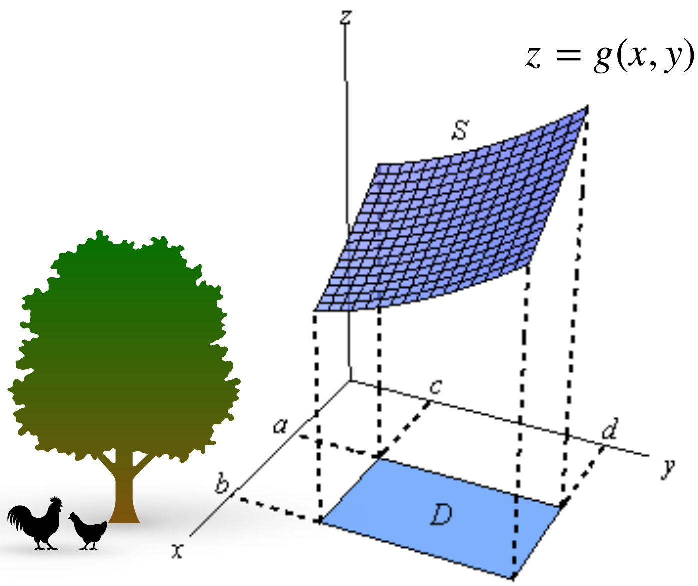
Surface Integrals



$$z = g(x,y) \implies f(x,y,z) = z - g(x,y) \implies dS = \frac{\|\nabla(f)\|}{\left|\frac{\partial f}{\partial z}\right|} dxdy$$

$$\|\nabla(f)\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2} \implies$$

$$\frac{\|\nabla(f)\|}{\left|\frac{\partial f}{\partial z}\right|} = \frac{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}}{\left|\frac{\partial f}{\partial z}\right|} = \sqrt{\frac{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}{\left(\frac{\partial f}{\partial z}\right)^2}}$$

$$\sqrt{\frac{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}{\left(\frac{\partial f}{\partial z}\right)^2}} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

$$z = g(x,y) \implies dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;\; dxdy$$

$$\iint_{(S)} f(x,y,z)\; dS = \iint_{D_{xy}} f(x,y,g(x,y))\,\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;\; dxdy$$

$$\text{Area of } S = \iint_{(S)} dS = \iint_{D_{xy}} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;\; dxdy$$

$$y = g(x,z) \implies dS = \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + 1 + \left(\frac{\partial y}{\partial z}\right)^2}\ dxdz$$

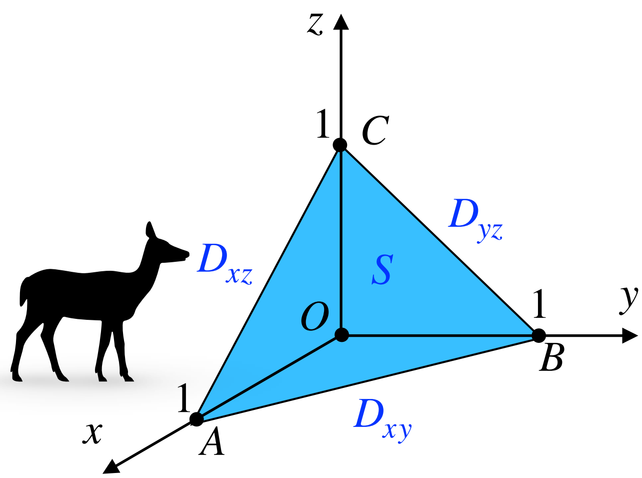
$$\iint_{(S)} f(x,y,z)\ dS = \iint_{D_{xz}} f(x,g(x,z),z)\ \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + 1 + \left(\frac{\partial y}{\partial z}\right)^2}\ dxdz$$

$$x = g(y,z) \implies dS = \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\ dydz$$

$$\iint_{(S)} f(x,y,z)\ dS = \iint_{D_{yz}} f(g(y,z),y,z)\ \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\ dydz$$

Example 1 Evaluate $\iint_{(S)} 6xy\ dS$ where S is the portion of the plane $x + y + z = 1$ that lies in the 1st octant and is in front of the yz-plane.

Solution. We have

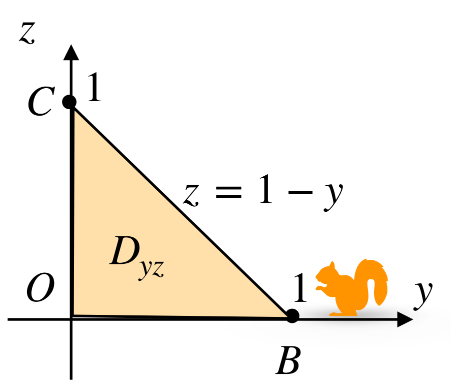


$$x = g(y, z) \quad \Longrightarrow \quad x = 1 - y - z, \ \ D_{yz} = \triangle\, BOC$$

$$y = h(x, z) \quad \Longrightarrow \quad y = 1 - x - z, \ \ D_{xz} = \triangle\, AOC$$

$$z = k(x, y) \quad \Longrightarrow \quad z = 1 - x - y, \ \ D_{xy} = \triangle\, AOB$$

Here is a sketch of the region $D_{yz} = \triangle\, BOC$:



$$D_{yz} \Longrightarrow \begin{cases} 0 \leqslant y \leqslant 1 \\ 0 \leqslant z \leqslant 1 - y \end{cases}$$

$$x = 1 - y - z \implies dS = \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\ dydz$$

$$\implies dS = \sqrt{1 + (-1)^2 + (-1)^2}\ dydz = \sqrt{3}\ dydz$$

$$\iint_{(S)} f(x,y,z)\, dS = \iint_{D_{yz}} f(1-y-z, y, z)\, \sqrt{3}\ dydz$$

$$= \sqrt{3} \int_0^1 \int_0^{1-y} 6(1-y-z)y\ dzdy$$

$$= 6\sqrt{3} \int_0^1 \int_0^{1-y} y - y^2 - yz\ dzdy$$

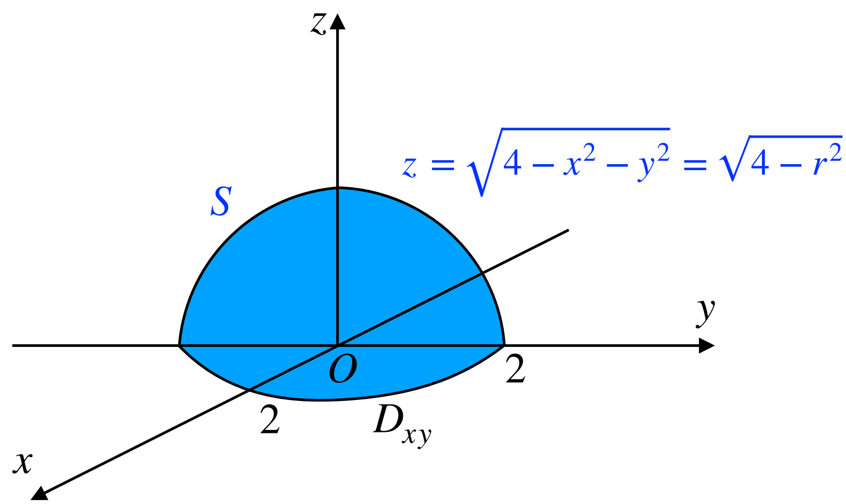
$$= 6\sqrt{3} \int_0^1 (y-y^2)z - yz^2/2 \Big|_0^{1-y} dy$$

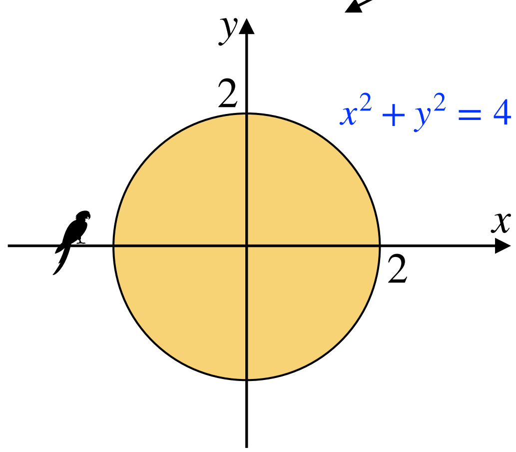
$$= 6\sqrt{3} \int_0^1 \frac{1}{2}y - y^2 + \frac{1}{2}y^3\ dy$$

$$= 6\sqrt{3} \left(\frac{1}{4}y^2 - \frac{1}{3}y^3 + \frac{1}{8}y^4\right)\Big|_0^1 = \frac{\sqrt{3}}{4}.$$

Example 2 Evaluate $\iint_{(S)} z\ dS$ where S is the upper half of a sphere of radius 2.

Solution. We have





$$D_{xy} \Longrightarrow \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant 2 \end{cases}$$

$$z = \sqrt{4 - r^2} \quad \Longrightarrow \quad dS = \sqrt{\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 + 1} \; r \, dr d\theta$$

$$\Longrightarrow \quad dS = \sqrt{\left(\frac{-r}{\sqrt{4 - r^2}}\right)^2 + 0^2 + 1} \; r \, dr d\theta$$

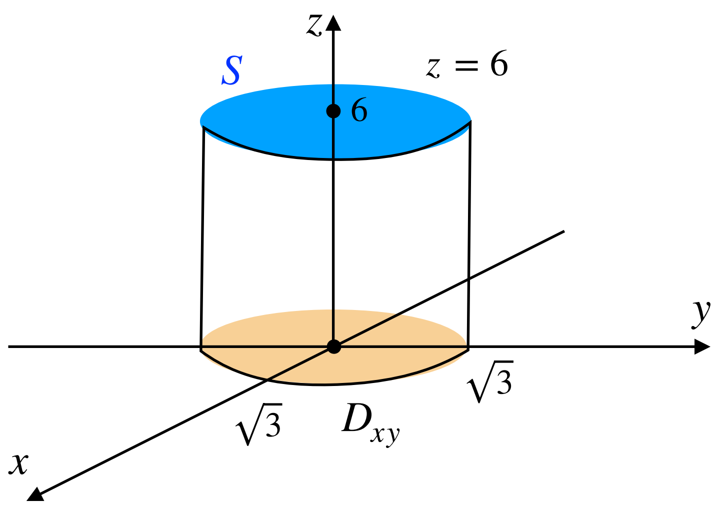
$$\Longrightarrow \quad dS = \sqrt{\frac{r^2}{4 - r^2} + 1} \; r \, dr d\theta = \frac{2}{\sqrt{4 - r^2}} \; r \, dr d\theta$$

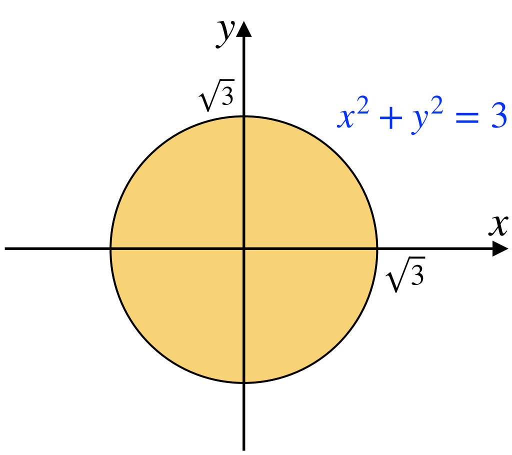
$$\iint_{(S)} f(x,y,z)\, dS = \iint_{D_{yz}} \sqrt{1-r^2}\, \frac{2}{\sqrt{1-r^2}}\, r\, dr d\theta$$

$$= \int_0^{2\pi}\int_0^2 2r\, dr\, d\theta = \int_0^{2\pi} r^2\Big|_0^2 d\theta = 4\int_0^{2\pi} d\theta = 8\pi.$$

Example 3 Evaluate $\iint_{(S)} y\, dS$ where S is the circle $x^2+y^2=3$ with $z=6$.

Solution. We have





$$D_{xy} \Longrightarrow \begin{cases} 0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant r \leqslant \sqrt{3} \end{cases}$$

$$z = 6 \quad \Longrightarrow \quad dS = \sqrt{\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 + 1}\; r\; drd\theta$$

$$\Longrightarrow \quad dS = \sqrt{0^2 + 0^2 + 1}\; r\; drd\theta = r\; drd\theta\,.$$

$$\iint_{(S)} f(x, y, z)\; dS = \iint_{D_{xy}} r \sin\theta\; r\; drd\theta$$

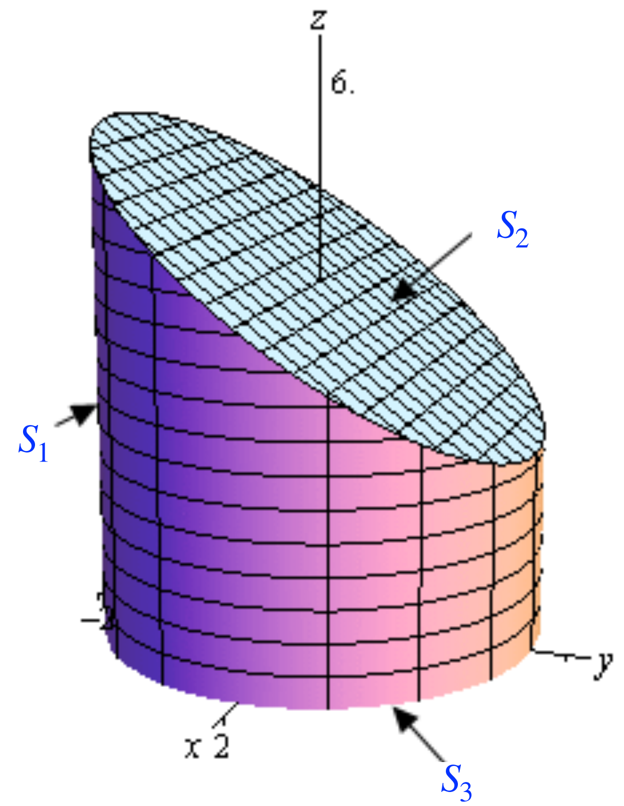
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} r^2 \sin\theta\; drd\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3} \sin\theta \Big|_0^{\sqrt{3}} d\theta$$

$$= \sqrt{3}(-\cos\theta)\Big|_0^{2\pi} = 0.$$

Example 4 Evaluate $\iint_{(S)} y + z\; dS$ where S is the surface whose side is the cylinder $x^2 + y^2 = 3$, whose bottom is the disk $x^2 + y^2 \leqslant 3$ *in the xy-plane and whose top is the plane $z = 4 - y$.*

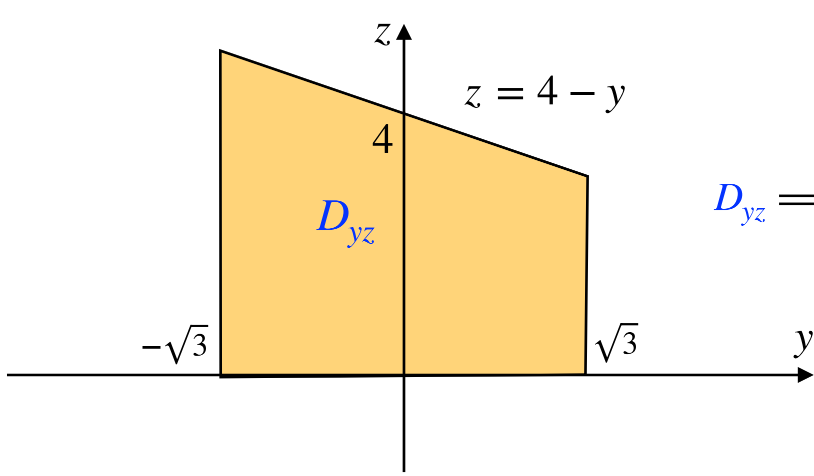
Solution. We have a sketch of the surface.



$$S = S_1 \cup S_2 \cup S_3$$

$$\iint_{(S)} y+z\ dS = \iint_{(S_1)} y+z\ dS + \iint_{(S_2)} y+z\ dS + \iint_{(S_3)} y+z\ dS$$

1. S_1 **:** *The Cylinder*



$$D_{yz} \Longrightarrow \begin{cases} -\sqrt{3} \leqslant y \leqslant \sqrt{3} \\ 0 \leqslant z \leqslant 4 - y \end{cases}$$

$$x^2 + y^2 = 3 \implies x = \pm\sqrt{3 - y^2} \implies$$

$$dS = \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\ dy\ dz = \sqrt{1 + \left(\frac{-y}{\sqrt{3-y^2}}\right)^2 + 0^2}\ dy\ dz$$

$$= \sqrt{1 + \frac{y^2}{3-y^2}}\ dy\ dz = \frac{\sqrt{3}}{\sqrt{3-y^2}}\ dy\ dz$$

$$\iint_{(S_1)} y + z\ dS = 2\iint_{D_{yz}} y + z\ \frac{\sqrt{3}}{\sqrt{3-y^2}}\ dy\ dz = 2\sqrt{3}\int_{-\sqrt{3}}^{\sqrt{3}}\int_0^{4-y} \frac{y+z}{\sqrt{3-y^2}}\ dz\ dy$$

$$= 2\sqrt{3}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{yz + z^2/2}{\sqrt{3-y^2}}\Big|_0^{4-y}\ dy = 2\sqrt{3}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{y(4-y) + (4-y)^2/2}{\sqrt{3-y^2}}\ dy$$

$$= 2\sqrt{3}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-\frac{1}{2}y^2 + 8}{\sqrt{3-y^2}}\ dy = 2\sqrt{3}\int_0^{\sqrt{3}} \frac{-y^2 + 16}{\sqrt{3-y^2}}\ dy$$

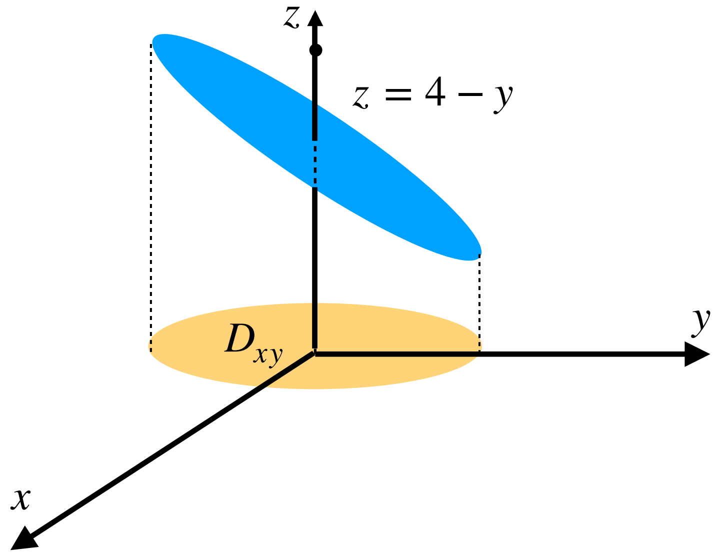
$$y = \sqrt{3}\sin\theta \implies \begin{cases} y = 0 \to \theta = 0 \\ y = \sqrt{3} \to \theta = \frac{\pi}{2} \end{cases} \quad dy = \sqrt{3}\cos\theta\ d\theta$$

$$= 2\sqrt{3}\int_0^{\frac{\pi}{2}} \frac{3\cos^2\theta + 13}{\sqrt{3}\cos\theta}\ \sqrt{3}\cos\theta\ d\theta = 2\sqrt{3}\int_0^{\frac{\pi}{2}} 3\cos^2\theta + 13\ d\theta$$

$$= 2\sqrt{3}\int_0^{\frac{\pi}{2}} 3\frac{1+\cos 2\theta}{2} + 13\ d\theta = 2\sqrt{3}\left(\frac{3}{2}\theta + \frac{3}{4}\sin 2\theta + 13\theta\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 2\sqrt{3}\left(\frac{3\pi}{4} + \frac{13\pi}{2}\right) = \sqrt{3}\frac{29\pi}{2}.$$

2. S_2 *: Plane on Top of the Cylinder*

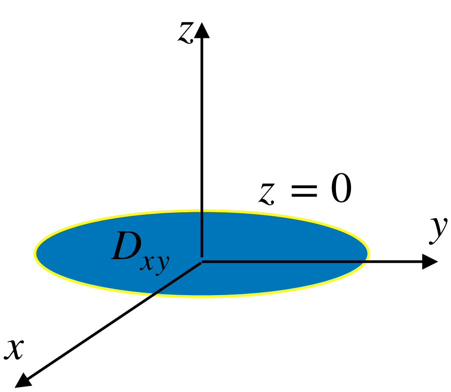


$$z = 4 - y \quad \Longrightarrow \quad dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dx\ dy = \sqrt{2}dx\ dy$$

$$\iint_{(S_2)} y + z\ dS = \iint_{D_{xy}} y + 4 - y\sqrt{2}\ dx\ dy = 4\sqrt{2}\iint_{D_{xy}} dxdy = 4\sqrt{2}(3\pi) = 12\sqrt{2}\pi\,.$$

3. S_3 *: Bottom of the Cylinder*

The equation for the bottom is given by $z = 0$ and D_{xy} is the disk of radius $\sqrt{3}$ centered at the origin.



$$z = 0 \implies dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dx\, dy = dx\, dy$$

$$\iint_{(S_3)} y + z\; dS = \iint_{D_{xy}} y + 0\; dx\, dy = \int_0^{2\pi}\int_0^{\sqrt{3}} r\sin\theta\; r\, dr d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\sin\theta \Big|_0^{\sqrt{3}} d\theta = \int_0^{2\pi} \sqrt{3}\sin\theta\; d\theta$$

$$= \sqrt{3}\int_0^{2\pi} \sin\theta\; d\theta = -\sqrt{3}\cos\theta\Big|_0^{2\pi} = 0.$$

Therefore, we obtain

$$\iint_{(S)} y + z\; dS = \iint_{(S_1)} y + z\; dS + \iint_{(S_2)} y + z\; dS + \iint_{(S_3)} y + z\; dS$$

$$= \sqrt{3}\frac{29\pi}{2} + 12\sqrt{2}\pi + 0 = \frac{\pi}{2}(29\sqrt{3} + 24\sqrt{2}).$$

Practice Problems

1. Evaluate $\iint_{(S)} z + 3y - x^2\; dS$ where S is the portion of $z = 2 - 3y + x^2$ that lies over the triangle in the xy- in the xy-plane with vertices $(0,0)$, $(2,0)$ and $(2,-4)$.

2. Evaluate $\displaystyle\iint_{(S)} 40y \ dS$ where S is the portion of $y = 3x^2 + 3z^2$ that lies behind $y = 6$.

3. Evaluate $\displaystyle\iint_{(S)} 2y \ dS$ where S is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$.

4. Evaluate $\displaystyle\iint_{(S)} xz \ dS$ where S is the portion of the sphere of radius 3 with $x \leqslant 0$, $y \geqslant 0$ and $z \geqslant 0$.

5. Evaluate $\displaystyle\iint_{(S)} yz + 4xy \ dS$ where S is the surface of the solid bounded by $4x + 2y + z = 8, z = 0, y = 0$ and $x = 0$. Note that all four surfaces of this solid are included in S.

6. Evaluate $\displaystyle\iint_{(S)} x - z \ dS$ where S is the surface of the solid bounded by $x^2 + y^2 = 4, z = x - 3$, and $z = x + 2$. Note that all three surfaces of this solid are included in S.