Limits, Continuity and Directional Derivatives

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Limit and Continuous Functions

Suppose $f(x,y)$ is a function. We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

if for every $\epsilon > 0$ there is a $\delta > 0$, so that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \implies |f(x,y) - L| < \epsilon.$$

A function $f(x,y)$ is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

- *Polynomials* are continuous everywhere.
- *Rational functions* are continuous everywhere they are defined.

Directional Derivative

Directional Derivative of $f(x, y)$ at (a, b) in the Direction of a Unit Vector $\vec{\mathbf{u}}$

If $\vec{\mathbf{u}} = u_1\vec{\mathbf{i}} + u_2\vec{\mathbf{j}}$ is a unit vector, we define the direction derivative $D_{\vec{\mathbf{u}}}f$ at the point (a, b) by

$$D_{\vec{\mathbf{u}}}f(a, b) = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided that the limit exists.

- If $\vec{\mathbf{u}} = \vec{\mathbf{i}}$, then $u_1 = 1$ and $u_2 = 0$, and we have

$$D_{\vec{\mathbf{i}}}f(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h} = f_x(a, b).$$

- If $\vec{\mathbf{u}} = \vec{\mathbf{j}}$, then $u_1 = 0$ and $u_2 = 1$, and we have

$$D_{\vec{\mathbf{j}}}f(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h} = f_y(a, b).$$

The Gradient Vector of $f(x, y)$

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The gradient vector of a differentiable function $f(x, y)$ at the point (a, b) is

$$\nabla f(a, b) = f_x(a, b)\vec{\mathbf{i}} + f_y(a, b)\vec{\mathbf{j}}.$$

Using this notation, we can define the directional derivative in terms of a unit vector in the desired direction and the gradient.

Directional Derivative of $f(x, y)$ at (a, b) in terms of the Gradient Vector

If $\vec{\mathbf{u}} = u_1\vec{\mathbf{i}} + u_2\vec{\mathbf{j}}$ is a unit vector and $f(x, y)$ is differentiable at the point (a, b), then we have that

$$D_{\vec{\mathbf{u}}} f(a, b) = \nabla f(a, b) \cdot \vec{\mathbf{u}} = f_x(a, b)u_1 + f_y(a, b)u_2.$$

Implications

We have another formula for the directional derivative, namely

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(a,b) &= \nabla f(a,b) \cdot \vec{\mathbf{u}} = ||\nabla f(a,b)|| \cdot ||\vec{\mathbf{u}}|| \cdot \cos\theta, \\ &= ||\nabla f(a,b)|| \cdot \cos\theta \end{aligned}$$

where θ is the angle between the gradient and the direction vector $\vec{\mathbf{u}}$ and we used the fact that the length of a unit vector is 1, i.e. $||\vec{\mathbf{u}}|| = 1$. Since $\cos\theta$ is always between -1 and $+1$:

- $\theta = 0$: The direction of maximum rate of increase. The rate of increase per unit distance is $||\nabla f(a,b)||$.
- $\theta = \pi$: The direction of minimum rate of increase. The rate of increase per unit distance is $-||\nabla f(a,b)||$.
- $\theta = \frac{\pi}{2}, \ \frac{3\pi}{2}$: The directions giving zero rate of increase.