# 4. Quadric Surfaces

### Section 1-1 : Summary

## 1. Vectors

1. $\overrightarrow{v} = \langle a_1, a_2, \ldots, a_n \rangle$, $\overrightarrow{w} = \langle b_1, b_2, \ldots, b_n \rangle$,

$$\overrightarrow{v} = \overrightarrow{w} \iff a_1 = b_1,\ a_2 = b_2,\ \ldots,\ a_n = b_n\,.$$

2. Given the two points

$$A = (a_1, a_2, a_3), \quad B = (b_1, b_2, b_3)$$

*the vector with the representation* $\overrightarrow{AB}$ is

$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle.$$

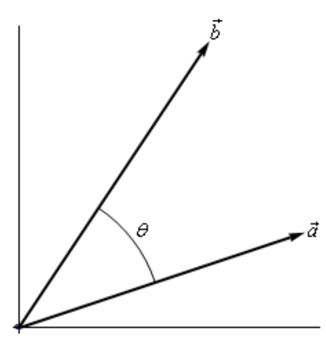
3. **Dot Product (Inner product)**. Given the vectors

$$\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle, \quad \overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$$

the dot product is,

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_1 b_1 + a_2 b_2 + a_3 b_3\,.$$

4. Suppose that $\theta$ is the angle between $\overrightarrow{a}, \overrightarrow{b}$ such that $0 \leqslant \theta \leqslant \pi$ as shown in the image below. Then




$$\overrightarrow{a} \cdot \overrightarrow{b} = \|\overrightarrow{a}\|\ \|\overrightarrow{b}\|\ \cos\theta\,.$$

5. $\overrightarrow{a} \perp \overrightarrow{b} \iff \overrightarrow{a} \cdot \overrightarrow{b} = 0.$

6. $\overrightarrow{a} \| \overrightarrow{b} \iff \overrightarrow{a} \cdot \overrightarrow{b} = \| \overrightarrow{a} \| \, \| \overrightarrow{b} \|$ or $\overrightarrow{a} \cdot \overrightarrow{b} = -\| \overrightarrow{a} \| \, \| \overrightarrow{b} \|$

7. $\operatorname{proj}_{\overrightarrow{a}} \overrightarrow{b} = \dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{\| \overrightarrow{a} \|^2} \overrightarrow{a}$ and $\operatorname{proj}_{\overrightarrow{b}} \overrightarrow{a} = \dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{\| \overrightarrow{b} \|^2} \overrightarrow{b}$

8. **Cross Product**. Given the vectors

$$\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle, \quad \overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$$

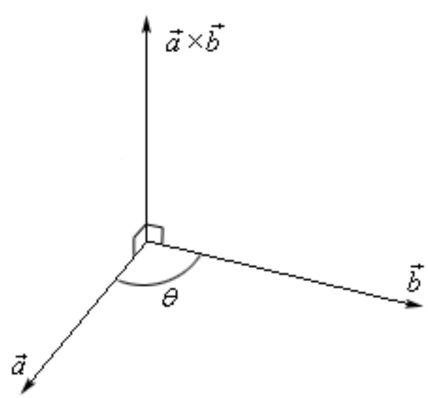
the cross product is given by the formula

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

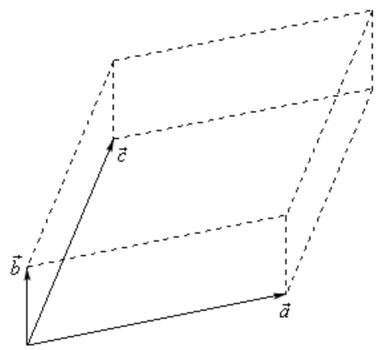
9. $\overrightarrow{u} \| \overrightarrow{v} \iff \overrightarrow{u} \times \overrightarrow{v} = \overrightarrow{0}$

10. Suppose that $\theta$ is the angle between $\overrightarrow{a}, \overrightarrow{b}$ such that $0 \leqslant \theta \leqslant \pi$ as shown in the image below. Then

$$\| \overrightarrow{a} \times \overrightarrow{b} \| = \| \overrightarrow{a} \| \, \| \overrightarrow{b} \| \ \sin \theta .$$



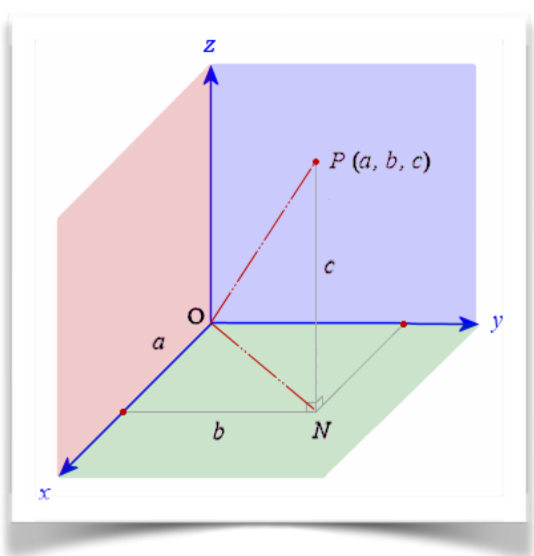
11. if $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle$, $\overrightarrow{b} = \langle b_1, b_2, b_3 \rangle$ and $\overrightarrow{c} = \langle c_1, c_2, c_3 \rangle$, then




$$\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

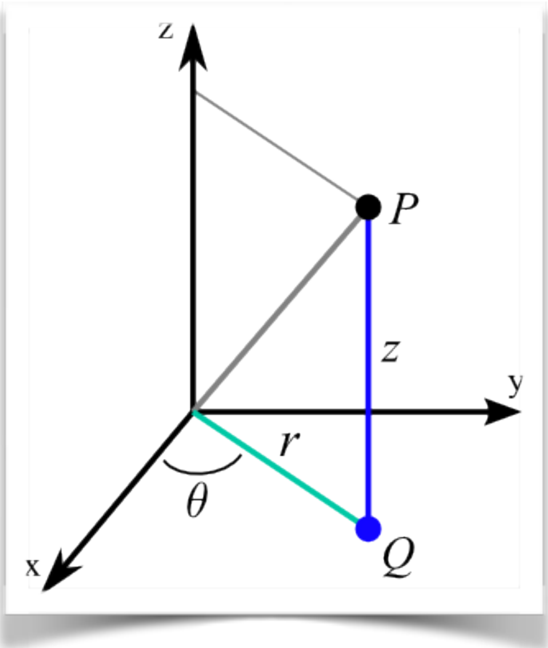
## 2. Three Dimensional Space

**Cartesian coordinates:**




$P = (a, b, c)$ *Cartesian coordinates*.

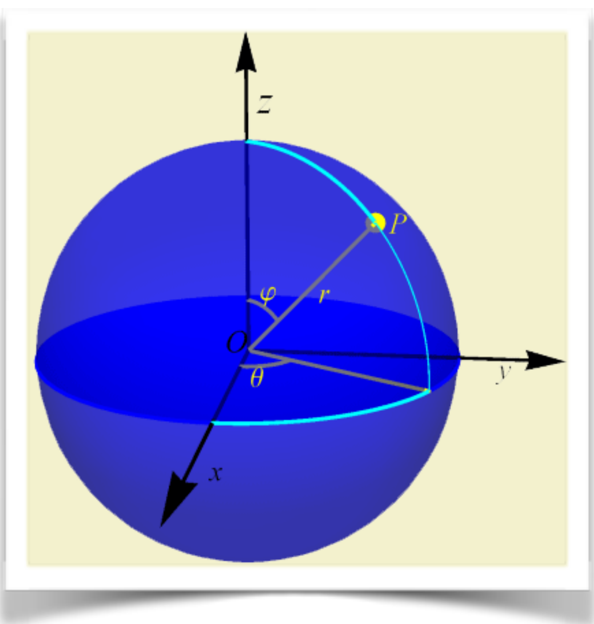
### Cylindrical Coordinates:




$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

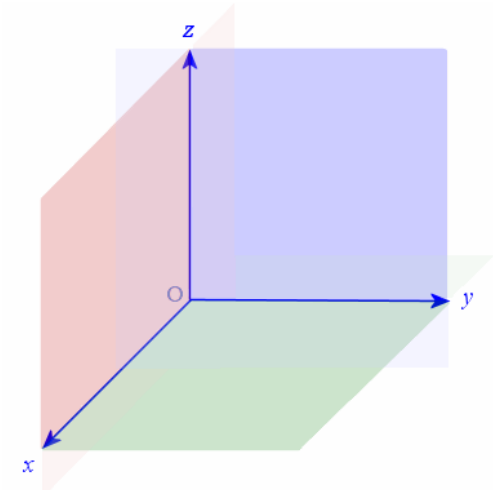
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

### Spherical Coordinates:




$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\phi$$

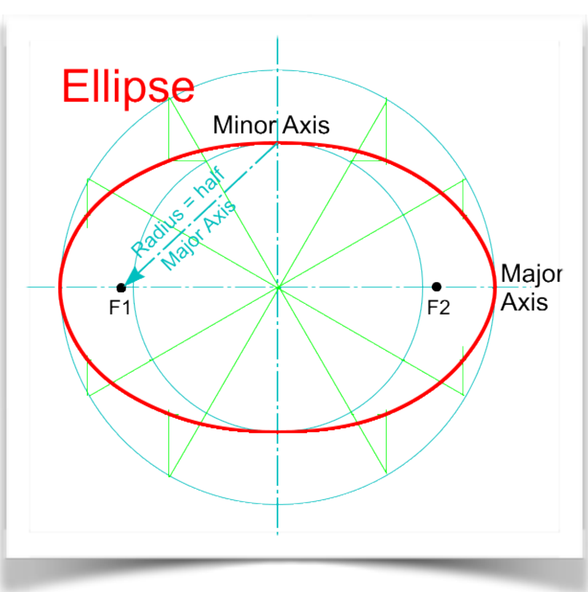
$$r^2 = x^2 + y^2 + z^2\,.$$

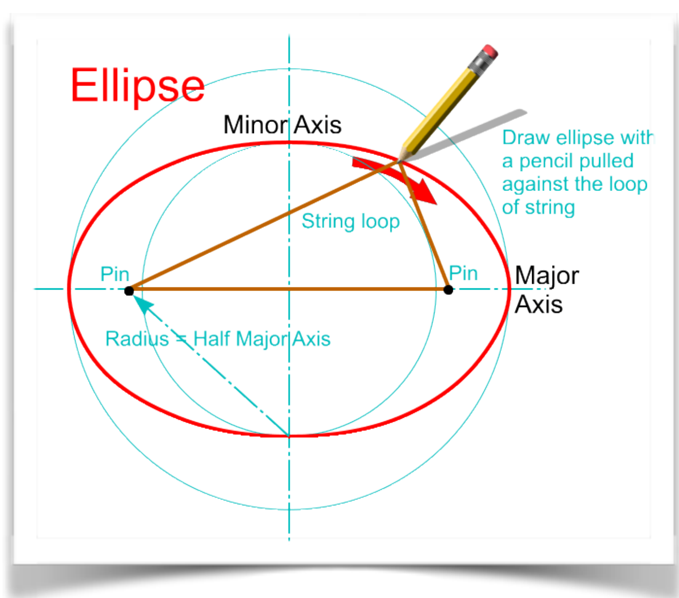



## Section 1-2 : Quadric Surfaces

### Ellipse

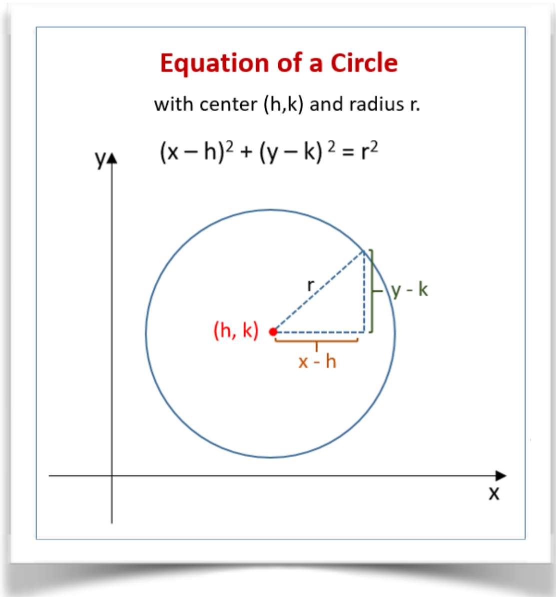
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \Longrightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$



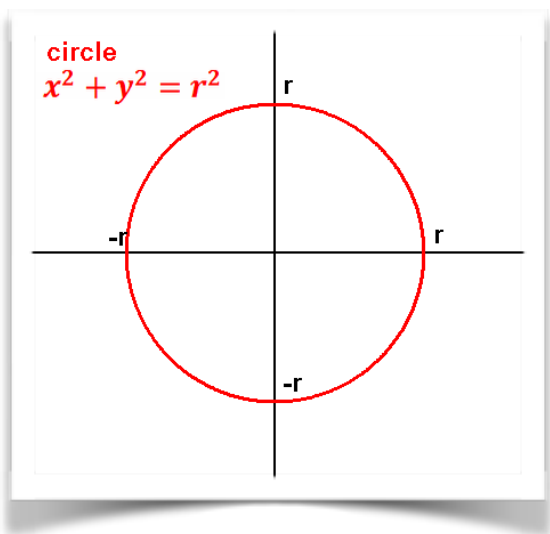



## Circle

$$a = b = r \implies \frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1 \implies (x-h)^2 + (y-k)^2 = r^2,$$




$$h = k = 0 \implies x^2 + y^2 = r^2.$$

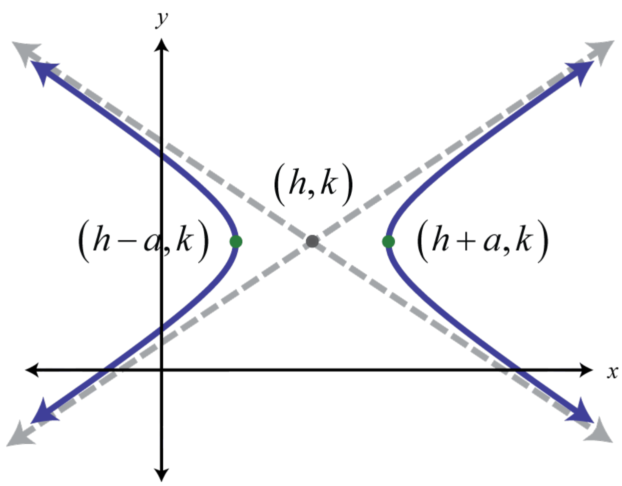


## Hyperbola

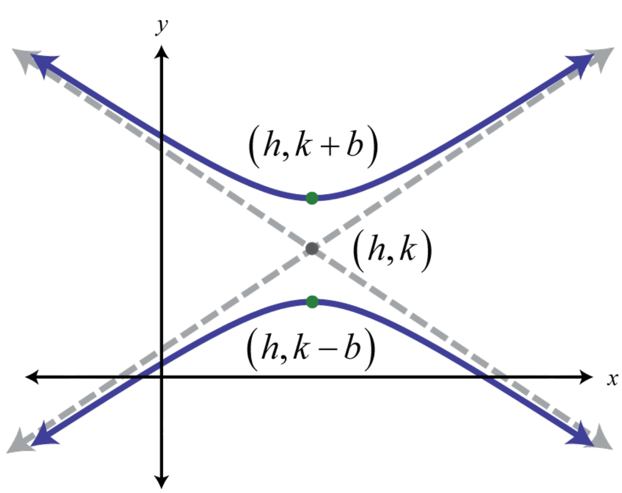
$$+\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad + \quad \Longrightarrow \quad -$$

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad +\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad -\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$




$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad 1 \implies 0,$$

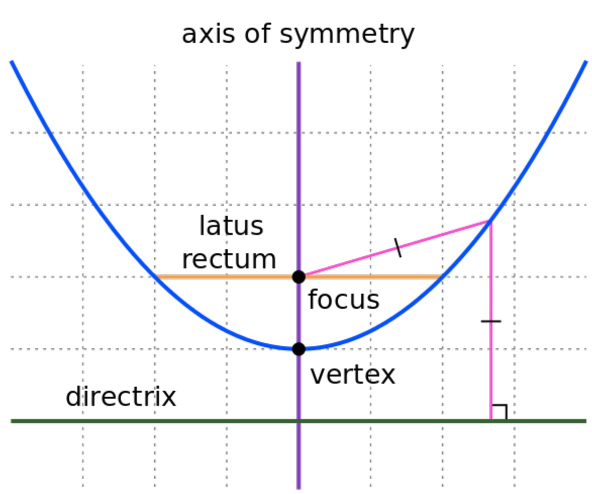
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0, \implies x = y = 0.$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad 1,+ \implies 0,-,$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \implies \left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0.$$ *Two lines*

## Parabola

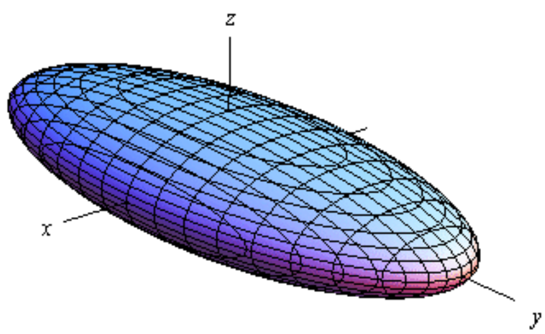
$$\frac{x^2}{a^2} + \frac{y}{b} = 0, \qquad \frac{x}{a} + \frac{y^2}{b^2} = 0.$$

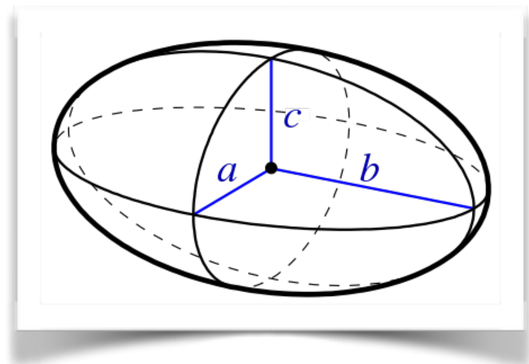


## Ellipsoid

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1, \quad \Longrightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

Here is a sketch of a typical ellipsoid.




## Hyperboloid of one sheet

$$+\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad + \quad \Longrightarrow \quad -$$

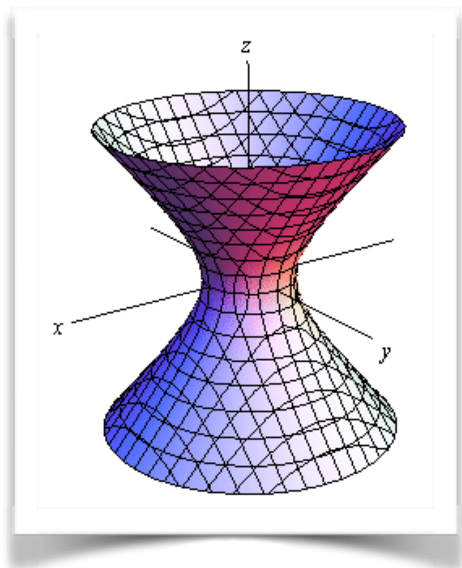
$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$+\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$+\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

Here is a sketch of a typical hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$




## Hyperboloid of two sheets

$$+\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad +,+ \quad \Longrightarrow \quad -,-$$

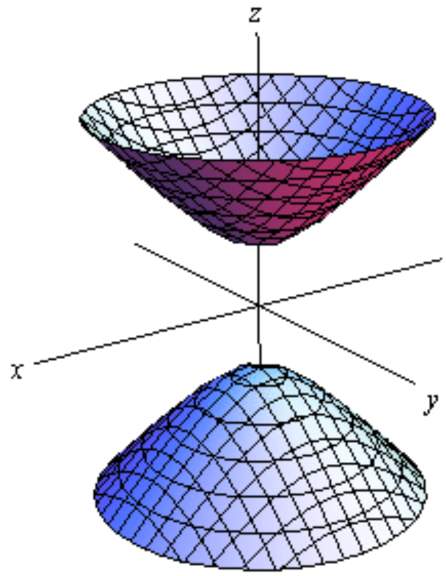
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

$$+\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

Here is a sketch of a typical hyperboloid of two sheets:

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$




$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0, \qquad 1 \implies 0,$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0, \implies x=y=z=0.$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0, \qquad 1,+ \implies 0,-, \qquad \textit{Cone}$$

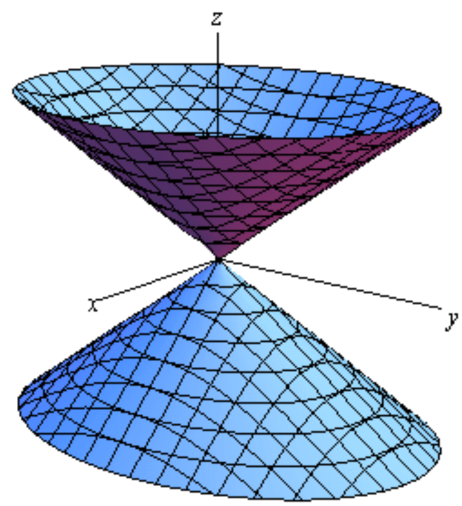
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$$

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$$

Here is a sketch of a typical cone:

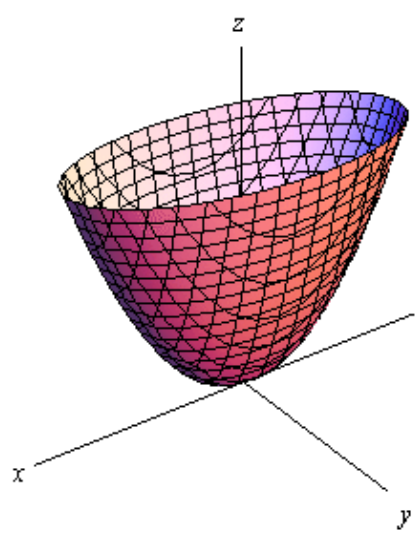
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$$




## Elliptic Paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0,$$

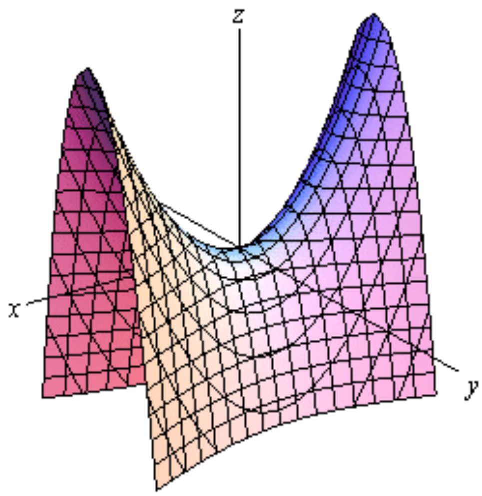
Here is a sketch of a typical elliptic paraboloid $c>0.$



## Hyperbolic Paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0,$$

Here is a sketch of a typical hyperbolic paraboloid.




## Cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Here is a sketch of typical cylinder with an ellipse cross section.