5. Vector Functions (I)

Section 5-1 : The concept of vector functions

A vector function of a single variable in $\mathbb{R}^n$ has the form,

$$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^n$$

$$\vec{r}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t) \rangle$$

where $f_1(t), f_2(t), \ldots, f_n(t)$ are called the *component functions*.

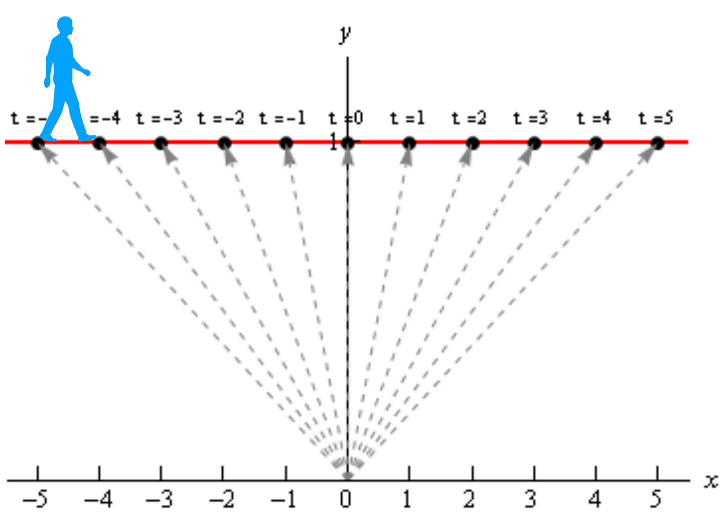
The vector functions of a single variable in $\mathbb{R}^2$ and $\mathbb{R}^3$ have the following forms:

$$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \text{and} \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

respectively, where $f(t), g(t), h(t)$ are called the component functions.

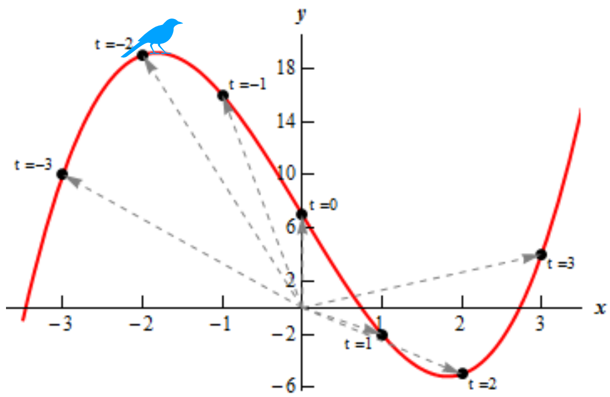
Example 1 Sketch the graph of each of the following vector functions.

- $\vec{r}(t) = \langle t, 1 \rangle$. Here is a sketch of this vector function.



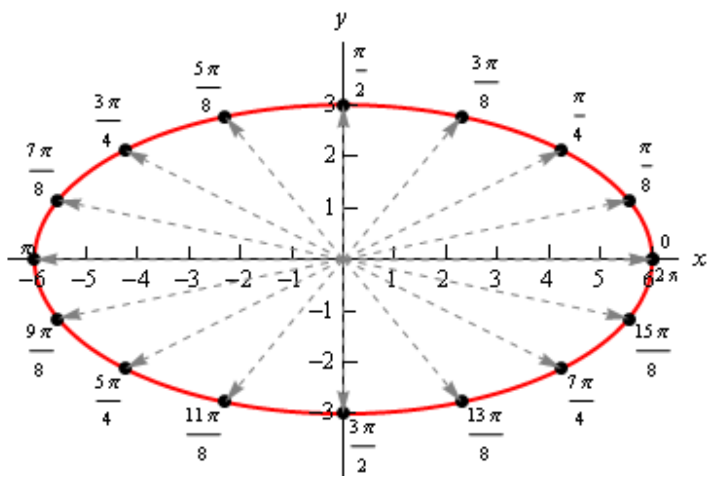
- $\vec{r}(t) = \langle t,\ t^3 - 10t + 7 \rangle$.

Here is a sketch of this vector function.



- $\vec{r}(t) = \langle 6\cos t,\ 3\sin t \rangle$.

Here is a sketch of this vector function.

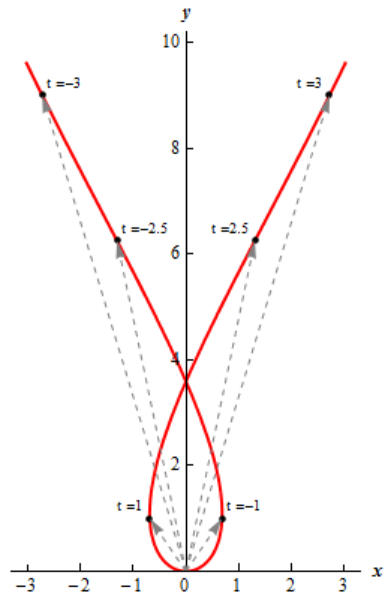


In this case it looks like we have got an ellipse:

$$x = 6\cos t,\ y = 3\sin t \implies \frac{x}{6} = \cos t,\ \frac{y}{3} = \sin t \implies \frac{x^2}{36} + \frac{y^2}{9} = 1.$$

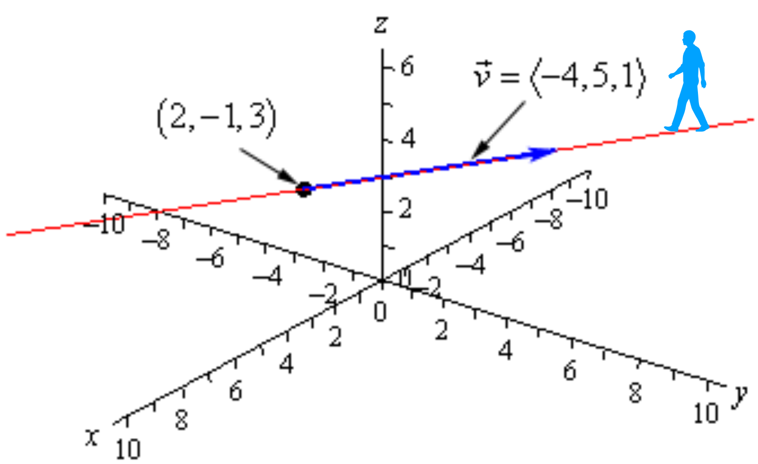
- $\vec{r}(t) = \langle t - 2\sin t,\ t^2 \rangle$.

Here's the sketch for this vector function.



- $\vec{r}(t) = \langle 2 - 4t,\ -1 + 5t,\ 3 + t \rangle$,

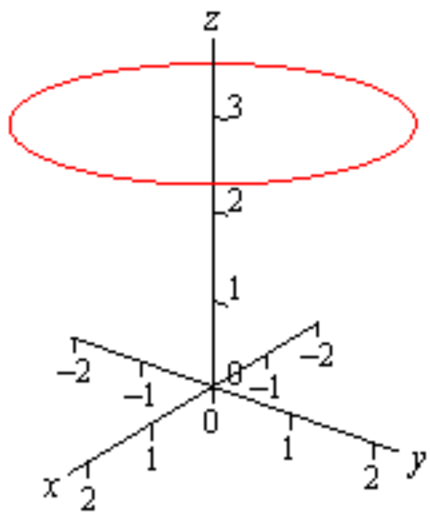
Here is a sketch.



$$x = 2 - 4t, y = -1 + 5t, z = 3 + 3 \implies \frac{x-2}{-4} = \frac{y+1}{5} = \frac{z-3}{3} = t\,.$$

- $\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ 3\rangle$,

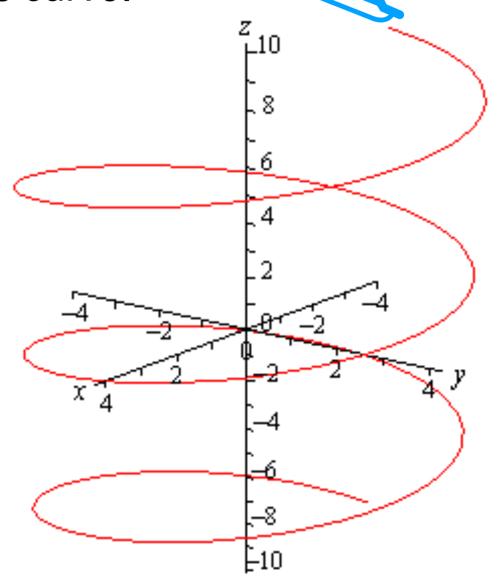
Here is a sketch:



$$x = 2\cos t, y = 2\sin t, \implies \frac{x^2}{4} + \frac{y^2}{4} = 1,\ z = 3, \implies x^2 + y^2 = 4,\ z = 3.$$

- $\vec{r}(t) = \langle 4\cos t,\ 4\sin t,\ t\rangle$,

Here is a sketch of this curve.



$$x^2 + y^2 = 16,\ z = t\,.$$

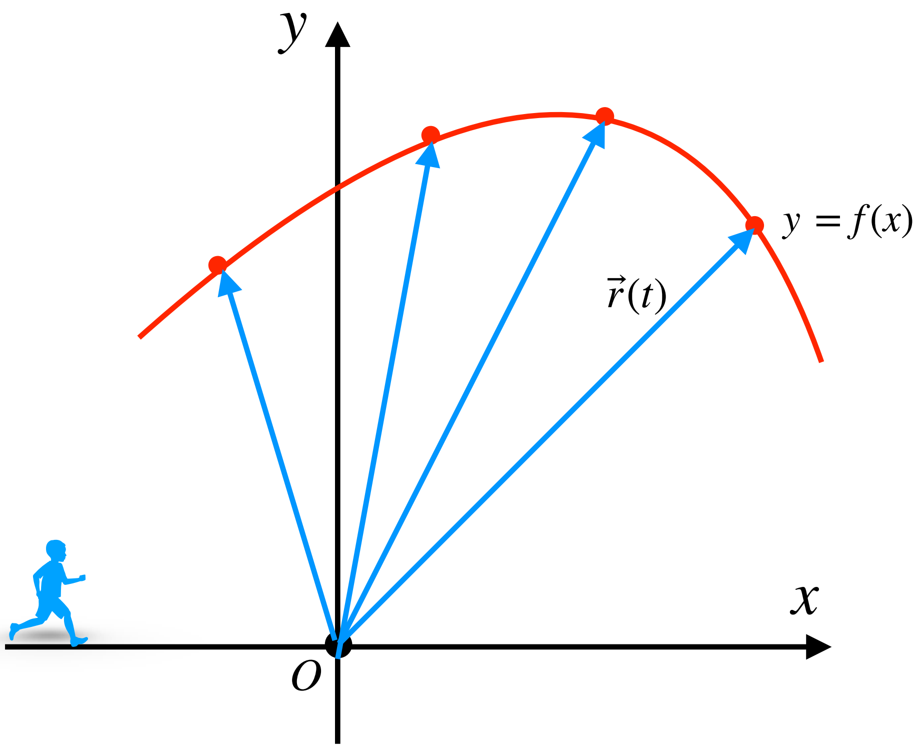
$$y = f(x) \quad \Longrightarrow \quad \vec{r}(t) = \langle t, f(t) \rangle = t\vec{i} + f(t)\vec{j}\,.$$

$$x = g(y) \quad \Longrightarrow \quad \vec{r}(t) = \langle g(t), t \rangle = g(t)\vec{i} + t\vec{j}\,.$$

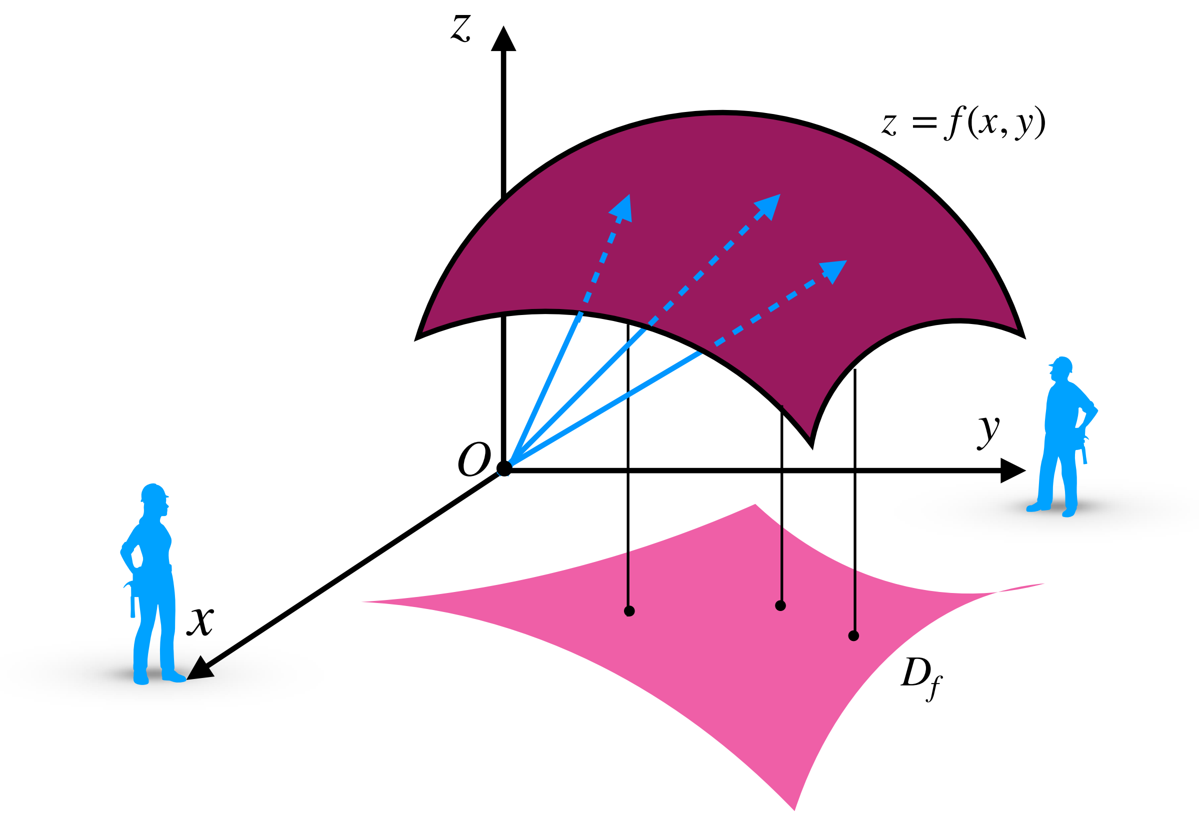
$$z = f(x, y) \quad \Longrightarrow \quad \vec{r}(x, y) = \langle x, y, f(x, y) \rangle = x\vec{i} + y\vec{j} + f(x, y)\vec{k}\,.$$

$$y = g(x, z) \quad \Longrightarrow \quad \vec{r}(x, z) = \langle x, g(x, z), z \rangle = x\vec{i} + g(x, z)\vec{j} + z\vec{k}\,.$$

$$x = h(y, z) \quad \Longrightarrow \quad \vec{r}(y, z) = \langle h(y, z), y, z \rangle = h(y, z)\vec{i} + y\vec{j} + z\vec{k}\,.$$



$$l \left| \begin{array}{l} \vec{T} = \langle a, b, c \rangle \\ P = (x_0, y_0, z_0) \end{array} \right. \quad \Longrightarrow \quad \vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle\,.$$



Section 5-2 : Calculus With Vector Functions

We will be doing all of the work in $\mathbb{R}^3$ but we can naturally extend the formulas/work in this section to $\mathbb{R}^n$ (i.e. n-dimensional space).

The limit of a vector function: If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, *then*

$$\begin{aligned}\lim_{t\to a} \vec{r}(t) &= \lim_{t\to a}\langle f(t), g(t), h(t)\rangle \\ &= \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle \\ &= \lim_{t\to a} f(t)\ \vec{i} + \lim_{t\to a} g(t)\ \vec{j} + \lim_{t\to a} h(t)\ \vec{k}\end{aligned}$$

The derivative of a vector function: If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, *then*

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\ \vec{i} + g'(t)\ \vec{j} + h'(t)\ \vec{k}\ .$$

Some facts about derivatives of vector functions:

1. $(\overrightarrow{u} + \overrightarrow{v})' = \overrightarrow{u}' + \overrightarrow{v}'$
2. $(c\overrightarrow{u})' = c\overrightarrow{u}'$
3. $(f(t)\overrightarrow{u})' = f'(t)\overrightarrow{u} + f(t)\overrightarrow{u}'$
4. $(\overrightarrow{u} \cdot \overrightarrow{v})' = \overrightarrow{u}' \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{v}'$
5. $(\overrightarrow{u} \times \overrightarrow{v})' = \overrightarrow{u}' \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{v}'$
6. $\overrightarrow{u}(f(t))' = f'(t)\ \overrightarrow{u}'(f(t))$.

A ***smooth curve*** $\vec{r}(t)$ is any curve for which $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$ for any t except possibly at the endpoints.

The integrals of vector functions: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, *then*

$$\int \vec{r}(t)dt = \left\langle \int f(t)dt, \int g(t)dt, \int h(t)dt \right\rangle$$

$$= \int f(t)dt\ \vec{i} + \int g(t)dt\ \vec{j} + \int h(t)dt\ \overrightarrow{k},$$

and

$$\int_a^b \vec{r}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$$

$$= \int_a^b f(t)dt\ \vec{i} + \int_a^b g(t)dt\ \vec{j} + \int_a^b h(t)dt\ \overrightarrow{k},$$

Example 2 Compute $\lim_{t\to 1} \vec{r}(t)$ where $\vec{r}(t) = \langle t^3, \dfrac{\sin(3t-3)}{t-1}, e^{2t}\rangle$.

We have

$$\lim_{t\to 1} \vec{r}(t) = \lim_{t\to 1}\langle t^3, \frac{\sin(3t-3)}{t-1}, e^{2t}\rangle$$

$$= \langle \lim_{t\to 1} t^3, \lim_{t\to 1} \frac{\sin(3t-3)}{t-1}, \lim_{t\to 1} e^{2t}\rangle = \langle 1,3,e^2\rangle\,.$$

Example 3 Compute $\vec{r}(t)'$ where

$$\vec{r}(t) = t^6\ \vec{i} + \sin 2t\ \vec{j} - \ln(t+1)\ \vec{k}\,.$$

We have

$$\vec{r}'(t) = 6t^5\ \vec{i} + 2\cos 2t\ \vec{j} - \frac{1}{t+1}\ \vec{k}\,.$$

Example 4 Compute $\int \vec{r}(t)dt$ and $\int_0^1 \vec{r}(t)dt$, where

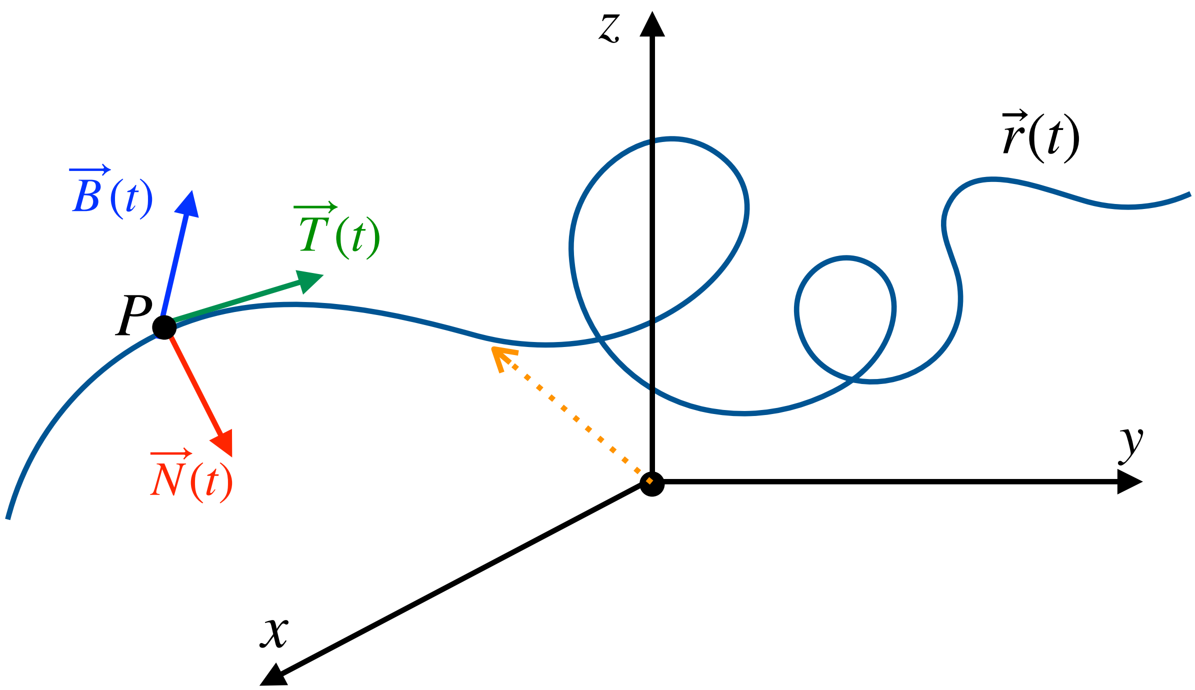
$$\vec{r}(t) = \langle \sin t, 6, 4t\rangle.$$

We have

$$\int \vec{r}(t)dt = \langle -\cos t, 6t, 2t^2\rangle + \vec{c}\,.$$

$$\int_0^1 \vec{r}(t)dt = \langle -\cos t, 6t, 2t^2 \rangle \Big|_0^1 = \langle -\cos 1, 6, 2\rangle - \langle 1, 0, 0\rangle = \langle 1 - \cos 1, 6, 2\rangle$$

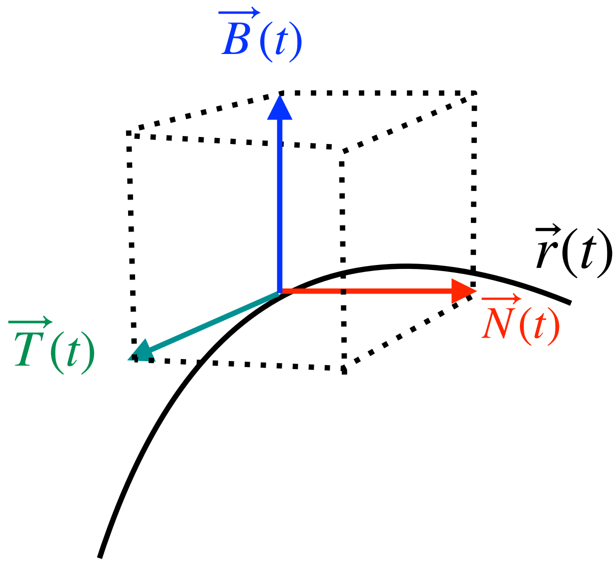
Section 5-3 : Tangent, Normal And Binormal Vectors



If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then we summarize the formulas for unit Tangent, unit Normal and unit Binormal vectors

$$\overrightarrow{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}, \qquad \overrightarrow{N}(t) = \frac{\overrightarrow{T}\,'(t)}{\|\overrightarrow{T}\,'(t)\|}, \qquad \overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t).$$

$\vec{r}\,'(t) \neq 0$



Example 5 Find the general formula for the tangent vector and unit tangent vector to the curve given by

$$\vec{r}(t) = t^2\ \vec{i} + 2\sin t\ \vec{j} + 2\cos t\ \overrightarrow{k}\ .$$

The tangent vector to the curve is

$$\vec{r}'(t) = 2t\ \vec{i} + 2\cos t\ \vec{j} - 2\sin t\ \overrightarrow{k}\ .$$

To get the unit tangent vector we need the length of the tangent vector:

$$\|\vec{r}'(t)\| = \sqrt{4t^2 + 4\cos^2 t + 4\sin^2 t} = \sqrt{4t^2+4} = 2\sqrt{t^2+1}\ .$$

The unit tangent vector is then,

$$\overrightarrow{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{2t\ \vec{i} + 2\cos t\ \vec{j} - 2\sin t\ \overrightarrow{k}}{2\sqrt{t^2+1}}$$

$$= \frac{t}{\sqrt{t^2+1}}\ \vec{i} + \frac{\cos t}{\sqrt{t^2+1}}\ \vec{j} - \frac{\sin t}{\sqrt{t^2+1}}\ \overrightarrow{k}\ .$$

Example 6 Find the vector equation of the tangent line to the curve given by $\vec{r}(t) = t^2\ \vec{i} + 2\sin t\ \vec{j} + 2\cos t\ \vec{k}$ at $t = \pi/3$.

Solution. Since

$$\vec{r}\,'(t) = 2t\ \vec{i} + 2\cos t\ \vec{j} - 2\sin t\ \vec{k},$$

we obtain

$$\vec{r}\,'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}\ \vec{i} + 2\cos\left(\frac{\pi}{3}\right)\vec{j} - 2\sin\left(\frac{\pi}{3}\right)\vec{k},$$

and so

$$\vec{r}\,'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}\ \vec{i} + \vec{j} - \sqrt{3}\ \vec{k}.$$

We'll also need the point on the line at $t = \pi/3$. Hence

$$\vec{r}\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}\ \vec{i} + \sqrt{3}\ \vec{j} + \vec{k}.$$

The vector equation of the line is then,

$$\frac{x - (\pi^2/9)}{2\pi/3} = \frac{y - \sqrt{3}}{1} = \frac{z - 1}{-\sqrt{3}}.$$

Example 7 Find the *normal* and *binormal* vectors for

$$\vec{r}(t) = t\ \vec{i} + 3\sin t\ \vec{j} + 3\cos t\ \vec{k}.$$

Solution. We first need the unit tangent vector so first get the tangent vector and its magnitude:

$$\vec{r}\,'(t) = \vec{i} + 3\cos t\ \vec{j} - 3\sin t\ \vec{k},$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10}\,.$$

The unit tangent vector is then,

$$\vec{T}(t) = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\cos t\ \vec{j} - \frac{3}{\sqrt{10}}\sin t\ \vec{k}\,.$$

The unit normal vector will now require the derivative of the unit tangent and its magnitude:

$$\vec{T}\,'(t) = 0\vec{i} - \frac{3}{\sqrt{10}}\sin t\ \vec{j} - \frac{3}{\sqrt{10}}\cos t\ \vec{k},$$

$$\|\vec{T}\,'(t)\| = \sqrt{\frac{9}{10}\cos^2 t + \frac{9}{10}\sin^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}\,.$$

The *unit normal vector* is then,

$$\vec{N}(t) = 0\vec{i} - \sin t\ \vec{j} - \cos t\ \vec{k},$$

Finally, the *binormal vector* is

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t & -\frac{3}{\sqrt{10}}\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix}$$

$$= -\frac{3}{\sqrt{10}}\vec{i} + \frac{1}{\sqrt{10}}\cos t\ \vec{j} - \frac{1}{\sqrt{10}}\sin t\ \vec{k}\,.$$

Example 8 Suppose that $\vec{r}(t)$ *is a vector such that* $\|\vec{r}(t)\| = c$ for all t. Then, $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$, that is $\vec{r}\,'(t) \perp \vec{r}(t)$.

Solution. For all t, we have

$$\vec{r}(t) \cdot \vec{r}(t) = \|\vec{r}(t)\|^2 = c^2.$$

Since, this is true for all t we can see that

$$\begin{aligned}
(\vec{r}(t) \cdot \vec{r}(t))' = 0 \quad &\Longrightarrow \quad \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0,\\
&\Longrightarrow \quad 2\vec{r}(t) \cdot \vec{r}\,'(t) = 0,\\
&\Longrightarrow \quad \vec{r}(t) \cdot \vec{r}\,'(t) = 0,
\end{aligned}$$

Therefore, $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$.

Practical Problems

For problems 1 & 2 find the domain of the given vector function.

1. $\vec{r}(t) = \langle t^2 + 1, \dfrac{1}{t+2}, \sqrt{t+4}\rangle$,
2. $\vec{r}(t) = \langle \ln(4 - t^2), \sqrt{t+1}\rangle$,

For problems $3 - 5$ sketch the graph of the given vector function.

3. $\vec{r}(t) = \langle 4t, 10 - 2t\rangle$,

4. $\vec{r}(t) = \langle t+1, \frac{1}{4}t^2 + 3 \rangle,$
5. $\vec{r}(t) = \langle 4\sin t, 8\cos t \rangle,$

For problems 6 & 7 identify the graph of the vector function without sketching the graph.

6. $\vec{r}(t) = \langle 3\cos(6t), -4, \sin(6t) \rangle,$
7. $\vec{r}(t) = \langle 2-t, 4+7t, -1-3t \rangle,$

For problems 8 – 10 evaluate the given limit.

8. $\lim_{t \to 1} \langle e^{t-1}, 4t, \frac{t-1}{t^2-1} \rangle.$
9. $\lim_{t \to -2} \left(\frac{1-e^{t+2}}{t^2+t-2} \vec{i} + \vec{j} + (t^2+6t)\,\vec{k} \right).$
10. $\lim_{t \to \infty} \left\langle \frac{1}{t^2}, \frac{2t^2}{1-t-t^2}, e^{-t} \right\rangle.$

For problems 11 – 13 compute the derivative of the given vector function.

11. $\vec{r}(t) = (t^3 - 1)\vec{i} + e^{2t}\,\vec{j} - \cos t\,\vec{k}.$
12. $\vec{r}(t) = \langle \ln(t^2+1), te^{-t}, 4 \rangle.$
13. $\vec{r}(t) = \left\langle \frac{t+1}{t-1}, \tan(4t), \sin^2 t \right\rangle.$

For problems 14 – 16 evaluate the given integral.

14. $\int \vec{r}(t)dt$, where $\vec{r}(t) = t^3\vec{i} - \frac{2t}{t^2+1}\ \vec{j} - \cos^2(3t)\ \vec{k}$.

15. $\int \vec{r}(t)dt$, where $\vec{r}(t) = \langle 6, 6t^2 - 4t, te^{2t}\rangle$.

16. $\int \vec{r}(t)dt$, where

$$\vec{r}(t) = \langle (1-t)\cos(t^2-2t), \cos t \sin t, \sec^2(4t)\rangle,$$

For problems 17 & 18 find the unit tangent vector for the given vector function.

17. $\vec{r}(t) = \langle t^2+1, 3-t, t^3\rangle$.

18. $\vec{r}(t) = te^{2t}\vec{i} + (2-t^2)\ \vec{j} - e^{2t}\ \vec{k}$.

For problems 19 & 20 find the tangent line to the vector function at the given point.

19. $\vec{r}(t) = \cos(4t)\vec{i} + 3\sin(4t)\ \vec{j} - t^3\ \vec{k}$ at $t = \pi$.

20. $\vec{r}(t) = \langle 7e^{2-t}, \frac{16}{t^3}, 5-t\rangle$ at $t = 2$.

21. Find the unit normal and the binormal vectors for the following vector function: $\vec{r}(t) = \langle \cos(2t), \sin(2t), 3\rangle$.