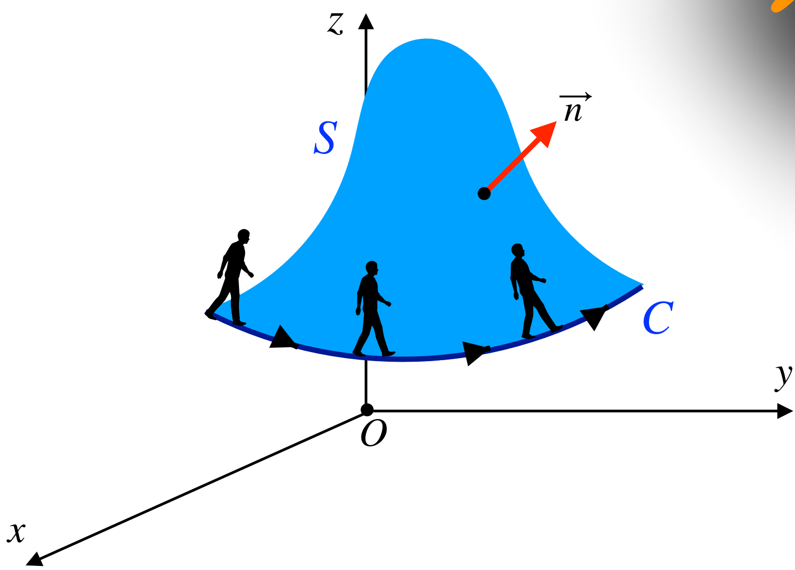
Stokes' Theorem



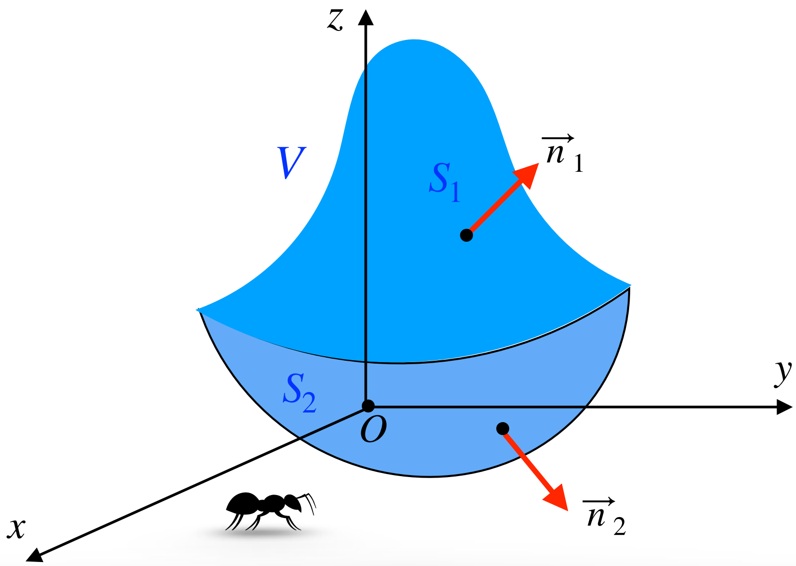
$$\vec{\mathbf{F}}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

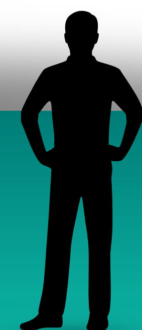
$$S : f(x,y,z) = 0 \quad \Longrightarrow \quad \vec{n} = \frac{\nabla(f)}{\|\nabla(f)\|}$$

Stokes' Theorem:

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{(S)} \operatorname{curl}(\vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\ dS$$

Divergence Theorem





Divergence Theorem:

$$\iint_{(S)} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS = \iiint_V \operatorname{div}(\vec{\mathbf{F}})\, dV$$

Example 1 Use the divergence theorem to evaluate $\iint_{(S)} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, dS$ where

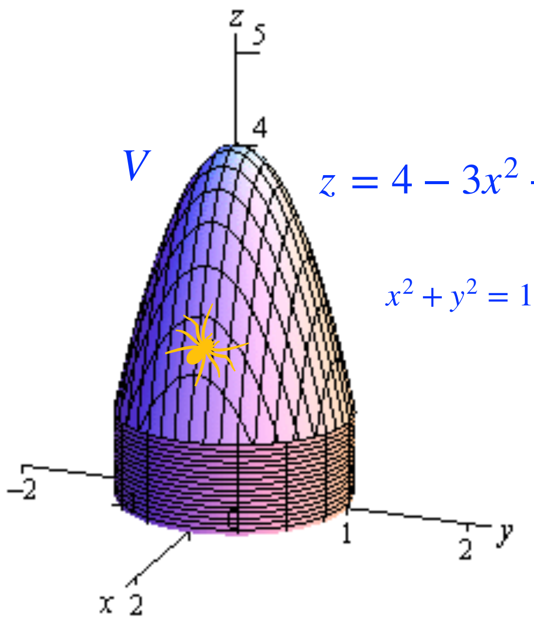
$$\vec{\mathbf{F}}(x, y, z) = xy\vec{i} - \frac{1}{2}y^2\vec{j} + z\vec{k}$$

and the surface consists of the three surfaces on the top,

$$z = 4 - 3x^2 - 3y^2,\ 1 \leqslant z \leqslant 4$$

on the sides $x^2 + y^2 = 1, 0 \leqslant z \leqslant 1$ and $z = 0$ on the bottom.

Solution. A sketch of the surface:



$$z = 4 - 3x^2 - 3y^2 = 4 - 3(x^2 + y^2) = 4 - 3r^2$$

$$V \Longrightarrow \begin{cases} 0 \leqslant r \leqslant 1 \\ 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant z \leqslant 4 - 3r^2 \end{cases}$$

$$\operatorname{div}(\overrightarrow{\mathbf{F}}) = y - y + 1 = 1.$$

$$\iint_{(S)} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{n}}\, dS = \iiint_V \operatorname{div}(\overrightarrow{\mathbf{F}})\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{4-3r^2} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 4r - 3r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{3}{4} r^4\right)\Big|_0^1 d\theta$$

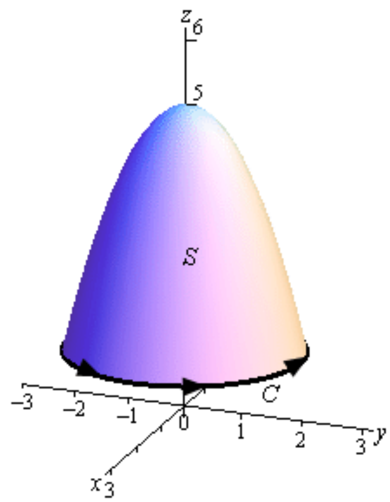
$$= \int_0^{2\pi} \frac{5}{4}\, d\theta = \frac{5}{2}\pi.$$

Example 2 Use Stokes' Theorem to evaluate $\displaystyle\iint_{(S)} \operatorname{curl}(\overrightarrow{\mathbf{F}}) \cdot \overrightarrow{\mathbf{n}}\ dS$ where

$$\overrightarrow{\mathbf{F}}(x, y, z) = z^2\vec{i} - 3xy\vec{j} + x^3y^3\vec{k}$$

and S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$.

Solution. A sketch of the surface:



the boundary curve C will be where the surface intersects the plane $z = 1$ and so will be the curve:

$$1 = 5 - x^2 - y^2 \implies C:\ x^2 + y^2 = 4,\ z = 1.$$

$$C: \vec{r}(t) = \langle 2\cos t, 2\sin t, 1\rangle,\ 0 \leqslant t \leqslant 2\pi.$$

$$\iint_{(S)} \operatorname{curl}(\overrightarrow{\mathbf{F}}) \cdot \overrightarrow{\mathbf{n}}\ dS = \int_0^{2\pi} \overrightarrow{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$$

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$$

$$\overrightarrow{\mathbf{F}}(\vec{r}(t)) = 1^2\vec{i} - 3(2\cos t)(2\sin t)\vec{j} + (2\cos 2)^3(2\sin t)^3\vec{k}$$

$$= \vec{i} - 12\cos t\sin t\vec{j} + 64\cos^3 t\ \sin^3 t\vec{k}$$

$$\overrightarrow{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -2\sin t - 24\sin t\cos^2 t$$

$$\iint_{(S)} \operatorname{curl}(\overrightarrow{\mathbf{F}}) \cdot \overrightarrow{\mathbf{n}}\ dS = \int_0^{2\pi} (-2\sin t - 24\sin t\cos^2 t)dt$$

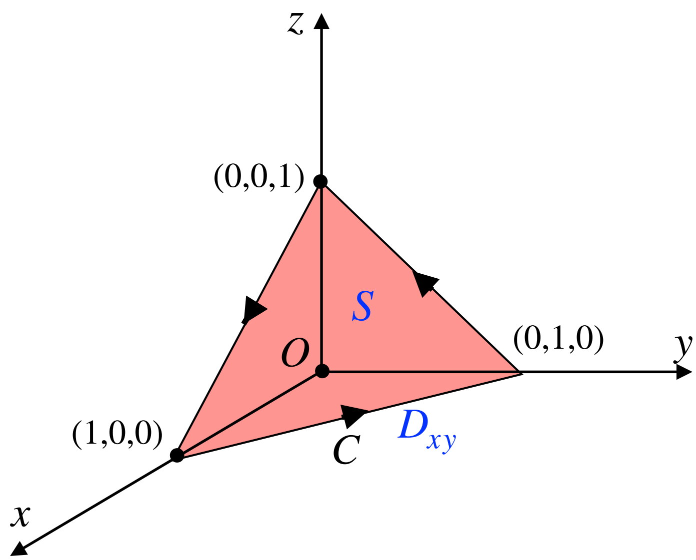
$$= (-2\sin t - 24\sin t\cos^2 t)\Big|_0^{2\pi} = 0.$$

Example 3 Use Stokes' Theorem to evaluate $\oint_C \overrightarrow{\mathbf{F}} \cdot d\,\overrightarrow{\mathbf{r}}$ where

$$\overrightarrow{\mathbf{F}}(x, y, z) = z^2\vec{i} + y^2\vec{j} + x\vec{k}$$

and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise rotation.

Solution. A sketch of the surface:



$$\operatorname{curl}(\overrightarrow{\mathbf{F}}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = (2z - 1)\vec{j}$$

$$S : x + y + z = 1 \quad \Longrightarrow \quad z = g(x, y) = 1 - x - y \quad \Longrightarrow \quad f(x, y, z) = z + x + y - 1$$

$$\overrightarrow{\mathbf{n}} = \frac{\nabla(f)}{\|\nabla(f)\|} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$$

Now, let's use Stokes' Theorem:

$$\oint_C \overrightarrow{\mathbf{F}} \cdot d\,\overrightarrow{\mathbf{r}} = \iint_{(S)} \operatorname{curl}(\overrightarrow{\mathbf{F}}) \cdot \overrightarrow{\mathbf{n}}\ dS$$

Since

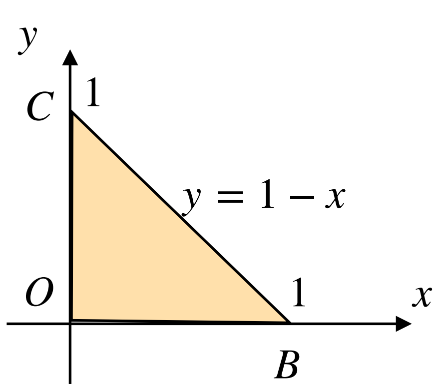
$$dS = \frac{\|\nabla(f)\|}{\left|\frac{\partial f}{\partial z}\right|} dxdy$$

we obtain

$$\operatorname{curl}(\overrightarrow{\mathbf{F}}) \cdot \overrightarrow{\mathbf{n}}\ dS = \operatorname{curl}(\overrightarrow{\mathbf{F}}) \cdot \frac{\nabla(f)}{\cancel{\|\nabla(f)\|}} \ \frac{\cancel{\|\nabla(f)\|}}{\left|\frac{\partial f}{\partial z}\right|} dxdy = \operatorname{curl}(\overrightarrow{\mathbf{F}}) \cdot \nabla(f)\ \frac{1}{\left|\frac{\partial f}{\partial z}\right|} dxdy$$

$$\Longrightarrow \quad \operatorname{curl}(\overrightarrow{\mathbf{F}}) \cdot \overrightarrow{\mathbf{n}}\ dS = \langle 0, 2z-1, 0\rangle \cdot \langle 1,1,1\rangle\ \frac{1}{|1|} dxdy = (2z-1)\ dx\ dy$$

$$= (2(1-x-y)-1)\ dx\ dy = (1-2x-2y)\ dx\ dy$$



$$D_{xy} \Longrightarrow \begin{cases} 0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1-x \end{cases}$$

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{(S)} \operatorname{curl}(\vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\ dS = \int_0^1 \int_0^{1-x} (1 - 2x - 2y)\ dy\ dx$$

$$= \int_0^1 (y - 2xy - y^2)\Big|_0^{1-x} dx = \int_0^1 (x^2 - x)\ dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1 = -\frac{1}{6}.$$

Practice Problems

1. Use Stokes' Theorem to evaluate $\iint_{(S)} \operatorname{curl}(\vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\ dS$ where

$$\vec{\mathbf{F}}(x, y, z) = y\vec{i} - x\vec{j} + yx^3\vec{k}$$

and S is the portion of the sphere of radius 4 with $z \geqslant 0$ and the upwards orientation.

2. Use Stokes' Theorem to evaluate $\iint_{(S)} \operatorname{curl}(\vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\ dS$ where

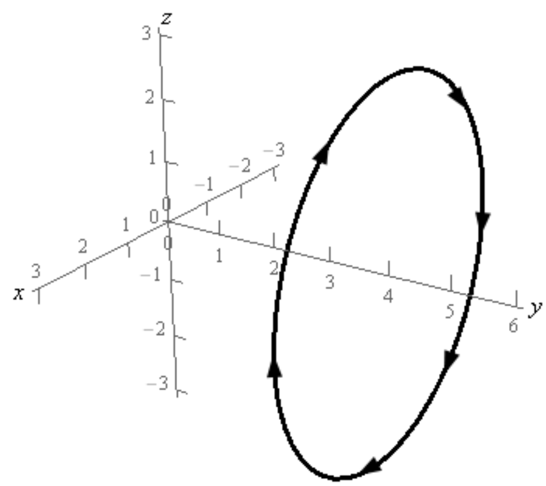
$$\vec{\mathbf{F}}(x, y, z) = (z^2 - 1)\vec{i} + (z + xy^3)\vec{j} + 6\vec{k}$$

and S is the portion of $x = 6 - 4y^2 - 4z^2$ in front of $x = -2$ with orientation in the negative x-axis direction.

3. Use Stokes' Theorem to evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where

$$\vec{\mathbf{F}}(x, y, z) = -yz\vec{i} + (4y + 1)\vec{j} + xy\vec{k}$$

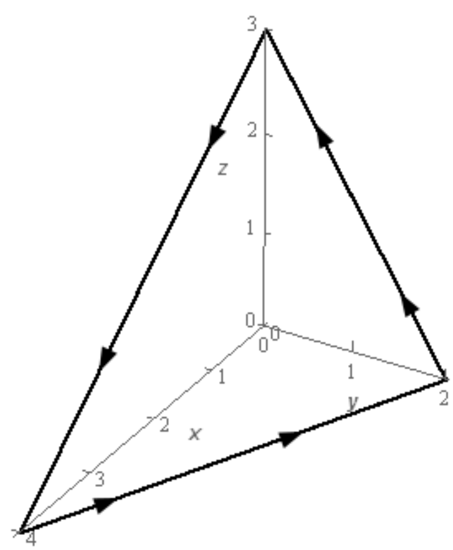
and C is the circle of radius 3 at $y = 4$ and perpendicular to the y-axis. C has a clockwise rotation:



4. Use Stokes' Theorem to evaluate $\oint_C \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}}$ where

$$\overrightarrow{\mathbf{F}}(x,y,z) = (3yx^2 + z^3)\vec{i} + (y^2)\vec{j} + 4yx^2\,\vec{k}$$

and C the triangle with vertices $(0,0,3)$, $(0,2,0)$ and $(4,0,0)$. C has a counter clockwise rotation:



5. Use the Divergence Theorem to evaluate $\iint_{(S)} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{n}}\, dS$ where

$$\overrightarrow{\mathbf{F}}(x,y,z) = (yx^2)\vec{i} + (xy^2 - 3z^4)\vec{j} + (x^3 + y^2)\vec{k}$$

and S is the surface of the sphere of radius 4 with $z \leqslant 0$ and $y \leqslant 0$. Note that all three surfaces of this solid are included in S.

6. Use the Divergence Theorem to evaluate $\displaystyle\iint_{(S)} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{n}}\, dS$ where

$$\overrightarrow{\mathbf{F}}(x, y, z) = \sin(\pi x)\vec{i} + zy^3\vec{j} + (z^2 + 4x)\vec{k}$$

and S is the surface of the box with $-1 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 1$ and $1 \leqslant z \leqslant 4$.

Note that all six sides of the box are included in S.

7. Use the Divergence Theorem to evaluate $\displaystyle\iint_{(S)} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{n}}\, dS$ where

$$\overrightarrow{\mathbf{F}}(x, y, z) = 2xz\vec{i} + (1 - 4xy^2)\vec{j} + (2z - z^2)\vec{k}$$

and S is the solid bounded by $z = 6 - 2x^2 - 2y^2$ and the plane $z = 0$. Note

that both of the surfaces of this solid included in S.