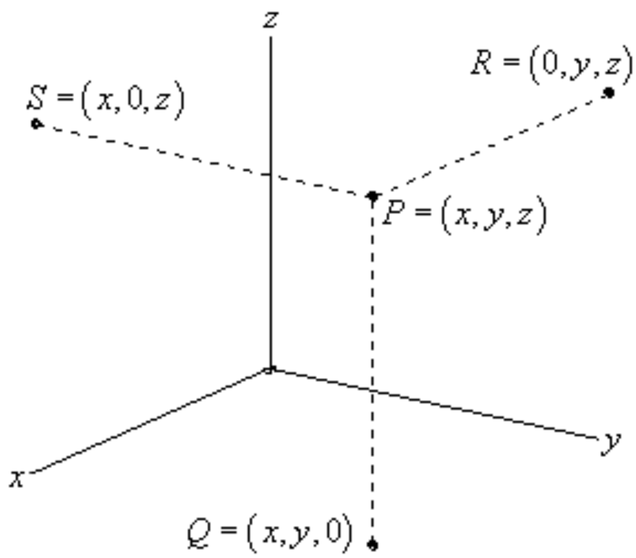
2. Three Dimensional Space

Section 1-1: The 3-D Coordinate System

- The *1-D coordinate system* is denoted by $\mathbb{R}$.
- The *2-D coordinate system* is often denoted by $\mathbb{R}^2$.
- The *3-D coordinate system* is often denoted by $\mathbb{R}^3$.
- A general n *dimensional coordinate system* is often denoted by $\mathbb{R}^n$.



- $P = (x, y, z)$ *Cartesian coordinates*.
- The xy, xz and yz-planes are called the *coordinate planes*.
- The point Q is referred to as the *projection* of P in the xy-plane.
- The point R is referred to as the *projection* of P in the yz-plane.
- The point S is referred to as the *projection* of P in the xz-plane.

● Many of the formulas that you are used to working with in $\mathbb{R}^2$ have natural extensions in $\mathbb{R}^3$.

- The *distance* between two points $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ in $\mathbb{R}^2$ is given by,

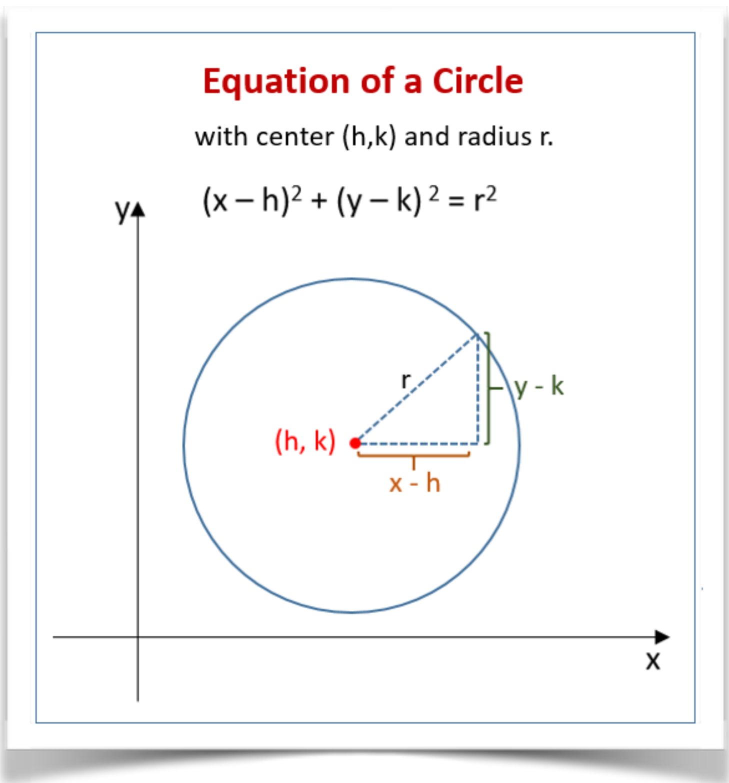
$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- The *distance* between two points $P_1 = (x_1, y_1, z_1), P_2 = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ is given by,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- A *circle* with center (h, k) and radius r is given by,

$$(x - h)^2 + (y - k)^2 = r^2 .$$

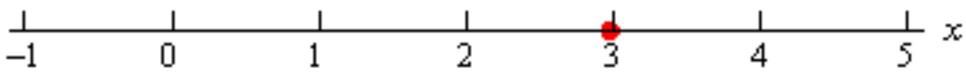


- The general equation for a *sphere* with center (h, k, l) and radius r is given by,

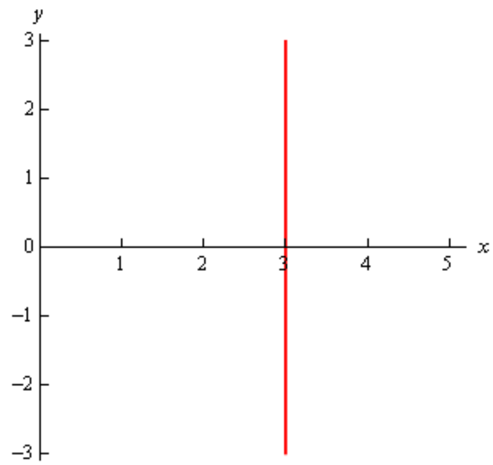
$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 .$$

Example 1 Graph $x = 3$ in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

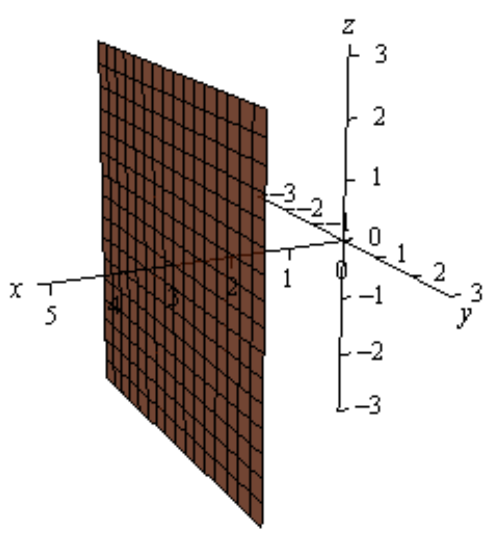
- The graph of $x = 3$ in $\mathbb{R}$:



- The graph of $x = 3$ in $\mathbb{R}^2$:



- The graph of $x = 3$ in $\mathbb{R}^3$:

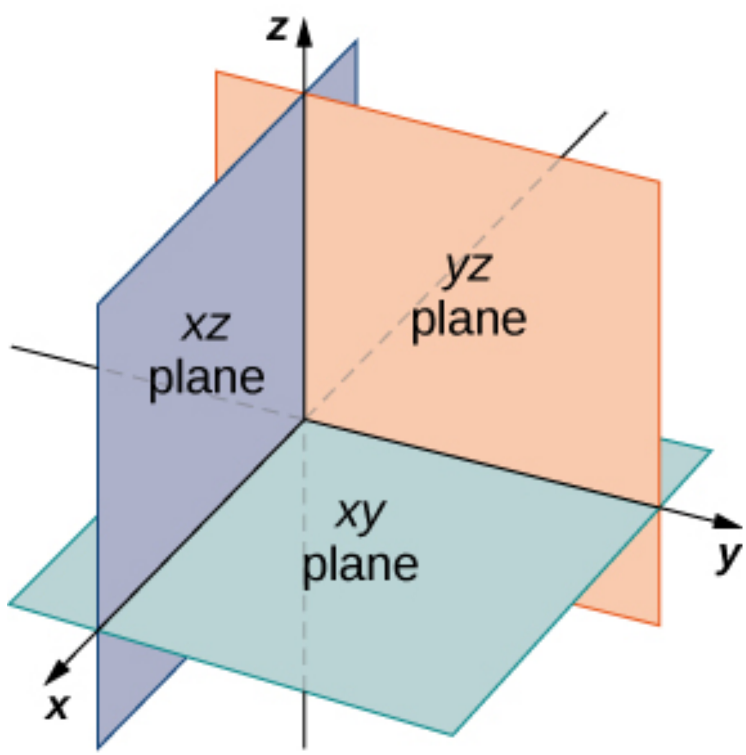


Example 2

$x = 0$ yz-plane

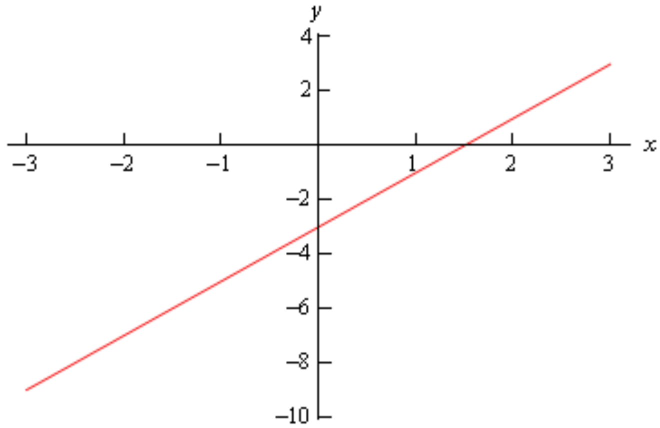
$y = 0$ xz-plane

$z = 0$ xy-plane

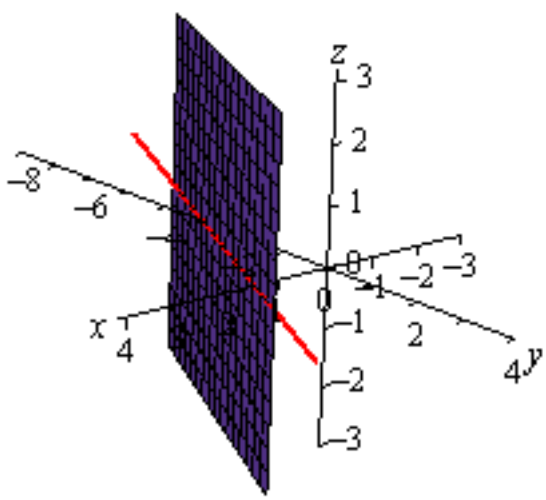


Example 3 Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

- In $\mathbb{R}^2$ this is a line with slope 2 and a intercept of -3.

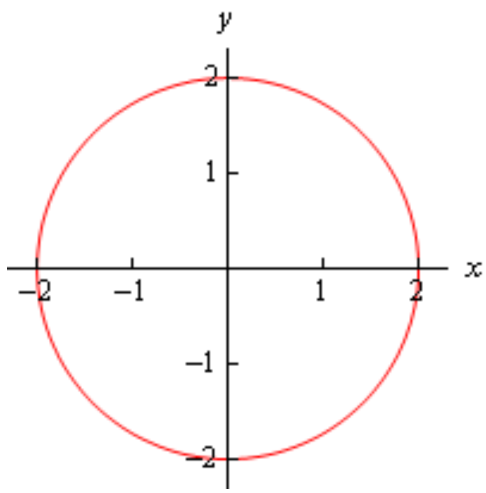


- In $\mathbb{R}^3$ the graph is a vertical plane that lies over the line given by $y = 2x - 3$ in the xy-plane.

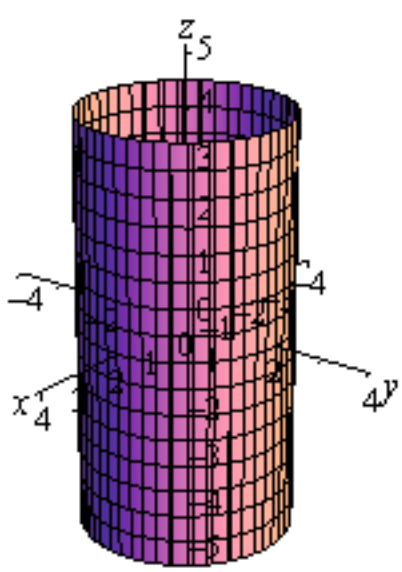


Example 4 Graph $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

- In $\mathbb{R}^2$ this is a circle centered at the origin with radius 2.

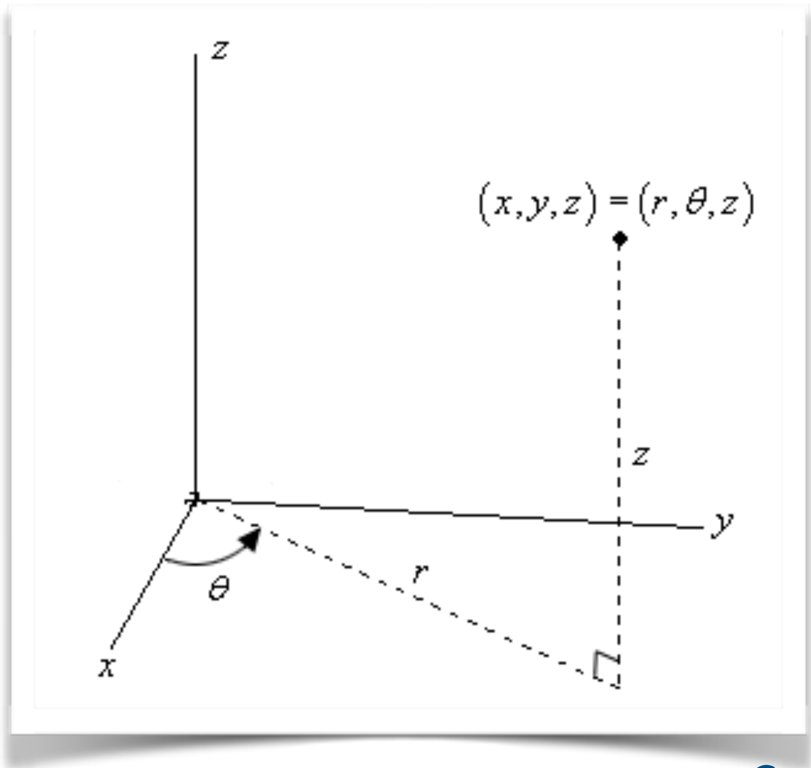


- In $\mathbb{R}^3$ this is a cylinder of radius 2 centered on the z-axis.

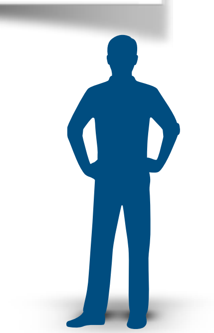


Section 1-2: Cylindrical Coordinates

This is an extension of *polar coordinates* into three dimensions.



- $P = (r, \theta, z)$ *Cylindrical coordinates*.



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$r^2 = x^2 + y^2.$ or $r = \sqrt{x^2 + y^2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$

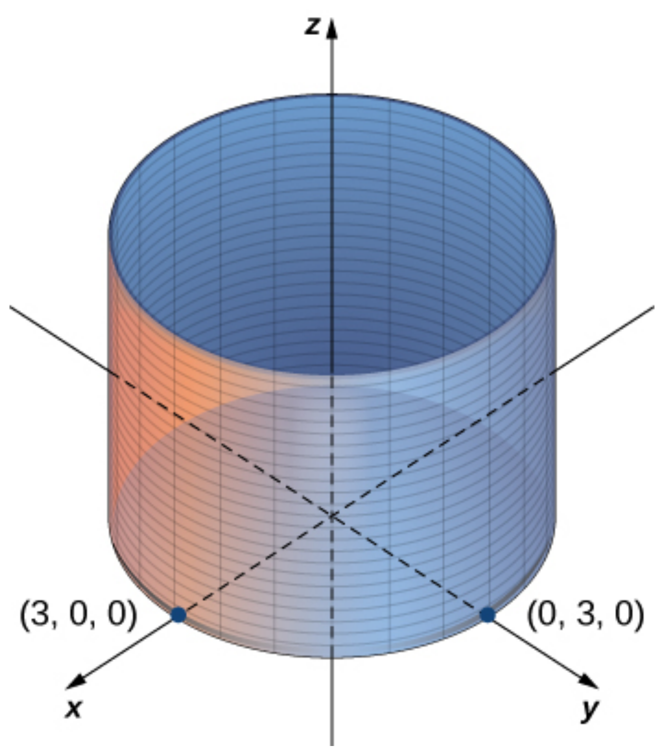
Example 5 Identify the surface for each of the following equations.

- $r = 3$

$$r = 3 \Longrightarrow r^2 = 9 \Longrightarrow x^2 + y^2 = 9.$$

In $\mathbb{R}^2$ this is a circle of radius 3.

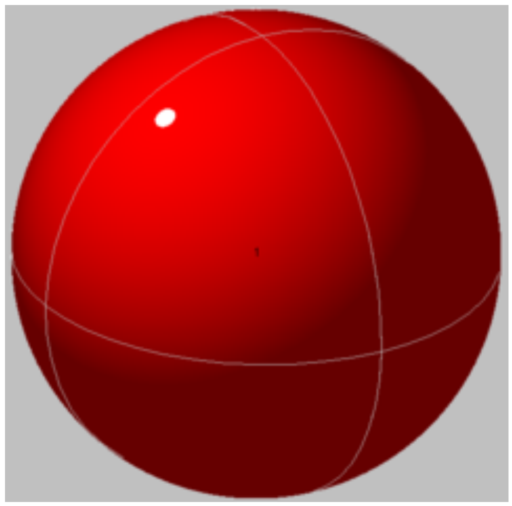
In $\mathbb{R}^3$ this is a cylinder of radius 3 centered on the z-axis.



- $r^2 + z^2 = 100.$

$$r^2 + z^2 = 1 \Longrightarrow x^2 + y^2 + z^2 = 100.$$

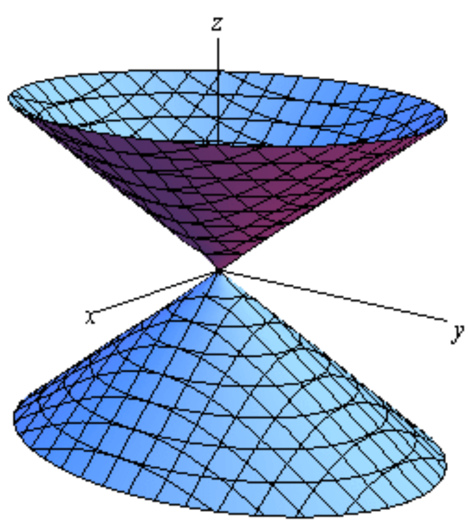
This is a sphere centered at the origin with radius 10.



- $z = r$.

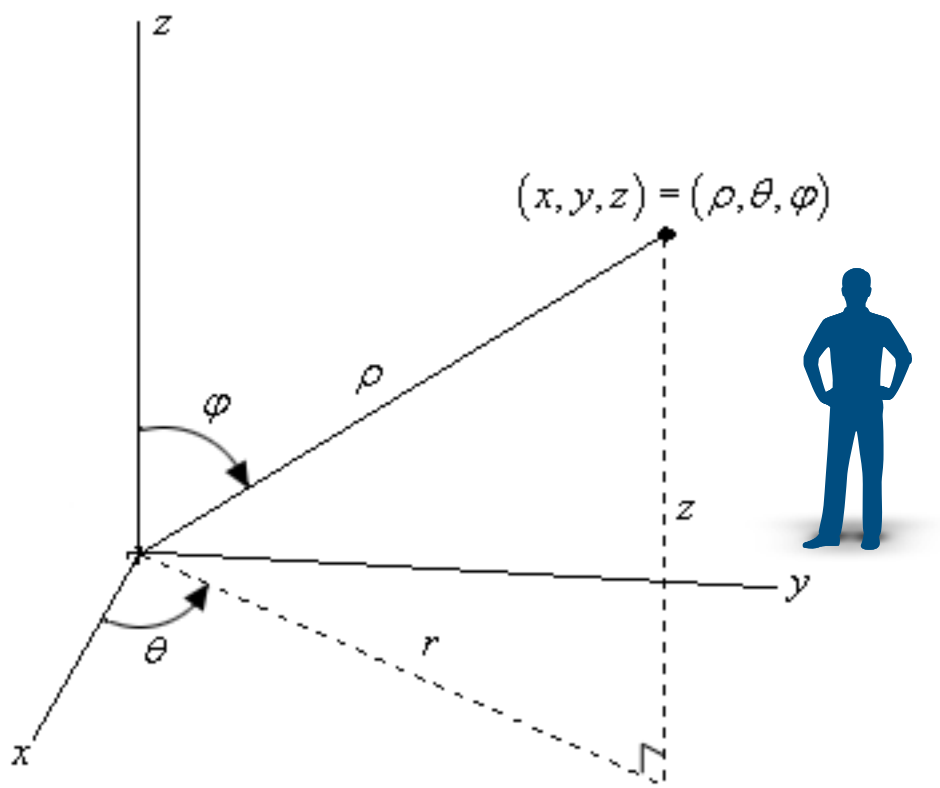
$$z = r \Longrightarrow z^2 = r^2 \Longrightarrow z^2 = x^2 + y^2.$$

This is the equation of a cone.



Section 1-3: Spherical Coordinates

Here, we introduce the spherical coordinates:



- ρ The distance from the origin to the point and we require $\rho \geqslant 0$.

- θ It is the angle between the positive x-axis and the line above denoted by r (which is also the same r as in polar/cylindrical coordinates). There are no restrictions on
- ϕ This is the angle between the positive z-axis and the line from the origin to the point. We will require $0 \leqslant \phi \leqslant \pi$.

- $P = (\rho, \theta, \phi)$ *Spherical coordinates.*

$$z = \rho\cos\phi$$
$$r = \rho\sin\phi \quad \Longrightarrow \quad x = r\cos\theta, \quad y = r\sin\theta$$

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

Also note that since we know that $r^2 = x^2 + y^2$ we obtain

$$\rho^2 = x^2 + y^2 + z^2 .$$

Example 6 Perform each of the following conversions.

- Convert the point $(\sqrt{6}, \frac{\pi}{4}, \sqrt{2})$ from *cylindrical* to *spherical* coordinates.

Find ρ :

$$\rho = \sqrt{r^2 + z^2} = \sqrt{6+2} = \sqrt{8} = 2\sqrt{2} .$$

Find ϕ :

$$z = \rho\cos\phi \quad \Longrightarrow \quad \cos\phi = \frac{z}{\rho} = \frac{\sqrt{2}}{2\sqrt{2}} \quad \Longrightarrow \quad \phi = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} .$$

So, the spherical coordinates of this point will are

$$(2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3})$$

- Convert the point $(-1,1,-\sqrt{2})$ from *Cartesian* to *spherical* coordinates.

Find ρ :

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{1+1+2}=2.$$

Find ϕ :

$$z=\rho\cos\phi \quad\Longrightarrow\quad \cos\phi=\frac{z}{\rho}=\frac{-\sqrt{2}}{2} \quad\Longrightarrow \phi=\cos^{-1}\left(\frac{-\sqrt{2}}{2}\right)=\frac{3\pi}{4}$$

Find θ :

$$y=\rho\sin\phi\sin\theta \quad\Longrightarrow\quad \sin\theta=\frac{y}{\rho\sin\phi}=\frac{1}{2\left(\frac{\sqrt{2}}{2}\right)}=\frac{\sqrt{2}}{2} \quad\Longrightarrow \theta=\frac{\pi}{4},\frac{3\pi}{4}$$

Since $(x,y)=(-1,1)$ lies in the second quadrant, $\theta=\dfrac{3\pi}{4}$.

The spherical coordinates of this point are

$$(2,\frac{3\pi}{4},\frac{3\pi}{4}).$$

Example 7 Identify the surface for each of the following equations.

- $\rho=5$.

$$\rho=5 \quad\Longrightarrow\quad \rho^2=25 \quad\Longrightarrow\quad x^2+y^2+z^2=25.$$

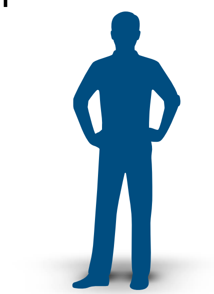
A *sphere* of radius 5 centered at the origin.

- $\theta=\dfrac{2\pi}{3}$.

Points in a vertical *plane* will do this. So, we have a vertical plane that forms an angle of $\frac{2\pi}{3}$ with the positive x-axis.

- $\phi = \frac{\pi}{3}$

This is a *cone*. All of the points on a cone are a fixed angle from the z-axis. So, we have a cone whose points are all at an angle of $\frac{\pi}{3}$ from the z-axis.



$\rho = a$: *sphere* of radius a centered at the origin.

$\theta = \alpha$: vertical *plane* that makes an angle of α with the positive x-axis

$\phi = \beta$: *cone* that makes an angle of β with the positive z-axis.

- $\rho \sin \phi = 2.$

$$\rho \sin \phi = 2 \implies r = 2 \implies r^2 = 4 \implies x^2 + y^2 = 4.$$

So, we have a cylinder of radius 2 centered on the z-axis.

Practical Problems

1. Give the projection of $P = (3, -4,6)$ onto the three coordinate planes.
2. Which of the points $P = (4, -2,6), Q = (-6, -3,2)$ is closest to the yz-plane?
3. Which of the points $P = (-1,4, -7), Q = (6, -1,5)$ is closest to the z-axis?
4. Convert the Cartesian coordinates for the point into Cylindrical coordinates: $(4, -5,2), (-4, -1,8)$.
5. Convert the following equation written in Cartesian coordinates into an equation in Cylindrical coordinates
$$x^3 + 2x^2 - 6z = 4 - 2y^2 .$$
6. convert the equation written in Cylindrical coordinates into an equation in Cartesian coordinates:

A. $zr = 2 - r^2$,

B. $4 \sin \theta - 2 \cos \theta = \dfrac{r}{z}$.

7. Identify the surface generated by the given equation:

I. $r^2 - 4r \cos \theta = 14.$

II. $z = 7 - 4r^2$

8. Convert the Cartesian coordinates for the point into Spherical coordinates: $(3, -4,1), (-2, -1, -7)$.
9. Convert the Cylindrical coordinates for the point $(2,0.345, -3)$, into Spherical coordinates.
10. Convert the following equation written in Cartesian coordinates into an equation in Spherical coordinates:
$$x^2 + y^2 = 4x + z - 2.$$
11. Convert the equation written in Spherical coordinates into an equation in Cartesian coordinates:

- $\rho^2 = 3 - \cos\phi$,
- $\csc\phi = 2\cos\theta + 4\sin\theta$

12. Identify the surface generated by the given equation:

- $\phi = \dfrac{4\pi}{5}$,
- $\rho = -2\sin\phi\cos\theta$.