Chapter 1

Implicit Differentiation

We know:

$$\frac{d}{dx}x^a = ax^{a-1} \quad | \quad a \in \mathbb{Z}$$

We will now extend this formula to cover $\mathbb{Q}$ as well:

$$a = \frac{m}{n} \to y = x^{\frac{m}{n}} \quad | \quad m, n \in \mathbb{Z}$$

We can start computing the derivative using the chain rule:

$$\begin{aligned}
y^n &= x^n \\
\frac{d}{dx}\left(y^n\right) &= \frac{d}{dx}\left(x^m\right) \\
\frac{d}{dy}\left(y^n\right) \cdot \frac{dy}{dx} &= mx^{m-1} \\
\frac{dy}{dx}\left(ny^{n-1}\right) &= mx^{m-1}
\end{aligned}$$

We finally have an expression for y':

$$\begin{aligned}
\frac{dy}{dx} &= \frac{mx^{m-1}}{ny^{n-1}} \\
&= \frac{m}{n} \cdot \frac{x^{m-1}}{y^{n-1}} \\
&= \frac{m}{n} \cdot \frac{x^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1}} \\
&= ax^{m-1-\frac{m}{n}(n-1)} \\
&= ax^{a-1} \\
\therefore \frac{d}{dx}x^a &= ax^{a-1} \quad | \quad a \in \mathbb{Q}
\end{aligned}$$

Example 1.1. The equation of a unit circle is:

$$y^2 = 1 - x^2$$

This can be rewritten as:

$$y = \left(1 - x^2\right)^{\frac{1}{2}}$$

We can compute the derivative using the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}\left(1 - x^2\right)^{\frac{1}{2}-1} \cdot (-2x) \\ &= \frac{-x}{(1-x^2)^{\frac{1}{2}}} \\ &= -\frac{x}{y}\end{aligned}$$

However, we can do the same thing using *implicit differentiation*:

$$\begin{aligned}x^2 + y^2 &= 1 \\ \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\ \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= 0 \\ 2x + \frac{d}{dy}\left(y^2\right) \cdot \frac{dy}{dx} &= 0 \\ 2x + 2yy' &= 0 \\ y' &= -\frac{x}{y}\end{aligned}$$

Example 1.2. In the following case, it is not so easy to solve for y:

$$y^3 + xy^2 + 1 = 0$$

We will need to use implicit differentiation to find the derivative:

$$\begin{aligned}3y^2y' + y^2 + 2xyy' + 0 &= 0 \\ y'\left(3y^2 + 2xy\right) &= -y^2 \\ y' &= -\frac{y^2}{3y^2 + 2xy}\end{aligned}$$

1.1 INVERSES

If $y = f(x)$ and $g(y) = x$, we call g the *inverse* of f, denoted f^{-1}:

$$x = g(y) = f^{-1}(y)$$

Now, we will use implicit differentiation to find the derivative of the inverse function:

$$
\begin{aligned}
y &= f(x) \\
f^{-1}(y) &= x \\
\frac{d}{dx}\left(f^{-1}(y)\right) &= \frac{d}{dx}(x) \\
\frac{d}{dy}\left(f^{-1}(y)\right) \cdot \frac{dy}{dx} &= 1 \\
\frac{d}{dy}\left(f^{-1}(y)\right) &= \frac{1}{\frac{dy}{dx}}
\end{aligned}
$$

Example 1.3. The derivative of $y = \tan^{-1}(x)$:

$$
\begin{aligned}
\tan y &= x \\
\frac{d}{dx}(\tan y) &= \frac{d}{dx}(x) \\
\frac{d}{dy}(\tan y) \cdot \frac{dy}{dx} &= 1 \\
\left(\csc^2 y\right) \cdot \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \cos^2 y \\
&= \cos^2\left(\tan^{-1}(x)\right)
\end{aligned}
$$

This form is messy but we can use geometry to simplify.

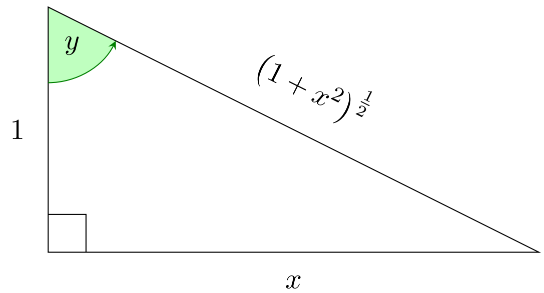

Figure 1.1: Triangle with angles and lengths corresponding to those in the example illustrating differentiation using the inverse function

In the triangle in Figure 1.1, $\tan y = x \Rightarrow y = \tan^{-1}(x)$. From this, we can find:

$$\begin{aligned}
\cos y &= \frac{1}{\sqrt{1+x^2}} \\
(\cos y)^2 &= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 \\
\cos^2 y &= \frac{1}{1+x^2} \\
\frac{dy}{dx} &= \frac{1}{1+x^2} \\
\frac{d}{dx}\left(\tan^{-1}(x)\right) &= \frac{1}{1+x^2}
\end{aligned}$$

Graphing

Suppose $y = f(x)$ and $g(y) = f^{-1}(y) = x$.

To graph f and g together, we need to write g as a function of x. If $g(x) = y$, then $x = f(y)$. What we have done is trade the variables x and y. This is illustrated in Figure 1.2:

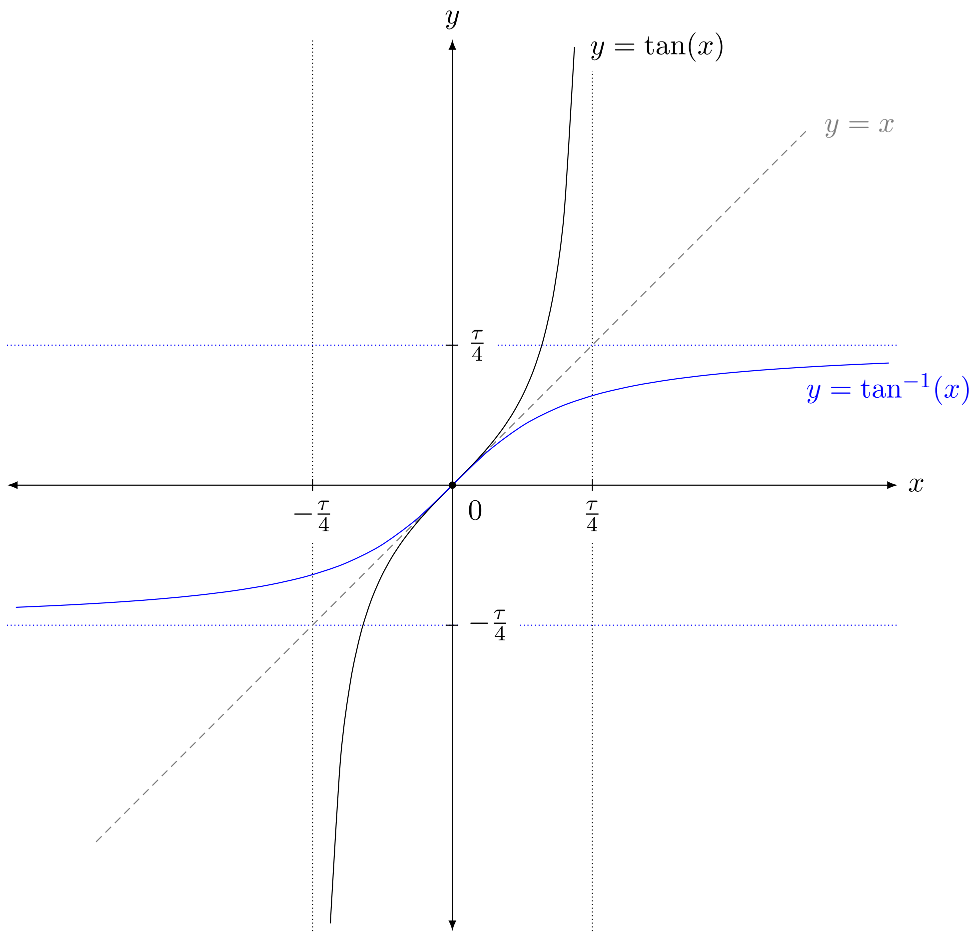

Figure 1.2: We can think about f^{-1} as the graph of f reflected about the line $y = x$