(3)(2)(1) = n!$$



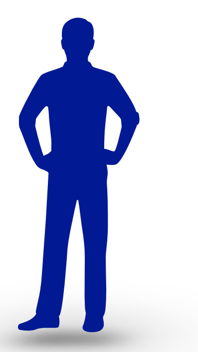
$0! = 1$

r-permutations

$$\begin{array}{rcccccccc} & S_1 & S_2 & S_3 & S_4 & & S_{r-1} & S_r \\ P(n,r) = & \boxed{n} & \boxed{n-1} & \boxed{n-2} & \boxed{n-3} & \cdots & \boxed{n-(r-2)} & \boxed{n-(r-1)} \end{array}$$

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)(n-3)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)(n-r)!}{(n-r)!} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

$$P(n,r) = \frac{n!}{(n-r)!}$$



Example 3.

1. How many permutations of letters $\{a, b, c\}$ are there?

$$P(3,3) = 3! = 6: \quad abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

2. The 2-permutations of set $\{a, b, c\}$ are:

$$ab,\ ac,\ ba,\ bc,\ ca,\ cb$$

The number of 2-permutations of this 3-element set is

$$P(3,2) = \frac{3!}{(3-2)!} = \frac{6}{1} = 6.$$

3. Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

Note that the runners are distinct and that the medals are ordered. The solution is

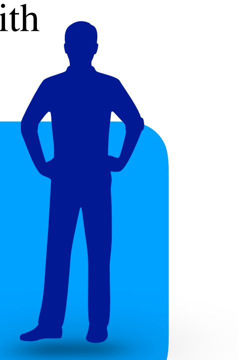
	Gold	Silver	Bronz
$P(8,3) =$	8	7	6

$$P(8,3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336.$$

Combinations

A k-*combination* of elements of a set is an unordered selection of k elements from the set. Thus, a k-combination is simply a subset of the set with k elements.

Theorem 1. The number of k-combinations of a set with n distinct elements, where n is a positive integer and k is an integer with $0 \leqslant k \leqslant n$ is:



$$C(n,k) = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!}$$

Proof. The k-permutations of the set can be obtained by first forming the x k-combinations of the set, and then ordering the elements in each k-combination, which can be done in $P(k,k)$ ways. Consequently,

	A k – combination of the set	ordering the elements
$P(n,k) =$	x	$P(k,k)$

$$P(n,k) = x \times P(k,k) = x \times k! \quad \Longrightarrow \quad x = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}. \ \square$$

Example 4. We need to create a team of 5 players for the competition out of 10 team members. How many different teams is it possible to create?

$$C(10,5) = \frac{10!}{5!(10-5)!} = 252.$$

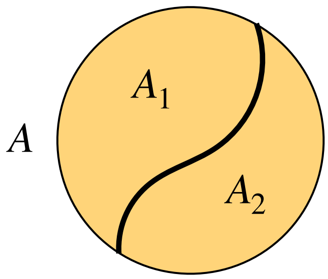
Example 5. Prove that

$$C(n,k) = C(n,n-k).$$

First Proof. We have

$$C(n,k) = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = C(n,n-k). \ \square$$

Second Proof. We have



$$\Longrightarrow \quad A = A_1 \cup A_2, \qquad A_1 \cap A_2 = \varnothing$$

$$|A_1| = k \quad \Longleftrightarrow \quad |A_2| = n - k\,.$$

$$\Longrightarrow \quad C(n,k) = C(n, n-k)\,. \square$$

Binomial coefficients

The number of k-combinations out of n elements $C(n,k)$ is often denoted as:

$$\binom{n}{k}$$

and reads *n choose k*. The number is also called a binomial coefficient.

Binomial coefficients occur as coefficients in the expansion of powers of binomial expressions such as:

Theorem 2. ***(Binomial Theorem)*** For all $n \geqslant 0$, we have

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Proof. Considering the following computations:

$$\begin{aligned}
(a+b)^n &= (a+b)(a+b)(a+b)(a+b)\cdots(a+b)(a+b)(a+b)\\
&= (a+b)(a+b)(a+b)(a+b)\cdots(a+b)(a+b)(a+b)\\
&= \binom{n}{n} a^n b^0 + \binom{n}{n-1} a^{n-1} b^1 + \cdots + \binom{n}{i} a^i b^{n-i} + \cdots + \binom{n}{0} a^0 b^n
\end{aligned}$$

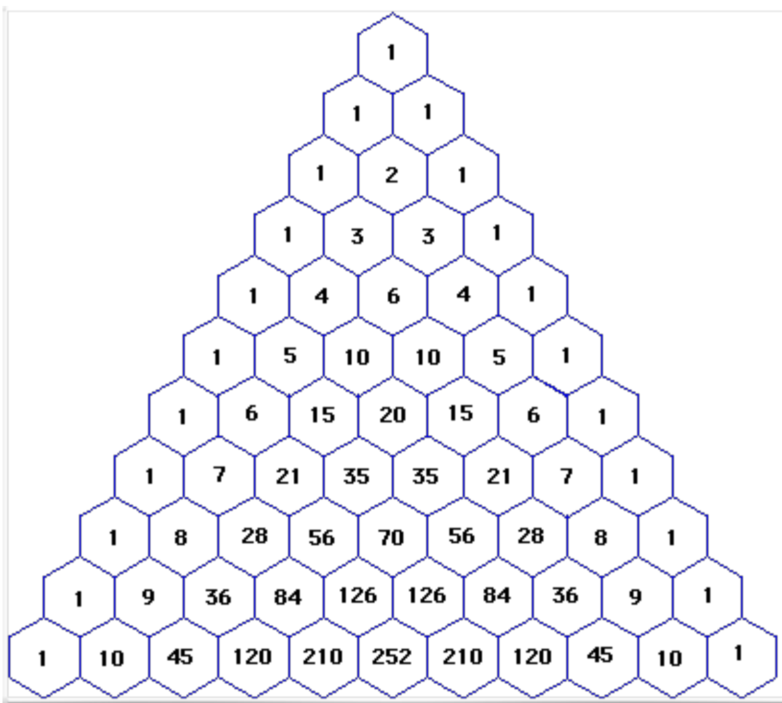
we obtain

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i},$$

as required. $\square$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Blaise Pascal



Pascal's triangle

Practical Problems

Prove that:

$$\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$$

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=2^{n-1}$$

$$\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots=2^{n-1}$$

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots(-1)^n\binom{n}{n}=0$$

$$\binom{n}{n}+\binom{n+1}{n}+\binom{n+2}{n}+\cdots+\binom{n+m}{n}=\binom{n+m+1}{n+1}$$

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}$$

$$\binom{m}{0}\binom{n}{p}+\binom{m}{1}\binom{n}{p-1}+\cdots+\binom{m}{p}\binom{n}{0}=\binom{m+n}{p}$$

$$1\cdot\binom{n}{1}+2\cdot\binom{n}{2}+3\cdot\binom{n}{3}+\cdots+n\cdot\binom{n}{n}=n2^{n-1}$$

$$1\cdot\binom{n}{1}-2\cdot\binom{n}{2}+3\cdot\binom{n}{3}-\cdots+n(-1)^{n+1}\cdot\binom{n}{n}=0$$

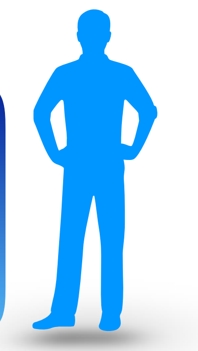
Arrangements with repetition

Permutation with repetition from the set S is an arrangement of elements from S in a sequence such that every element from S occurs a given number of times. Denote the number of them by

$$P(n; n_1, n_2, \ldots, n_k).$$

The number of all permutations with repetition from a k-element set, where the i-th element is repeated in n_i identical copies ($i = 1, \ldots, k$):

$$P(n; n_1, n_2, \ldots, n_k) = \binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$$



Example 6. How many anagrams of the word *MISSISSIPPI* exist? (anagram is a word obtained by rearranging all characters of a given word).

Solution. If no character in

"*MISSISSIPPI*"

would repeat, the calculation would rely on permutation:

$$M\ I_1\ S_1\ S_2\ I_2\ S_3\ S_4\ I_3\ P_1\ P_2\ I_4 \quad \Longrightarrow \quad 11!$$

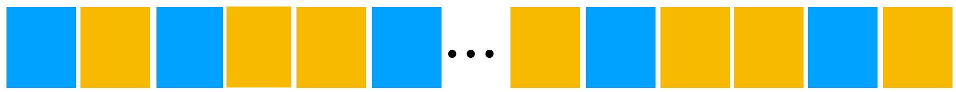
But they repeat: S 4 times, I 4 times, P 2 times.

$$\begin{array}{l} M\ I_1\ S_1\ S_2\ I_2\ S_3\ S_4\ I_3\ P_1\ P_2\ I_4 \\ M\ I_1\ S_1\ S_2\ I_2\ S_3\ S_4\ I_3\ P_2\ P_1\ I_4 \end{array} \quad \Longrightarrow \quad M\ I_1\ S_1\ S_2\ I_2\ S_3\ S_4\ I_3\ P\ P\ I_4$$

$$P(11; 1,4,4,2) = \frac{11!}{1!4!4!2!} = 34650.$$

Example 7. Permutation with repetition of 2 elements, one element occurs in k copies the other in $(n-k)$ copies:

$$P(n;k,n-k) = \frac{n!}{k!(n-k)!} = \binom{n}{k,n-k} = \binom{n}{k} = \binom{n}{n-k}.$$

$P(n;k,n-k) =$ 

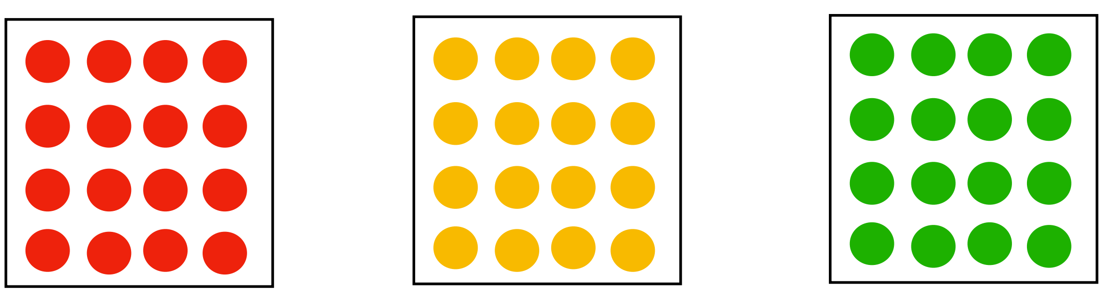
Theorem 3. ***(Multinomial Theorem)*** For all $n \geqslant 0$, and $k \geqslant 1$, we have

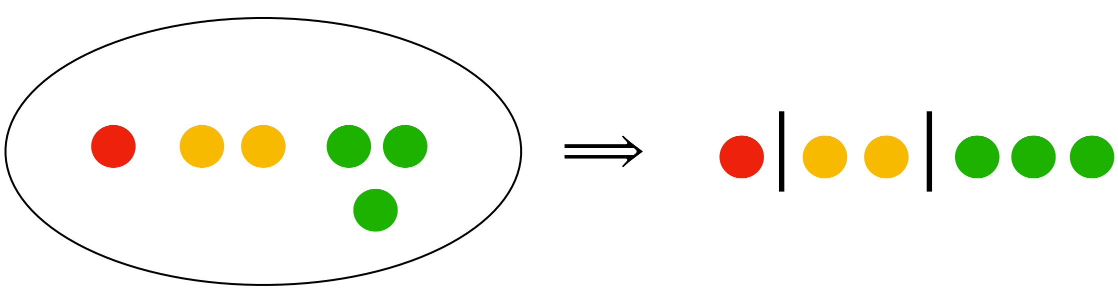
$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

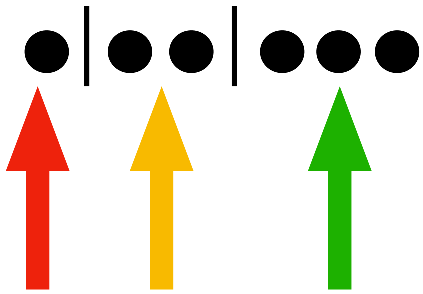
Combination with repetition

Example 8. How man ways are there to select 6 balls of 3 colors, provided we have an unlimited supply of balls of each color?

Solution. We present a beautiful trick, how to count the total number of selections. Suppose we pick







$$\Longrightarrow \quad P(8;2,6) = \frac{(6+2)!}{6!2!}.$$

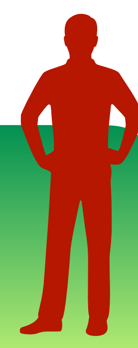
This selection we can order based on colors

$$P(8;2,6) = \frac{(6+2)!}{6!2!} = \binom{6+2}{2} = \binom{6+3-1}{3-1}.$$

$$P(8;2,6) = \frac{(6+2)!}{6!2!} = \binom{6+2}{6} = \binom{6+3-1}{6}.$$

An r-combination with repetition from an n-element set S is a selection of r elements from S, while each element can occur in an arbitrary number of identical copies. The number of them we denote by $C^*(n,r)$. The total number of all r-element selections with repetition from n possibilities is

$$C^*(n,r) = \binom{r+n-1}{n-1} = \binom{n+r-1}{r}$$



Problems solved using k-combinations with repetition

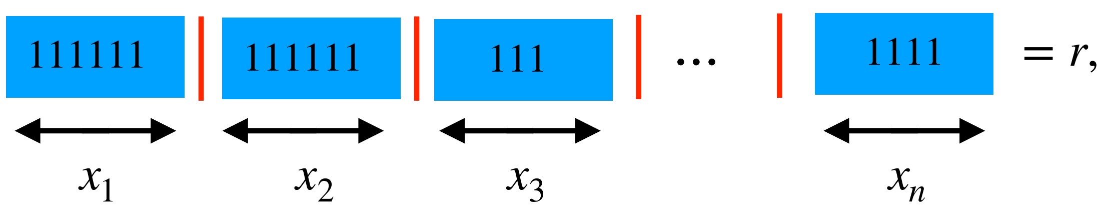
1. Number of ways how to write r using n nonnegative integer summands.
2. Drawing r elements of n kinds provided after each draw we return the elements back to the polling urn.

Example 9. How many ways are there to write r as the sum of n nonnegative integer summands? We distinguish the order of summands!

Solution. We have

$$x_1 + x_2 + \cdots + x_n = r,$$

We will select (draw) r ones and distribute them into n boxes (with the possibility of tossing more ones into each box). some boxes can remain empty. we can toss all ones into one box we repeat boxes, not ones! (a different problem):



Now the answer is: $\binom{r+n-1}{n-1} = \binom{n+r-1}{r}$.

Some Questions.

1. How many ways are there to write k as the sum of n positive summands?

2. How many ways are there to write k as the sum of at least n natural summands?

3. How many ways are there to write k as the sum of at most n natural summands?

An r-permutation with repetition from an n-element set S is an arrangement of r elements from S, while elements can repeat in an arbitrary number of identical copies. The number of them we denote by $P^*(n,r)$. The arrangement is a sequence. The number of all r-permutations with repetition from n possibilities is

$$P^*(n,r) = \underbrace{n \times n \times \cdots \times n}_{r\text{–times}} = n^r.$$

Problems solved by r-permutation with repetition

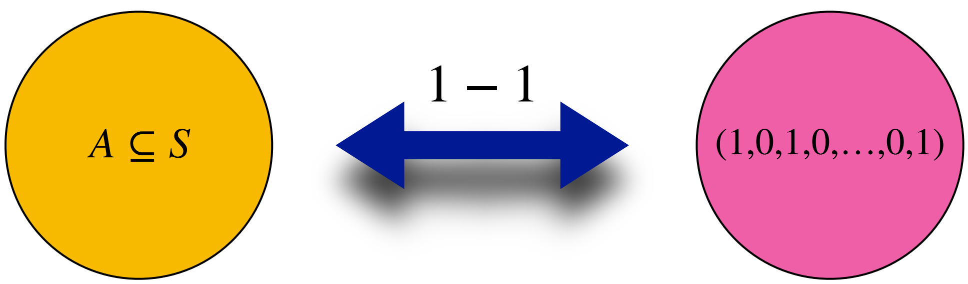
1. Number of mappings of an r-element set to an n-element set.
2. Cardinality of the Cartesian power $|A^r|$.

Question How many odd-sized subsets has a given set of $n$elements?

Number of Subsets of a Finite Set

Example 10. A finite set, S, has $2^{|S|}$ distinct subsets.

***Solution*:** Let $S = \{x_1, x_2, \ldots, x_n\}$. There is a one-to-one correspondence (bijection), between subsets of S and bit strings of length $|S| = n$.



The bit string of length $|S|$ we associate with a subset $A \subseteq S$ has a 1 in position i if $s_i \in A$, and 0 in position i if $s_i \notin A$, for all $i \in \{1,2,\ldots,n\}$.

$$\{s_2, s_4, s_5, \ldots, s_n\} \longleftrightarrow \underbrace{(0,1,0,1,1,\ldots,1)}_{n-times}$$

By the rule of product, there are $2^{|S|}$ such bit strings.

Number of Functions

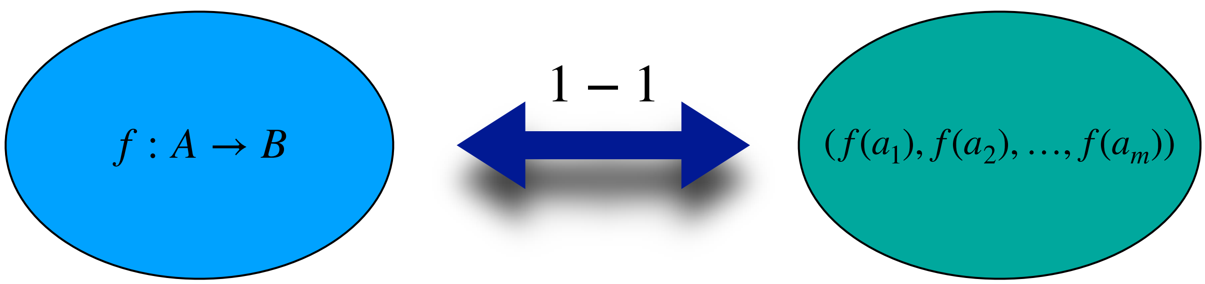
Example 11. For all finite sets A and B, the number of distinct functions,

$$f : A \to B,$$

mapping A to B, is $|B|^{|A|}$.

Solution: Suppose $A = \{a_1, a_2, \ldots, a_m\}$. There is a one-to-one correspondence between functions $f : A \to B$, and strings (sequences) of length $|A| = m$ over an alphabet of size $|B| = n$:

$$f : A \to B \quad \longleftrightarrow \quad (f(a_1), f(a_2), \ldots, f(a_m))$$



By the rule of product rule, there are n^m such strings of length m.