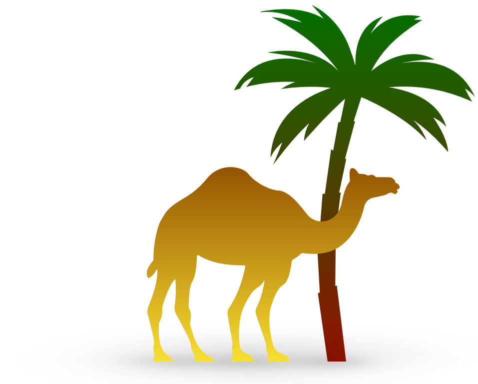
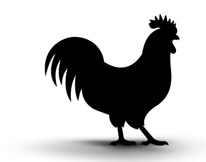
Problems and Solutions:



26. *Prove that*

$$\frac{\left[\binom{n+1}{r+1}-\binom{n}{r}\right]\binom{n-1}{r-1}}{\binom{n}{r}^2-\binom{n+1}{r+1}\binom{n-1}{r-1}}=r.$$

Solution. Since



$$\boxed{\binom{n+1}{r+1}=\frac{n+1}{r+1}\binom{n}{r}}$$

and

$$\binom{n}{r}=\frac{n}{r}\binom{n-1}{r-1},$$

we obtain

$$\frac{\left[\binom{n+1}{r+1}-\binom{n}{r}\right]\binom{n-1}{r-1}}{\binom{n}{r}^2-\binom{n+1}{r+1}\binom{n-1}{r-1}}=\frac{\left[\binom{n}{r}\frac{n+1}{r+1}-\binom{n}{r}\right]\binom{n-1}{r-1}}{\binom{n}{r}^2-\frac{n+1}{r+1}\binom{n}{r}\binom{n-1}{r-1}}=$$

$$=\frac{\cancel{\binom{n}{r}}\left[\frac{n+1}{r+1}-1\right]\binom{n-1}{r-1}}{\cancel{\binom{n}{r}}\left[\binom{n}{r}-\frac{n+1}{r+1}\binom{n-1}{r-1}\right]}=\frac{\left[\frac{n+1}{r+1}-1\right]\binom{n-1}{r-1}}{\left[\frac{n}{r}\binom{n-1}{r-1}-\frac{n+1}{r+1}\binom{n-1}{r-1}\right]}=$$

$$= \frac{\left[\frac{n+1}{r+1} - 1\right] \cancel{\binom{n-1}{r-1}}}{\cancel{\binom{n-1}{r-1}} \left[\frac{n}{r} - \frac{n+1}{r+1}\right]} = \frac{\frac{n+1}{r+1} - 1}{\frac{n}{r} - \frac{n+1}{r+1}} = \frac{\frac{n-r}{r+1}}{\frac{n-r}{r(r+1)}} = r\,.$$

27. *Prove that*

(a) $1 - 3\binom{n}{2} + 9\binom{n}{4} - 27\binom{n}{6} + \cdots = (-1)^n 2^n \cos\frac{2n\pi}{3}$.

(b) $\binom{n}{1} - 3\binom{n}{3} + 9\binom{n}{5} - \cdots = \frac{(-1)^{n+1}\cdot 2^n}{\sqrt{3}} \sin\frac{2n\pi}{3}$.

Solution. We have

$$\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^n = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\,.$$

It follows by *Binomial Theorem* that

$$\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n = \left[\left(-\frac{1}{2}\right)(1 - i\sqrt{3})\right]^n = \frac{(-1)^n}{2^n}(1 - i\sqrt{3})^n$$

$$= \frac{(-1)^n}{2^n}\left[1 + \binom{n}{1}(-i\sqrt{3}) + \binom{n}{2}(-i\sqrt{3})^2 + \binom{n}{3}(-i\sqrt{3})^3 + \cdots\right]$$

$$= \frac{(-1)^n}{2^n}\left(\left[1 - 3\binom{n}{2} + 9\binom{n}{4} - \cdots\right] - i\sqrt{3}\left[\binom{n}{1} - 3\binom{n}{3} + 9\binom{n}{5} - \cdots\right]\right)$$

$$\Rightarrow \begin{cases} 1 - 3\binom{n}{2} + 9\binom{n}{4} - 27\binom{n}{6} + \cdots = (-1)^n 2^n \cos\frac{2n\pi}{3}\,. \\[2ex] \binom{n}{1} - 3\binom{n}{3} + 9\binom{n}{5} - \cdots = \frac{(-1)^{n+1}\cdot 2^n}{\sqrt{3}} \sin\frac{2n\pi}{3}\,. \end{cases}$$

28. *Prove that for $m > n$,*

$$\sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{m(m-1)\cdots(m-k+1)} = \frac{m+1}{m-n+1},$$

and

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{n}{r}}{\binom{2n}{k+r}} = \frac{2n+1}{n+1}.$$

Solution. We have

$$\sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)(n-k)!(m-k)!}{m(m-1)\cdots(m-k+1)(m-k)!(n-k)!} = \sum_{k=0}^{n} \frac{n!(m-k)!}{m!(n-k)!}$$

$$= \sum_{k=0}^{n} \frac{(m-k)!}{\frac{m!}{n!}(n-k)!} = \sum_{k=0}^{n} \frac{(m-k)!}{\frac{m!}{n!(m-n)!}(n-k)!(m-n)!} = \sum_{k=0}^{n} \frac{\binom{m-k}{m-n}}{\binom{m}{n}}$$

$$= \frac{1}{\binom{m}{n}} \sum_{k=0}^{n} \binom{m-k}{m-n} = \frac{1}{\binom{m}{n}} \left[\binom{m}{m-n} + \binom{m-1}{m-n} + \cdots + \binom{m-n}{m-n}\right]$$

$$= \frac{\binom{m+1}{m-n+1}}{\binom{m}{n}} = \frac{\frac{(m+1)!}{(m-n+1)!n!}}{\frac{m!}{n!(m-n)!}} = \frac{m+1}{m-n+1}.$$

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{n}{r}}{\binom{2n}{k+r}} = \frac{n!\binom{n}{r}}{(2n)!} \sum_{k=0}^{n} \frac{(k+r)!(2n-k-r)!}{k!(n-k)!}$$

$$= \frac{(n!)^2}{(2n)!}\sum_{k=0}^{n}\frac{(k+r)!(2n-k-r)!}{r!(n-r)!k!(n-k)!} = \frac{(n!)^2}{(2n)!}\sum_{k=0}^{n}\frac{(k+r)!}{r!k!}\cdot\frac{(2n-k-r)!}{(n-r)!(n-k)!}$$

$$= \frac{(n!)^2}{(2n)!}\sum_{k=0}^{n}\binom{k+r}{r}\binom{2n-k-r}{n-r} = \frac{(n!)^2}{(2n)!}\binom{2n+1}{n+1} = \frac{2n+1}{n+1}.$$

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29. *Consider the following numerical triangle*

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 3 & 2 & 1 & & \\
 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

and prove that in every row, beginning with the third, there is an even number.

Solution. We have

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & 1 & 1 & & & & & \\
 & & & & 1 & 2 & 3 & 2 & 1 & & & & \\
 & & & 1 & 3 & 6 & 7 & 6 & 3 & 1 & & & \\
 & & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 & & \\
 & 1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 & \\
1 & 6 & 21 & 50 & 80 & 126 & 141 & 126 & 80 & 50 & 21 & 6 & 1
\end{array}$$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & 1 & 1 & & & & & \\
 & & & & O & E & O & E & 1 & & & & \\
 & & & O & O & E & O & 6 & 3 & 1 & & & \\
 & & O & E & E & E & 19 & 16 & 10 & 4 & 1 & & \\
 & O & O & O & E & 45 & 51 & 45 & 30 & 15 & 5 & 1 & \\
O & E & O & E & 80 & 126 & 141 & 126 & 80 & 50 & 21 & 6 & 1
\end{array}$$

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30. *Consider the following numerical triangle*

$$\begin{array}{cccccccccccccc}
0 & 1 & 2 & 3 & 4 & . & . & . & . & 1955 & 1956 & 1957 & 1958 \\
 & 1 & 3 & 5 & 7 & . & . & . & . & . & 3911 & 3913 & 3915 \\
 & & 4 & 8 & 12 & . & . & . & . & . & . & 7824 & 7828 \\
 & & & 12 & 20 & . & . & . & . & . & . & . & 15652 \\
 & & & & . & . & . & . & . & . & . & . & .
\end{array}$$

and prove that the element of the last row of the triangle is divisible by 1958.

Solution. We will show that:

(1) each row of the triangle is an arithmetic progression, and

(2) the sum of elements equidistant from the ends is divisible by 1958.

By induction on the number of the row.

1. The initial step

1th row $\Rightarrow$ $0 + 1958 = 1958,\ 1 + 1957 = 1958,\ 2 + 1956 = 1958,\ \ldots,$

2th row $\Rightarrow$ $1 + 3915 = 3916 = 2 \times 1958,\ 3 + 3913 = 2 \times 1958,\ \ldots,$

2. Inductive step

nth row $\Rightarrow$ $a, a + d, a + 2d, a + 3d, \ldots, a + (k-1)d$

$$1958 \,|\, a + (a + (k-1)d) = 2a + (k-1)d,$$
$$1958 \,|\, a + d + (a + (k-2)d) = 2a + (k-1)d,$$
$$\vdots$$

nth row $\Rightarrow$ $a + sd, \quad a + (s+1)d, \quad a + (s+2)d$

$(n+1)$th row $\Rightarrow$ $2a + (2s+1)d, \ 2a + (2s+3)d$

$$[2a + (2s+3)d] - [2a + (2s+1)d] = 2d$$

The sum of elements equidistant from the ends of $(n+1)$th row:

nth row $\Rightarrow$ $r,\ r+d,\ \ldots\ ,\ s-d,\ s$

$1958 \mid (r+d)+(s-d) = r+s$

$1958 \mid r+s$

$(n+1)$th row $\Rightarrow$ $2r+d,\ \ldots\ ,\ 2s-d$

$$(2r+d)+(2s-d) = 2(r+s) \Rightarrow 1958 \mid 2(r+s)$$