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Problems 2 - Calculus II

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1. Determine the domain of the function $\vec{r}(t) = \langle \cos t, \ln(4-t), \sqrt{t+1} \rangle$.

Answer: $[-1,4)$.

2. Evaluate the following limit $\lim_{t\to\infty} \left\langle \frac{1}{t^2}, \frac{2t^2}{1-t-t^2}, e^{-t} \right\rangle$.

Answer: $\langle 0, -2, 0 \rangle$.

3. Compute the derivative of the given vector function $\vec{r}(t) = \langle \ln(t^2+1), te^{-t}, 4 \rangle$.

Answer: $\vec{r}'(t) = \left\langle \frac{2t}{t^2+1}, e^{-t} - te^{-t}, 0 \right\rangle$.

4. Evaluate $\int_{-1}^{2} \vec{r}(t)dt$ where $\vec{r}(t) = \langle 6, 6t^2 - 4t, te^{2t} \rangle$.

Answer: $\left\langle 18, 12, \frac{3}{4}(e^4 + e^{-2}) \right\rangle$.

5. Find the tangent line to the vector function $\vec{r}(t) = \langle \cos(4t), 3\sin(4t), t^3 \rangle$ and $t = \pi$.

Answer: $\vec{r}(t) = \langle 1, 0, \pi^3 \rangle + t\langle 0, 12, 3\pi^2 \rangle = \langle 1, 12t, (\pi^3 + 3\pi^2 t) \rangle$.

6. Find the unit tangent, the unit normal and the binormal vectors for the vector function $\vec{r}(t) = \langle \cos(2t), \sin(2t), 3 \rangle$.

Answer:

$\overrightarrow{T}(t) = \langle -sin(2t), \cos(2t), 0 \rangle$, $\overrightarrow{N}(t) = \langle -cos(2t), -sin(2t), 0 \rangle$, $\overrightarrow{B}(t) = \langle 0, 0, 1 \rangle$
.

7. Determine the length of the vector function $\vec{r}(t) = \langle \frac{1}{3}t^3, 4t, \sqrt{2}\ t^2 \rangle$ on the interval $[0,2]$.

Answer: $\frac{32}{3}$.

8. Find the curvature for $\vec{r}(t) = \langle \cos(2t), -\sin(2t), 4t \rangle$.

Answer: $\frac{1}{5}$.

9. Evaluate the limit $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3}$.

Answer: This limit does not exist.

10. Evaluate the limit $\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$.

Answer: $\frac{1}{2}$.

11. Find all the first order partial derivatives of the function

$$f(x,y) = \frac{x^2}{y^2+1} - \frac{y^2}{x^2+y}.$$

Answer: $f_x(x,y) = \dfrac{2x}{y^2+1} + \dfrac{2xy^2}{(x^2+y)^2}$, $f_y(x,y) = -\dfrac{2yx^2}{(y^2+1)^2} - \dfrac{2yx^2+y^2}{(x^2+y)^2}$.

12. Find $\frac{\partial z}{\partial x}$ for the function $x^2\sin(y^3) + xe^{3z} - \cos(z^2) = 3y - 6z + 8$.

Answer: $\dfrac{\partial z}{\partial x} = \dfrac{2x\sin(y^3) + e^{3z}}{-6 - 3xe^{3z} - 2z\sin(z^2)}$.

13. Find all second order derivatives for the function $f(s,t) = s^2t + \ln(t^2 - s)$.

Answer: $f_{ss} = 2t - \dfrac{1}{(t^2-s)^2}$, $f_{st} = f_{ts} = 2s + \dfrac{2t}{(t^2-s)^2}$, $f_{tt} = \dfrac{-2t^2-2s}{(t^2-s)^2}$.

14. Determine the gradient of the function $f(x, y, z) = x\cos(xy) + z^2y^4 - 7xz$.

Answer:

$$\nabla(f) = \langle \cos(xy) - xy\sin(xy) - 7z,\ -x^2\sin(xy) + 4z^2y^3,\ 2zy^4 - 7x\rangle.$$

15. Determine $D_{\vec{u}}f$ for the function $f(x, y, z) = x^2y^3 - 4xz$ in the direction of $\vec{v} = \langle -1,2,0\rangle$.

Answer: $D_{\vec{u}}f = \dfrac{1}{\sqrt{5}}(4z - 2xy^3 + 6x^2y^2)$.

16. Find the equation of the tangent plane to $z = x\sqrt{x^2 + y^2} + y^3$ at $(-4,3)$.

Answer: $(z - 7) - \dfrac{41}{5}(x + 4) - \dfrac{123}{5}(y - 3) = 0$.

17. Find the tangent plane and normal line to $x^2y = 4ze^{x+y} - 35$ at $(3, -3,2)$.

Answer: $-26x + y - 4z = -89$, $\langle 3 - 26t, -3 + t, 2 - 4t\rangle$.

18. Find $D_{\vec{u}}f$ for $f(x, y) = e^x\cos y$ in the direction 30 degrees from the positive x axis at the point $(1, \dfrac{\pi}{4})$.

Answer: $e\sqrt{2}(\sqrt{3} - 1)/4$.

19. Find the equation of the plane perpendicular to $\vec{r}(t) = \langle \cos t, \sin t, \cos(6t)\rangle$ when $t = \pi/4$.

Answer: $-x/\sqrt{2} + y/\sqrt{2} + 6z = 0$.

20. Find the line of intersection of the plane given by $3x + 6y - 5z = -3$ and the plane given $-2x + 7y - z = 24$.

Answer: $\vec{r}(t) = \langle -5,2,0\rangle + t\langle 29,13,33\rangle = \langle -5 + 29t, 2 + 13t, 33t\rangle$.