Generating Function (Ordinary):

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

$1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots,$

$x \Longrightarrow -x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots,$$

$1, -1, 1, -1, 1, -1, 1, -1, 1, \ldots,$

$x \Longrightarrow x^2$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots + x^{2n} + \cdots,$$

$1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots,$

$\vdots$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

1, 1, 1, 1, 1, 1, 1, 1, 1,...,

$x \Longrightarrow 2x$

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots + 2^n x^n + \cdots,$$

$1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots, 2^n, \ldots$

$x \Longrightarrow ax$

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \cdots + a^nx^n + \cdots,$$

$1, a, a^2, a^3, a^4, \ldots, a^n, \ldots$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,$$

1, 1, 1, 1, 1, 1, 1, 1, 1,...,

$$\frac{1}{1-x} \Longrightarrow \frac{1}{1-x} - x^2 = \frac{1 - x^2 + x^3}{1-x}$$

$$\frac{1 - x^2 + x^3}{1-x} = 1 + x + x^3 + \cdots + x^n + \cdots,$$

1, 1, 0, 1, 1, 1, 1, 1, 1,...,

$$\frac{1}{1-x} \Longrightarrow \frac{1}{1-x} + 2x^3 = \frac{1 + 2x^3 - 2x^4}{1-x}$$

$$\frac{1 + 2x^3 - 2x^4}{1-x} = 1 + x + x^2 + 3x^3 + \cdots + x^n + \cdots,$$

1, 1, 1, 3, 1, 1, 1, 1, 1,...,

$$\frac{x(x+1)}{(1-x)^3} \Rightarrow 0^2,\ 1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2,\dots,\ n^2,\ \dots$$

$$+$$

$$\frac{x}{(1-x)^2} \Rightarrow 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \dots,\ n,\ \dots$$

$$=$$

$$\frac{x(x+1)}{(1-x)^3}+\frac{x}{(1-x)^2} \Rightarrow 0^2+0,\ 1^2+1,\ 2^2+2,\ 3^2+3,\dots,\ n^2+n,\ \dots$$

$$=$$

$$\frac{2x}{(1-x)^3} \Rightarrow 0,\ 2,\ 6,\ 12,\ 20,\ \dots,\ n^2+n,\ \dots$$

For any $n \in \mathbb{Z}^+$, we have

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n .$$

$n \in \mathbb{Z}^-$

$n \in \mathbb{R}$

$\Rightarrow$?

With $n, r \in \mathbb{Z}^+$ and $n \geqslant r \geqslant 0$, we have

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

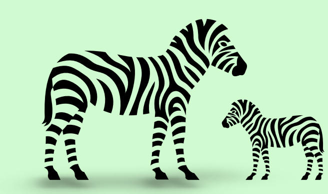
$$\binom{n}{r} = \qquad = \boxed{\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}}$$

$n \in \mathbb{R}$ $\Rightarrow$ $$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$$\binom{n}{0} = 1$$

$n \in \mathbb{Z}^+$

$$\begin{aligned}
\binom{-n}{r} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!} \\
&= \frac{(-1)^r(n)(n+1)(n+2)\cdots(n+r-1)}{r!} \\
&= \frac{(-1)^r(n-1)!(n)(n+1)(n+2)\cdots(n+r-1)}{r!(n-1)!} \\
&= \frac{(-1)^r(n+r-1)!}{r!(n-1)!} \\
&= (-1)^r\binom{n+r-1}{r}
\end{aligned}$$



For any $n \in \mathbb{Z}^+$, the Maclaurin series expansion for $(1+x)^{-n}$ is given by:

$$(1+x)^{-n} = 1 + \frac{(-n)}{1!}x + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}x^3 + \cdots$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)(-n-2)\cdots(-n+r-1)}{r!}x^r$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{-n}{r} x^r = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots$$

$$\Downarrow$$

$$(1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r$$

Example 1. Find the coefficient of x^5 in $(1-2x)^{-7}$.

Solution. We have

$$(1+y)^{-n} = \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} y^r \qquad y=-2x,\ n=7$$

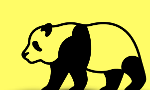
$$(1-2x)^{-7} = \sum_{r=0}^{\infty} (-1)^r \binom{7+r-1}{r} (-2x)^r = \sum_{r=0}^{\infty} (-1)^r \binom{r+6}{r} (-2)^r x^r$$

$$r=5 \qquad (-1)^5 \binom{5+6}{5} (-2)^5 = 32 \binom{11}{5} = 14{,}784.$$

For any $n \in \mathbb{R}$, the Maclaurin series expansion for $(1+x)^n$ is given by:

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$= 1 + \sum_{r=1}^{\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} x^r$$



Example 2. Find the Maclaurin series expansion for $(1+3x)^{-\frac{1}{3}}$.

Solution. We have

$$(1+3x)^{-\frac{1}{3}} = 1 + \sum_{r=1}^{\infty} \frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)\cdots(-\frac{1}{3}-r+1)}{r!}(3x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})\cdots(\frac{-3r+2}{3})}{r!}3^r x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-4)(-7)\cdots(-3r+2)}{r!} x^r$$

$$\boxed{(1+3x)^{-\frac{1}{3}}}$$

$$\Downarrow$$

$$1,\ -1,\ \frac{(-1)(-4)}{2!},\ \frac{(-1)(-4)(-7)}{3!},\ \ldots,\ \frac{(-1)(-4)(-7)\cdots(-3r+2)}{r!},\ \ldots$$

Example 3. Find the coefficient of x^{15} in $f(x) = (x^2 + x^3 + x^4 + \cdots)^4$.

Solution. We have

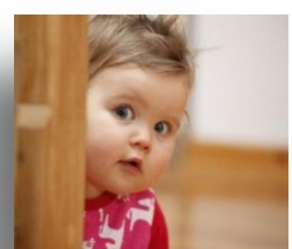
$$\begin{aligned} f(x) = (x^2 + x^3 + x^4 + \cdots)^4 &= [x^2(1 + x + x^2 + x^3 + \cdots)]^4 \\ &= x^8(1 + x + x^2 + x^3 + \cdots)^4 \\ &= x^8\left(\frac{1}{1-x}\right)^4 \\ &= x^8(1-x)^{-4} \end{aligned}$$

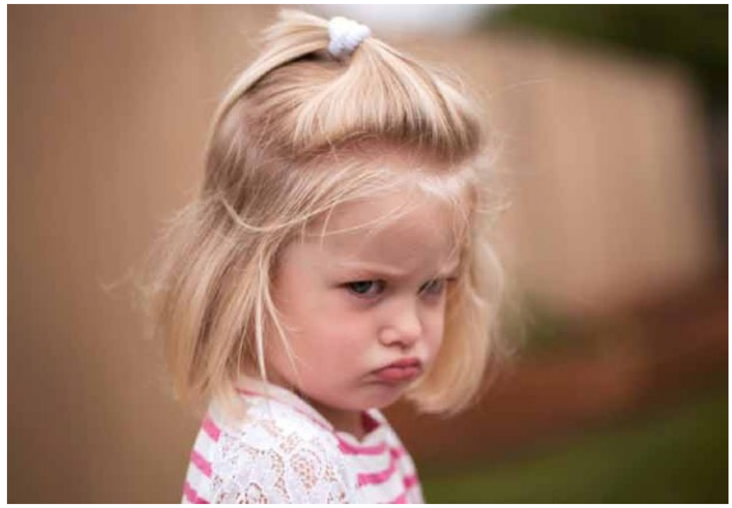
$$(1-x)^{-4} = \sum_{r=0}^{\infty} (-1)^r \binom{4+r-1}{r} (-x)^r = \sum_{r=0}^{\infty} \binom{r+3}{r} x^r$$

$$r = 7 \qquad \binom{7+3}{7} = \binom{10}{7} = \binom{10}{3} = 120.$$

In general, for $n \in \mathbb{Z}^+$, the coefficient of x^n in $f(x)$ is

$$\begin{cases} 0 & \text{if } n \leqslant 7, \\ \binom{n-5}{n-8} & \text{if } n \geqslant 8, \end{cases}$$





Example 4. How many integer solutions are there for the equation

$$x_1 + x_2 + x_3 + x_4 + \cdots + x_n = r, \quad x_i \geqslant 0, \;\; \text{for all } 1 \leqslant i \leqslant n?$$

Solution. In this example, the generating function is:

$$x_i \Rightarrow 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$\Rightarrow f(x) = \left(\frac{1}{1-x}\right)^n = (1-x)^{-n}$$

$$f(x) = \cdots + \boxed{?}\, x^r + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i = \cdots + \binom{n+r-1}{r} x^r + \cdots$$

$$\Rightarrow \binom{n+r-1}{r}$$

$$f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^4 + \cdots + a_i x^i + \cdots$$

$$g(x) = \sum_{i=0}^{\infty} b_i x^i = b_0 + b_1 x + b_2 x^2 + b_3 x^4 + \cdots + b_i x^i + \cdots$$

$$f(x)g(x) = \sum_{i=0}^{\infty} c_i x^i = c_0 + c_1 x + c_2 x^2 + c_3 x^4 + \cdots + c_i x^i + \cdots$$

$$c_0 = a_0 b_0$$

$$c_1 = a_0 b_1 + a_1 b_0$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$

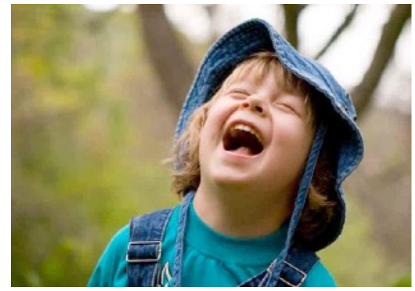
$$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0$$

$$\vdots$$

$$c_i = a_0 b_i + a_1 b_{i-1} + a_2 b_{i-2} + \cdots + a_{i-1} b_1 + a_i b_0$$

$$\vdots$$

Example 4. How many integer solutions are there for the equation



$$x_1 + x_2 + x_3 + x_4 = 24, \quad 3 \leqslant x_i \leqslant 8, \quad \text{for all } 1 \leqslant i \leqslant 4?$$

Solution. In this example, the generating function is:

x_1 $\quad x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

x_2 $\quad x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

x_3 $\quad x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

x_4 $\quad x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

$$f(x) = (x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4 .$$

$$f(x) = x^{12}(1 + x + x^2 + x^3 + x^4 + x^5)^4 = x^{12}\left(\frac{1 - x^6}{1 - x}\right)^4$$

$$= x^{12}(1 - x^6)^4(1 - x)^{-4}$$

$$(1 - x^6)^4 = \sum_{i=0}^{4} a_i x^i = \binom{4}{0} - \binom{4}{1}x^6 + \binom{4}{2}x^{12} - \binom{4}{3}x^{18} + \binom{4}{4}x^{24}$$

$$(1 - x)^{-4} = \sum_{i=0}^{\infty} b_i x^i = \binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \cdots$$

$$(1 - x^6)^4(1 - x)^{-4} = \quad \cdots + \boxed{c_{12}}\; x^{12} + \cdots$$

$$c_{12} = a_0 b_{12} + a_1 b_{11} + a_2 b_{10} + \cdots + a_{11} b_1 + a_{12} b_0$$

$$= a_0 b_{12} + a_6 b_6 + a_{12} b_0$$

$$c_{12} = \binom{4}{0}\binom{-4}{12}(-1)^{12} - \binom{4}{1}\binom{-4}{6}(-1)^{6} + \binom{4}{2}\binom{-4}{0}$$

$$= \binom{15}{12} - \binom{4}{1}\binom{9}{6} + \binom{4}{2} = 125.$$

Example 5. Verify that for all $n \in \mathbb{Z}^+$,

$$\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2$$

Solution. Since

$$(1+x)^{2n} = (1+x)^n(1+x)^n$$

by comparing of coefficients, the coefficient of x^n in $(1+x)^{2n}$ which is

$$(1+x)^{2n} \Rightarrow \binom{2n}{n}$$

must equal the coefficients of x^n in

$$(1+x)^n(1+x)^n = \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]^2$$

and this is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$$

$$\binom{n}{i} = \binom{n}{n-i}$$

$$= \binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \binom{n}{2}\binom{n}{2} + \cdots + \binom{n}{n}\binom{n}{n}$$

$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2$$

Therefore, we obtain

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2. \ \square$$


$\sqrt{-1}\ 2^3 \sum \pi$
...and it was DELICIOUS!

Thank you so much for your attention and patient.