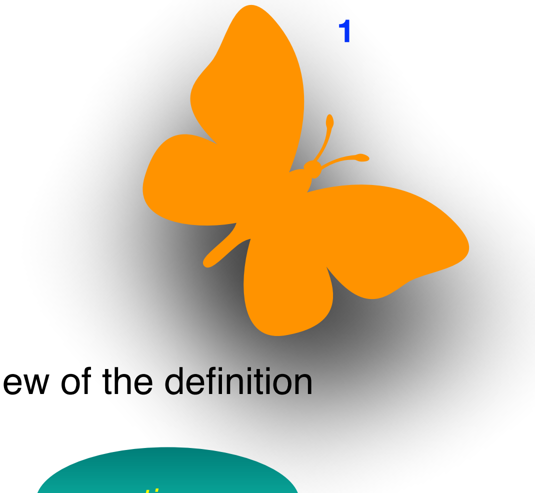
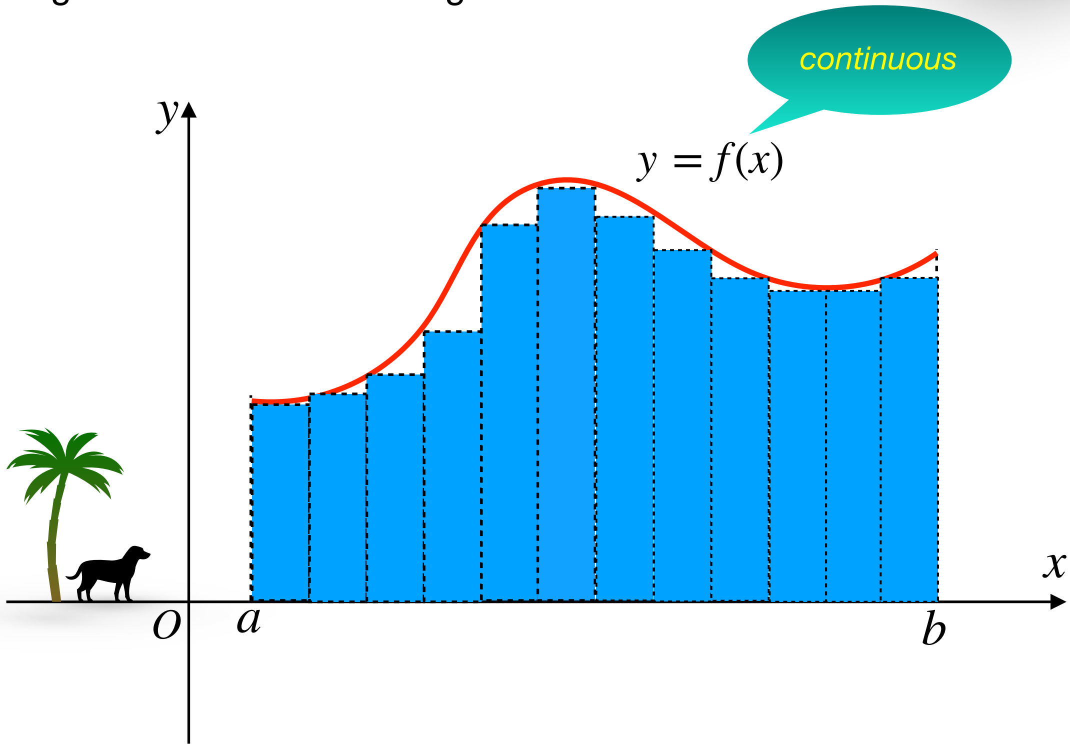
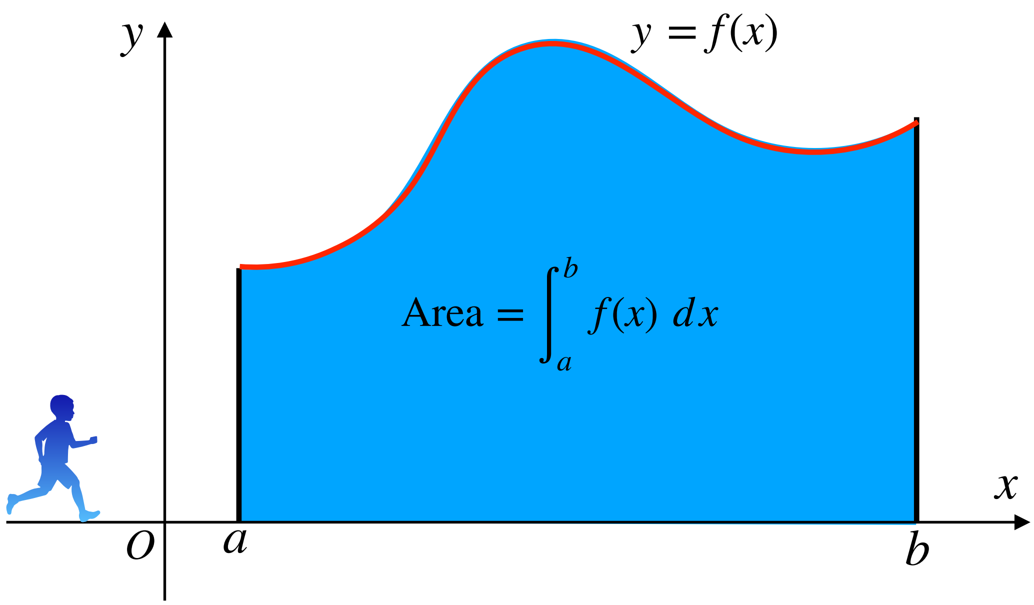


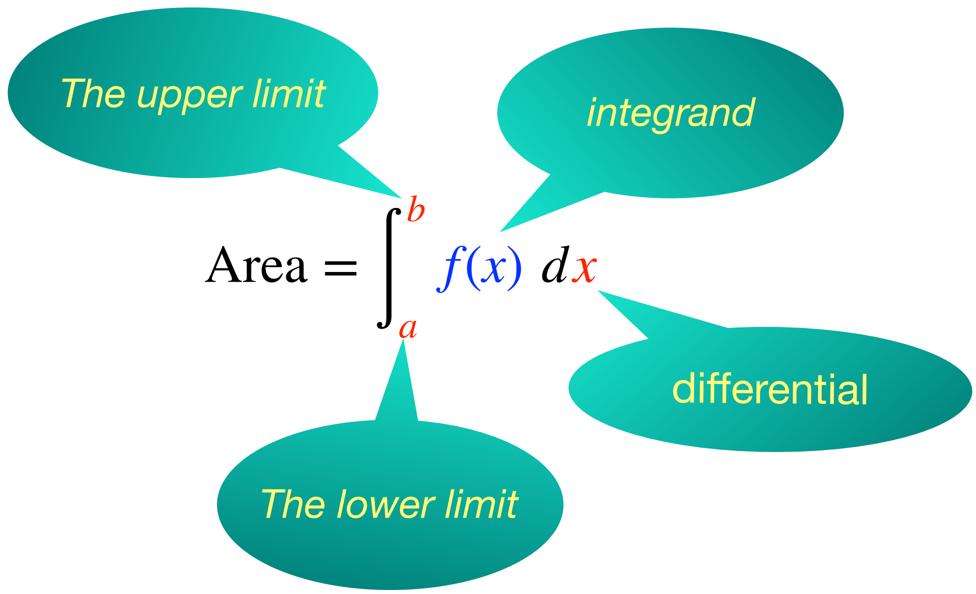
Multiple Integrals

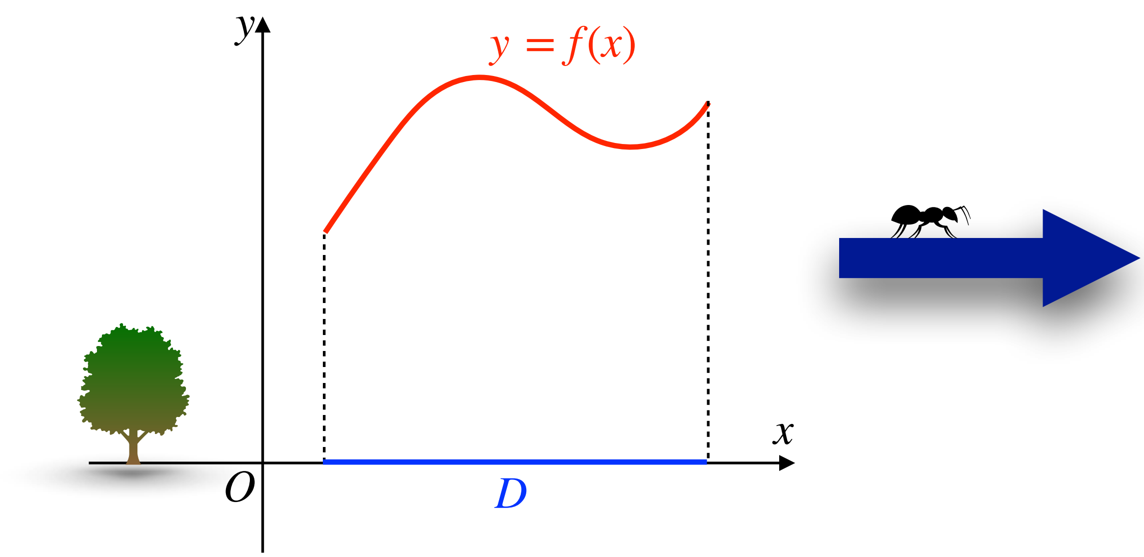
1. Double Integrals

Before starting on double integrals let’s do a quick review of the definition of definite integrals for functions of single variables.

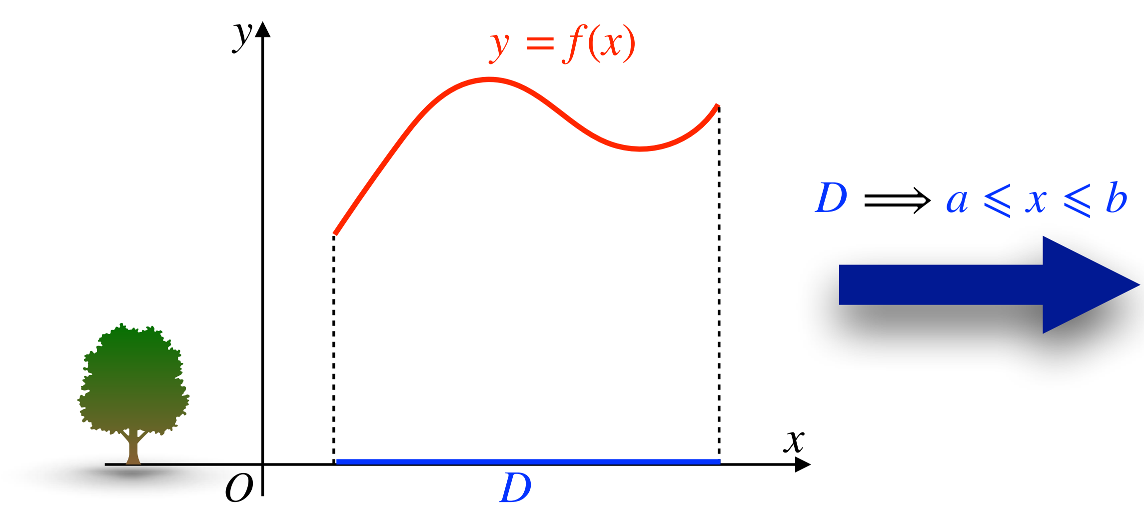








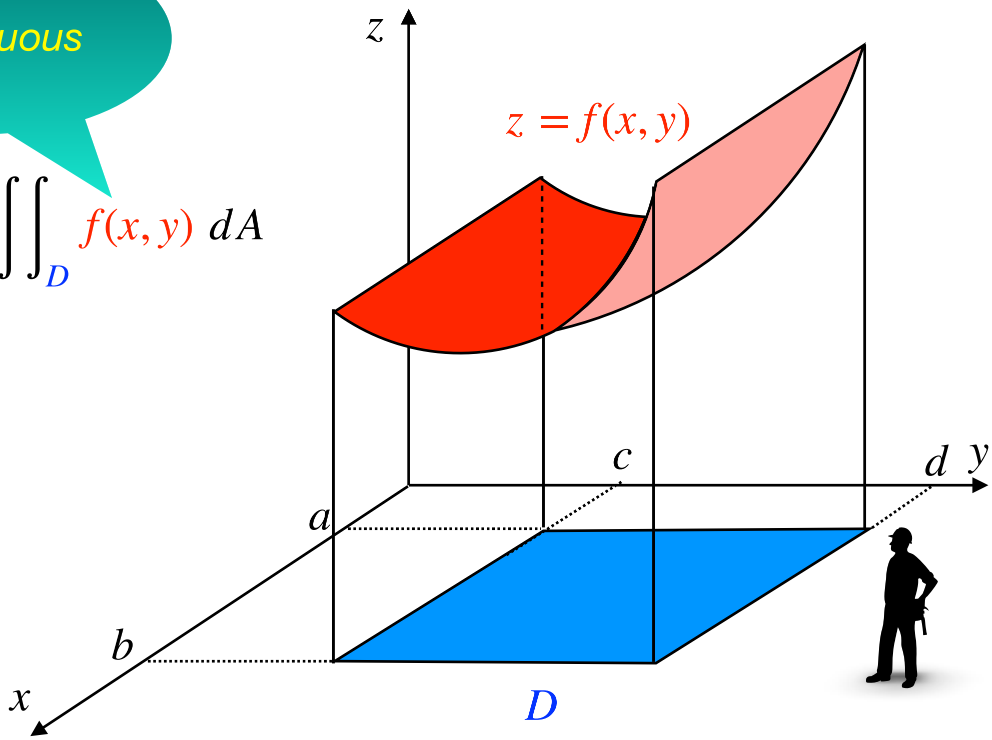
$$\int_D f(x)\,dx$$



$$\int_a^b f(x)\,dx$$

continuous

$$\text{Volume} = \iint_D f(x,y)\ dA$$



$$D \Longrightarrow \begin{cases} a \leqslant x \leqslant b \\ c \leqslant y \leqslant d \end{cases}$$

$$\text{Volume} = \int_a^b \int_c^d f(x,y)\ dy\ dx$$

$$\text{Volume} = \int_c^d \int_a^b f(x,y)\ dx\ dy$$

Fubini's Theorem. If $f(x,y)$ is continuous on $D=[a,b]\times[c,d]$, then

$$\iint_D f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy\,.$$

These integrals are called *iterated integrals*.

$$\iint_D f(x,y)\,dA=\int_a^b\left(\int_c^d f(x,y)\,dy\right)\,dx$$

$$\iint_D f(x,y)\,dA=\int_c^d\left(\int_a^b f(x,y)\,dx\right)\,dy\,.$$

Example 1 Compute each of the following double integrals over the indicated rectangles:

1. $\displaystyle\iint_D 6xy^2\,dA,\ D=[2,4]\times[1,2],$

Solution. We have

$$\iint_D 6xy^2\,dA=\int_2^4\left(\int_1^2 6xy^2\,dy\right)\,dx=\int_2^4\left(2xy^3\Big|_1^2\right)\,dx$$

$$=\int_2^4\left(16x-2x\right)\,dx=\int_2^4\left(14x\right)\,dx=7x^2\Big|_2^4=112-28=84.$$

2. $\displaystyle\iint_D 2x-4y^3\,dA,\ D=[-5,4]\times[0,3],$

Answer: $-756.$

3. $\displaystyle\iint_D x^2y^2 + \cos(\pi x) + \sin(\pi y)\ dA,\ D = [-2, -1] \times [0,1],$

Answer: $\dfrac{7}{9} + \dfrac{2}{\pi}$.

4. $\displaystyle\iint_D \frac{1}{(2x+3y)^2}\ dA,\ D = [0,1] \times [1,2],$

Answer: $-\dfrac{1}{6}(2\ln 2 - \ln 5)$.

5. $\displaystyle\iint_D xe^{xy}\ dA,\ D = [-1,2] \times [0,1].$

Answer: $e^2 - e^{-1} - 3.$

6. $\displaystyle\iint_D 6y\sqrt{x} - 2y^3\ dA,\ D = [1,4] \times [0,3].$

Answer: $\dfrac{9}{2}$.

7. $\displaystyle\iint_D \frac{e^x}{2y} - \frac{4x-1}{y^2}\ dA,\ D = [-1,0] \times [1,2].$

Answer: $\dfrac{1}{2}(\ln(2) - \ln(2)e^{-1} + 3)$.

8. $\displaystyle\iint_D \sin(2x) - \frac{1}{1+6y}\ dA,\ D = [\frac{\pi}{4}, \frac{\pi}{2}] \times [0,1].$

Answer: $\dfrac{1}{2} - \dfrac{\pi}{24}\ln(7)$.

9. $\displaystyle\iint_D ye^{y^2-4x}\ dA,\ D = [0,2] \times [0,\sqrt{8}].$

Answer: $\frac{1}{8}(e^{8}+e^{-8}-2)$.

10. $\displaystyle\iint_D xy^2\sqrt{x^2+y^3}\; dA,\;\; D=[0,3]\times[0,2].$

Answer: $\frac{2}{45}(17^{\frac{5}{2}}-243-128\sqrt{2})$.

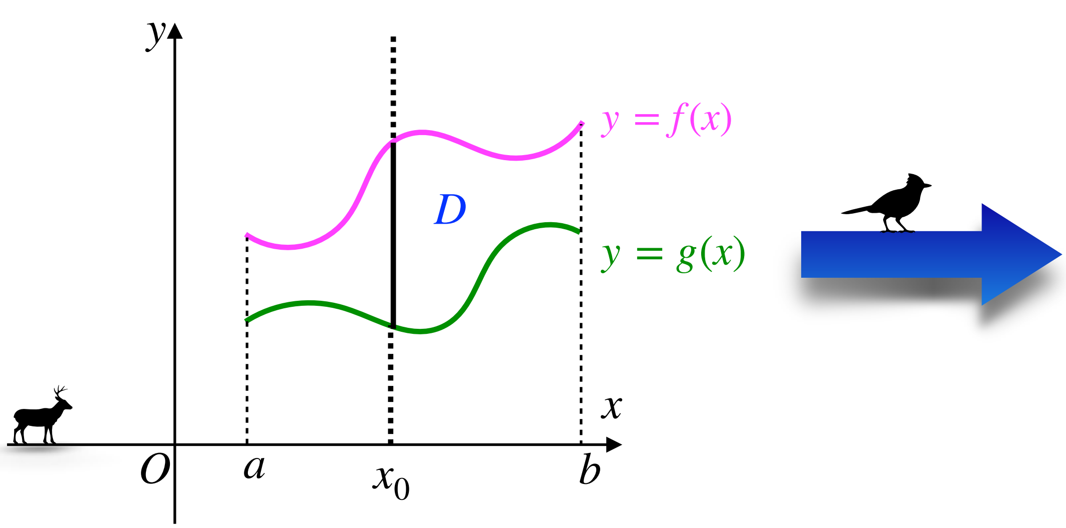
11. $\displaystyle\iint_D xy\cos(yx^2)\; dA,\;\; D=[-2,3]\times[-1,1].$

Answer: 0.

12. $\displaystyle\iint_D xy\cos(y)-x^2\; dA,\;\; D=[1,2]\times[\frac{\pi}{2},\pi].$

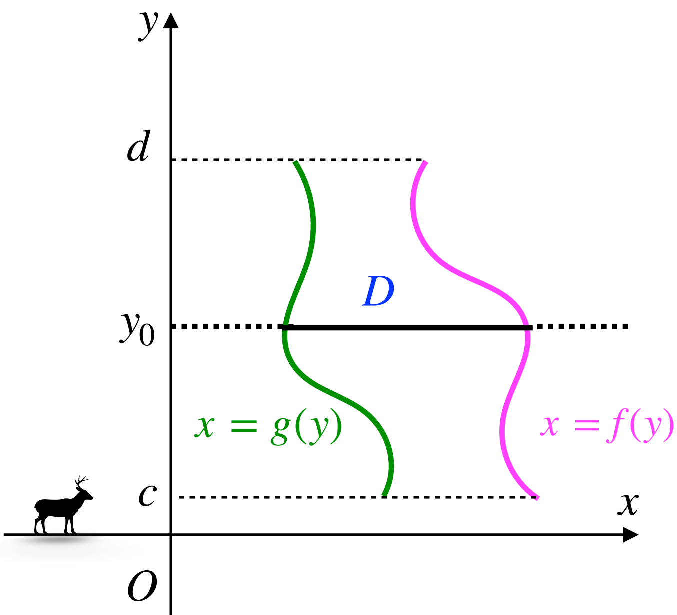
Answer: $-\frac{3}{2}-\frac{23}{12}\pi$.

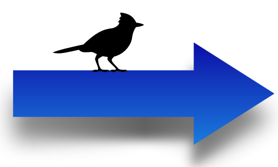
2. Double Integrals Over General Regions



$$\begin{cases} a \leqslant x \leqslant b \\ g(x) \leqslant y \leqslant f(x) \end{cases}$$

$$\iint_D f(x,y)\, dA = \int_a^b \left(\int_{g(x)}^{f(x)} f(x,y)\, dy \right) dx$$





$$\begin{cases} c \leqslant y \leqslant d \\ g(y) \leqslant x \leqslant f(y) \end{cases}$$

$$\iint_D f(x,y)\, dA = \int_c^d \left(\int_{g(y)}^{f(y)} f(x,y)\, dx \right) dy$$

Some Properties

1. $$\iint_D f(x,y) + g(x,y)\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

2. $\displaystyle\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA$, where c is any constant.

3. If $D = D_1 \uplus D_2$, then
$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA.$$

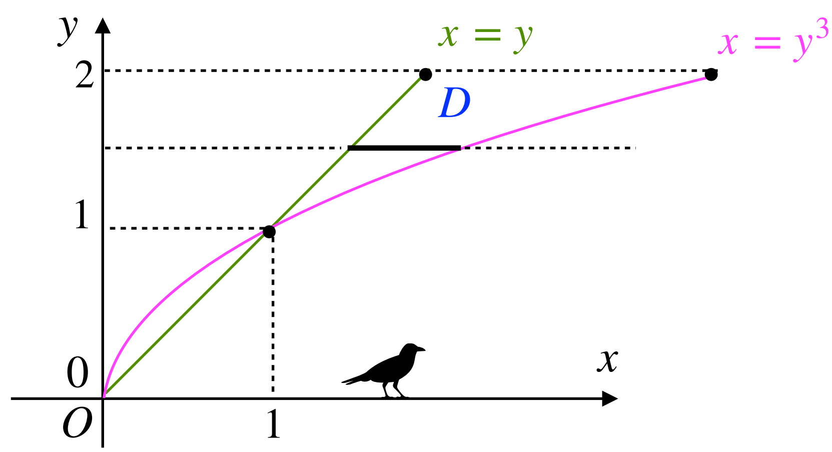
Example 2 Evaluate each of the following integrals over the given region D.

1. $\displaystyle\iint_D e^{\frac{x}{y}}\, dA,\ D = \{(x,y) \mid 1 \leqslant y \leqslant 2,\ \ y \leqslant x \leqslant y^3.$

Solution. We have

$$\iint_D e^{\frac{x}{y}}\, dA = \int_1^2 \left(\int_y^{y^3} e^{\frac{x}{y}}\, dx \right) dy = \int_1^2 \left(y e^{\frac{x}{y}} \Big|_y^{y^3} \right) dx$$

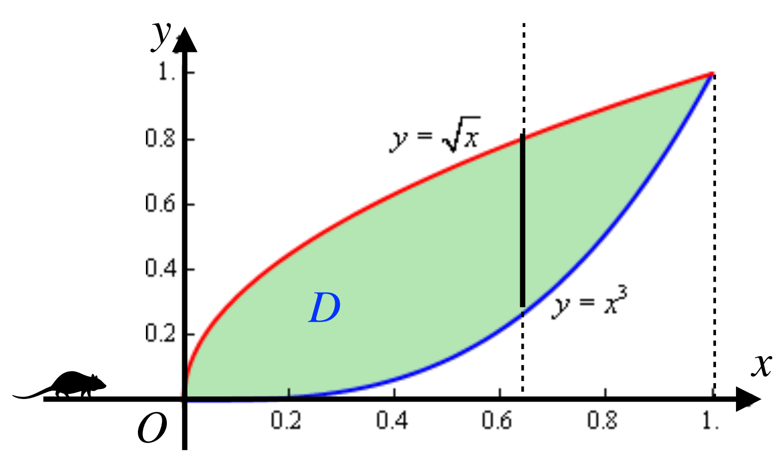
$$= \int_1^2 y e^{y2} - y e^1\, dy = \left(\frac{1}{2} e^{y^2} - \frac{1}{2} y^2 e^1 \right) \Big|_1^2 = \frac{1}{2} e^4 - 2e\,.$$



2. $\displaystyle\iint_D 4xy - y^3\, dA$, D is the region bounded by $y = \sqrt{x}$ and $y = x^3$.

Solution. We have

$$\begin{cases} 0 \leqslant x \leqslant 1 \\ x^3 \leqslant y \leqslant \sqrt{x} \end{cases}$$

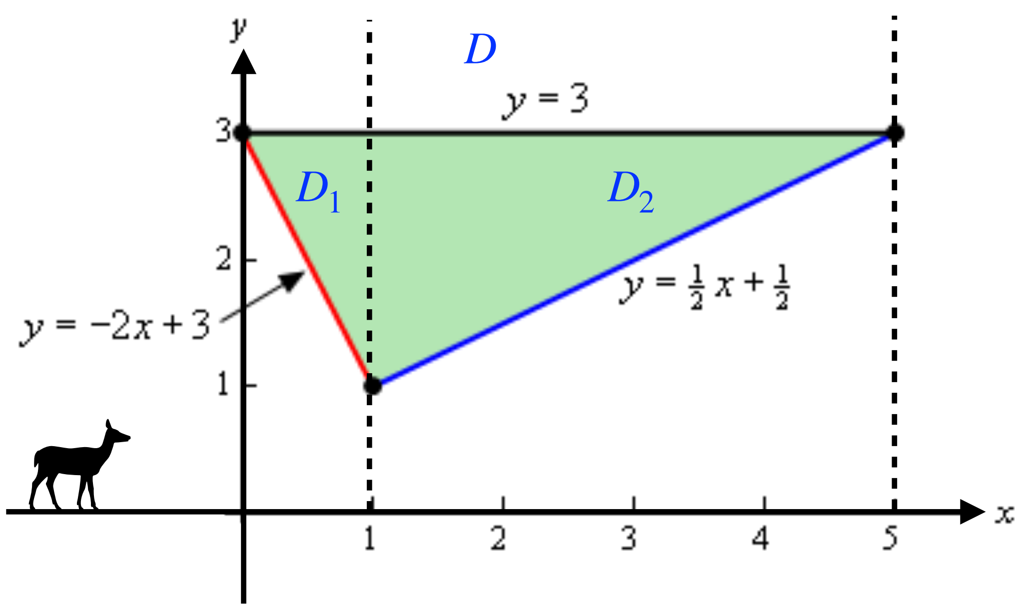


$$\iint_D 4xy - y^3 \, dA = \int_0^1 \left(\int_{x^3}^{\sqrt{x}} 4xy - y^3 \, dy \right) dx = \int_0^1 \left(2xy^2 - \frac{1}{4} y^4 \Big|_{x^3}^{\sqrt{x}} \right) dx$$

$$= \int_0^1 \frac{7}{4} x^2 - 2x^7 + \frac{1}{4} x^{12} \, dx = \left(\frac{7}{12} x^3 - \frac{1}{4} x^8 + \frac{1}{52} x^{13} \right) \Big|_0^1 = \frac{55}{156}.$$

3. $\iint_D 6x^2 - 40y \, dA$, D is the triangle with vertices $(0,3)$, $(1,1)$ and $(5,3)$.

Solution. We have



$$D = D_1 \uplus D_2 \Longrightarrow \quad D_1 \begin{cases} 0 \leqslant x \leqslant 1 \\ -2x + 3 \leqslant y \leqslant 3 \end{cases}, \qquad D_2 \begin{cases} 1 \leqslant x \leqslant 5 \\ \frac{1}{2}x + \frac{1}{2} \leqslant y \leqslant 3 \end{cases}$$

$$D\begin{cases} 1 \leqslant y \leqslant 3 \\ -\frac{1}{2}y+\frac{3}{2} \leqslant x \leqslant 2y-1 \end{cases}$$

First Solution.

$$\iint_D 6x^2-40y\ dA = \iint_{D_1} 6x^2-40y\ dA + \iint_{D_2} 6x^2-40y\ dA$$

$$= \int_0^1 \left(\int_{-2x+3}^{3} 6x^2-40y\ dy \right) dx + \int_1^5 \left(\int_{\frac{1}{2}x+\frac{1}{2}}^{3} 6x^2-40y\ dy \right) dx$$

$$= \cdots = -\frac{935}{3}.$$

Second Solution.

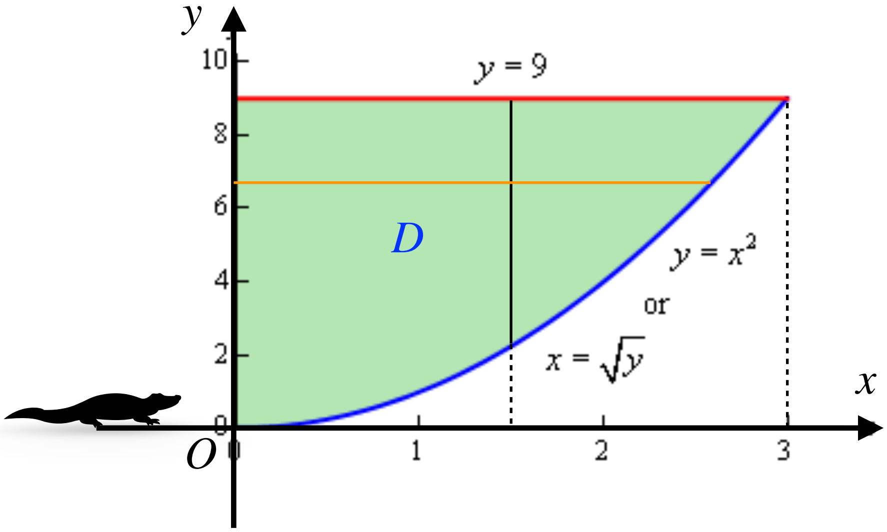
$$\iint_D 6x^2-40y\ dA = \int_1^3 \left(\int_{-\frac{1}{2}x+\frac{3}{2}}^{2y-1} 6x^2-40y\ dx \right) dy = \cdots = -\frac{935}{3}.$$

Example 3 Evaluate the following integrals by first reversing the order of integration:

1. $\displaystyle\int_0^3 \left(\int_{x^2}^{9} x^3 e^{y^3}\ dy \right) dx$

Solution. We have

$$\int_0^3 \left(\int_{x^2}^{9} x^3 e^{y^3}\ dy \right) dx \qquad D \Longrightarrow \begin{cases} 0 \leqslant x \leqslant 3 \\ x^2 \leqslant y \leqslant 9 \end{cases}$$



$$D \Longrightarrow \begin{cases} 0 \leqslant y \leqslant 9 \\ 0 \leqslant x \leqslant \sqrt{y} \end{cases}$$

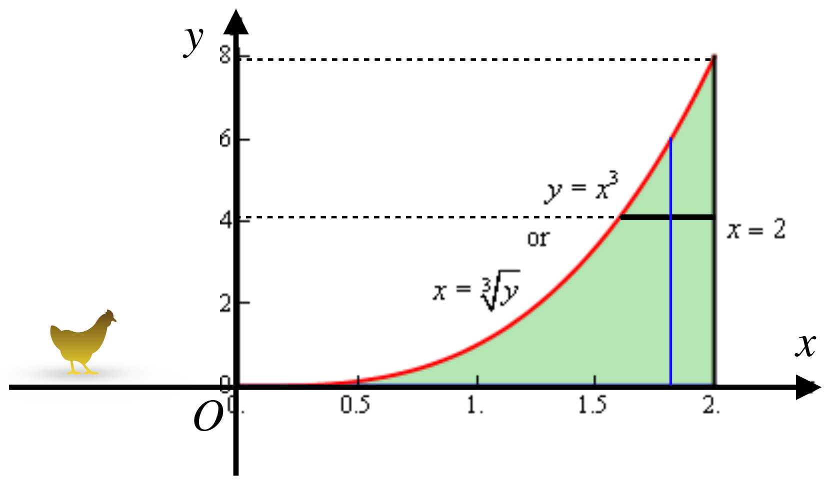
$$\int_0^3 \left(\int_{x^2}^9 x^3 e^{y^3}\, dy \right) dx = \int_0^9 \left(\int_0^{\sqrt{y}} x^3 e^{y^3}\, dx \right) dy$$

$$= \int_0^9 \left(\frac{x^4}{4} e^{y^3} \Big|_0^{\sqrt{y}} \right) dy = \int_0^9 \frac{y^2}{4} e^{y^3}\, dy = \frac{1}{12} e^{y^3} \Big|_0^9 = \frac{1}{12}(e^{729} - 1)\,.$$

2. $$\int_0^8 \left(\int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx \right) dy$$

Solution. We have

$$\int_0^8 \left(\int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx \right) dy \qquad D \Longrightarrow \begin{cases} 0 \leqslant y \leqslant 8 \\ \sqrt[3]{y} \leqslant x \leqslant 2 \end{cases}$$



$$D \Longrightarrow \begin{cases} 0 \leqslant x \leqslant 2 \\ 0 \leqslant y \leqslant x^3 \end{cases}$$

$$\int_0^8 \left(\int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx \right) dy = \int_0^2 \left(\int_0^{x^3} \sqrt{x^4+1}\, dy \right) dx = \int_0^2 y\sqrt{x^4+1}\,\Big|_0^{x^3} dx$$

$$= \int_0^2 x^3\sqrt{x^4+1}\, dx = \frac{1}{6}(x^4+1)^{\frac{3}{2}}\Big|_0^2 = \frac{1}{6}(17^{\frac{3}{2}}-1).$$

Practice Problems

1. Evaluate $\displaystyle\iint_D 42y^2 - 12x \; dA$ where

$D = \{(x, y) \,|\, 0 \leqslant x \leqslant 4, \;\; (x-2)^2 \leqslant y \leqslant 6\,.$

Solution. $11136.$

2. Evaluate $\displaystyle\iint_D 2yx^2 + 9y^3 \; dA$ where D is the region bounded by

$y = \dfrac{2}{3}x$ and $y = 2\sqrt{x}$.

Solution. $\dfrac{24057}{5}.$

3. Evaluate $\displaystyle\iint_D 10x^2y^3 - 6 \; dA$ where D is the region bounded by

$x = -2y^2$ and $x = y^3$.

Solution. $-\dfrac{8296}{13}.$

4. Evaluate $\displaystyle\iint_D x(y-1) \; dA$ where D is the region bounded by

$y = 1 - x^2$ and $y = x^2 - 3$.

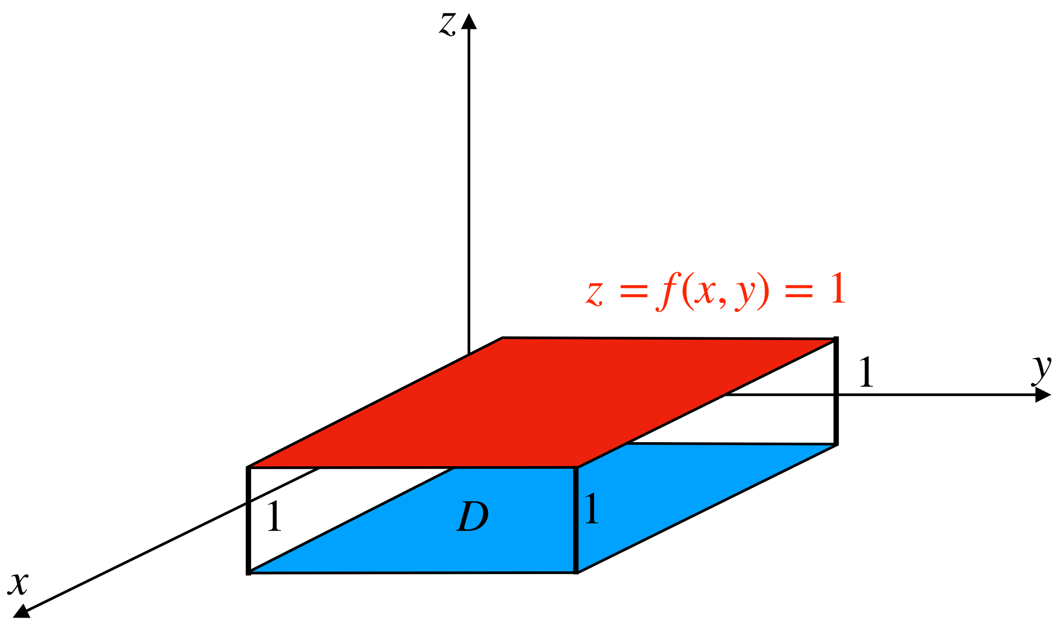
Solution. $0.$

5. Evaluate $\iint_D 5x^3\cos(y^3)\ dA$ where D is the region bounded by $y = 2$ and $y = \frac{1}{4}x^2$ and the x-axis.

Solution. $\frac{20}{3}\sin(8)$.

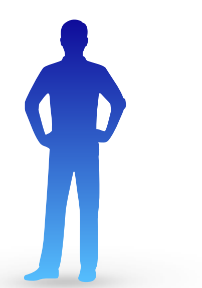
6. Evaluate $\int_0^3 \int_{2x}^{6} \sqrt{y^2+2}\ dy\ dx$ by reversing the order of integration.

7. Evaluate $\int_0^1 \int_{-\sqrt{y}}^{y^2} 6x - y\ dx\ dy$ by reversing the order of integration.



$$\text{Volume} = \iint_D f(x,y)\ dA = \iint_D dA = \text{Area}$$

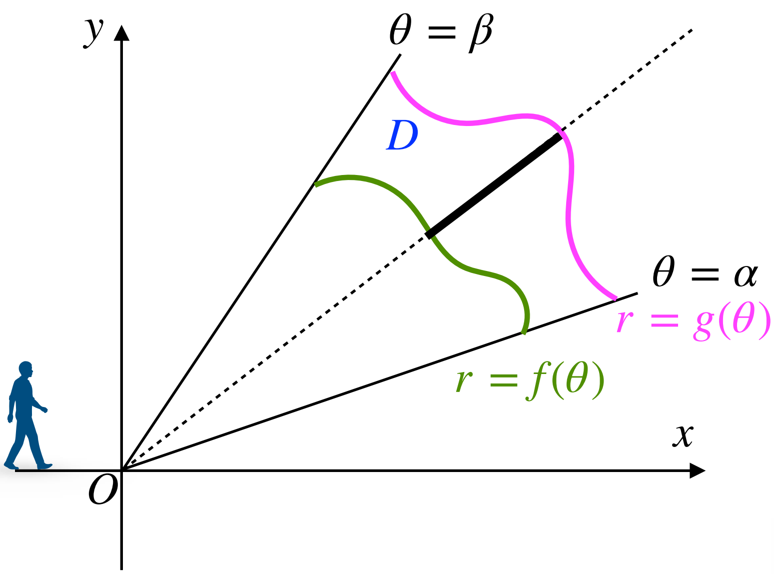
$$\text{Area of } D = \iint_D dA$$



8. Use a double integral to determine the area of the region bounded by $y = 1 - x^2$ and $y = x^2 - 1$.

Solution. $\dfrac{16\sqrt{2}}{3}$.

3. Double Integrals In Polar Coordinates

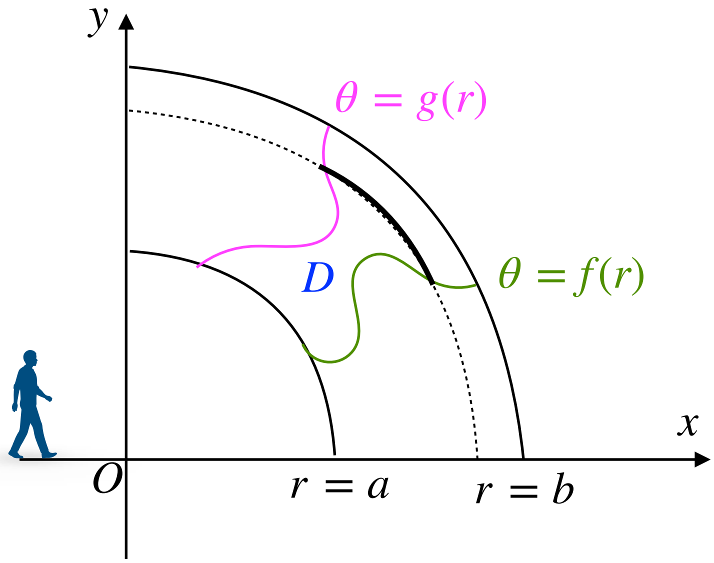


$$D \Longrightarrow \begin{cases} \alpha \leqslant \theta \leqslant \beta \\ f(\theta) \leqslant r \leqslant g(\theta) \end{cases}$$

Jacobian

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \left(\int_{f(\theta)}^{g(\theta)} f(r\cos\theta, r\sin\theta)\; r\; dr \right) d\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Longrightarrow \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$



$$D \Longrightarrow \begin{cases} a \leqslant r \leqslant b \\ f(r) \leqslant \theta \leqslant g(r) \end{cases}$$

$$\iint_D f(x,y)\, dA = \int_a^b \left(\int_{f(r)}^{g(r)} f(r\cos\theta, r\sin\theta)\; r\; d\theta \right) dr$$