Problem 1. Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

where $a_0 = 1$ and $a_1 = 4$.

Solution. The characteristic equation of the recurrence relation is

$x^2 - 5x + 6 = 0,$ ➡ $(x-3)(x-2) = 0,$

➡ $x_1 = 3, \quad x_2 = 2,$

λ_0, λ_1 ➡ $a_n = \lambda_0 3^n + \lambda_1 2^n$

$$\boxed{\begin{array}{lll} n = 0 & \Longrightarrow & \left\{\begin{array}{l} 1 = a_0 = \lambda_0 + \lambda_1, \\ 4 = a_1 = 3\lambda_0 + 2\lambda_1 . \end{array}\right. \\ n = 1 & \Longrightarrow & \end{array}}$$

➡ $\lambda_0 = 2, \quad \lambda_1 = -1$

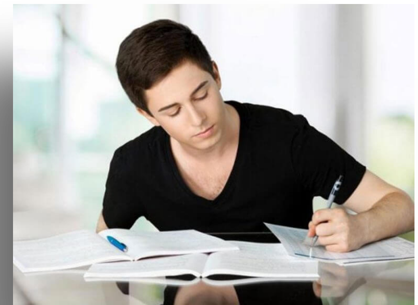
➡ $a_n = 2 \cdot 3^n - 2^n$

➡ 1, 4, 14, 46, … □

Problem 2. Solve the recurrence relation

$$a_n = 10a_{n-1} - 25a_{n-2}$$

where $a_0 = 3$ and $a_1 = 17$.



Solution. The characteristic equation of the recurrence relation is

$$x^2 - 10x + 25 = 0 \quad \Rightarrow \quad (x-5)^2 = 0,$$

$$\Rightarrow \quad x_1 = 5,$$

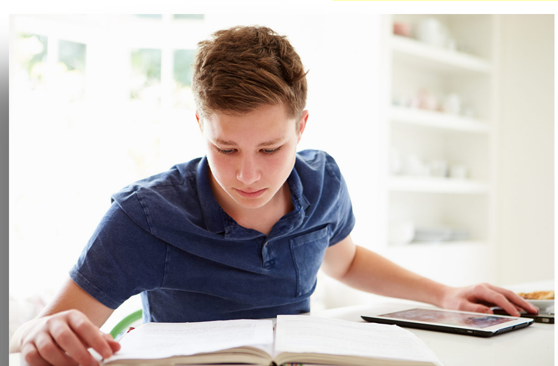
$$\lambda_0, \lambda_1 \quad \Rightarrow \quad a_n = (\lambda_0 + \lambda_1 n)5^n$$

$$\begin{array}{lcl} n = 0 & \Longrightarrow & \left\{ \begin{array}{l} 3 = a_0 = \lambda_0 \\ 17 = a_1 = (\lambda_0 + \lambda_1)5. \end{array} \right. \\ n = 1 & \Longrightarrow & \end{array}$$

$$\Rightarrow \quad \lambda_0 = 3, \quad \lambda_1 = \frac{2}{5}$$

$$\Rightarrow \quad a_n = (3 + \frac{2}{5}n)5^n = (15 + 2n)5^{n-1}$$

$$\Rightarrow \quad 3,\ 17,\ 95,\ 525,\ \ldots \quad \square$$



Problem 3. Solve the recurrence relation

$$\sqrt{a_n} = \sqrt{a_{n-1}} + 2\sqrt{a_{n-2}},$$

with initial conditions $a_0 = a_1 = 1$ by making the substitution $b_n = \sqrt{a_n}$.

Solution. Let $b_n = \sqrt{a_n}$ We obtain

$$b_n = b_{n-1} + 2b_{n-2}, \quad b_0 = b_1 = 1$$

The characteristic equation of the recurrence relation is

$$x^2 - x - 2 = 0 \quad \Rightarrow \quad (x-2)(x+1) = 0,$$

$$\Rightarrow \quad x_1 = 2, \quad x_2 = -1$$

$$\lambda_0, \lambda_1 \quad \Rightarrow \quad b_n = \lambda_0 2^n + \lambda_1 (-1)^n$$

$$\boxed{\begin{array}{lll} n = 0 & \Longrightarrow & \begin{cases} 1 = b_0 = \lambda_0 + \lambda_1 \\ 1 = b_1 = 2\lambda_0 - \lambda_1 . \end{cases} \\ n = 1 & \Longrightarrow & \end{array}}$$

$$\Rightarrow \quad \lambda_0 = \frac{2}{3}, \quad \lambda_1 = \frac{1}{3}$$

$$\Rightarrow \quad b_n = \frac{2^{n+1} + (-1)^n}{3}$$

$$\Rightarrow \quad b_n : \quad 1, 1, 3, 5, 11, 21, \ldots$$

$$\Rightarrow \quad a_n = b_n^2 : \quad 1, 1, 9, 25, 121, 441, \ldots \quad \square$$

Problem 4. Solve the recurrence relation

$$a_n = \sqrt{\frac{a_{n-2}}{a_{n-1}}}$$

with initial conditions $a_0 = 8$, $a_1 = \dfrac{1}{2\sqrt{2}}$

by making the logarithm of both sides and making the substitution $b_n = \log a_n$.

Solution. Let $b_n = \log a_n$. We obtain

$$a_n = \sqrt{\frac{a_{n-2}}{a_{n-1}}} \Longrightarrow \log a_n = \log\sqrt{\frac{a_{n-2}}{a_{n-1}}} = \frac{1}{2}\log a_{n-2} - \frac{1}{2}\log a_{n-1}$$

$$\Longrightarrow b_n = \frac{1}{2}b_{n-2} - \frac{1}{2}b_{n-1}$$

$$\Longrightarrow b_0 = \log a_0 = \log 8 = 3\log 2$$

$$\Longrightarrow b_1 = \log a_1 = \log\frac{1}{2\sqrt{2}} = -\frac{3}{2}\log 2$$

The characteristic equation of the recurrence relation is

$$x^2 + \frac{1}{2}x - \frac{1}{2} = 0 \Longrightarrow 2x^2 + x - 1 = 0$$

$$\Longrightarrow x = -1, \quad x = \frac{1}{2}$$

$$\lambda_0, \lambda_1 \Longrightarrow b_n = \lambda_0(-1)^n + \lambda_1\left(\frac{1}{2}\right)^n$$

$$\begin{array}{lll} n = 0 & \Longrightarrow & \left\{\begin{array}{l} 3\log 2 = b_0 = \lambda_0 + \lambda_1 \\ -\dfrac{3}{2}\log 2 = b_1 = -\lambda_0 + \dfrac{1}{2}\lambda_1. \end{array}\right. \\ n = 1 & \Longrightarrow & \end{array}$$

$$\Longrightarrow \lambda_0 = 2\log 2, \quad \lambda_1 = \log 2$$

$$\Rightarrow \quad b_n = (-1)^n 2 \log 2 + \left(\frac{1}{2}\right)^n \log 2$$

$$\Rightarrow \quad b_n = \log\left(2^{2(-1)^n} \times 2^{\left(\frac{1}{2}\right)^n}\right) = \log\left(2^{2(-1)^n + \left(\frac{1}{2}\right)^n}\right)$$

$$\Rightarrow \quad \log a_n = \log\left(2^{2(-1)^n + \left(\frac{1}{2}\right)^n}\right)$$

$$\Rightarrow \quad a_n = 2^{2(-1)^n + \left(\frac{1}{2}\right)^n}$$

$$\Rightarrow \quad a_n : \boxed{2, \ \frac{1}{2\sqrt{2}}, \ 4\sqrt[4]{2}, \ \ldots} \qquad \square$$

Inhomogeneous Recurrence Relation and Particular Solutions

homogeneous

$\{a_n\}_{n=0}^{\infty} \Rightarrow a_0, a_1, a_2, a_3, a_4, a_5, \ldots, a_n, \cdots$

$$\begin{cases} a_0, a_1, a_2, \ldots, a_{k-1} \\ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \quad n \geqslant k \end{cases}$$

$\boxed{a_0, \ a_1, \ \ldots, \ a_{k-1},} \ a_k, \ a_{k+1}, \ a_{k+2}, \ a_{k+3}, \ a_{k+4}, \ \ldots$

A recurrence relation is called *inhomogeneous* if it is in the form:

inhomogeneous

$\{a_n\}_{n=0}^{\infty} \Rightarrow a_0, a_1, a_2, a_3, a_4, a_5, \ldots, a_n, \cdots$

$$\begin{cases} a_0, a_1, a_2, \ldots, a_{k-1} \\ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n), \quad f(n) \neq 0, \quad n \geqslant k \end{cases}$$

$\boxed{a_0, \ a_1, \ \ldots, \ a_{k-1},} \ a_k, \ a_{k+1}, \ a_{k+2}, \ a_{k+3}, \ a_{k+4}, \ \ldots$

Two Examples:

(1) $a_n = 5a_{n-1} - 6a_{n-2} + 6(4)^n$ $\Longrightarrow$ $f(n) = 6(4)^n$.

(2) $a_n = a_{n-1} + 12n^2$ $\Longrightarrow$ $f(n) = 12n^2$.

inhomogeneous

$\{a_n\}_{n=0}^{\infty} \Longrightarrow a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots,\ a_n,\ \cdots$

$$\begin{cases} a_0,\ a_1, a_2,\ \ldots,\ a_{k-1} \\ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n), \quad f(n) \neq 0, \quad n \geqslant k \end{cases}$$

$$\begin{cases} a_0,\ a_1, a_2,\ \ldots,\ a_{k-1} \\ a_n = h_n + f(n), \quad f(n) \neq 0, \quad n \geqslant k \end{cases}$$

$\boxed{a_0,\ a_1,\ \ldots,\ a_{k-1}}$ $a_k,\ a_{k+1},\ a_{k+2},\ a_{k+3},\ a_{k+4},\ \ldots$

Unlike the homogeneous case there is no general method to obtain a particular solution for an arbitrary inhomogeneous problem. However, there are techniques available for certain special cases. We have two such special cases:

(1) $f(n) = cn^k$, where k is a nonnegative integer, and

(2) $f(n) = cq^n$, where q is a rational number not equal to 1.

Step 1.

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \Longrightarrow p(x) = x^k - c_1 x^{k-1} - \cdots - c_k = 0$

$\Longrightarrow r_1,\ \ldots,\ r_k \Longrightarrow \lambda_1, \ldots, \lambda_k, \Longrightarrow a_h = \lambda_1 r_1^n + \cdots + \lambda_k r_k^n,$

$\boxed{a_n = a_h + a_p}$ $a_p = ?$

Case (1) $f(n) = cn^k$

I. $p(1) \neq 0 \Rightarrow a_p = A_0 + A_1 n + A_2 n^2 + \cdots + A_k n^k$

II. $p(x) = \cdots (x-1)^t \cdots \Rightarrow a_p = (A_0 + A_1 n + A_2 n^2 + \cdots + A_k n^k) n^t$

Case (2) $f(n) = cq^n$

I. $p(q) \neq 0 \Rightarrow a_p = Aq^n$

II. $p(x) = \cdots (x-q)^t \cdots \Rightarrow a_p = An^t q^n$

If k consecutive initial conditions of the inhomogeneous relation are known, these initial conditions can be used to define a linear system of k equations in k variables giving a *unique solution*.

Some Examples:

Problem 5. Evaluate the sum of the squares of the first n positive integers:

$$0^2 + 1^2 + 2^2 + \cdots + n^2 = ?$$

Solution. We put

$$a_n := 0^2 + 1^2 + 2^2 + \cdots + n^2 .$$

Then, we have

$$a_n = \underbrace{a_{n-1}}_{a_h} + \underbrace{n^2}_{f(n)}, \qquad a_0 = 0.$$

The homogeneous part gives the solution:

$$a_n = a_{n-1} \Longrightarrow p(x) = x - 1 \Longrightarrow r_1 = 1$$

$$\Longrightarrow \lambda_1 \Longrightarrow a_h = \lambda_1 1^n = \lambda_1 .$$

The choice for the particular solution is: $a_p = (A_0 + A_1 n + A_2 n^2)n$

Therefore, we obtain

$$a_n = a_h + a_p = \lambda_1 + (A_0 + A_1 n + A_2 n^2)n$$

$$\boxed{\begin{array}{lll} n = 0 & \Longrightarrow & \left\{\begin{array}{l} 0 = a_0 = \lambda_1 \\ 1 = a_1 = \lambda_1 + A_0 + A_1 + A_2 \\ 5 = a_2 = \lambda_1 + 2A_0 + 4A_1 + 8A_2 \\ 13 = a_3 = \lambda_1 + 3A_0 + 9A_1 + 27A_2 \end{array}\right. \\ n = 1 & \Longrightarrow & \\ n = 2 & \Longrightarrow & \\ n = 3 & \Longrightarrow & \end{array}} \qquad \begin{array}{l} \lambda_1 = 0, \\ A_0 = \frac{1}{6} \\ A_1 = \frac{1}{2} \\ A_0 = \frac{1}{3} \end{array}$$

$$\Longrightarrow a_n = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 \Longrightarrow a_n = \frac{n(n+1)(2n+1)}{6}. \quad \square$$

Problem 6. Solve $a_n = a_{n-1} + 2a_{n-2} + 4(3)^n$ with the initial conditions $a_0 = 11,\ a_1 = 28$.

Solution. The homogeneous part gives the solution:

$$a_n = a_{n-1} + 2a_{n-2} \Longrightarrow p(x) = x^2 - x - 2$$

$$\Longrightarrow r_1 = 2,\quad r_2 = -1$$

$$\Longrightarrow \lambda_1,\ \lambda_2 \qquad a_h = \lambda_1 2^n + \lambda_2(-1)^n .$$

The choice for the particular solution is: $a_p = \lambda_3(3)^n$

Therefore, we obtain

$$a_n = a_h + a_p = \lambda_1 2^n + \lambda_2(-1)^n + \lambda_3(3)^n$$

$$\begin{array}{lcl} n = 0 & \Longrightarrow & \\ n = 1 & \Longrightarrow & \\ n = 2 & \Longrightarrow & \end{array} \begin{cases} 11 = a_0 = \lambda_1 + \lambda_2 + \lambda_3 \\ 28 = a_1 = 2\lambda_1 - \lambda_2 + 3\lambda_3 \\ 86 = a_2 = 4\lambda_1 + \lambda_2 + 9\lambda_3 \end{cases} \qquad \begin{array}{l} \lambda_1 = 1, \\ \lambda_2 = 1, \\ \lambda_3 = 9, \end{array}$$

$$\Longrightarrow a_n = 2^n + (-1)^n + 3^{n+2} . \quad \square$$

a_n : 11, 28, 86, 250, …