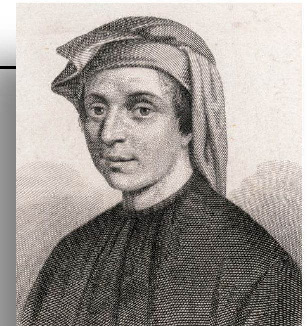
1. Recurrence Relations

Familiar examples:



Fibonacci numbers

$\{F_n\}_{n=0}^{\infty}$ → $F_0,\ F_1,\ F_2,\ F_3,\ F_4,\ F_5,\ \ldots,\ F_n,\ \cdots$

A000045

$$\begin{cases} F_0 = 0, \quad F_1 = 1 \\ F_n = F_{n-1} + F_{n-2}, \quad n \geqslant 2 \end{cases}$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, …

Lucas numbers

$\{L_n\}_{n=0}^{\infty}$ → $L_0,\ L_1,\ L_2,\ L_3,\ L_4,\ L_5,\ \ldots,\ L_n,\ \cdots$

A000032

$$\begin{cases} L_0 = 2, \quad L_1 = 1 \\ L_n = L_{n-1} + L_{n-2}, \quad n \geqslant 2 \end{cases}$$

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, …

Pell numbers

$\{P_n\}_{n=0}^{\infty}$ → $P_0,\ P_1,\ P_2,\ P_3,\ P_4,\ P_5,\ \ldots,\ F_n,\ \cdots$

A000129

$$\begin{cases} P_0 = 0, \quad P_1 = 1 \\ P_n = 2P_{n-1} + P_{n-2}, \quad n \geqslant 2 \end{cases}$$

0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, …

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founded in 1964 by N. J. A. Sloane

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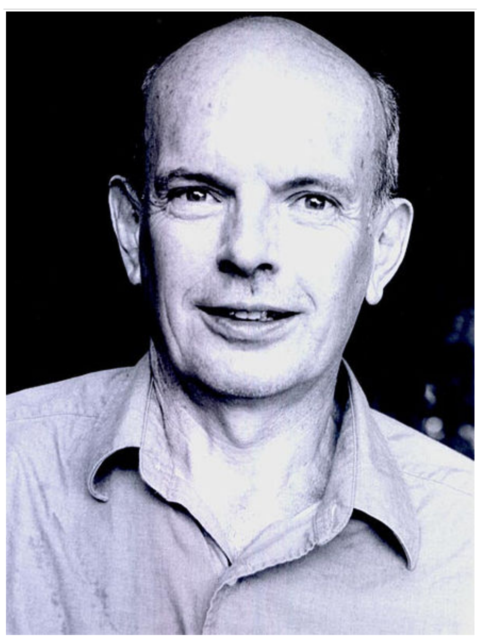
Neil Sloane



1997

Neil James Alexander Sloane (born October 10, 1939) is a *British-American* mathematician. His major contributions are in the fields of *combinatorics*, *error-correcting codes*, and *sphere packing*. Sloane is best known for being the creator and maintainer of the *On-Line Encyclopedia of Integer Sequences* (OEIS).

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Solving homogeneous linear recurrence relations with constant coefficients

An *order-k homogeneous linear recurrence with constant coefficients* is an equation of the form,

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},}$$

where the k coefficients c_i (for all i) are *constants*, and

$$\boxed{c_k \neq 0.}$$

A *constant-recursive sequence* is a sequence satisfying a recurrence of this form. There are k degrees of *freedom for solutions* to this recurrence, i.e., the initial values

$$\boxed{a_0,\ a_1,\ \ldots,\ a_{k-1}}$$

can be taken to be any values but then the recurrence determines the sequence uniquely. The same coefficients yield the *characteristic polynomial* (also *auxiliary polynomial*)

$$p(x) = x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k$$

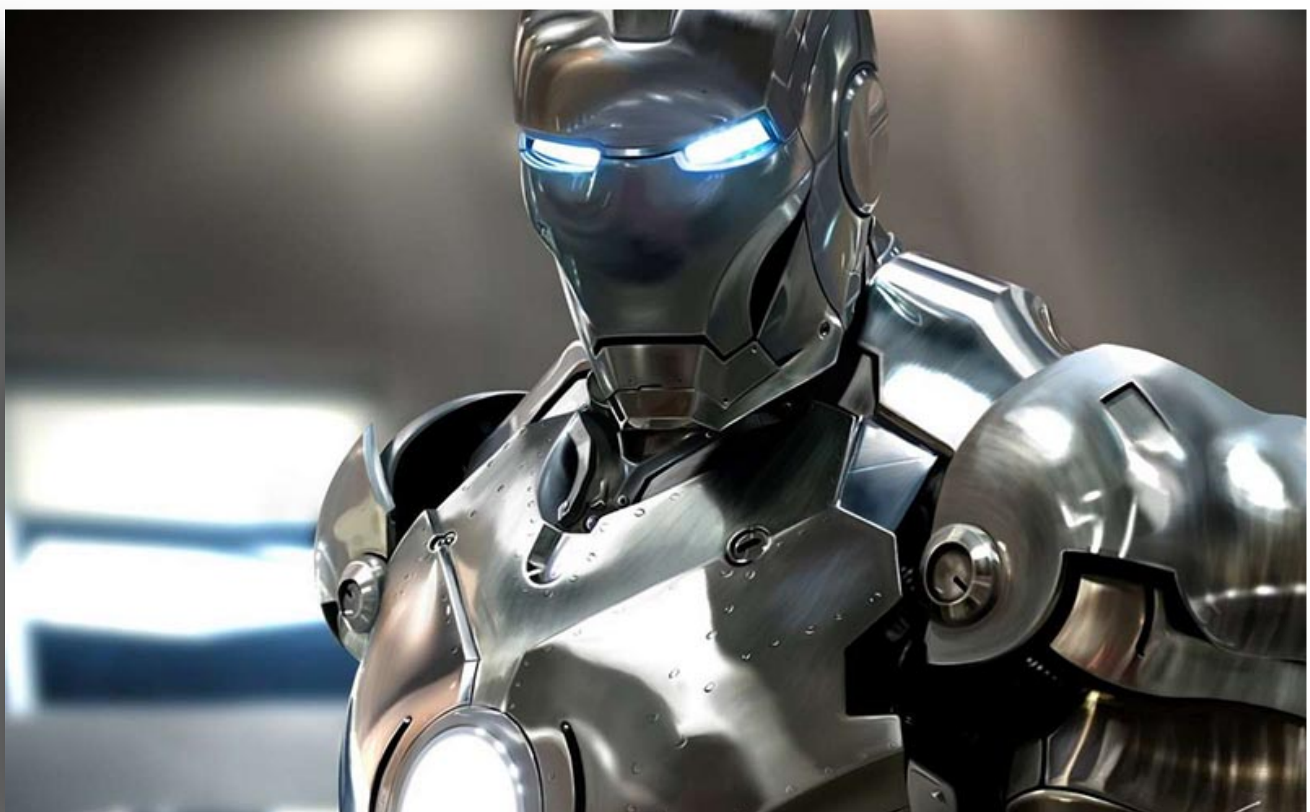
whose k roots play a crucial role in finding and understanding the sequences satisfying the recurrence. If the roots $r_1,\ r_2,\ \ldots,\ r_k$ are *all distinct*, then each solution to the recurrence takes the form

$$a_n = \lambda_1 r_1^n + \lambda_2 r_2^n + \cdots + \lambda_k r_k^n,$$

where the coefficients λ_i are determined in order to fit the initial conditions of the recurrence. When the same roots occur *multiple times*, the terms in this formula corresponding to the second and later occurrences of the same root are multiplied by *increasing powers of* n. For instance, if the characteristic polynomial is divisible by $(x-r)^m$, with the same root r occurring m times, then the solution would take the form

$$a_n = \cdots + (b_1 + b_2 n + b_3 n^2 + \cdots + b_m n^{m-1}) r^n + \cdots .$$

Equate each $a_0, a_1, \ldots, a_k$ (*plugging* $n = 0, 1, \ldots, k$ into the general solution of the recurrence relation) with the known values
$a_0, a_1, \ldots, a_k$ from the original recurrence relation. This process will produce a linear system of k equations with d unknowns. Solving these equations for the unknown coefficients $\lambda_1, \lambda_2, \ldots, \lambda_k$ of the general solution and plugging these values back into the general solution will produce the particular solution to the original recurrence relation that fits the original recurrence relation's initial conditions (as well as all subsequent values $a_0, a_1, a_2, \ldots$ of the original recurrence relation).



Summary

$$\{a_n\}_{n=0}^{\infty} \Rightarrow a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots,\ a_n,\ \cdots$$

$$\begin{cases} a_0,\ a_1, a_2,\ \ldots,\ a_{k-1} \\ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \qquad n \geqslant k \end{cases}$$

$$\boxed{a_0,\ a_1,\ \ldots,\ a_{k-1},}\ a_k,\ a_{k+1},\ a_{k+2},\ a_{k+3},\ a_{k+4},\ \ldots$$

Step 1.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \xRightarrow{a_i \to x^i} x^n = c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_k x^{n-k}$$

$$x^n - c_1 x^{n-1} - c_2 x^{n-2} - \cdots - c_k x^{n-k} = 0 \Rightarrow \boxed{x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k = 0}$$

Characteristic Polynomial

Step 2.

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k = 0 \Rightarrow r_1,\ r_2,\ \ldots,\ r_k$$

Case 1. The roots $r_1, r_2, \ldots, r_k$ are *all distinct.*

Step 3.

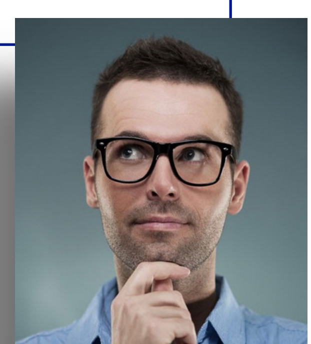
$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k = (x - r_1)(x - r_2)\cdots(x - r_k) = 0$$

$$\lambda_1, \lambda_2, \ldots, \lambda_k, \quad \Longrightarrow \quad a_n = \lambda_1 r_1^n + \lambda_2 r_2^n + \cdots + \lambda_k r_k^n,$$

Step 4.

$$\begin{array}{ccl} n = 0 & \Longrightarrow & \left\{\begin{array}{l} a_0 = \lambda_1 + \lambda_2 + \cdots + \lambda_k, \\ a_1 = \lambda_1 r_1 + \lambda_2 r_2 + \cdots + \lambda_k r_k, \\ \quad \vdots \qquad\qquad \vdots \\ a_{k-1} = \lambda_1 r_1^{k-1} + \lambda_2 r_2^{k-1} + \cdots + \lambda_k r_k^{k-1}, \end{array}\right. \\ n = 1 & \Longrightarrow & \\ \vdots & \vdots & \\ n = k-1 & \Longrightarrow & \end{array}$$

$$\Longrightarrow \quad \lambda_1, \lambda_2, \ldots, \lambda_k, \quad \Longrightarrow \quad a_n = \lambda_1 r_1^n + \lambda_2 r_2^n + \cdots + \lambda_k r_k^n,$$



Case 2. The characteristic equation has t *distinct roots* $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geqslant 1$for $i = 1,2,\ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$.

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k = (x - r_1)^{m_1}(x - r_2)^{m_2}\cdots(x - r_t)^{m_t} = 0$$

Step 3.

$$\lambda_{i,j}, \quad 1 \leqslant i \leqslant t, \quad 0 \leqslant j \leqslant m_i - 1 \Rightarrow$$

$$a_n = (\lambda_{1,0} + \lambda_{1,1} n + \cdots + \lambda_{1,m_1-1} n^{m_1-1}) r_1^n + (\lambda_{2,0} + \lambda_{2,1} n + \cdots + \lambda_{2,m_2-1} n^{m_2-1}) r_2^n$$
$$+ \cdots + (\lambda_{t,0} + \lambda_{t,1} n + \cdots + \lambda_{t,m_t-1} n^{m_t-1}) r_t^n$$

Step 4.

$$\begin{array}{lcl} n = 0 & \Longrightarrow & \\ n = 1 & \Longrightarrow & \\ \vdots & \vdots & \\ n = k-1 & \Longrightarrow & \end{array} \begin{cases} a_0 = \sum_{i=1}^{t} \lambda_{i,0}, \\ a_1 = \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} \lambda_{i,j} r_i \\ \vdots \qquad \vdots \\ a_{k-1} = \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} \lambda_{i,j} (k-1)^j r_i, \end{cases}$$

$$\Rightarrow \lambda_{i,j}, \quad 1 \leqslant i \leqslant t, \quad 0 \leqslant j \leqslant m_i - 1 \Rightarrow$$

$$a_n = (\lambda_{1,0} + \lambda_{1,1} n + \cdots + \lambda_{1,m_1-1} n^{m_1-1}) r_1^n + (\lambda_{2,0} + \lambda_{2,1} n + \cdots + \lambda_{2,m_2-1} n^{m_2-1}) r_2^n$$
$$+ \cdots + (\lambda_{t,0} + \lambda_{t,1} n + \cdots + \lambda_{t,m_t-1} n^{m_t-1}) r_t^n$$

Some Examples:

Example 1. *Find an explicit formula for the Fibonacci sequence.*

Solution. We have

$$\begin{cases} F_0 = 0, \quad F_1 = 1 \\ F_n = F_{n-1} + F_{n-2}, \quad n \geqslant 2 \end{cases}$$

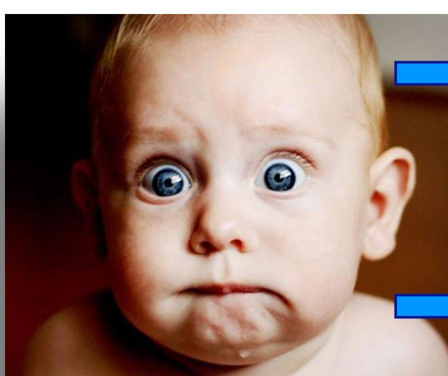
$$F_n = F_{n-1} + F_{n-2} \Rightarrow x^n = x^{n-1} + x^{n-2}$$

$$\Rightarrow x^2 - x - 1 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1, \lambda_2 \Rightarrow F_n = \lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

$$\boxed{\begin{array}{lll} n = 0 & \Longrightarrow & \begin{cases} 0 = F_0 = \lambda_1 + \lambda_2, \\ 1 = F_1 = \lambda_1 \left(\frac{1+\sqrt{5}}{2}\right) + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right). \end{cases} \\ n = 1 & \Longrightarrow & \end{array}}$$



$$\Rightarrow \lambda_1 = \frac{1}{\sqrt{5}}, \quad \lambda_2 = -\frac{1}{\sqrt{5}}$$

$$\Rightarrow \boxed{F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]}$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, …

Example 2. *Find the solution to the recurrence relation*

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, *and* $a_2 = 15$.

Solution. We have

$$\begin{cases} a_0 = 2, \quad a_1 = 5, \quad a_2 = 15, \\ a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}, \quad n \geqslant 3 \end{cases}$$

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \quad \Longrightarrow \quad x^n = 6x^{n-1} - 11x^{n-2} + 6x^{n-3}$$

$$\Longrightarrow \quad x^3 - 6x^2 + 11x - 6 = 0$$

$$\Longrightarrow \quad x = 1,\ 2,\ 3$$

$$\lambda_1,\ \lambda_2,\ \lambda_3 \quad \Longrightarrow \quad a_n = \lambda_1 + \lambda_2 2^n + \lambda_3 3^n,$$

$$\boxed{\begin{array}{lll} n = 0 & \Longrightarrow & 2 = a_0 = \lambda_1 + \lambda_2 + \lambda_3, \\ n = 1 & \Longrightarrow & 5 = a_1 = \lambda_1 + 2\lambda_2 + 3\lambda_3\,. \\ n = 2 & \Longrightarrow & 15 = a_2 = \lambda_1 + 4\lambda_2 + 9\lambda_3\,. \end{array}}$$

$$\Longrightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 2.$$

$$\Longrightarrow \quad \boxed{a_n = 1 - 2^n + 2(3^n)}$$

$2,\ 5,\ 15,\ 47,\ 147,\ 455,\ \ldots$

Theorem 1. Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k = 0$$

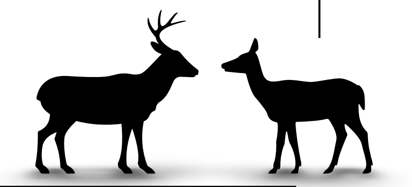
has k *distinct roots* $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \lambda_1 r_1^n + \lambda_2 r_2^n + \cdots + \lambda_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are constants.



Theorem 2. Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k = 0$$

has t *distinct roots* $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geqslant 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = (\lambda_{1,0} + \lambda_{1,1} n + \cdots + \lambda_{1,m_1-1} n^{m_1-1}) r_1^n$$

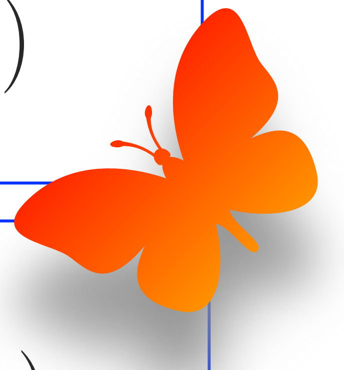
$$+ (\lambda_{2,0} + \lambda_{2,1} n + \cdots + \lambda_{2,m_2-1} n^{m_2-1}) r_2^n$$

$$+ \cdots + (\lambda_{t,0} + \lambda_{t,1} n + \cdots + \lambda_{t,m_t-1} n^{m_t-1}) r_t^n$$

for $n = 0, 1, \ldots$, where $\lambda_{i,j}$ are constants for $1 \leqslant i \leqslant t$ and $0 \leqslant j \leqslant m_i - 1$.

1. *Compute*

$$\prod_{k=2}^{100}\left(1-\frac{1}{k}\right)=\left(1-\frac{1}{2}\right)\times\left(1-\frac{1}{3}\right)\times\cdots\times\left(1-\frac{1}{100}\right)$$



2. *Compute*

$$\prod_{k=2}^{100}\left(1-\frac{1}{k^2}\right)=\left(1-\frac{1}{2^2}\right)\times\left(1-\frac{1}{3^2}\right)\times\cdots\times\left(1-\frac{1}{100^2}\right)$$

$$\prod_{k=2}^{100}\left(1-\frac{1}{k}\right)=\prod_{k=2}^{100}\frac{k-1}{k}=\frac{1}{\cancel{2}}\times\frac{\cancel{2}}{\cancel{3}}\times\frac{\cancel{3}}{\cancel{4}}\times\frac{\cancel{4}}{\cancel{5}}\times\cdots\times\frac{\cancel{98}}{\cancel{99}}\times\frac{\cancel{99}}{100}=\frac{1}{100}.$$

$$\prod_{k=2}^{100}\left(1-\frac{1}{k^2}\right)=\prod_{k=2}^{100}\frac{k^2-1}{k^2}=\prod_{k=2}^{100}\left(\frac{k-1}{k}\times\frac{k+1}{k}\right)$$

$$=\frac{1}{2}\times\frac{3}{2}\times\frac{2}{3}\times\frac{4}{3}\times\frac{3}{4}\times\frac{5}{4}\times\cdots\times\frac{98}{99}\times\frac{100}{99}\times\frac{99}{100}\times\frac{101}{100}=\frac{101}{200}.$$