Chapter 1

Higher Derivatives

Higher derivatives are derivatives of derivatives. For instance, if y' is the derivative of y', then y'' is the derivative of y'.

Table 1.1: All the different notations for higher derivates

y'	$\frac{d}{dx}y$	$\frac{dy}{dx}$	Dy
y''	$\left(\frac{d}{dx}\right)^2 y$	$\frac{d^2y}{dx^2}$	D^2y
y'''	$\left(\frac{d}{dx}\right)^3 y$	$\frac{d^3y}{dx^3}$	D^3y
$y^{(4)}$	$\left(\frac{d}{dx}\right)^4 y$	$\frac{d^4y}{dx^4}$	D^4y
$y^{(n)}$	$\left(\frac{d}{dx}\right)^n y$	$\frac{d^ny}{dx^n}$	D^ny

Higher derivatives are pretty straightforward — just keep taking the derivative!

Example 1.1. Let us see what happens if we keep taking the derivative of $f(x) = \sin x$:

$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)} &= \sin x \end{aligned}$$

We have, somehow, arrived back at the original function, $f''''(x) = f(x)$. The sine and cosine functions, both, have this property.

Example 1.2. What is $D^n x^n$?
We will start small and look for a pattern. We know:

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d^2}{dx^2}x^n = n(n-1)x^{n-2}$$
$$\frac{d^3}{dx^3}x^n = n(n-1)(n-2)x^{n-3}$$

We can reasonably extend this pattern to deduce:

$$\frac{d^{n-1}}{dx^{n-1}}x^n = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot x$$

Finally, we get:

$$\frac{d^n}{dx^n}x^n = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

There is a name for this pattern of products; *factorials* ($n!$). Therefore:

$$\frac{d^n}{dx^n}x^n = n!$$

Now, we can also see:

$$\frac{d^{n+1}}{dx^{n+1}}x^n = 0$$

We just (unwittingly) did a proof by *mathematical induction*! It is an extremely useful tool in every mathematician's toolbox.