7. Partial Derivatives

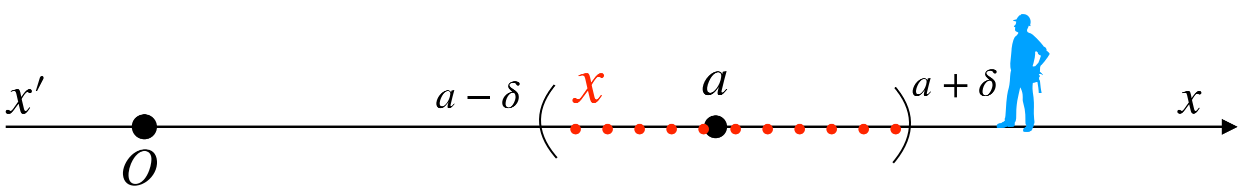
Section 7-1 : Limits

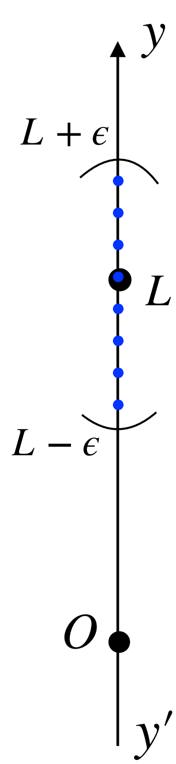
$$\lim_{x \to a} f(x) = L,$$

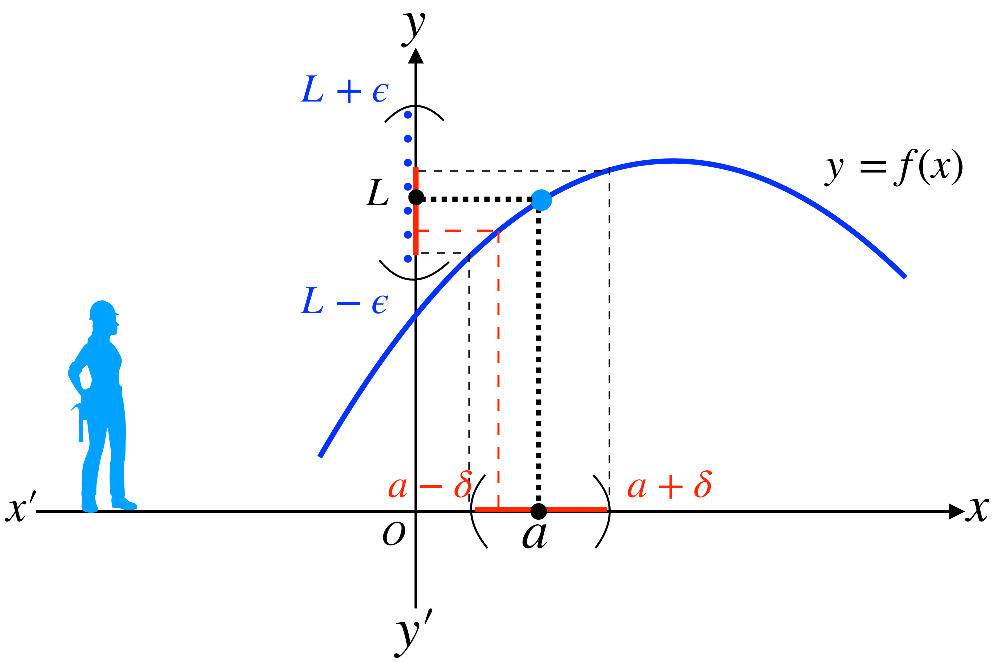
$$\forall \epsilon > 0 \quad \exists \delta > 0, \quad \forall x \in D_f \quad (|x - a| \leqslant \delta \implies |f(x) - L| \leqslant \epsilon).$$

If $x \to a$, then $f(x) \to L$.

$$x \to a, \quad \forall \delta > 0, \quad (a - \delta, a + \delta) \cap D_f$$

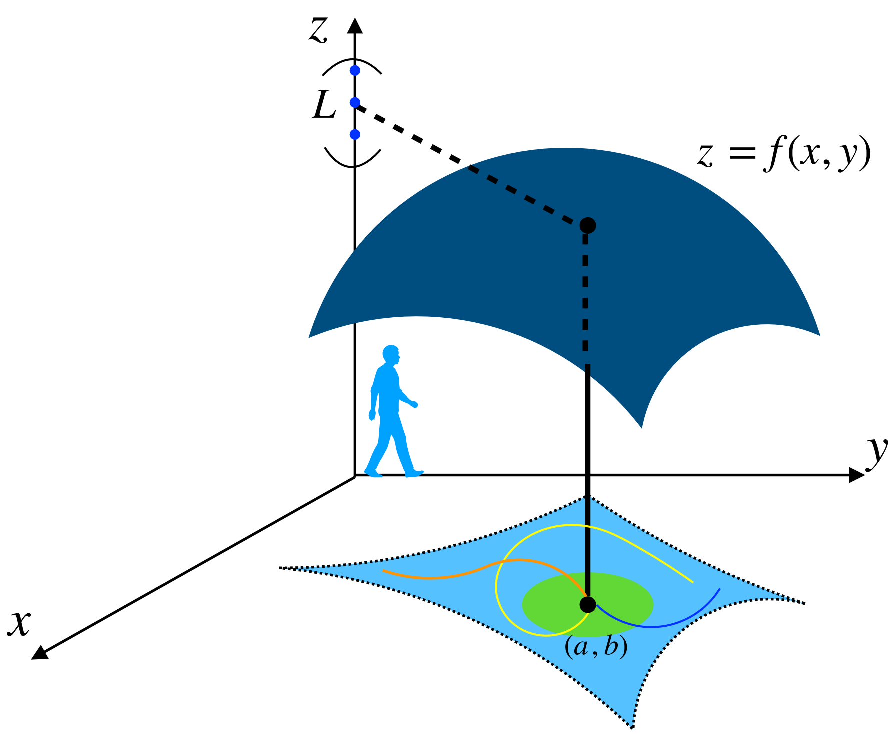






$$\forall \epsilon > 0 \;\; \exists \delta > 0, \;\; \forall x \in D_f \;\; (|x-a| \leqslant \delta \implies |f(x)-L| \leqslant \epsilon).$$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$



$$\forall \epsilon > 0 \ \ \exists \delta > 0, \ \ \forall (x, y) \in D_f \ \ (\|(x, y) - (a, b)\| \leqslant \delta \implies |f(x, y) - L| \leqslant \epsilon).$$

$$\forall \epsilon > 0 \ \ \exists \delta > 0, \ \ \forall (x, y) \in D_f \ \ (\sqrt{(x-a)^2 + (y-b)^2} \leqslant \delta \implies |f(x, y) - L| \leqslant \epsilon).$$

Definition. A function $f(x, y)$ is continuous at the point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

Example 1 Determine if the following limits exist or not. If they do exist give the value of the limit.

1. $$\lim_{(x,y,z)\to(2,1,-1)} 3x^2z + yx\cos(\pi x - \pi z).$$

$$\lim_{(x,y,z)\to(2,1,-1)} 3x^2z + yx\cos(\pi x - \pi z) = 3(2)^2(-1) + (1)(2)\cos(2\pi + \pi) = -14.$$

2. $$\lim_{(x,y)\to(5,1)} \frac{xy}{x+y}.$$

$$\lim_{(x,y)\to(5,1)} \frac{xy}{x+y} = \frac{5}{6}.$$

3. $$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}.$$

Along the x-axis, $y = 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(x,0)\to(0,0)} \frac{x^2(0)^2}{x^4+3(0)^4} = \lim_{(x,0)\to(0,0)} 0 = 0.$$

Along the y-axis, $x = 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(0,y)\to(0,0)} \frac{(0)^2y^2}{(0)^4+3(y)^4} = \lim_{(0,y)\to(0,0)} 0 = 0.$$

The path $y = x$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(x,x)\to(0,0)} \frac{(x)^2x^2}{(x)^4+3(x)^4} = \lim_{(x,x)\to(0,0)} \frac{x^4}{4x^4} = \frac{1}{4}.$$

The limit does not exist.

4. $$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}.$$

The path $y = x$.

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^3x}{x^6+x^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^4+1} = 0.$$

The path $y = x^3$.

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^3x^3}{x^6+x^6} = \lim_{(x,x)\to(0,0)} \frac{1}{2} = \frac{1}{2}.$$

The limit does not exist.

Practical Problems

Evaluate each of the following limits.

1. $$\lim_{(x,y,z)\to(-1,0,4)} \frac{x^3 - ze^{2y}}{6x + 2y - 3z}.$$

2. $$\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}.$$

3. $$\lim_{(x,y)\to(0,0)} \frac{x - 4y}{6y + 7x}.$$

4. $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3}.$$

Section 7-2 : Partial Derivatives

The formal definitions of the two partial derivatives. Given the function $z = f(x, y)$:

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

and

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

The possible alternate notation for partial derivatives:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f(x, y)) = z_x = \frac{\partial z}{\partial x} = D_x f,$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f(x, y)) = z_y = \frac{\partial z}{\partial y} = D_y f.$$

Example 2 Find all of the first order partial derivatives for the following functions.

1. $f(x, y) = x^4 + \sqrt{6y} - 10.$

$$f_x(x, y) = 4x^3, \quad f_y(x, y) = \frac{3}{\sqrt{6y}}.$$

2. $w = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$.

$$\frac{\partial w}{\partial x} = 2xy + 43, \quad \frac{\partial w}{\partial y} = x^2 - 20yz^3 - 28\sec^2(4y), \quad \frac{\partial w}{\partial z} = -30y^2z^2.$$

3. $h(s, t) = t^7 \ln(s^2) + \dfrac{9}{t^3} - \sqrt[7]{s^4}$.

4. $f(x, y) = \cos\left(\dfrac{4}{x}\right) e^{x^2y - 5y^3}$.

Example 3 Find $\dfrac{dy}{dx}$ for $3y^4 + x^7 = 5x$.

We have

$$12y^3\frac{dy}{dx} + 7x^6 = 5 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{5 - 7x^6}{12y^3}.$$

Example 4 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for each of the following functions.

1. $x^3z^2 - 5xy^5z = x^2 + y^3$.

We have

$$3x^2z^2 + 2x^3z\frac{\partial z}{\partial x} - 5y^5z - 5xy^5\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial x} = \frac{2x - 3x^2z^2 + 5y^5z}{2x^3z - 5xy^5}.$$

and

$$2x^3z\frac{\partial z}{\partial y} - 25xy^4z - 5xy^5\frac{\partial z}{\partial y} = 3y^2,$$

$$\frac{\partial z}{\partial y} = \frac{3y^2 + 25xy^4z}{2x^3z - 5xy^5}.$$

2. $x^2\sin(2y - 5z) = 1 + y\cos(6zx)$.

Practical Problems

For problems $1 - 8$ find all the 1st order partial derivatives.

1. $f(x, y, z) = 4x^3y^2 - e^zy^4 + \dfrac{z^3}{x^2} + 4y - x^{16}$.
2. $w = \cos(x^2 + 2y) - e^{4x - z^4y} + y^3$.
3. $f(u, v, p, t) = 8u^2t^3p - \sqrt{v}p^2t^{-5} + 2u^2t + 3p^4 - v$.
4. $f(u, v) = u^2\sin(u + v^3) - \sec(4u)\tan^{-1}(2v)$.
5. $f(x, z) = e^{-x}\sqrt{z^4 + x^2} - \dfrac{2x + 3z}{4z - 7x}$.
6. $g(s, t, v) = t^2\ln(s + 2t) - \ln(3v)(s^3 + t^2 - 4v)$.

7. $R(x,y) = \dfrac{x^2}{y^2+1} - \dfrac{y^2}{x^2+y}$.

8. $z = \dfrac{p^2(r+1)}{t^3} + pre^{2p+3r+4t}$.

9. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the following function:

$x^2\sin(y^3) + xe^{3z} - \cos(z^2) =$ [illegible] $z + 8.$