Gradient Vector, Tangent Planes And Normal Lines

A. R. Moghaddamfar

K. N. Toosi University of Technology

E-mail: moghadam@kntu.ac.ir

Gradient Vector

- The gradient vector $\nabla f(x_0, y_0)$ is *orthogonal* (or *perpendicular*) to the curve $f(x, y) = 0$ at the point (x_0, y_0). Likewise, the gradient vector $\nabla f(x_0, y_0, z_0)$ is *orthogonal* to the surface $f(x, y, z) = 0$ at the point (x_0, y_0, z_0).
- Recall that the gradient vector is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle f_x, f_y, f_z \rangle .$$

- the gradient vector is always orthogonal, or *normal*, to the surface at a point:

$$A = (x_0, y_0, z_0) \qquad \text{(where } f(x_0, y_0, z_0) = 0\text{)},$$

$$\vec{N} = \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \rangle$$

The Tangent Plane to a Surface

Tangent Plane

The tangent plane to the surface given by $f(x, y, z) = 0$ at the point $A = (x_0, y_0, z_0)$ has the equation:

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

- Note that if $z = \varphi(x, y)$, then we define a new function

$$f(x, y, z) = \varphi(x, y) - z.$$

The equation of the tangent plane is then,

$$\varphi_x(x_0, y_0)(x - x_0) + \varphi_y(x_0, y_0)(y - y_0) - (z - z_0) = 0,$$

where $z_0 = \varphi(x_0, y_0)$.

The Orthogonal Line to a Surface

Normal Line

The normal line to the surface given by $f(x, y, z) = 0$ at the point $A = (x_0, y_0, z_0)$ has the equation:

$$\vec{\mathbf{r}}(t) = \langle x_0, y_0, z_0 \rangle + t\, \nabla f(x_0, y_0, z_0)$$

Some Examples

Find the tangent plane and normal line to $f(x, y, z) = 0$ at the point $A = (x_0, y_0, z_0)$:

- $f(x, y, z) = x^2 + y^2 + z^2 - 30$, $A = (1, -2, 5)$:

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle \quad \Longrightarrow \quad \nabla f(1, -2, 5) = \langle 2, -4, 10 \rangle.$$

$$\text{Tangent Plane}: \quad 2(x-1) - 4(y+2) + 10(z-5) = 0.$$

$$2x - 4y + 10z - 60 = 0.$$

$$\text{Normal Line}: \quad \vec{\mathbf{r}}(t) = \langle 1, -2, 5 \rangle + t\langle 2, -4, 10 \rangle.$$

$$\vec{\mathbf{r}}(t) = \langle 2t + 1, -4t - 2, 10t + 5 \rangle.$$

Some Examples

- $x^2y = 4ze^{x+y} - 35$, $A = (3, -3, 2)$:
- $f(x, y, z) = 4ze^{x+y} - x^2y - 35 \quad \Longrightarrow$

$$\nabla f(x, y, z) = \langle 2xy - 4ze^{x+y}, x^2 - 4ze^{x+y}, -4e^{x+y} \rangle$$

$$\Longrightarrow \quad \nabla f(3, -3, 2) = \langle -26, 1, -4 \rangle.$$

Tangent Plane : $$-26(x-3)+1(y+3)-4(z-2) = 0.$$

$$-26x + y - 4z + 89 = 0.$$

Normal Line : $$\vec{\mathbf{r}}(t) = \langle 3, -3, 2 \rangle + t\langle -26, 1, -4 \rangle.$$

$$\vec{\mathbf{r}}(t) = \langle 3 - 26t, -3 + t, 2 - 4t \rangle.$$

Some Examples

- $\ln\left(\frac{x}{2y}\right) = z^2(x-2y)+3z+3$, $A=(4,2,-1)$:

- $f(x,y,z) = \ln\left(\frac{x}{2y}\right) - z^2(x-2y)-3z-3=0 \quad \Longrightarrow$

$$\nabla f(x,y,z) = \langle \frac{1}{x} - z^2, -\frac{1}{y}+2z^2, -2z(x-2y)-3\rangle$$

$$\Longrightarrow \quad \nabla f(4,2,-1) = \langle -\tfrac{3}{4}, \tfrac{3}{2}, -3\rangle.$$

Tangent Plane : $$-\frac{3}{4}(x-4)\frac{3}{2}(y-2)-3(z+1)=0.$$

$$-\frac{3}{4}x+\frac{3}{2}y-3z-3=0.$$

Normal Line : $$\vec{\mathbf{r}}(t) = \langle 4,2,-1\rangle + t\langle -\frac{3}{4}, \frac{3}{2}, -3\rangle.$$

$$\vec{\mathbf{r}}(t) = \langle 4-\frac{3}{4}t, 2+\frac{3}{2}t, -1-3t\rangle.$$