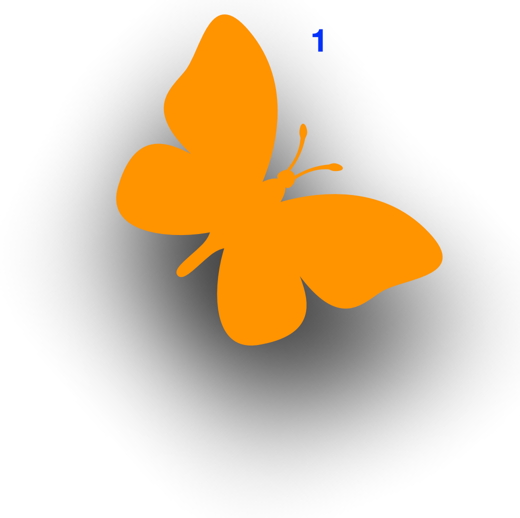
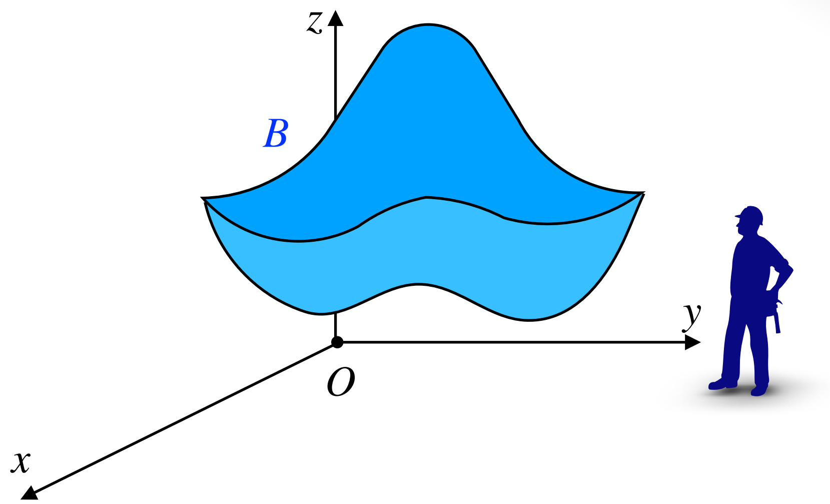


Triple Integrals

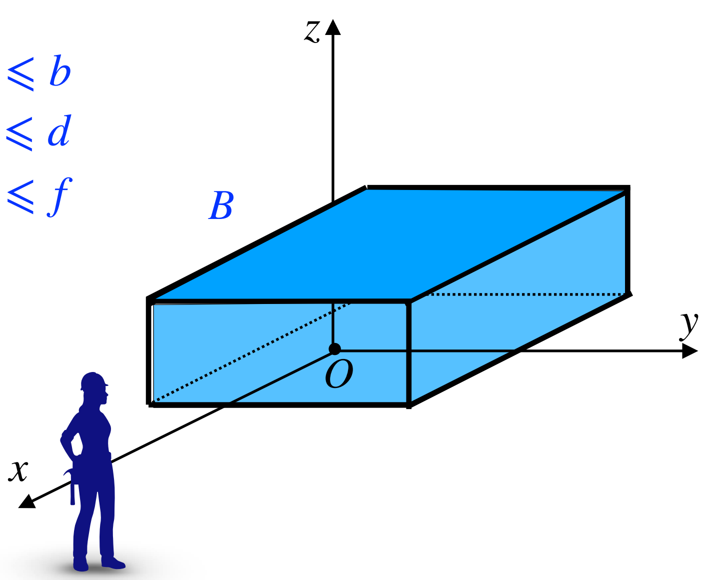
The notation for the general triple integrals is

$$\iiint_B f(x,y,z)\, dV$$



Let's start simple by integrating over the box: $B = [a,b] \times [c,d] \times [e,f]$

$$B \Longrightarrow \begin{cases} a \leqslant x \leqslant b \\ c \leqslant y \leqslant d \\ e \leqslant z \leqslant f \end{cases}$$



$$\iiint_B f(x,y,z)\, dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\, dz\, dy\, dx$$

Example 1 Evaluate the following integral:

$$\iiint_B 8xyz\, dV, \qquad B = [2,3] \times [1,2] \times [0,1]$$

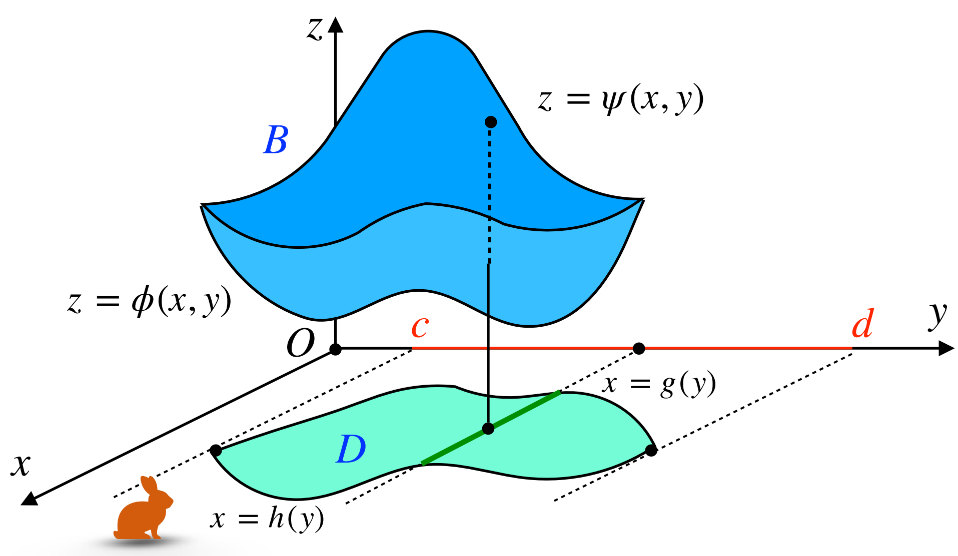
Solution. We do the integral in the following order:

$$\begin{aligned}
\iiint_B 8xyz\, dV &= \int_0^1 \int_1^2 \int_2^3 8xyz\, dx\, dy\, dz \\
&= \int_0^1 \left(\int_1^2 \left(\int_2^3 8xyz\, dx \right) dy \right) dz \\
&= \int_0^1 \left(\int_1^2 \left(4x^2yz \Big|_2^3 \right) dy \right) dz \\
&= \int_0^1 \left(\int_1^2 20yz\, dy \right) dz \\
&= \int_0^1 \left(10y^2z \Big|_1^2 \right) dz = \int_0^1 30z\, dz = 15z^2 \Big|_0^1 = 15.
\end{aligned}$$

$$\iiint_B 1 \, dV = \text{Volume of } B$$

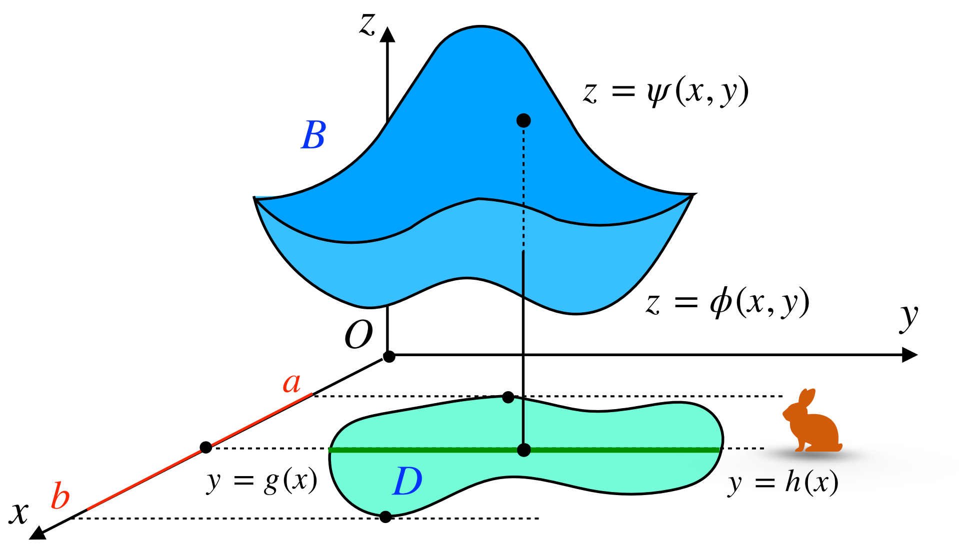
General three-dimensional regions

We have six different possibilities for a general region.



$$B \Longrightarrow \begin{cases} c \leqslant y \leqslant d \\ g(y) \leqslant x \leqslant h(y) \\ \phi(x, y) \leqslant z \leqslant \psi(x, y) \end{cases}$$

$$\iiint_B f(x, y, z) \, dV = \int_c^d \int_{g(y)}^{h(y)} \int_{\phi(x,y)}^{\psi(x,y)} f(x, y, z) \, dz \, dx \, dy$$

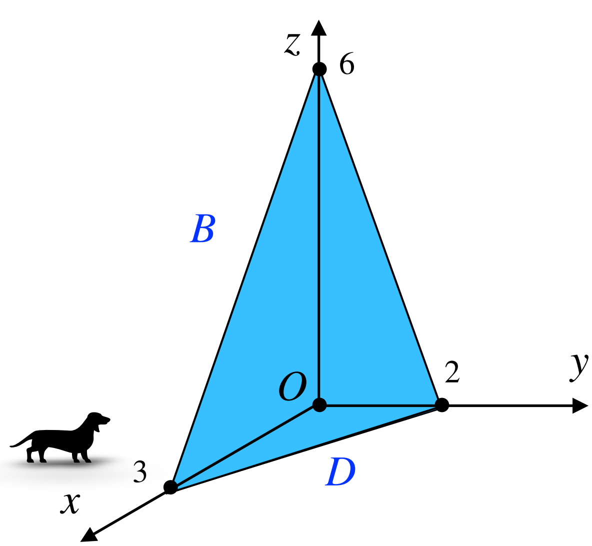


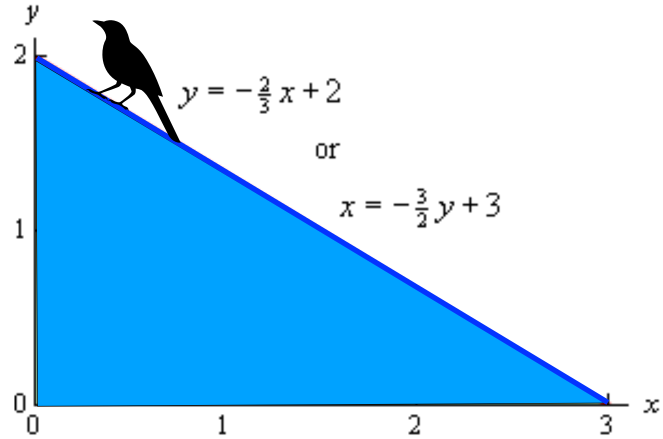
$$B \Longrightarrow \begin{cases} a \leqslant x \leqslant b \\ g(x) \leqslant y \leqslant h(x) \\ \phi(x,y) \leqslant z \leqslant \psi(x,y) \end{cases}$$

$$\iiint_B f(x,y,z)\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)\, dz\, dy\, dx$$

Example 2 Evaluate $\iiint_B 2x\, dV$ where B is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

Solution. We have





$$B \Longrightarrow \begin{cases} 0 \leqslant x \leqslant 3 \\ 0 \leqslant y \leqslant -\frac{2}{3}x + 2 \\ 0 \leqslant z \leqslant 6 - 2x - 3y \end{cases} \qquad B \Longrightarrow \begin{cases} 0 \leqslant y \leqslant 2 \\ 0 \leqslant x \leqslant -\frac{3}{2}y + 3 \\ 0 \leqslant z \leqslant 6 - 2x - 3y \end{cases}$$

$$\begin{aligned} \iiint_B 2x \, dV &= \int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{6-2x-3y} 2x \, dz \, dy \, dx \\ &= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2xz \Big|_0^{6-2x-3y} dy \, dx \\ &= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x(6 - 2x - 3y) \, dy \, dx \\ &= \int_0^3 (12xy - 4x^2y - 3y^2) \Big|_0^{-\frac{2}{3}x+2} dx \\ &= \int_0^3 (\frac{4}{3}x^3 - 8x^2 + 12x) \, dx \\ &= (\frac{1}{3}x^4 - \frac{8}{3}x^3 + 6x^2) \Big|_0^3 = 9. \end{aligned}$$

Triple Integrals In Cylindrical Coordinates

The following are the conversion formulas for cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

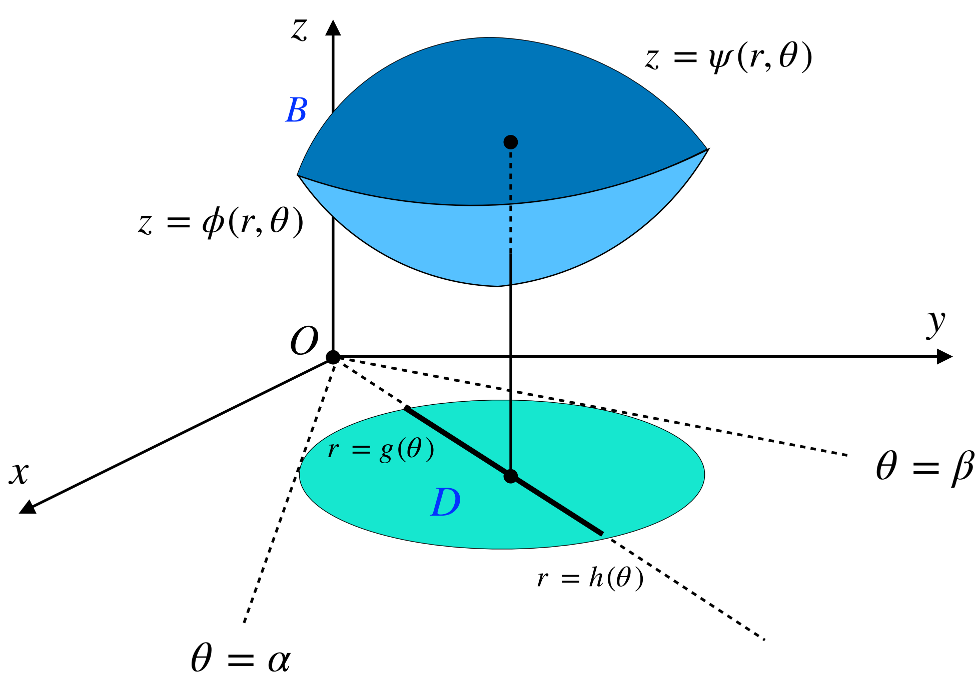
and the Jacobian is:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r\,.$$

Jacobian

So, we have

$$\iiint_B f(x,y,z)\; dV = \iiint_B f(r\cos\theta, r\sin\theta, z)\; r\; d\overline{V}$$

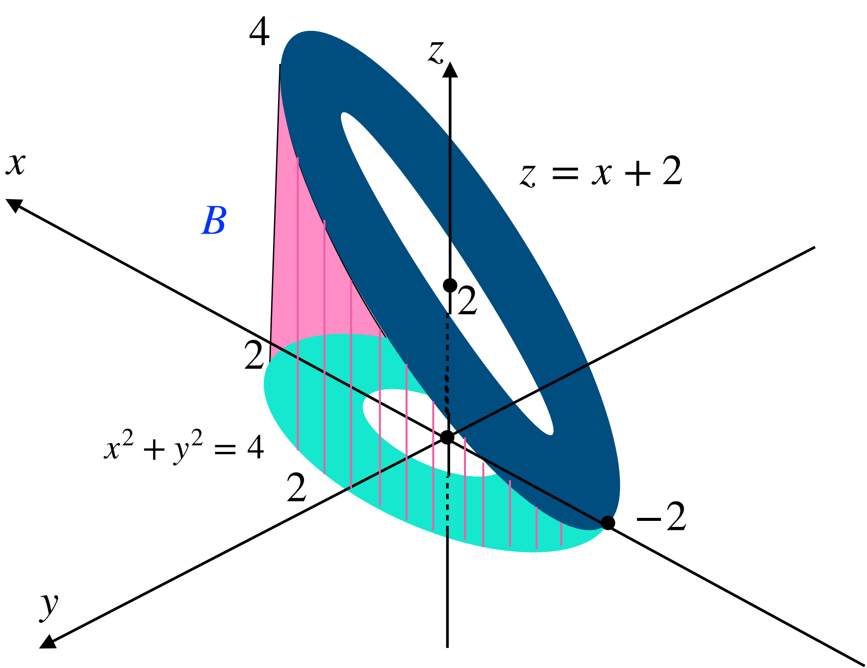


$$\iiint_B f(x,y,z)\ dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{\phi(r,\theta)}^{\psi(r,\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ dr\ d\theta$$

Example 3 Evaluate $\iiint_B y\ dV$ where B is the region under the plane $z = x + 2$ above the xy-plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution. We have

$$B \Longrightarrow \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 1 \leqslant r \leqslant 2 \\ 0 \leqslant z \leqslant r\cos\theta + 2 \end{cases}$$



$$
\begin{aligned}
\iiint_B y\, dV &= \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r\sin\theta\; r\; dz\; dr\; d\theta \\
&= \int_0^{2\pi}\int_1^2 r^2\sin\theta\;(r\cos\theta+2)\; dr\; d\theta \\
&= \int_0^{2\pi}\int_1^2 \frac{1}{2}r^3\sin 2\theta + 2r^2\sin\theta\; dr\; d\theta \\
&= \int_0^{2\pi} \frac{1}{8}r^4\sin 2\theta + \frac{2}{3}r^3\sin\theta\Big|_1^2\; d\theta \\
&= \int_0^{2\pi} \frac{15}{8}\sin 2\theta + \frac{14}{3}\sin\theta\; d\theta \\
&= \left(-\frac{15}{16}\cos 2\theta - \frac{14}{3}\cos\theta\right)\Big|_0^{2\pi} = 0.
\end{aligned}
$$

Triple Integrals In Spherical Coordinates

The following are the conversion formulas for spherical coordinates:

$$
\begin{cases}
x = \rho\sin\phi\cos\theta \\
y = \rho\sin\phi\sin\theta \\
z = \rho\cos\phi
\end{cases}
$$

and the Jacobian is:

$$
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = -\,\rho^2\sin\phi\,.
$$

Jacobian

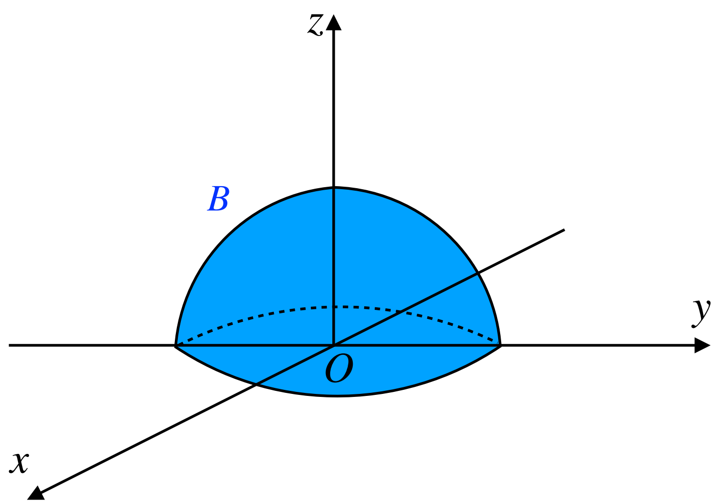
So, we have

$$\iiint_B f(x,y,z)\ dV = \iiint_B f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\ |-\rho^2 \sin\phi|\ d\overline{V}$$

$$= \iiint_B f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\ \rho^2 \sin\phi\ d\overline{V}$$

Example 4 Evaluate $\iiint_B 16z\ dV$ where B is the upper half of the sphere $x^2+y^2+z^2=1$.

Solution. We have



$$B \Longrightarrow \begin{cases} 0 \leqslant \rho \leqslant 1 \\ 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant \phi \leqslant \frac{\pi}{2} \end{cases}$$

$$\iiint_B 16z\ dV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 16\rho\cos\phi\ \rho^2 \sin\phi\ d\rho d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 8\rho^3 \sin 2\phi\ d\rho d\theta d\phi$$

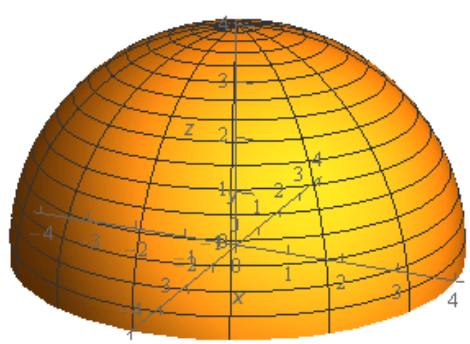
$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 2\sin 2\phi\ d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} 4\pi \sin 2\phi\ d\phi$$

$$= -2\pi\cos 2\phi \Big|_0^{\frac{\pi}{2}} = 4\pi.$$

Example 5 Evaluate $\iiint_B 10xz + 3\ dV$ where B is the region portion of $x^2 + y^2 + z^2 = 16$ with $z \geqslant 0$.

Solution. We have a quick sketch of the region B:



$$B \Longrightarrow \begin{cases} 0 \leqslant \rho \leqslant 4 \\ 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant \phi \leqslant \frac{\pi}{2} \end{cases}$$

$$\iiint_B 10xz + 3 \; dV$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^4 [10(\rho \sin\phi \cos\theta)(\rho \cos\phi) + 3] \; \rho^2 \sin\phi \; d\rho d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^4 [10\rho^4 \sin^2\phi \cos\phi \cos\theta + 3\rho^2 \sin\phi \; d\rho d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 2048 \sin^2\phi \cos\phi \cos\theta + 64 \sin\phi \; d\theta d\phi$$

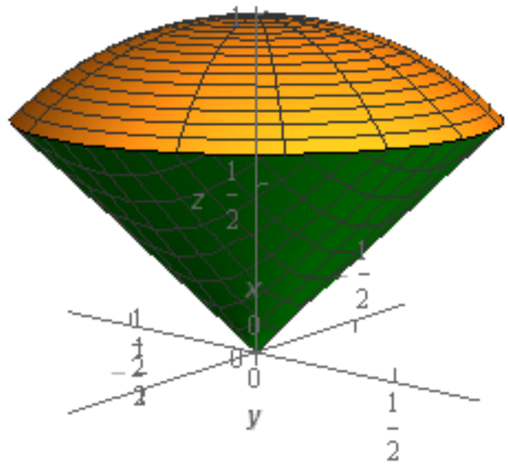
$$= \int_0^{\frac{\pi}{2}} 2048 \sin^2\phi \cos\phi \sin\theta + 64\theta \sin\phi \Big|_0^{2\pi} d\phi$$

$$= \int_0^{\frac{\pi}{2}} 128\pi \sin\phi \; d\phi$$

$$= -128\pi \cos\phi \Big|_0^{\frac{\pi}{2}} = 128\pi \, .$$

Example 6 Evaluate $\iiint_B 3z\ dV$ where B is the region below $x^2+y^2+z^2=1$ and inside $z=\sqrt{x^2+y^2}$.

Solution. We have a quick sketch of the region B:



$$z=\sqrt{x^2+y^2} \implies \rho\cos\phi=\rho\sin\phi \implies \tan\phi=1 \implies \phi=\frac{\pi}{4}.$$

$$B \implies \begin{cases} 0 \leqslant \rho \leqslant 1 \\ 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant \phi \leqslant \frac{\pi}{4} \end{cases}$$

$$\iiint_B 3z\ dV=\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 (3\rho\cos\phi)\ \rho^2\sin\phi\ d\rho d\theta d\phi$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^1 (3\rho^3 \cos\phi \sin\phi) \, d\rho d\theta d\phi$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} (\frac{3}{4}\rho^4 \cos\phi \sin\phi)\Big|_0^1 d\theta d\phi$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} (\frac{3}{4} \cos\phi \sin\phi) d\theta d\phi$$

$$= \int_0^{\frac{\pi}{4}} (\frac{3\pi}{4} \sin 2\phi) d\phi$$

$$= \left(-\frac{3\pi}{8} \cos 2\phi\right)\Big|_0^{\frac{\pi}{4}} = \frac{3}{8}\pi .$$