Problems and Solutions:

17. *If n is even, then $\phi(2n) = 2\phi(n)$, and if n is odd, then $\phi(2n) = \phi(n)$.*

Solution. The Euler ϕ-function ϕ is an *arithmetic function*, for any m, n satisfying $\gcd(m, n) = 1$, we have .

$$\boxed{\phi(m \cdot n) = \phi(m) \cdot \phi(n)}$$

Now let $n = 2^k \cdot m$ where m is an odd natural number. Then

$$\phi(2n) = \phi(2^{k+1} \cdot m) = \phi(2^{k+1}) \cdot \phi(m)$$

If n is even then $k \geqslant 1$, thus

$$\phi(2^{k+1}) = 2^{k+1}\left(1 - \frac{1}{2}\right) = 2^{k+1} - 2^k = 2 \cdot (2^k - 2^{k-1}) = 2 \cdot \phi(2^k)$$

$$\phi(2n) = \phi(2^{k+1}) \cdot \phi(m) = 2 \cdot \phi(2^k) \cdot \phi(m) = 2 \cdot \phi(2^k \cdot m) = 2\phi(n).$$

If n is odd, then $k = 0$, and so

$$\phi(2^{k+1}) = 1,$$

$$\phi(2n) = \phi(2^{k+1}) \cdot \phi(m) = 1 \cdot \phi(m) = \phi(m) = \phi(n). \square$$

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$$

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_t}\right)$$

$$= p_1^{a_1-1} p_2^{a_2-1} \cdots p_t^{a_t-1} (p_1 - 1)(p_2 - 1)\cdots(p_t - 1)$$

$$\phi(p^m) = p^m\left(1 - \frac{1}{p}\right) = p^m - p^{m-1}.$$

$$\phi(p) = p - 1.$$

$$(m, n) = 1 \implies \phi(mn) = \phi(m)\phi(n)$$

$$\phi(n) = \phi(p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}) = \phi(p_1^{a_1})\phi(p_2^{a_2})\cdots\phi(p_t^{a_t})$$

18. *Show that $\phi(n)$ is even for all $n \geqslant 3$.*

Solution.

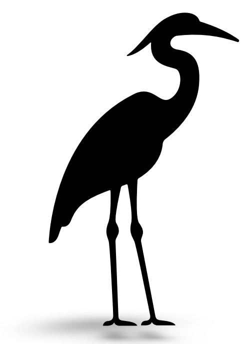
Even

$$n = 2^k \geqslant 4 \quad \Longrightarrow \quad \phi(2^k) = 2^k\left(1 - \frac{1}{2}\right) = 2^{k-1}.$$

$$\exists p \in \mathbb{P},\ p \geqslant 3,\ \ n = p^k \cdot m,\ \ \gcd(p, m) = 1 \quad \Longrightarrow$$

$$\phi(n) = \phi(p^k \cdot m) = \phi(p^k)\phi(m) = p^{k-1}(p-1)\phi(m)$$

Even



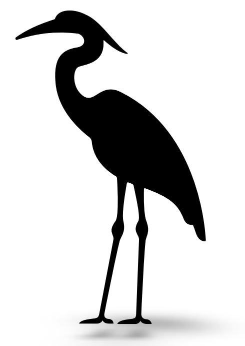
For which $n \in \mathbb{Z}^+$ is $\phi(n)$ odd? $\Longrightarrow$ $n = 1,2$

19. *If $m, n \in \mathbb{Z}^+$, prove that $\phi(n^m) = n^{m-1}\phi(n)$.*

Solution.

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t} \quad \Rightarrow \quad n^m = \left(p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}\right)^m = p_1^{ma_1} p_2^{ma_2} \cdots p_t^{ma_t}$$

$$\begin{aligned} \phi(n^m) &= n^m \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_t}\right) \\ &= n^{m-1} n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_t}\right) \\ &= n^{m-1}\phi(n)\,. \end{aligned}$$



20. *Find four values for* $n \in \mathbb{Z}^+$ *where* $\phi(n) = 16$.

Solution.

$n = 2^a \cdot p_1^{a_1} \cdots p_t^{a_t}$ $\Rightarrow$

$\phi(n) = 2^{a-1} \cdot p_1^{a_1-1} p_2^{a_2-1} \cdots p_t^{a_t-1} \left(p_1 - 1\right)\left(p_2 - 1\right)\cdots\left(p_t - 1\right) = 16.$

$\Rightarrow$ $a_1 = a_2 = \cdots = a_t = 1$

$\Rightarrow$ $\phi(n) = 2^{a-1}\left(p_1 - 1\right)\left(p_2 - 1\right)\cdots\left(p_t - 1\right) = 16.$

$\Rightarrow$ $p_i - 1 = 2^{k_i}, \quad i = 1,2,\ldots,t$

$\Rightarrow$ $p_i = 2^{k_i} + 1, \quad i = 1,2,\ldots,t$ $\Rightarrow$ $k_i = 2^{m_i} \quad i = 1,2,\ldots,t$

$\Rightarrow$ $p_i = 2^{2^{m_i}} + 1, \quad i = 1,2,\ldots,t$

$n = 2^a \cdot p_1 p_2 \cdots p_t \qquad p_i =$ Fermat prime $\qquad i = 1,2,\ldots,t$

$\Rightarrow$ $n = 2^a$ $\Rightarrow$ $\phi(n) = 2^{a-1} = 16$ $\Rightarrow$ $n = 32$

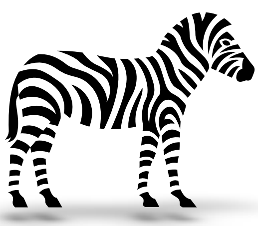
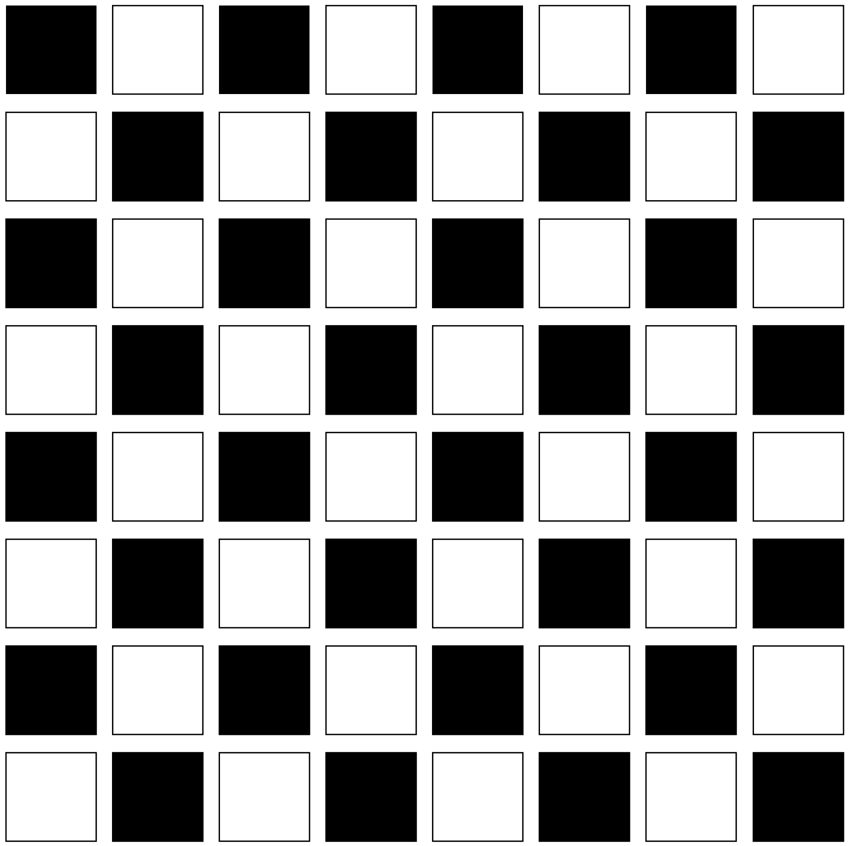
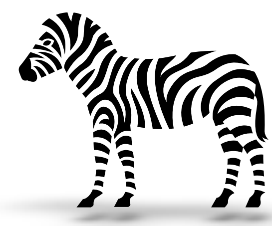
$\Rightarrow$ $n = p_1$ $\Rightarrow$ $\phi(n) = p_1 - 1 = 16$ $\Rightarrow$ $n = 17$

$\Rightarrow$ $n = 2^a p_1$ $\Rightarrow$ $\phi(n) = 2^{a-1}(p_1 - 1) = 16$

$\Rightarrow$ $(a, p_1) = (3,5),\ (4,3)$ $\Rightarrow$ $n = 20,\ 48$

21. *Find the rook polynomial for the standard* 8×8 *chessboard.*

Solution.

$$
\begin{aligned}
r(C,x) = & \binom{8}{0} + \binom{8}{1} 8x + \binom{8}{2} (8\times 7)x^2 + \binom{8}{3} (8\times 7\times 6)x^3 \\
& + \binom{8}{4}(8\times 7\times 6\times 5)x^4 + \binom{8}{5}(8\times 7\times 6\times 5\times 4)x^5 \\
& + \binom{8}{6}(8\times 7\times 6\times 5\times 4\times 3)x^6 + \binom{8}{7}(8\times 7\times 6\times 5\times 4\times 3\times 2)x^7 \\
& + \binom{8}{8}(8\times 7\times 6\times 5\times 4\times 3\times 2\times 1)x^8 = \sum_{i=0}^{8}\binom{8}{i} P(8,i)x^i
\end{aligned}
$$

22. *Find the rook polynomial for the standard $n \times n$ chessboard.*

Solution. $r(C, x) = \sum_{i=0}^{n} \binom{n}{i} P(n, i) x^i$.

23. For $n \in \mathbb{Z}^+$, *n is prime if and only if $\phi(n) = n - 1$.*

Solution.

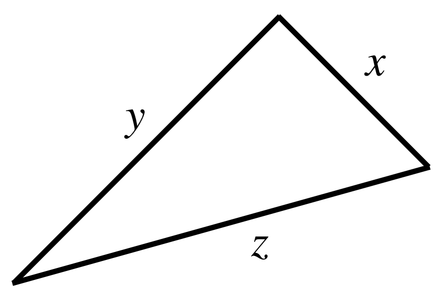
If $n = p$ is prime, then $\phi(p) = p\left(1 - \frac{1}{p}\right) = p - 1$.

If $n \in \mathbb{Z}^+$ and $\phi(n) = n - 1$, then all of the numbers less than n are *co-prime* to n, which means n is prime.

24. *How many triangles are there with integral sides and perimeter 40?*

Solution. $x + y + z = 40, \quad x, y, z \in \mathbb{N},$

$x \leqslant y \leqslant z$ $\quad$ $x + y > z$



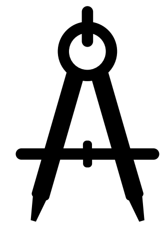
$14 \leqslant z \leqslant 19$

$z = 19 \implies x + y = 21, \quad x \leqslant y \leqslant 19$ $\quad$ $11 \leqslant y \leqslant 19$

Therefore, we have 9 triangles with $z = 19$.

In exactly the same way, we find that

		The number of triangles
$z = 18$	$\Longrightarrow$	8
$z = 17$	$\Longrightarrow$	6
$z = 16$	$\Longrightarrow$	5
$z = 15$	$\Longrightarrow$	3
$z = 14$	$\Longrightarrow$	2



Thus, we find 33 triangles in all. $\square$

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25. *Prove that*

$$\binom{n}{1}^2 + 2\binom{n}{2}^2 + 3\binom{n}{3}^2 + \cdots + n\binom{n}{n}^2 = \frac{(2n-1)!}{[(n-1)!]^2}.$$

Solution. We have

$$f(x) = (1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{i}x^i + \cdots + \binom{n}{n}x^n.$$

$$f'(x) = n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \cdots + i\binom{n}{i}x^{i-1} + \cdots + n\binom{n}{n}x^{n-1}.$$

$$f(x)\cdot f'(x) = n(1+x)^{2n-1}$$

$$= \left[1 + \binom{n}{1}x + \cdots + \binom{n}{n}x^n\right]\left[\binom{n}{1} + 2\binom{n}{2}x + \cdots + n\binom{n}{n}x^{n-1}\right].$$

The coefficient of x^{n-1}

LHS: $n\binom{2n-1}{n-1} = \cancel{n}\dfrac{(2n-1)!}{(n-1)!\cancel{n}!} = \dfrac{(2n-1)!}{[(n-1)!]^2}$

RHS: $\binom{n}{1}^2 + 2\binom{n}{2}^2 + 3\binom{n}{3}^2 + \cdots + n\binom{n}{n}^2$

$$\binom{n}{1}^2 + 2\binom{n}{2}^2 + 3\binom{n}{3}^2 + \cdots + n\binom{n}{n}^2 = \frac{(2n-1)!}{[(n-1)!]^2}.$$