Problem 1. *Find the solution to the recurrence relation*

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}, \quad n \geqslant 3,$$

with initial conditions $a_0 = 1$, $a_1 = -2$ *and* $a_2 = -1$.

Solution. We have $1, \ -2, \ -1, 8, \ \ldots$

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} \Rightarrow x^n = -3x^{n-1} - 3x^{n-2} - x^{n-3},$$

$$x^3 + 3x^2 + 3x + 1 = 0 \Rightarrow (x+1)^3 = 0 \Rightarrow x = -1$$

$$b_0, b_1, b_2 \Rightarrow a_n = (b_0 + b_1 n + b_2 n^2)(-1)^n$$

$$\boxed{\begin{array}{ll} n = 0 \implies & \left\{\begin{array}{l} 1 = a_0 = b_0, \\ -2 = a_1 = -(b_0 + b_1 + b_2), \\ -1 = a_2 = b_0 + 2b_1 + 4b_2 . \end{array}\right. \\ n = 1 \implies & \\ n = 2 \implies & \end{array}} \Rightarrow b_0 = 1, \ b_1 = 3, \ b_2 = -2$$

$$a_n = (1 + 3n - 2n^2)(-1)^n \Rightarrow 1, \ -2, \ -1, 8, \ \ldots \quad \square$$

Problem 2. *Solve the recurrence relation*

$$a_n = 6a_{n-1} - 9a_{n-2}, \quad n \geqslant 2$$

with initial conditions $a_0 = 1, a_1 = 6$.

Solution. We have

$$\begin{cases} a_0 = 1, \quad a_1 = 6, \\ a_n = 6a_{n-1} - 9a_{n-2}, \quad n \geqslant 2 \end{cases}$$

1,6,27,108,405,…

$$a_n = 6a_{n-1} - 9a_{n-2} \Rightarrow x^n = 6x^{n-1} - 9x^{n-2}$$

$$\Rightarrow x^2 - 6x + 9 = 0$$

$$\Rightarrow (x-3)^2 = 0 \Longrightarrow x = 3$$

$$b_0, b_1 \Rightarrow a_n = (b_0 + b_1 n)3^n$$

$$\begin{array}{ll} n = 0 \Longrightarrow & \\ n = 1 \Longrightarrow & \end{array} \begin{cases} 1 = a_0 = b_0, \\ 6 = a_1 = (b_0 + b_1)3. \end{cases}$$

$$\Rightarrow b_0 = 1, \quad b_1 = 1.$$

$$\Rightarrow a_n = (1+n)3^n,$$

Question: *How can you double check this answer is right?*

$$\begin{array}{lll} n = 0 \Longrightarrow a_0 = (1+0)3^0 = 1, & & n = 3 \Longrightarrow a_3 = (1+3)3^3 = 108, \\ n = 1 \Longrightarrow a_1 = (1+1)3^1 = 6, & & n = 4 \Longrightarrow a_4 = (1+4)3^4 = 405, \\ n = 2 \Longrightarrow a_2 = (1+2)3^2 = 27, & & n = 5 \Longrightarrow a_5 = (1+5)3^5 = 1458, \end{array}$$

Problem 3. *Solve the recurrence relation*

$$a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3},\ n \geqslant 3,$$

with initial conditions $a_0 = 2$, $a_1 = 5$, $a_2 = 13$.

Solution. We have $2, 5, 13, 35, 97, \ \ldots$

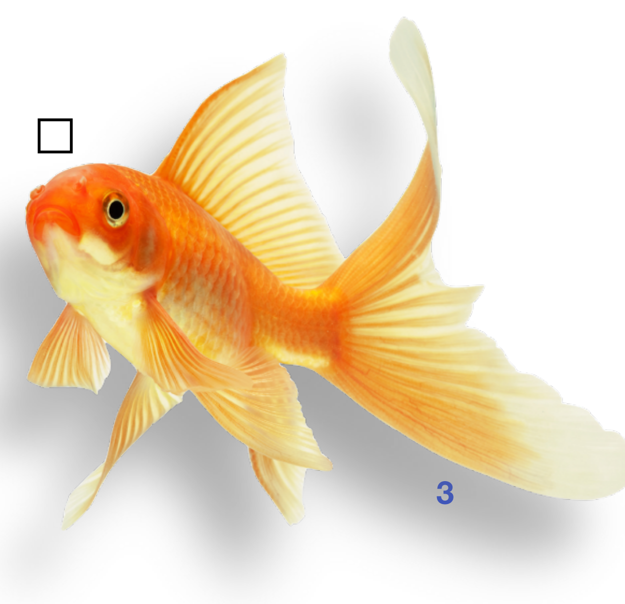
$$a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3}, \Rightarrow x^n = 3x^{n-1} + 4x^{n-2} - 12x^{n-3},$$

$$x^3 - 3x^2 - 4x + 12 = 0 \Rightarrow x = 2, \ -2, 3 \Rightarrow$$

$$\lambda_0, \lambda_1, \lambda_2 \Rightarrow a_n = \lambda_0 2^n + \lambda_1(-2)^n + \lambda_2 3^n$$

$$\boxed{\begin{array}{lll} n=0 & \Longrightarrow & \\ n=1 & \Longrightarrow & \\ n=2 & \Longrightarrow & \end{array} \begin{cases} 2 = a_0 = \lambda_0 + \lambda_1 + \lambda_2, \\ 5 = a_1 = 2\lambda_0 - 2\lambda_1 + 3\lambda_2, \\ 13 = a_2 = 4\lambda_0 + 4\lambda_1 + 9\lambda_2. \end{cases}} \Rightarrow \lambda_0 = 1,\ \lambda_1 = 0,\ \lambda_2 = 1$$

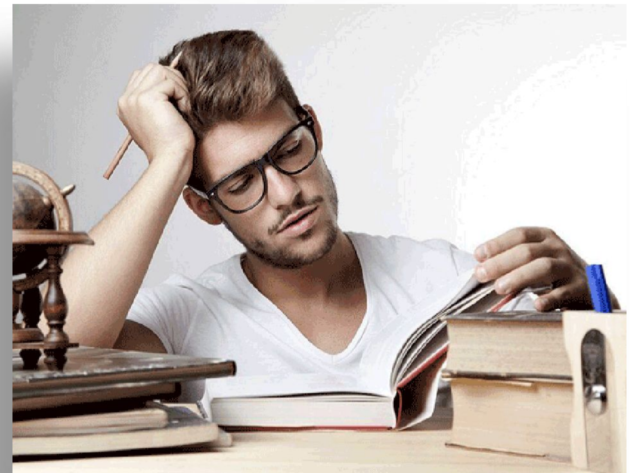
$$a_n = 2^n + 3^n \Rightarrow 2, 5, 13, 35, 97, \ \ldots \ \square$$



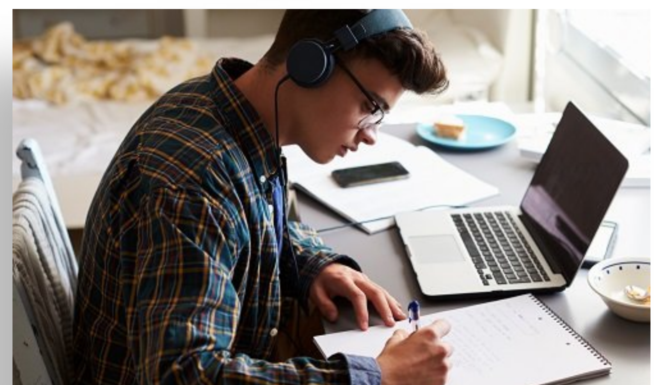
Problem 4. *Solve the recurrence relation*

$$a_{n+3} = 4a_{n+2} - 5a_{n+1} + 2a_n, \quad n \geqslant 3$$

with initial conditions $a_0 = 2, a_1 = 4, a_2 = 7.$



Solution. We have $a_n = 1 + n + 2^n$ → 2, 4, 7, 12, 21, … □

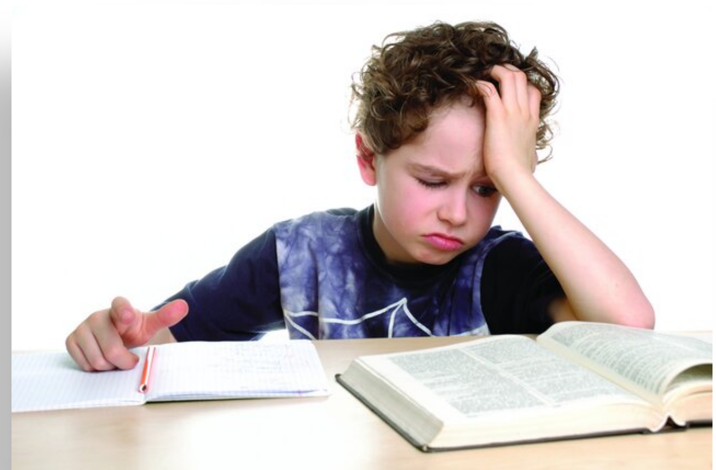


Problem 5. *Solve the recurrence relation*

$$a_{n+2} = 3a_{n+1} - 2a_n, \quad n \geqslant 0$$

with initial conditions $a_0 = 1, a_1 = 2.$

Solution. We have $a_n = 2^n$ → 1, 2, 4, 8, 16, … □



Problem 6. *Solve the recurrence relation*

$$a_n = 2(a_{n-1} - a_{n-2}), \quad n \geqslant 2$$

with initial conditions $a_0 = 1$, $a_1 = 2$.

Solution. We have

$$a_n = 2(a_{n-1} - a_{n-2}) \Longrightarrow x^2 - 2x + 2 = 0 \Longrightarrow x = 1 \pm i$$

$$\lambda_0, \lambda_1 \Longrightarrow a_n = \lambda_0(1+i)^n + \lambda_1(1-i)^n$$

$$\begin{cases} 1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)) \\ 1 - i = \sqrt{2}(\cos(\pi/4) - i\sin(\pi/4)) \end{cases}$$

$$\begin{aligned} a_n &= \lambda_0\left[\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))\right]^n + \lambda_1\left[\sqrt{2}(\cos(\pi/4) - i\sin(\pi/4))\right]^n \\ &= \lambda_0\left[\sqrt{2}^n(\cos(n\pi/4) + i\sin(n\pi/4))\right] + \lambda_1\left[\sqrt{2}^n(\cos(n\pi/4) - i\sin(n\pi/4))\right] \\ &= \sqrt{2}^n\left[(\lambda_0 + \lambda_1)\cos(n\pi/4) + (\lambda_0 - i\lambda_1)\sin(n\pi/4))\right] \\ &= \sqrt{2}^n\left[k_0\cos(n\pi/4) + k_1\sin(n\pi/4))\right] \end{aligned}$$

$$\boxed{\begin{array}{ll} n = 0 \Longrightarrow & \left\{\begin{array}{l} 1 = a_0 = [k_0\cos(0) + k_1\sin(0))] = k_0, \\ 2 = a_1 = \sqrt{2}\,[k_0\cos(\pi/4) + k_1\sin(\pi/4))] = 1 + k_1\,. \end{array}\right. \\ n = 1 \Longrightarrow & \end{array}}$$

$$a_n = \sqrt{2}^n\left[\cos(n\pi/4) + \sin(n\pi/4))\right], \quad n \geqslant 0. \ \square$$

Problem 7. Solve the given recurrence relation for the initial conditions given.

1. $a_n = 6a_{n-1} - 8a_{n-2}$; $a_0 = 1$, $a_1 = 0$.
2. $2a_n = 7a_{n-1} - 3a_{n-2}$; $a_0 = a_1 = 1$.
3. $a_n = -8a_{n-1} - 16a_{n-2}$; $a_0 = 2$, $a_1 = -20$.

Solution. We have

1. $a_n = 2^{n+1} - 4^n$.
2. $a_n = (2^{2-n} + 3^n)/5$.
3. $a_n = 2(-4)^n + 3n(-4)^n$.