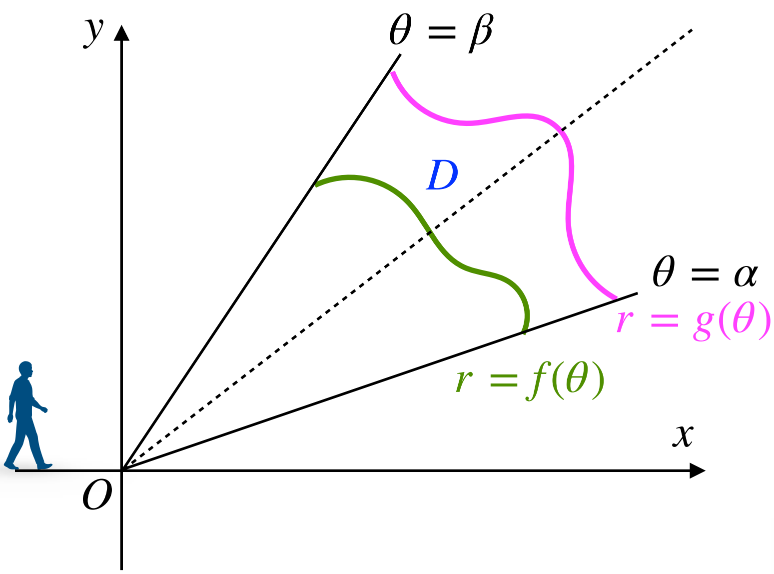
Multiple Integrals

1. Double Integrals In Polar Coordinates

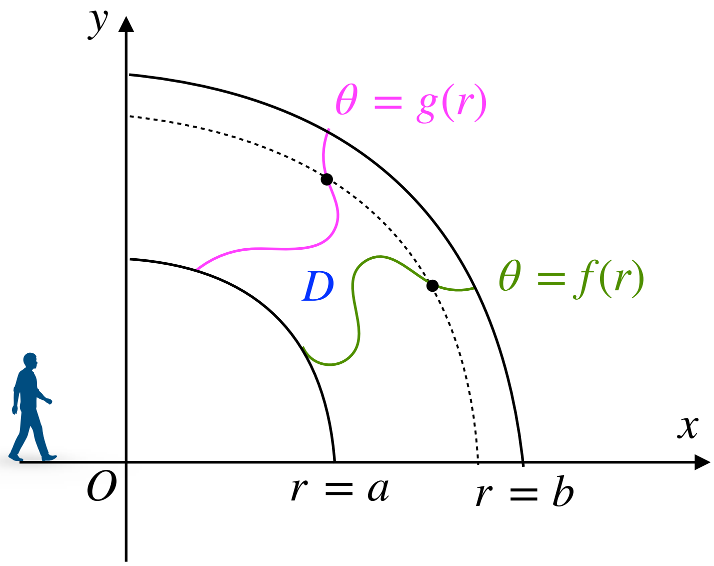


$$D \Longrightarrow \begin{cases} \alpha \leqslant \theta \leqslant \beta \\ f(\theta) \leqslant r \leqslant g(\theta) \end{cases}$$

Jacobian

$$\iint_D f(x,y)\; dA = \int_\alpha^\beta \left(\int_{f(\theta)}^{g(\theta)} f(r\cos\theta, r\sin\theta)\; r\; dr \right) d\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \Longrightarrow \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$



$$D \Longrightarrow \begin{cases} a \leqslant r \leqslant b \\ f(r) \leqslant \theta \leqslant g(r) \end{cases}$$

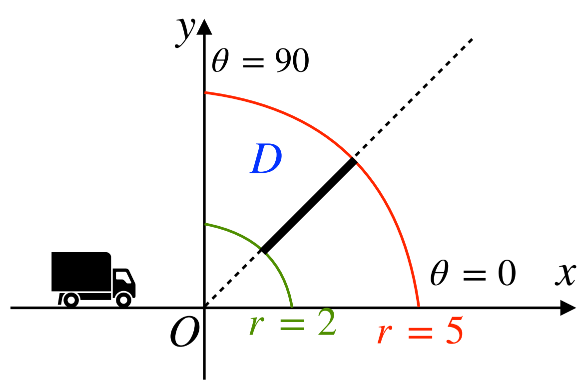
$$\iint_D f(x,y)\, dA = \int_a^b \left(\int_{f(r)}^{g(r)} f(r\cos\theta, r\sin\theta)\ r\ d\theta \right) dr$$

Example 1 Evaluate the following integrals by converting them into polar coordinates.

1. $\displaystyle\iint_D 2xy\ dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

Solution.

$$D \Longrightarrow \begin{cases} 0 \leqslant \theta \leqslant \pi/2 \\ 2 \leqslant r \leqslant 5 \end{cases}$$



$$\iint_D 2xy\, dA = \int_0^{\frac{\pi}{2}} \int_2^5 2(r\cos\theta)(r\sin\theta)\ r\, dr\, d\theta$$

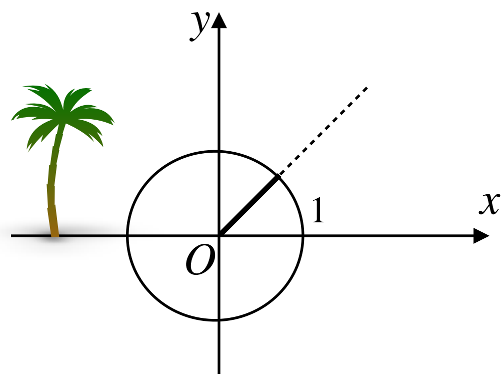
$$= \int_0^{\frac{\pi}{2}} \int_2^5 2r^3 \cos\theta \sin\theta\ dr\, d\theta = \int_0^{\frac{\pi}{2}} \frac{r^4}{4} \sin 2\theta \Big|_2^5 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{609}{4} \sin 2\theta d\theta = -\frac{609}{8} \cos 2\theta \Big|_0^{\frac{\pi}{2}} = \frac{609}{4}.$$

2. $\iint_D e^{x^2+y^2}\ dA$, D is the unit disk centered at the origin.

Solution.

$$D \Longrightarrow \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant 1 \end{cases}$$

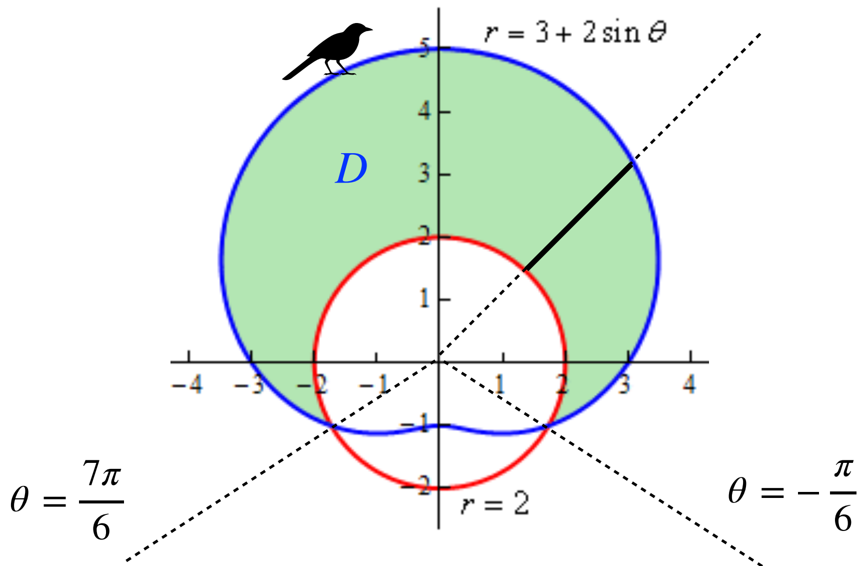


$$\iint_D e^{x^2+y^2}\ dA = \int_0^{2\pi} \int_0^1 e^{r^2}\ r\ dr\ d\theta = \int_0^{2\pi} \frac{1}{2} e^{r^2} \Big|_0^1\ d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(e-1)\ d\theta = \pi(e-1).$$

Example 2 Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution. The sketch of the region D:



$$3 + 2\sin\theta = 2 \implies \sin\theta = -\frac{1}{2} \implies \theta = -\frac{\pi}{6}, \frac{7\pi}{6}.$$

$$D \implies \begin{cases} -\frac{\pi}{6} \leqslant \theta \leqslant \frac{7\pi}{6} \\ 2 \leqslant r \leqslant 3 + 2\sin\theta \end{cases}$$

$$\text{Area} = \iint_D dA = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \int_2^{3+2\sin\theta} r\ dr\ d\theta = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{r^2}{2}\Big|_2^{3+2\sin\theta} d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} [(\frac{9}{2} + 6\sin\theta + 2\sin^2\theta) - 2]\ d\theta = \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (\frac{7}{2} + 6\sin\theta - \cos 2\theta)\ d\theta$$

$$= \left(\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$= \left(\frac{49\pi}{12} - 6\cos\frac{7\pi}{6} - \frac{1}{2}\sin\frac{7\pi}{3}\right) - \left(-\frac{7\pi}{12} - 6\cos\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right)$$

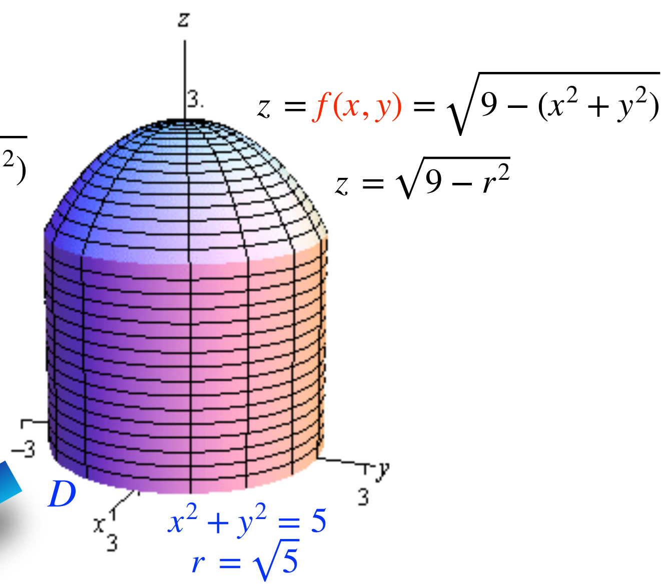
$$= \frac{14\pi}{3} + \frac{11\sqrt{3}}{2}.$$

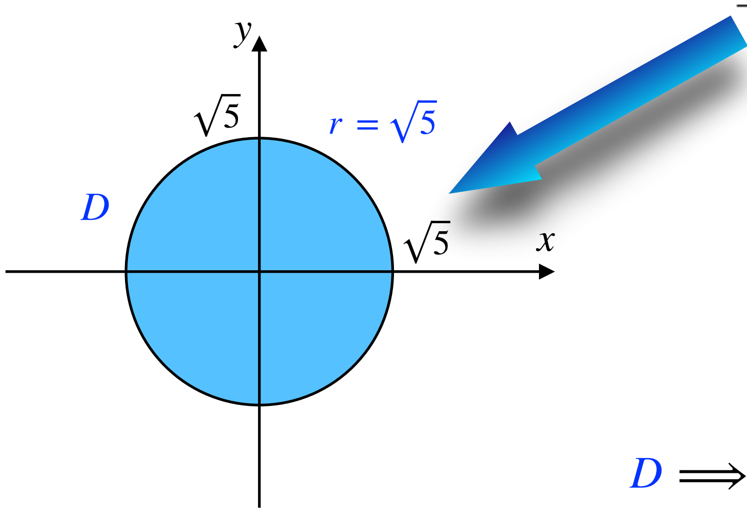
Example 3 Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$ above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

Solution. $\text{Volume} = \iint_D f(x,y)\, dA$

$$x^2 + y^2 + z^2 = 9 \implies z = f(x,y) = \sqrt{9-(x^2+y^2)}$$

$$\implies z = \sqrt{9-r^2}$$





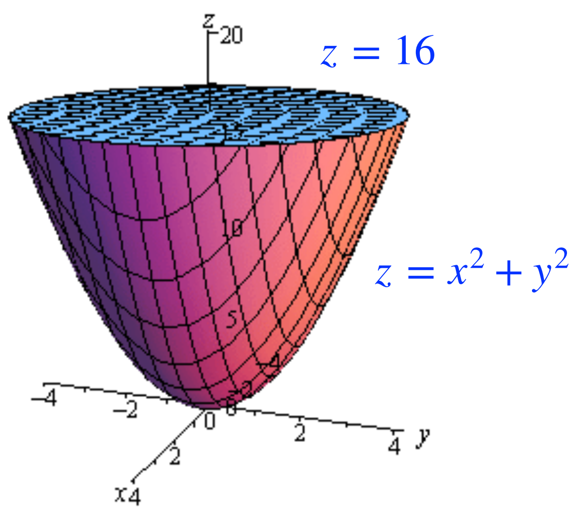
$$D \implies \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant \sqrt{5} \end{cases}$$

$$\text{Volume} = \iint_D \sqrt{9-(x^2+y^2)}\; dA = \int_0^{2\pi}\int_0^{\sqrt{5}} \sqrt{9-r^2}\; r\; dr\; d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}(9-r^2)^{\frac{3}{2}}\Big|_0^{\sqrt{5}}\; d\theta = \int_0^{2\pi} \frac{19}{3}\; d\theta = \frac{38\pi}{3}.$$

Example 4 Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.

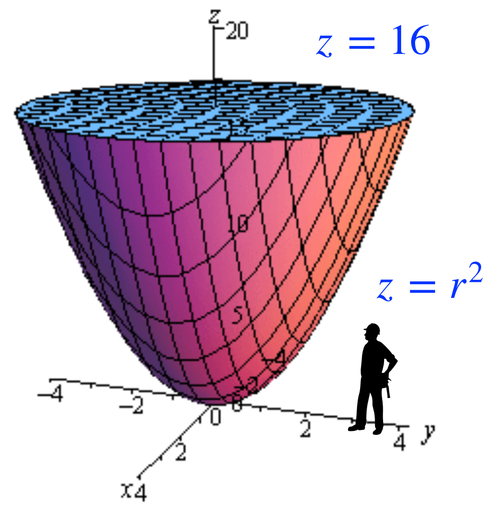
Solution.



$$\text{Volume} = \iint_D f(x,y)\; dA = \iint_D 16\; dA - \iint_D x^2+y^2\; dA$$

$$= \iint_D 16-(x^2+y^2)\; dA$$

$$D \Longrightarrow \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant 4 \end{cases}$$



$$\text{Volume} = \iint_D f(x,y)\; dA = \iint_D 16 - (x^2+y^2)\; dA = \int_0^{2\pi}\int_0^4 16 - r^2\; r\; dr\; d\theta$$

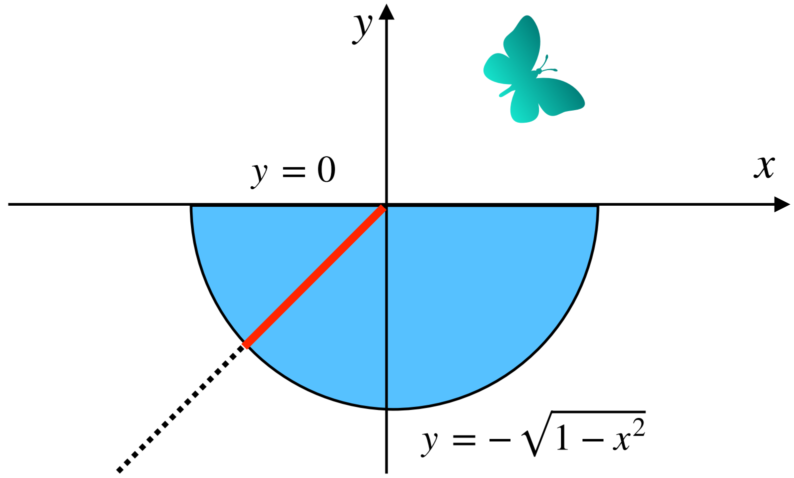
$$= \int_0^{2\pi} \left(8r^2 - \frac{1}{4}r^4\right)\Big|_0^4\; d\theta = \int_0^{2\pi} 64\; d\theta = 128\pi\,.$$

Example 5 Evaluate the following integral by converting to polar coordinates:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \cos(x^2+y^2)\; dy\; dx$$

Solution. $D \Longrightarrow \begin{cases} \pi \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant 1 \end{cases}$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \cos(x^2+y^2)\; dy\; dx = \int_{\pi}^{2\pi}\int_0^1 \cos(r^2)\; r\; dr\; d\theta$$



$$= \int_{\pi}^{2\pi} \frac{1}{2}\sin(r^2)\Big|_0^1 \, d\theta = \int_{\pi}^{2\pi} \frac{1}{2}\sin(1)\, d\theta = \frac{\pi}{2}\sin(1).$$

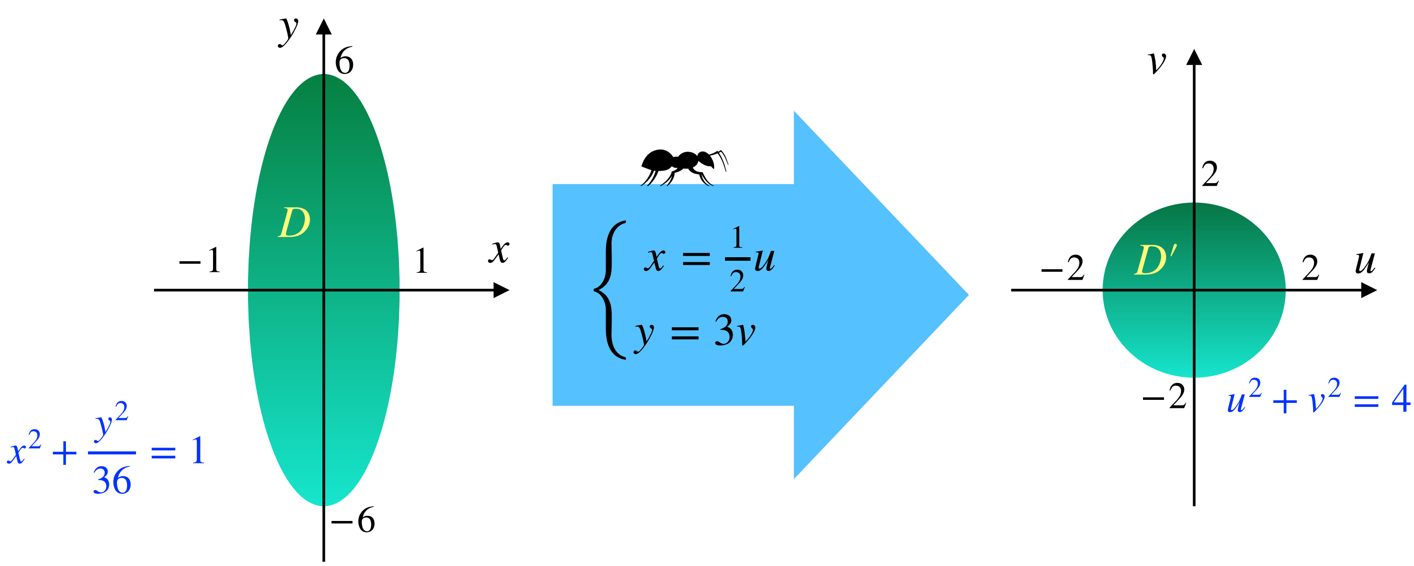
Practice Problems

1. Evaluate $\iint_D y^2 + 3x \; dA$ where D is the region in the 3rd quadrant between $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.
2. Evaluate $\iint_D \sqrt{1 + 4x^2 + 4y^2} \; dA$ where D is the bottom half of $x^2 + y^2 = 16$.
3. Evaluate $\iint_D 4xy - 7 \; dA$ where D is the portion of $x^2 + y^2 = 2$ in the 1st quadrant.
4. Use a double integral to determine the area of the region that is inside $r = 4 + 2\sin\theta$ and outside $r = 3 - \sin\theta$.
5. Evaluate the following integral by first converting to an integral in polar coordinates.

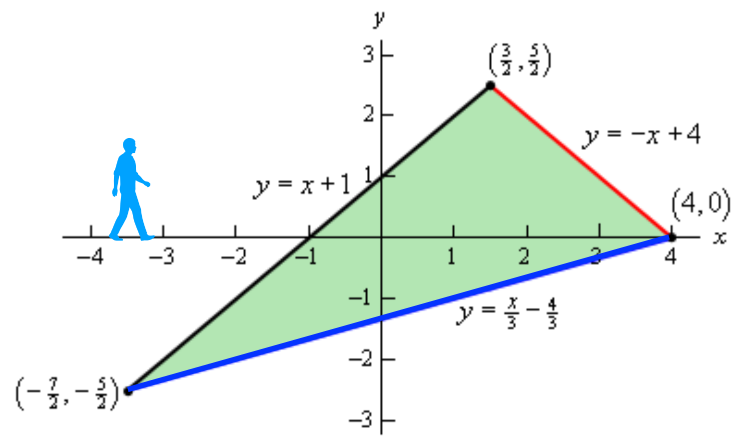
$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 e^{x^2+y^2}\, dy\, dx.$$

6. Use a double integral to determine the volume of the solid that is inside the cylinder $x^2+y^2=16$, below $z=2x^2+2y^2$ and above the xy-plane.
7. Use a double integral to determine the volume of the solid that is bounded by $z=8-x^2-y^2$ and $z=3x^2+3y^2-4$.

2. Change of variables



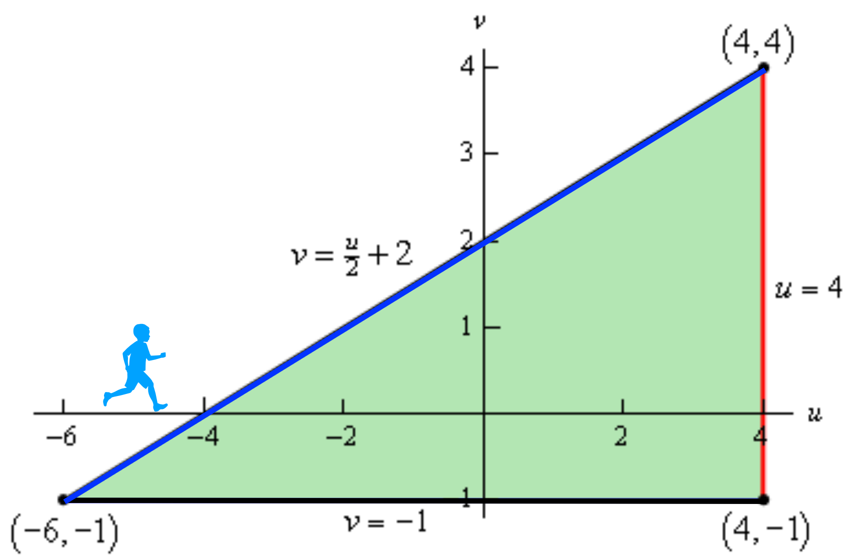
$$x^2+\frac{y^2}{36}=1 \implies \left(\frac{u}{2}\right)^2+\frac{(3v)^2}{36}=1 \implies \frac{u^2}{4}+\frac{v^2}{4}=1 \implies u^2+v^2=4.$$



$$
\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}
$$

$$
\begin{aligned}
y &= x + 1 \\
\tfrac{1}{2}(u-v) &= \tfrac{1}{2}(u+v) + 1 \\
u - v &= u + v + 2 \\
-2v &= 2 \\
v &= -1
\end{aligned}
$$

$$
\begin{aligned}
y &= -x + 4 \\
\tfrac{1}{2}(u-v) &= -\tfrac{1}{2}(u+v) + 4 \\
u - v &= -u - v + 8 \\
2u &= 8 \\
u &= 4
\end{aligned}
$$



Definition 1. The ***Jacobian*** of the transformation

$$\begin{cases} x = g(u, v) \\ y = h(u, v) \end{cases}$$

is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Definition 2. The ***Jacobian*** of the transformation

$$\begin{cases} x = g(u, v, w) \\ y = h(u, v, w) \\ z = k(u, v, w) \end{cases}$$

is

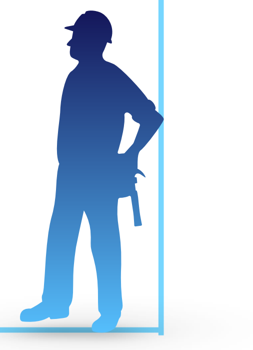
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

3. Change of variables for a double integral

$z = f(x, y), \quad D$

$$\begin{cases} x = g(u, v) \\ y = h(u, v) \end{cases} \Longrightarrow \begin{cases} D \Longrightarrow \overline{D} \\ f(x, y) \Longrightarrow f(g(u, v), h(u, v)) \end{cases}$$

$$\iint_D f(x, y)\, dA = \iint_{\overline{D}} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| d\overline{A}$$



$t = f(x, y, z), \quad D$

$$\begin{cases} x = g(u, v, w) \\ y = h(u, v, w) \\ z = k(u, v, w) \end{cases} \Longrightarrow \begin{cases} D \Longrightarrow \overline{D} \\ f(x, y, z) \Longrightarrow f(g(u, v, w), h(u, v, w), k(u, v, w)) \end{cases}$$

$$\iiint_D f(x, y, z)\, dV = \iiint_{\overline{D}} f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| d\overline{V}$$

Spherical coordinates

The transformation is

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

and the Jacobian is:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cdots = -\rho^2 \sin\phi\,.$$

Cylindrical coordinates

The transformation is

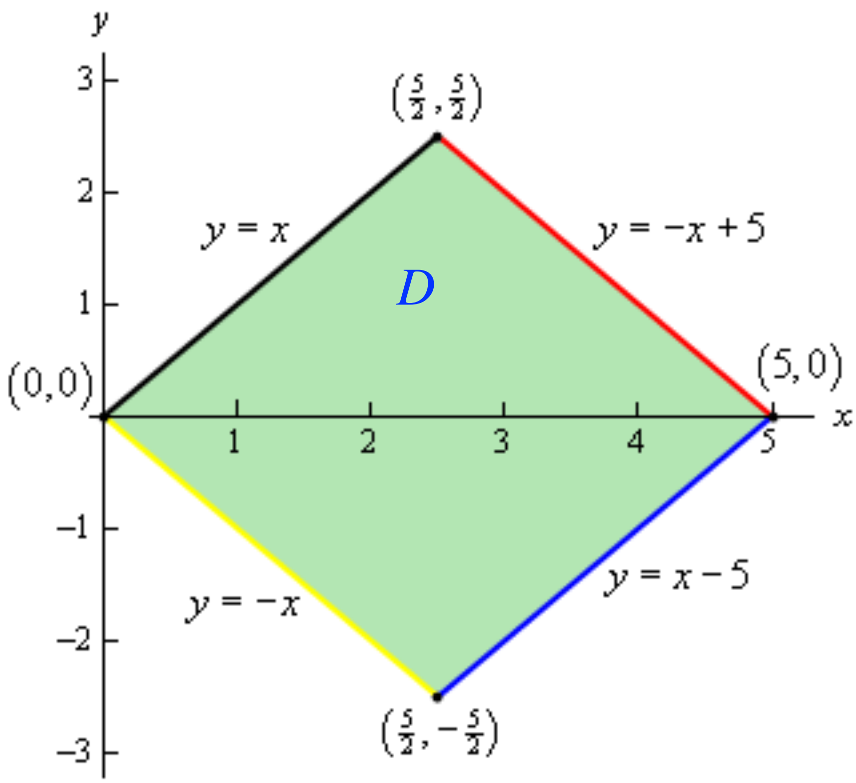
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

and the Jacobian is:

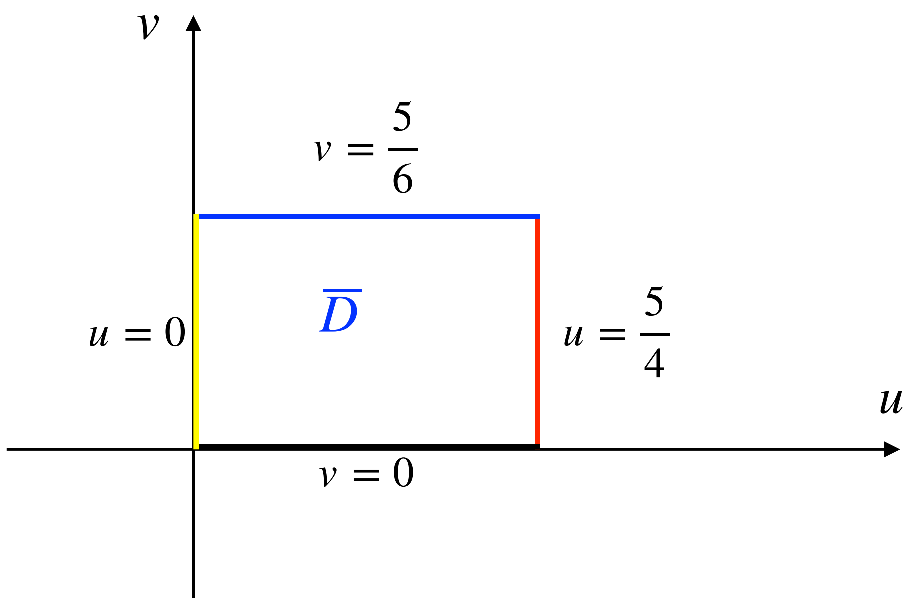
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cdots = r$$

Example 6 Evaluate $\displaystyle\iint_D x + y \; dA$ where D is the trapezoidal region with vertices given by $(0,0)$, $(5,0)$, $\left(\frac{5}{2}, \frac{5}{2}\right)$, $\left(\frac{5}{2}, -\frac{5}{2}\right)$ using the transformation $x = 2u + 3$ and $y = 2u - 3v$.

Solution. First, let's sketch the region D and determine equations for each of the sides:



$$\begin{cases} x = 2u + 3v \\ y = 2u - 3v \end{cases}$$



$$\overline{D} \Longrightarrow \begin{cases} 0 \leqslant u \leqslant \frac{5}{4} \\ 0 \leqslant v \leqslant \frac{5}{6} \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -12$$

$$\iint_D x + y \; dA = \int_0^{\frac{5}{6}} \int_0^{\frac{5}{4}} ((2u+3v)+(2u-3v))\,|-12|\; du\; dv$$

$$= \int_0^{\frac{5}{6}} \int_0^{\frac{5}{4}} 48u\; du\; dv = \int_0^{\frac{5}{6}} 24u^2 \Big|_0^{\frac{5}{4}} dv$$

$$= \int_0^{\frac{5}{6}} \frac{75}{2} dv = \frac{75}{2} v \Big|_0^{\frac{5}{6}} = \frac{125}{4}.$$

Practice Problems

For problems $1-3$ compute the Jacobian of each transformation.

1. $x = 4u - 3v^2,\ y = u^2 - 6v$.
2. $x = u^2v^3,\ y = 4 - 2\sqrt{u}$.
3. $x = \dfrac{v}{u},\ y = u^2 - 4v^2$.
4. If D is the region inside $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$ determine the region we would get applying the transformation $x = 2u, y = 6v$ to D.
5. If D is the parallelogram with vertices $(1,0), (4,3), (1,6)$ and $(-2,3)$ determine the region we would get applying the transformation $x = \dfrac{1}{2}(v-u), y = \dfrac{1}{2}(v+u)$ to D.
6. If D is the region bounded by $xy = 1, xy = 3, y = 2$ and $y = 6$ determine the region we would get applying the transformation $x = \dfrac{v}{6u}, y = 2u$ to D.
7. Evaluate $\displaystyle\iint_D xy^3\,dA$ where D is the region bounded by $xy = 1, xy = 3, y = 2$ and $y = 6$ using the transformation $x = \dfrac{v}{6u}, y = 2u$.

8. Evaluate $\iint_D 6x - 3y dA$ where D is the parallelogram with vertices $(2,0), (5,3), (6,7)$ and $(3,4)$ using the transformation $x = \frac{1}{3}(v - u),\ y = \frac{1}{3}(4v - u)$ to D.

9. Evaluate $\iint_D x + 2y dA$ where D is the triangle with vertices $(0,3), (4,1)$ and $(2,6)$ using the transformation $x = \frac{1}{2}(u - v),\ y = \frac{1}{4}(3u + v + 12)$ to D.