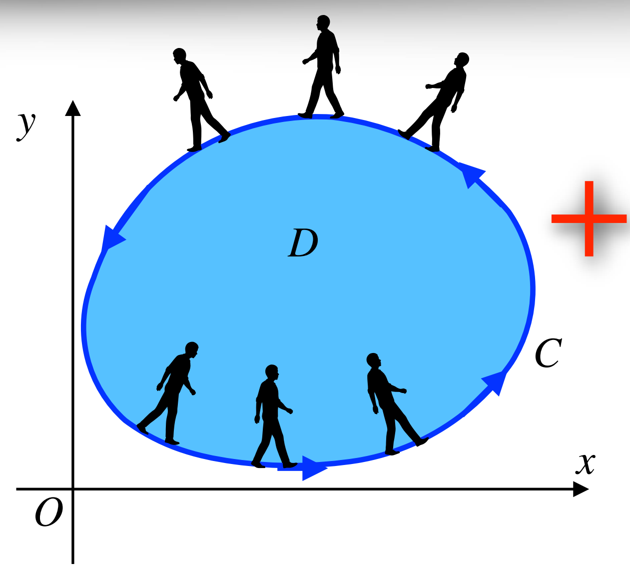
Green's Theorem

$$\vec{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y)\rangle = M(x,y)\vec{i} + N(x,y)\vec{j}\,.$$

A **simple closed** curve C

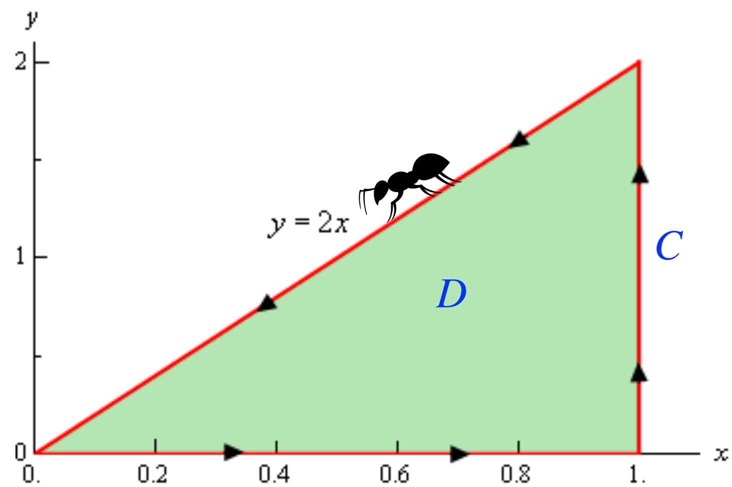
The **region enclosed** D



$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C \vec{\mathbf{F}} \cdot d\vec{r}$$

Example 1 Use Green's Theorem to evaluate $\oint_C xydx + x^2y^3dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Solution.



$$D \Longrightarrow \begin{cases} 0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 2x \end{cases}$$

We can identify $M(x,y)$ and $N(x,y)$ from the line integral. Here they are:

$$M(x,y) = xy, \qquad N(x,y) = x^2y^3.$$

So, using Green's Theorem the line integral becomes,

$$\oint_C xydx + x^2y^3dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_D 2xy^3 - x \; dA$$

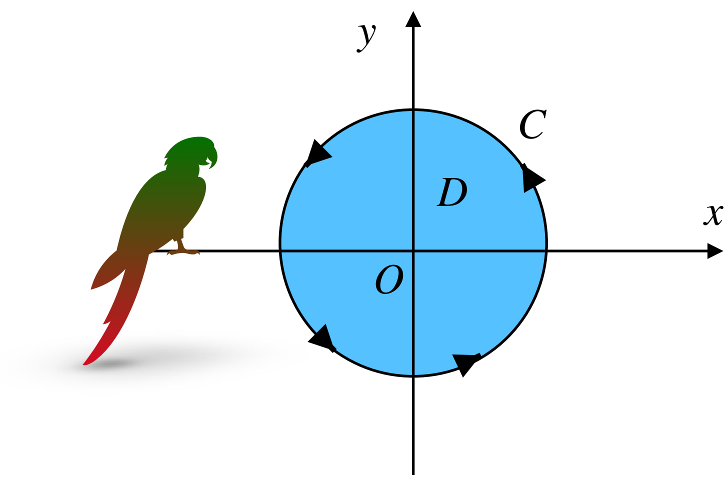
$$= \int_0^1 \int_0^{2x} 2xy^3 - x \; dy \; dx = \int_0^1 \left(\frac{1}{2}xy^4 - xy\right)\Big|_0^{2x} dx = \int_0^1 (8x^5 - 2x^2) \; dx$$

$$= \left(\frac{4}{3}x^6 - \frac{2}{3}x^3\right)\Big|_0^1 = \frac{2}{3}.$$

Example 2 Evaluate $\oint_C y^3 dx - x^3 dy$ where C is the positively oriented circle of radius 2 centered at the origin..

Solution. Let's first identify $M(x, y)$ and $N(x, y)$ from the line integral:

$$M(x, y) = y^3, \quad N(x, y) = -x^3.$$

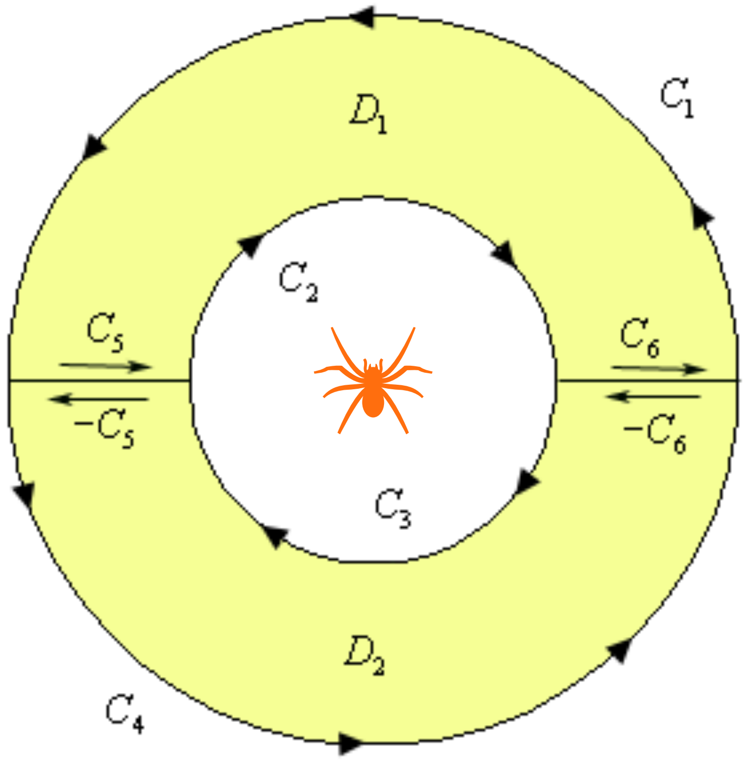


Now, using Green's Theorem on the line integral gives:

$$\oint_C y^3 dx - x^3 dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_D -3x^2 - 3y^2 \, dA$$

$$\iint_D -3(x^2 + y^2) \, dA = -3\int_0^{2\pi}\int_0^2 r^2 \, r \, dr \, d\theta = -3\int_0^{2\pi} \frac{1}{4}r^4\Big|_0^2 \, d\theta$$

$$= -3\int_0^{2\pi} 4 \, d\theta = -24\pi.$$

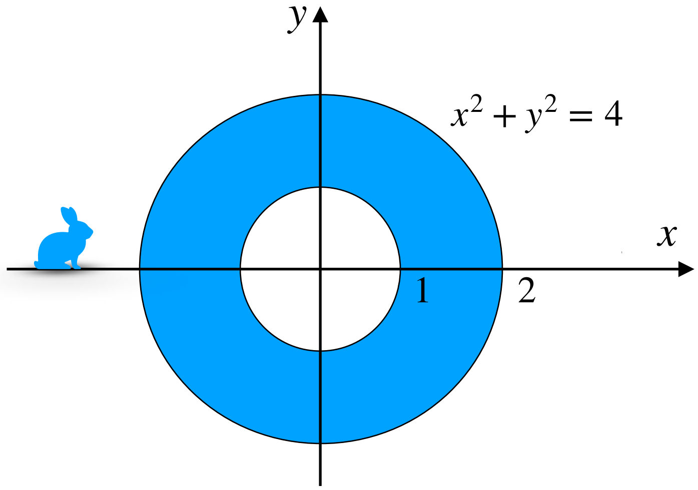


$$\iint_D \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA=\iint_{D_1}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA+\iint_{D_2}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA$$

$$=\oint_{C_1\cup C_5\cup C_2\cup C_6}\vec{\mathbf{F}}\cdot d\vec{r}+\oint_{C_4\cup(-C_6)\cup C_3\cup(-C_5)}\vec{\mathbf{F}}\cdot d\vec{r}$$

Example 2 Evaluate $\oint_C y^3dx-x^3dy$ where C is the union of two circles of radius 2 and radius 1 centered at the origin with positive orientation.

Solution.

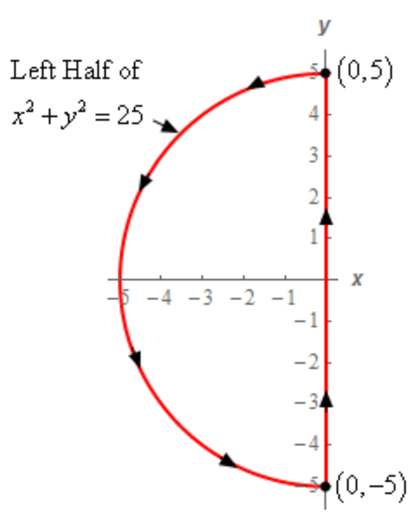


$$\oint_C y^3 dx - x^3 dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = -3 \iint_D (x^2 + y^2)\, dA$$

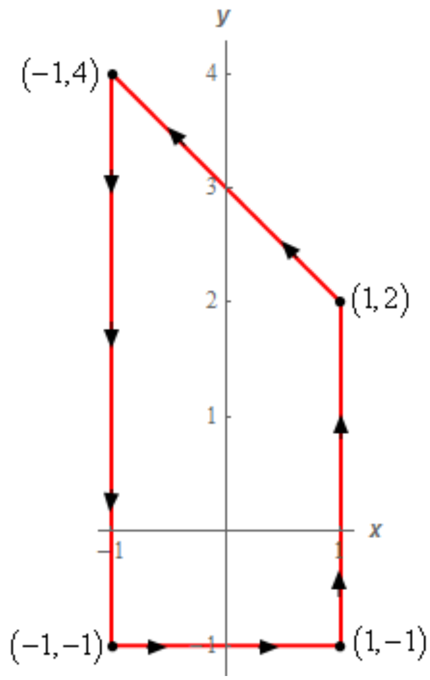
$$= -3 \int_0^{2\pi} \int_1^2 r^3\, dr\, d\theta = -3 \int_0^{2\pi} \frac{1}{4} r^4 \Big|_1^2 d\theta = -3 \int_0^{2\pi} \frac{15}{4}\, d\theta = -\frac{45\pi}{2}.$$

Practice Problems

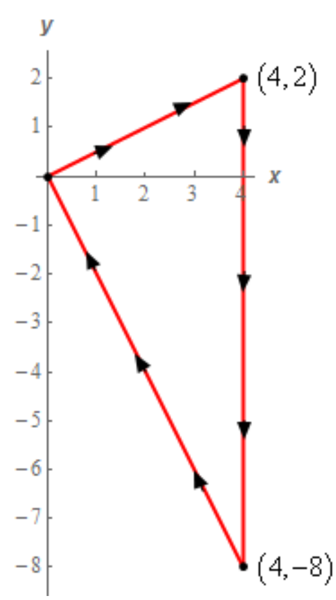
1. Use Green's Theorem to evaluate $\oint_C yx^2 dx - x^2 dy$ where C is shown below:



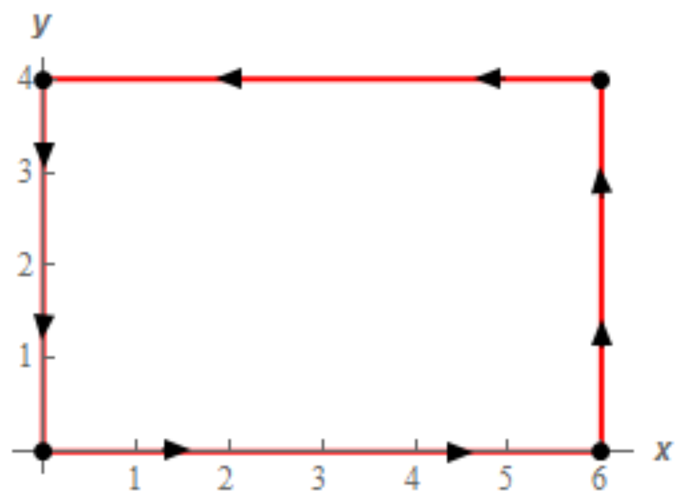
2. Use Green's Theorem to evaluate $\oint_C (6y - 9x)dy - (yx - x^3)dx$ where C is shown below:



3. Use Green's Theorem to evaluate $\oint_C x^2y^2dx + (yx^3 + y^2)dx$ where C is shown below:



4. Use Green's Theorem to evaluate $\oint_C (y^4 - 2y)dx - (6x - 4xy^3)dy$ where C is shown below:



5. Verify Green's Theorem for $\oint_C (xy^2 + x^2)dx + (4x - 1)dy$ where C is shown below by **(a)** computing the line integral directly and **(b)** using Green's Theorem to compute the line integral.