Chapter 1

Derivatives

1.1 Geometric Interpretation

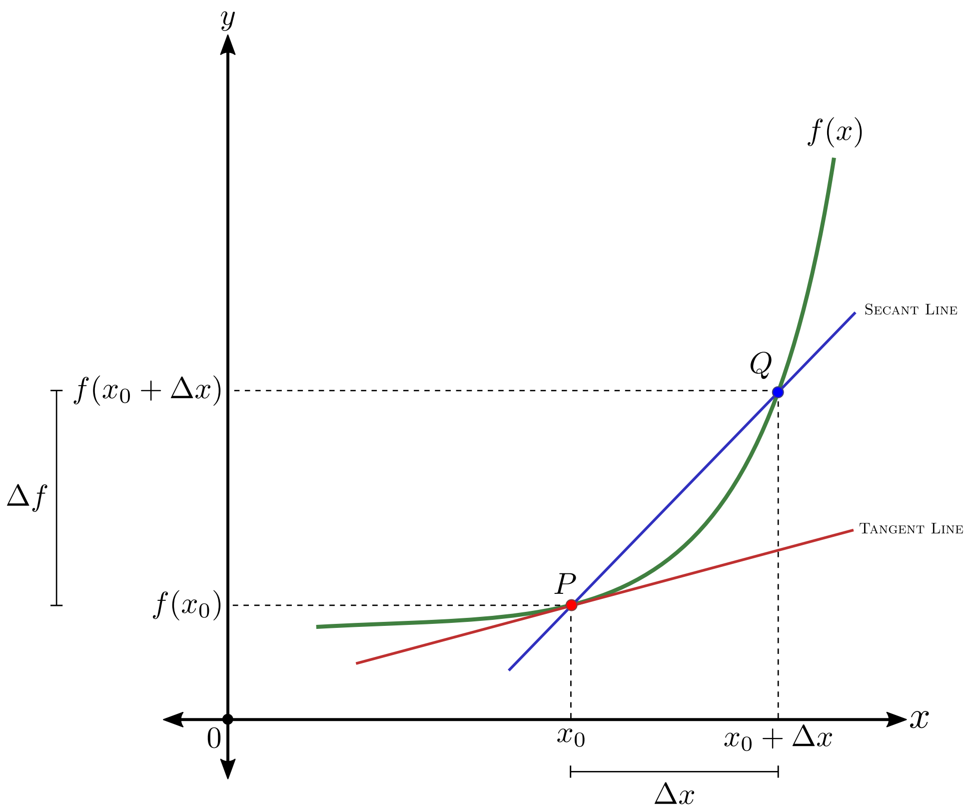

Figure 1.1: A function with secant and tangent lines

This is the graph of a function, $f(x)$, and P and Q are points on that curve.

Geometrically, the line $\overline{PQ}$ is known as the *secant* line — a line that passes through at least two points on a curve. We can see that the slope of $\overline{PQ}$ is $\frac{\Delta f}{\Delta x}$ and it is different than the slope (m) of the *tangent* line.

Definition 1.1. The derivative of f at the point x_0, denoted $f'(x_0)$, is the slope of the tangent line to $y = f(x)$ at the point P.

Although that is a pretty solid definition of a derivative, we need to be careful about how we think about the tangent line. It is not just any line that passes through P; it is the *limit* of the secant line as the distance between P and Q goes to 0.

Similarly, the slope of $\overline{PQ}$ *approaches* the slope of the tangent line as $Q \to P$. We can imagine this by dragging Q along the path of the curve down to P and think about how that would change the slope of $\overline{PQ}$. In the *limiting* case, Q would be right next to P and the slope of $\overline{PQ}$ would be identical to the slope of the tangent line. This idea of limits is mathematically denoted in the following way:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = m \tag{1.1}$$

Example 1.1. $f(x) = x^2$

We need to find the general equation of the slope of the tangent line at any point x on the curve of $y = x^2$ in terms of x.

From Equation 1.1, we have:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

By plugging in the given function, we get:

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x (2x + \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x) \\
&= 2x
\end{aligned}$$

Notations

Calculus, rather like English or any other language, was developed by several people. As a result, just as there are many ways to express the same thing in English, there are many notations for the derivative.

Since $y = f(x)$, it is natural to write:

$$\Delta y = \Delta f = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

If we divide both sides by Δx, we get two expressions for the *difference quotient*:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x}$$

As $\Delta x \to 0$:

$$\frac{\Delta y}{\Delta x} \to \left.\frac{dy}{dx}\right|_{x=x_0} \qquad \text{(Leibniz' Notation)}$$

$$\frac{\Delta f}{\Delta x} \to f'(x_0) \qquad \text{(Newton's Notation)}$$

Other, equally valid notations for the derivative of a function f include:

$$f'(x) = f' = Df = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x)$$

The *dot notation* (also introduced by Newton) is another convention used to denote derivatives with respect to t:

$$\frac{dy}{dt} = \dot{y}$$

As we have seen from Example 1.1, at any point x, the slope of the parabola, x^2, is $2x$. We can show that this can be generalised to the following formula:

$$\frac{d}{dx}(x^n) = nx^{n-1} \tag{1.2}$$

Example 1.2.

$$\begin{aligned} \frac{d}{dx}\left(x^3 + 3x^{10}\right) &= (3)x^{3-1} + 3(10)x^{10-1} \\ &= 3x^2 + 30x^9 \end{aligned}$$

1.2 Physical Interpretation

When something is changing with respect some other thing, it can be useful to know fast it's changing at a particular instant.

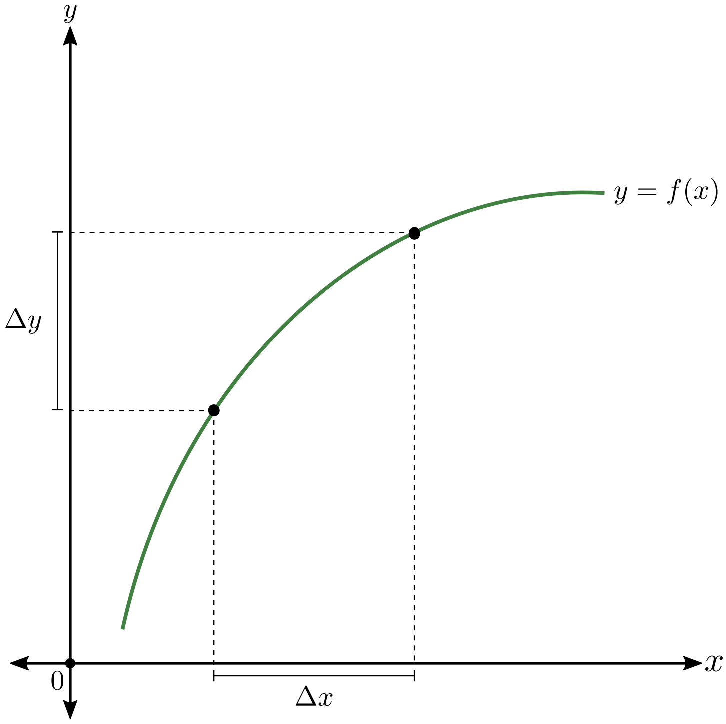

Figure 1.2: Graph of a function with Δx and Δy labelled

Speed is a perfect example of this. When you are moving, the distance (s) you travel changes with time (t). The *rate of change* of distance with respect to time (what we usually call *speed*), is a measure of how fast you're moving. Sometimes, it is useful to know what your *average* speed ($\bar{v}$) was over the whole journey, which is given by:

$$\bar{v} = \frac{\Delta s}{\Delta t}$$

Other times, your speed at a particular instant is more important – this is what is known as your *instantaneous* speed (v). You don't get a speeding ticket for having a high average speed over the whole journey; you get one for having a high speed at the *instant* you crossed a detection point. Similarly, the speed shown on a car's speedometer is not your average speed for the whole trip, it is your instantaneous speed at that particular instant. Derivates do a great job at finding these instantaneous rates of change:

$$v = \frac{ds}{dt}$$

Example 1.3. The distance (s) travelled by any free-falling object over time (t) is, approximately:

$$s = 5t^2$$

In other words, after falling for one second, free-falling objects usually travel five meters. After two seconds, twenty meters, and so on.

If we drop a ball from top of a five-hundred-meter tall building, we know it will take about ten seconds to fall:

$$\begin{aligned} 500 &= 5t^2 \\ \therefore t &= 10\,\mathrm{s} \end{aligned}$$

Since we know $\Delta s = 500$ and $\Delta t = 10$, we can find the average speed of the ball over its entire fall:

$$\begin{aligned} \bar{v} &= \frac{\Delta s}{\Delta t} \\ &= \frac{500}{10} \\ &= 50\,\mathrm{m\,s^{-1}} \end{aligned}$$

However, using differentiation, we can also find its instantaneous speed at any instant during its ten-second fall. We just need the derivative of s with respect to t:

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{d}{dt}\left(5t^2\right) \\ &= 10t \end{aligned}$$

All we need to do now is plug in any value of t we want, and this derivative will tell us the speed of the ball at that particular instant in time. After, say, six seconds of falling ($t = 6\,\mathrm{s}$), we know the ball was travelling at $v = 60\,\mathrm{m\,s^{-1}}$.