Gradient, Divergence, Curl, Nabla (Del) Operator, Laplacian Operator

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Gradient and Nabla (Del) Operator

- Let $f(x, y, z)$ be a *scalar field*. The *gradient* of f is the vector field defined by

$$\operatorname{grad}(f) = \nabla(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x}\vec{\mathbf{i}} + \frac{\partial f}{\partial y}\vec{\mathbf{j}} + \frac{\partial f}{\partial z}\vec{\mathbf{k}}.$$

- The vector differential operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \frac{\partial}{\partial x}\vec{\mathbf{i}} + \frac{\partial}{\partial y}\vec{\mathbf{j}} + \frac{\partial}{\partial z}\vec{\mathbf{k}},$$

  is called *nabla* or *del* .

Divergence and Curl of a Vector Field

- Let
$$\vec{\mathbf{F}} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle ,$$
be a *vector field*, continuously differentiable with respect to x, y and z. Then the *divergence* of $\vec{\mathbf{F}}$ is the scalar field defined by
$$\operatorname{div}(\vec{\mathbf{F}}) = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$
- The *curl* of $\vec{\mathbf{F}}$ is the vector field defined by
$$\operatorname{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

Summary

- Scalar Field $\Longrightarrow$ Vector Field

$$\nabla(f) = \frac{\partial f}{\partial x}\vec{\mathbf{i}} + \frac{\partial f}{\partial y}\vec{\mathbf{j}} + \frac{\partial f}{\partial z}\vec{\mathbf{k}}.$$

- Vector Field $\Longrightarrow$ Scalar Field

$$\mathrm{div}(\vec{\mathbf{F}}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

- Vector Field $\Longrightarrow$ Vector Field

$$\mathrm{curl}(\vec{\mathbf{F}}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{\mathbf{k}}.$$

Conservative Vector Fields

Definition 1 (Conservative Vector Fields)

Let $\vec{\mathbf{F}} : D \to \mathbb{R}^n$ be a vector field with domain $D \subseteq \mathbb{R}^n$. The vector field $\vec{\mathbf{F}}$ is said to be *conservative* if it is the gradient of a scalar field, in other words:

$$\vec{\mathbf{F}} \text{ is conservative} \iff \exists \text{ a differentiable function } f \text{ s.t. } \vec{\mathbf{F}} = \nabla(f)$$

Such a function f is called a *potential function* for $\vec{\mathbf{F}}$.

Two Examples:

- $\vec{\mathbf{F}}(x, y, z) = \langle 2xyz^3,\ x^2z^3,\ 3x^2yz^2 \rangle$ is conservative, since $\vec{\mathbf{F}} = \nabla(f)$ for the function $f(x, y, z) = x^2yz^3$.
- $\vec{\mathbf{F}}(x, y, z) = \langle 2z,\ z^2,\ 2x + 2yz \rangle$ is conservative, since $\vec{\mathbf{F}} = \nabla(f)$ for the function $f(x, y, z) = 2xz + yz^2$.

Laplacian Operator

The *Laplacian operator*

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

is defined for a scalar field $f(x, y, z)$ by:

$$\nabla \cdot (\nabla(f)) = \nabla^2(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

and for a *vector field*

$$\vec{\mathbf{F}}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle,$$

by:

$$\nabla^2(\vec{\mathbf{F}}) = \langle \nabla^2(P), \nabla^2(Q), \nabla^2(R) \rangle.$$

Some Properties of Divergence and Curl

Result 1

If $f, g : \mathbb{R}^3 \to \mathbb{R}$ *are scalar fields and* $\vec{\mathbf{F}}, \vec{\mathbf{G}} : \mathbb{R}^3 \to \mathbb{R}^3$ *are vector fields, then we have:*

(a) $\nabla(f + g) = \nabla(f) + \nabla(g)$.

(b) $\nabla \cdot (\vec{\mathbf{F}} + \vec{\mathbf{G}}) = \nabla \cdot \vec{\mathbf{F}} + \nabla \cdot \vec{\mathbf{G}}$.

(d) $\nabla \times (\vec{\mathbf{F}} + \vec{\mathbf{G}}) = \nabla \times \vec{\mathbf{F}} + \nabla \times \vec{\mathbf{G}}$.

(e) $\nabla \cdot (f\vec{\mathbf{F}}) = \nabla(f) \cdot \vec{\mathbf{F}} + f\nabla \cdot \vec{\mathbf{F}}$.

(f) $\nabla \times (f\vec{\mathbf{F}}) = \nabla(f) \times \vec{\mathbf{F}} + f\nabla \times \vec{\mathbf{F}}$.