Chapter 1

Implicit Differentiation and Inverses

1.1 Implicit Differentiation

We know:

$$\frac{d}{dx}x^a = ax^{a-1} \quad | \quad a \in \mathbb{Z}$$

We will now extend this formula to cover $\mathbb{Q}$ as well:

$$a = \frac{m}{n} \to y = x^{\frac{m}{n}} \quad | \quad m, n \in \mathbb{Z}$$

We can start computing the derivative using the chain rule:

$$\begin{aligned} y^n &= x^n \\ \frac{d}{dx}\left(y^n\right) &= \frac{d}{dx}\left(x^m\right) \\ \frac{d}{dy}\left(y^n\right) \cdot \frac{dy}{dx} &= mx^{m-1} \\ \frac{dy}{dx}\left(ny^{n-1}\right) &= mx^{m-1} \end{aligned}$$

We finally have an expression for y':

$$\begin{aligned}
\frac{dy}{dx} &= \frac{mx^{m-1}}{ny^{n-1}} \\
&= \frac{m}{n} \cdot \frac{x^{m-1}}{y^{n-1}} \\
&= \frac{m}{n} \cdot \frac{x^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1}} \\
&= ax^{m-1-\frac{m}{n}(n-1)} \\
&= ax^{a-1} \\
\therefore \frac{d}{dx}x^a &= ax^{a-1} \quad \mid \quad a \in \mathbb{Q}
\end{aligned}$$

Example 1.1. The equation of a unit circle is:

$$y^2 = 1 - x^2$$

This can be rewritten as:

$$y = \left(1 - x^2\right)^{\frac{1}{2}}$$

We can compute the derivative using the chain rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\left(1 - x^2\right)^{\frac{1}{2}-1} \cdot (-2x) \\
&= \frac{-x}{(1 - x^2)^{\frac{1}{2}}} \\
&= -\frac{x}{y}
\end{aligned}$$

However, we can do the same thing using *implicit differentiation*:

$$\begin{aligned}
x^2 + y^2 &= 1 \\
\frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\
\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= 0 \\
2x + \frac{d}{dy}\left(y^2\right) \cdot \frac{dy}{dx} &= 0 \\
2x + 2yy' &= 0 \\
y' &= -\frac{x}{y}
\end{aligned}$$

Example 1.2. In the following case, it is not so easy to solve for y:

$$y^3 + xy^2 + 1 = 0$$

We will need to use implicit differentiation to find the derivative:

$$\begin{aligned} 3y^2y' + y^2 + 2xyy' + 0 &= 0 \\ y'\left(3y^2 + 2xy\right) &= -y^2 \\ y' &= -\frac{y^2}{3y^2 + 2xy} \end{aligned}$$

1.2 Inverses

If $y = f(x)$ and $g(y) = x$, we call g the *inverse* of f, denoted f^{-1}:

$$x = g(y) = f^{-1}(y)$$