SINGLE-VARIABLE CALCULUS

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Contents

Preface

Before Newton ever formulated a description of universal gravitation or his famous laws of motion that changed physics forever, he did something arguably more important; changing mathematics forever too. I am talking, of course, about infinitesimal calculus.

Much like ourselves, Isaac Newton had to live through a deadly pandemic – the Bubonic plague. Much like today, universities were closed, students were sent home, and people were told to self-isolate. It was sometime around 1665, while he was in quarantine at his home in London, that he figured out a way to analyse the motion of objects using infinitesimals – that taking smaller and smaller increments of time give a more and more accurate calculation for an object's speed at a particular instant. He was thinking in terms of distance and speed at the time, but these same ideas would later be generalised to apply to all sorts of functions in mathematics and physics – what we now call differential calculus.

Incredibly, at around exactly the same time, a German mathematician, Gottfried Wilhelm Leibniz, was also thinking about and developing a different branch of calculus, and would soon arrive at the same fundamentals of calculus as Newton, completely independently!

Leibniz, working at Leipzig University (my own alma mater), was thinking about the area under an arbitrary curve, and how it can be approximated using smaller and smaller shapes. He was deriving what we now call integral calculus, and much of the notation we use comes directly from him.

Newton and Leibniz would later have many disputes over who came up with calculus first; both claiming one copied the other. It is somewhat surprising and hard to believe at first that two people would come up with such a novel idea at exactly the same time in history, but all retrospective analysis shows that this was indeed a coincidence. The invention (or discovery?) of this magical branch of mathematics was inevitable – and the world was ready for it in 1665.

Not long after, the ideas of calculus would lead to the mathematical descriptions of gravity by Newton, and of electromagnetism by James Clerk Maxwell.

These ideas sparked the first industrial revolution in England. They gave us the manufacturing capabilities to wage world wars. Electricity dramatically increased the quality of life for billions of people. A few centuries later, calculus put the first human being on the Moon and helps epidemiologists fight off the pandemics that plague us today.

Part I

Differentiation

Chapter 1

Derivatives

1.1 Geometric Interpretation

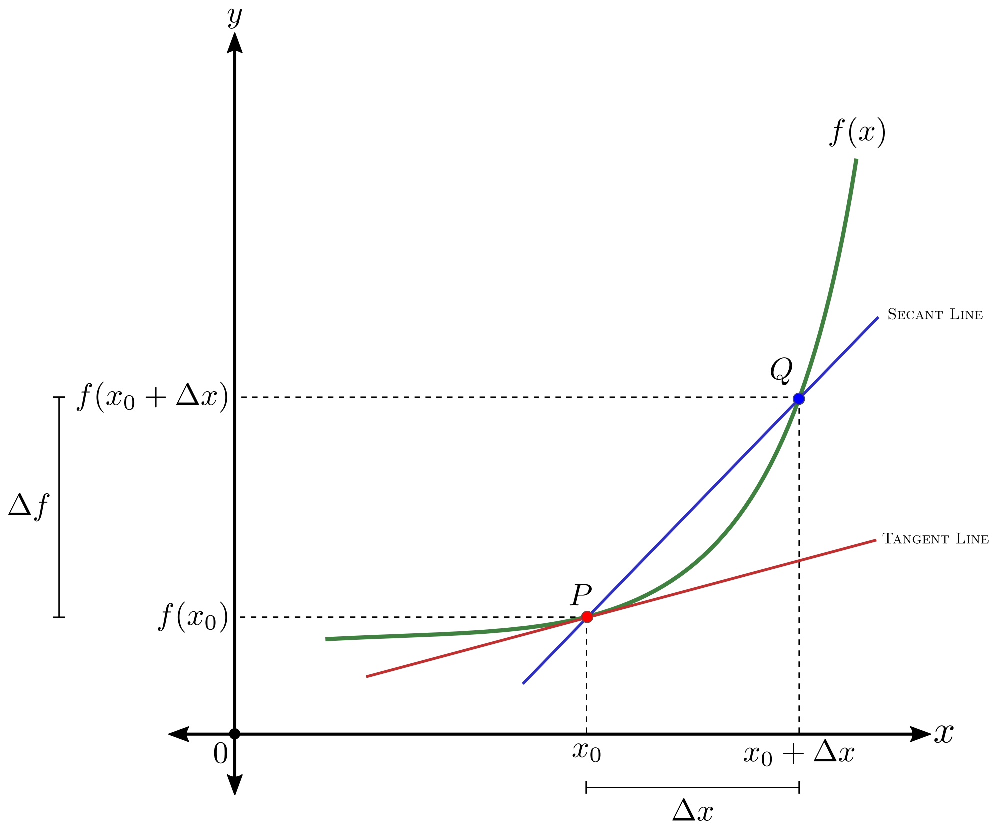

Figure 1.1: A function with secant and tangent lines

This is the graph of a function, $f(x)$, and P and Q are points on that curve. Geometrically, the line $\overline{PQ}$ is known as the *secant* line — a line that passes through at least two points on a curve. We can see that the slope of $\overline{PQ}$ is $\frac{\Delta f}{\Delta x}$ and it is different than the slope (m) of the *tangent* line.

Definition 1.1. The derivative of f at the point x_0, denoted $f'(x_0)$, is the slope of the tangent line to $y = f(x)$ at the point P.

Although that is a pretty solid definition of a derivative, we need to be careful about how we think about the tangent line. It is not just any line that passes through P; it is the *limit* of the secant line as the distance between P and Q goes to 0.

Similarly, the slope of $\overline{PQ}$ *approaches* the slope of the tangent line as $Q \to P$. We can imagine this by dragging Q along the path of the curve down to P and think about how that would change the slope of $\overline{PQ}$. In the *limiting* case, Q would be right next to P and the slope of $\overline{PQ}$ would be identical to the slope of the tangent line. This idea of limits is mathematically denoted in the following way:

$$f'\left(x_0\right) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f\left(x_0 + \Delta x\right) - f\left(x_0\right)}{\Delta x} = m \tag{1.1}$$

Example 1.1. $f(x) = x^2$

We need to find the general equation of the slope of the tangent line at any point x on the curve of $y = x^2$ in terms of x.

From Equation 1.1, we have:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f\left(x_0 + \Delta x\right) - f\left(x_0\right)}{\Delta x}$$

By plugging in the given function, we get:

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x\left(2x + \Delta x\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(2x + \Delta x\right) \\
&= 2x
\end{aligned}$$

Notations

Calculus, rather like English or any other language, was developed by several people. As a result, just as there are many ways to express the same thing in English, there are many notations for the derivative.

Since $y = f(x)$, it is natural to write:

$$\Delta y = \Delta f = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

If we divide both sides by Δx, we get two expressions for the *difference quotient*:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x}$$

As $\Delta x \to 0$:

$$\frac{\Delta y}{\Delta x} \to \left.\frac{dy}{dx}\right|_{x=x_0} \qquad \text{(Leibniz' Notation)}$$

$$\frac{\Delta f}{\Delta x} \to f'(x_0) \qquad \text{(Newton's Notation)}$$

Other, equally valid notations for the derivative of a function f include:

$$f'(x) = f' = Df = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x)$$

The *dot notation* (also introduced by Newton) is another convention used to denote derivatives with respect to t:

$$\frac{dy}{dt} = \dot{y}$$

As we have seen from Example 1.1, at any point x, the slope of the parabola, x^2, is $2x$. We can show that this can be generalised to the following formula:

$$\frac{d}{dx}(x^n) = nx^{n-1} \tag{1.2}$$

Example 1.2.

$$\begin{aligned}\frac{d}{dx}\left(x^3 + 3x^{10}\right) &= (3)x^{3-1} + 3(10)x^{10-1} \\ &= 3x^2 + 30x^9\end{aligned}$$

1.2 Physical Interpretation

When something is changing with respect some other thing, it can be useful to know fast it's changing at a particular instant.

Speed is a perfect example of this. When you are moving, the distance (s) you travel changes with time (t). The *rate of change* of distance with respect

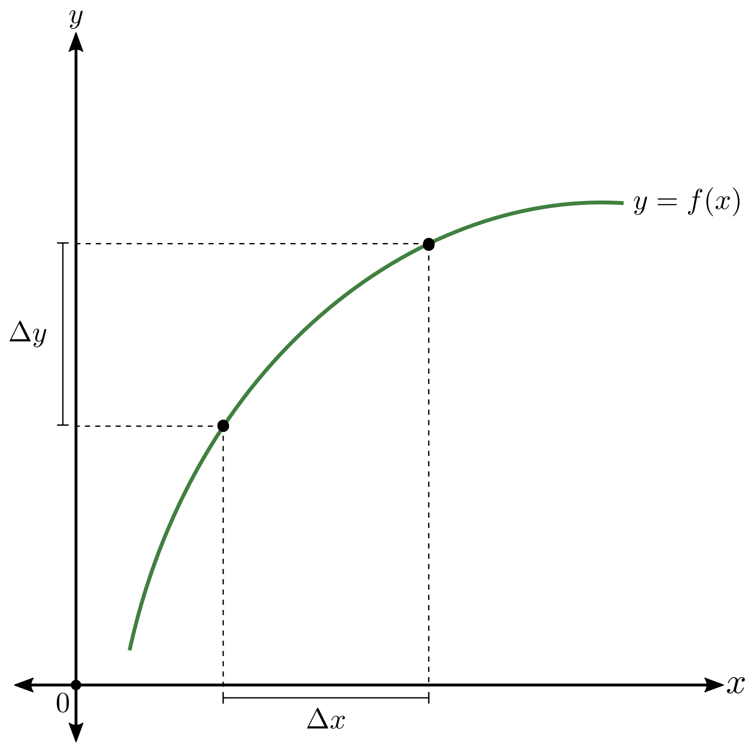

Figure 1.2: Graph of a function with Δx and Δy labelled

to time (what we usually call *speed*), is a measure of how fast you're moving. Sometimes, it is useful to know what your *average* speed ($\bar{v}$) was over the whole journey, which is given by:

$$\bar{v} = \frac{\Delta s}{\Delta t}$$

Other times, your speed at a particular instant is more important – this is what is known as your *instantaneous* speed (v). You don't get a speeding ticket for having a high average speed over the whole journey; you get one for having a high speed at the *instant* you crossed a detection point. Similarly, the speed shown on a car's speedometer is not your average speed for the whole trip, it is your instantaneous speed at that particular instant. Derivates do a great job at finding these instantaneous rates of change:

$$v = \frac{ds}{dt}$$

Example 1.3. The distance (s) travelled by any free-falling object over time (t) is, approximately:

$$s = 5t^2$$

In other words, after falling for one second, free-falling objects usually travel five meters. After two seconds, twenty meters, and so on.

If we drop a ball from top of a five-hundred-meter tall building, we know it will take about ten seconds to fall:

$$\begin{aligned}500 &= 5t^2\\ \therefore t &= 10\,\mathrm{s}\end{aligned}$$

Since we know $\Delta s = 500$ and $\Delta t = 10$, we can find the average speed of the ball over its entire fall:

$$\begin{aligned}\bar{v} &= \frac{\Delta s}{\Delta t}\\ &= \frac{500}{10}\\ &= 50\,\mathrm{m\,s^{-1}}\end{aligned}$$

However, using differentiation, we can also find its instantaneous speed at any instant during its ten-second fall. We just need the derivative of s with respect to t:

$$\begin{aligned}v &= \frac{ds}{dt}\\ &= \frac{d}{dt}\left(5t^2\right)\\ &= 10t\end{aligned}$$

All we need to do now is plug in any value of t we want, and this derivative will tell us the speed of the ball at that particular instant in time. After, say, six seconds of falling ($t = 6\,\mathrm{s}$), we know the ball was travelling at $v = 60\,\mathrm{m\,s^{-1}}$.

Chapter 2

Limits and Continuity

2.1 Easy Limits

With easy limits, you can get the answer simply by plugging in the limiting value:

$$\lim_{x \to 3} \frac{x^2 + x}{x + 1} = \frac{(3)^2 + (3)}{(3) + 1} = 3$$

2.2 Continuity

Definition 2.1. We say $f(x)$ is continuous at x_0 when:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Example 2.1. Consider the following *piecewise* function:

$$f(x) = \begin{cases} x + 1 & : \ x > 0 \\ -x & : \ x \leq 0 \end{cases}$$

It is said to be *discontinuous* (see Figure 2.1) at $x = 0$ since we know $f(0) = 0$, but for $x > 0$:

$$\lim_{x \to 0} f(x) = 1$$

We can say f is continuous from the left at $x = 0$, but not the right.

Definition 2.2 (Right-Hand Limit)**.**

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} f(x) \ : \ x > 0$$

Definition 2.3 (Left-Hand Limit)**.**

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) \ : \ x < 0$$

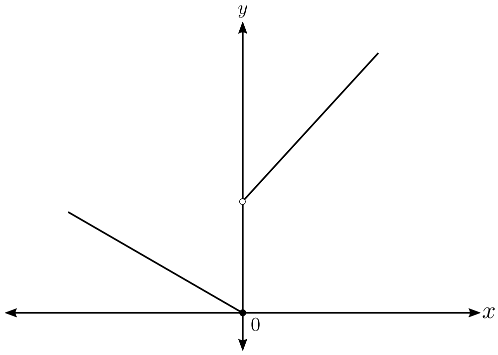

Figure 2.1: Graph of a discontinuous function

Discontinuities

Definition 2.4. A discontinuity is removable if:

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) \neq f(x_0)$$

Case. The function $\frac{\sin(x)}{x}$ is undefined for $x = 0$. However, its limit as $x \to 0$ still exists:

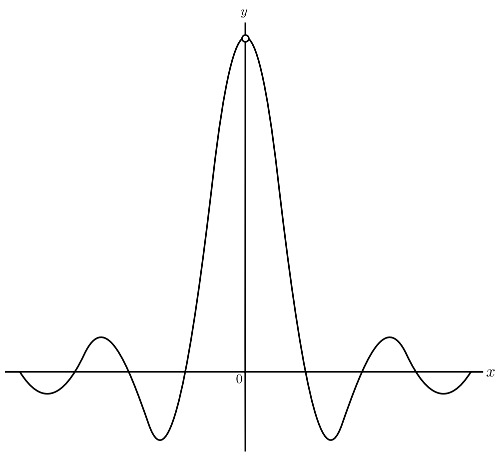

Figure 2.2: A removable discontinuity — the function is continuous everywhere except one point

Definition 2.5. A jump discontinuity is when $\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$ even if they both exist.

Definition 2.6. There is an infinite discontinuity when the right- and left-hand limits are both infinite, but in opposite directions.

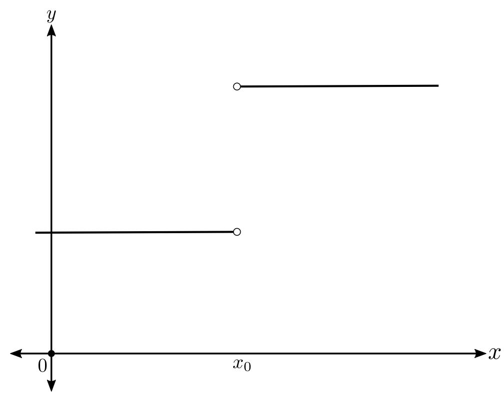

Figure 2.3: An example of a jump discontinuity

Case. $f(x) = \frac{1}{x}$ has an infinite discontinuity (*singularity*) at $x = 0$:

$$\lim_{x \to x_0^+} \frac{1}{x} = -\infty \quad , \quad \lim_{x \to x_0^+} \frac{1}{x} = \infty$$

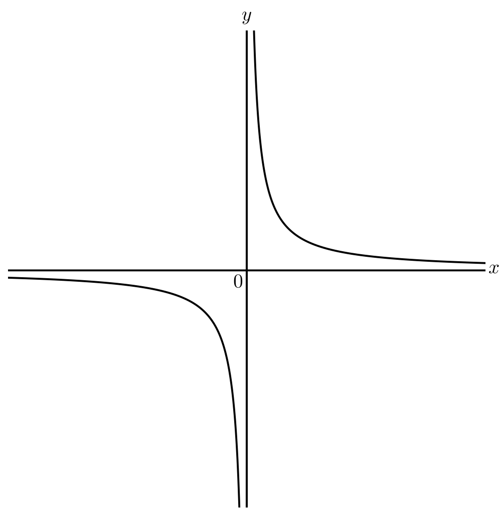

Figure 2.4: $f(x) = \frac{1}{x}$ is an example of an infinite discontinuity

Ugly Discontinuities

The function shown in Figure 2.5 doesn't even go to $\pm\infty$ — it doesn't make sense to say it "goes to" anything. For something like this, we say the limit does not exist (DNE).

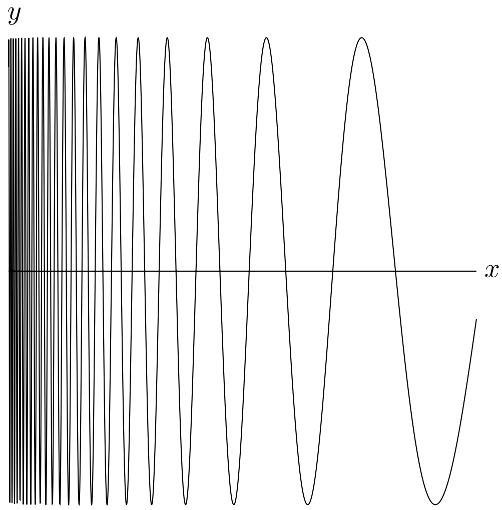

Figure 2.5: An example of an ugly discontinuity: a function that oscillates a lot as it approaches the origin

Theorem 2.1. *If f is differentiable at x_0, then f is continuous at x_0*

Proof. From Definition 2.1, we need to show:

$$\begin{aligned} &\lim_{x \to x_0} f(x) = f(x_0) \\ \Rightarrow &\lim_{x \to x_0} f(x) - f(x_0) = 0 \end{aligned}$$

The LHS can be rewritten as:

$$\begin{aligned} \lim_{x \to x_0} f(x) - f(x_0) &= \lim_{x \to x_0} f(x) - f(x_0) \cdot \frac{x - x_0}{x - x_0} \quad \because \quad \frac{x - x_0}{x - x_0} = 1 \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) \end{aligned}$$

If f is differentiable, we can use Equation 1.1 and rearrange to get:

$$\begin{aligned} \lim_{x \to x_0} f(x) - f(x_0) &= \lim_{x \to x_0} (x - x_0) \cdot f'(x_0) \\ &= (x_0 - x_0) \cdot f'(x_0) \\ &= 0 \cdot f'(x_0) \\ &= 0 \end{aligned}$$

□

Note. You can never divide by zero! The first step was to multiply by $\frac{x-x_0}{x-x_0}$. It looks as if this is illegal because we are multiplying by $\frac{0}{0}$ when $x = x_0$. But, when computing the limit as $x \to x_0$, we always assume $x \neq x_0$. In other words, $x - x_0 \neq 0$. Therefore, the proof is valid.

Chapter 3

Derivative Formulæ

3.1 General Functions

For any two functions u and v:

$$(u+v)' = u' + v'$$

When there is a constant, c:

$$(c \cdot u)' = c \cdot u'$$

3.2 Trigonometric Functions

$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x \\ \frac{d}{dx}\cos x &= -\sin x \end{aligned}$$

3.3 Product Rule

$$(u \cdot v)' = v \cdot u' + u \cdot v' \tag{3.1}$$

Example 3.1. To differentiate $f(x) = x^3 \sin x$, we let $u = x^3$ and $v = \sin x$:

$$\begin{aligned} \therefore \quad u' &= 3x^2 \\ v' &= \cos x \end{aligned}$$

From Equation 3.1, we know:

$$\begin{aligned} f'(x) &= vu' + uv' \\ &= 3x^2 \sin x + x^3 \cos x \end{aligned}$$

3.4 Quotient Rule

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

3.5 Chain Rule

The *chain rule* (in Leibniz's notation) can be written in the following way:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Example 3.2. To differentiate $y = \sin^{10} t$, we let $x = \sin t$:

$$\begin{aligned} y &= x^{10} \\ \therefore \quad \frac{dy}{dx} &= 10x^9 \end{aligned}$$

From the chain rule, we get:

$$\frac{dy}{dt} = 10x^9 \cos t \quad \because \quad \frac{dx}{dt} = \cos t$$

Finally, we need to substitute $x = \sin t$:

$$\frac{dy}{dt} = 10 \sin^9 t \cos t$$

Example 3.3. To differentiate $\sin(10t)$, we let $x = 10t$ and $y = \sin x$: We now have:

$$\frac{dx}{dt} = 10 \quad , \quad \frac{dy}{dx} = \cos x$$

$$\begin{aligned} \frac{d}{dt}\sin(10t) &= \frac{dy}{dt} \\ \frac{dy}{dt} &= \frac{dy}{dx} \cdot \frac{dx}{dt} \\ &= 10 \cos x \\ &= 10 \cos(10t) \end{aligned}$$

Chapter 4

Higher Derivatives

Higher derivatives are derivatives of derivatives. For instance, if y' is the derivative of y', then y'' is the derivative of y'.

Table 4.1: All the different notations for higher derivates

y'	$\frac{d}{dx}y$	$\frac{dy}{dx}$	Dy
y''	$\left(\frac{d}{dx}\right)^2 y$	$\frac{d^2y}{dx^2}$	D^2y
y'''	$\left(\frac{d}{dx}\right)^3 y$	$\frac{d^3y}{dx^3}$	D^3y
$y^{(4)}$	$\left(\frac{d}{dx}\right)^4 y$	$\frac{d^4y}{dx^4}$	D^4y
$y^{(n)}$	$\left(\frac{d}{dx}\right)^n y$	$\frac{d^ny}{dx^n}$	D^ny

Higher derivatives are pretty straightforward — just keep taking the derivative!

Example 4.1. Let us see what happens if we keep taking the derivative of $f(x) = \sin x$:

$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)} &= \sin x \end{aligned}$$

We have, somehow, arrived back at the original function, $f''''(x) = f(x)$. The sine and cosine functions, both, have this property.

Example 4.2. What is $D^n x^n$?

We will start small and look for a pattern. We know:

$$\begin{aligned}
\frac{d}{dx}x^n &= nx^{n-1} \\
\frac{d^2}{dx^2}x^n &= n(n-1)x^{n-2} \\
\frac{d^3}{dx^3}x^n &= n(n-1)(n-2)x^{n-3}
\end{aligned}$$

We can reasonably extend this pattern to deduce:

$$\frac{d^{n-1}}{dx^{n-1}}x^n = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot x$$

Finally, we get:

$$\frac{d^n}{dx^n}x^n = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

There is a name for this pattern of products; *factorials* ($n!$). Therefore:

$$\frac{d^n}{dx^n}x^n = n!$$

Now, we can also see:

$$\frac{d^{n+1}}{dx^{n+1}}x^n = 0$$

We just (unwittingly) did a proof by *mathematical induction*! It is an extremely useful tool in every mathematician's toolbox.

Chapter 5

Implicit Differentiation and Inverses

5.1 Implicit Differentiation

We know:

$$\frac{d}{dx}x^a = ax^{a-1} \quad | \quad a \in \mathbb{Z}$$

We will now extend this formula to cover $\mathbb{Q}$ as well:

$$a = \frac{m}{n} \to y = x^{\frac{m}{n}} \quad | \quad m, n \in \mathbb{Z}$$

We can start computing the derivative using the chain rule:

$$\begin{aligned} y^n &= x^n \\ \frac{d}{dx}\left(y^n\right) &= \frac{d}{dx}\left(x^m\right) \\ \frac{d}{dy}\left(y^n\right) \cdot \frac{dy}{dx} &= mx^{m-1} \\ \frac{dy}{dx}\left(ny^{n-1}\right) &= mx^{m-1} \end{aligned}$$

We finally have an expression for y':

$$\begin{aligned}
\frac{dy}{dx} &= \frac{mx^{m-1}}{ny^{n-1}} \\
&= \frac{m}{n} \cdot \frac{x^{m-1}}{y^{n-1}} \\
&= \frac{m}{n} \cdot \frac{x^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1}} \\
&= ax^{m-1-\frac{m}{n}(n-1)} \\
&= ax^{a-1} \\
\therefore \frac{d}{dx}x^a &= ax^{a-1} \quad | \quad a \in \mathbb{Q}
\end{aligned}$$

Example 5.1. The equation of a unit circle is:

$$y^2 = 1 - x^2$$

This can be rewritten as:

$$y = \left(1 - x^2\right)^{\frac{1}{2}}$$

We can compute the derivative using the chain rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\left(1 - x^2\right)^{\frac{1}{2}-1} \cdot (-2x) \\
&= \frac{-x}{(1 - x^2)^{\frac{1}{2}}} \\
&= -\frac{x}{y}
\end{aligned}$$

However, we can do the same thing using *implicit differentiation*:

$$\begin{aligned}
x^2 + y^2 &= 1 \\
\frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\
\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= 0 \\
2x + \frac{d}{dy}\left(y^2\right) \cdot \frac{dy}{dx} &= 0 \\
2x + 2yy' &= 0 \\
y' &= -\frac{x}{y}
\end{aligned}$$

Example 5.2. In the following case, it is not so easy to solve for y:

$$y^3 + xy^2 + 1 = 0$$

We will need to use implicit differentiation to find the derivative:

$$\begin{aligned} 3y^2y' + y^2 + 2xyy' + 0 &= 0 \\ y'\left(3y^2 + 2xy\right) &= -y^2 \\ y' &= -\frac{y^2}{3y^2 + 2xy} \end{aligned}$$

5.2 Inverses

If $y = f(x)$ and $g(y) = x$, we call g the *inverse* of f, denoted f^{-1}:

$$x = g(y) = f^{-1}(y)$$

Now, we will use implicit differentiation to find the derivative of the inverse function:

$$\begin{aligned} y &= f(x) \\ f^{-1}(y) &= x \\ \frac{d}{dx}\left(f^{-1}(y)\right) &= \frac{d}{dx}(x) \\ \frac{d}{dy}\left(f^{-1}(y)\right) \cdot \frac{dy}{dx} &= 1 \\ \frac{d}{dy}\left(f^{-1}(y)\right) &= \frac{1}{\frac{dy}{dx}} \end{aligned}$$

Example 5.3. The derivative of $y = \tan^{-1}(x)$:

$$\begin{aligned} \tan y &= x \\ \frac{d}{dx}(\tan y) &= \frac{d}{dx}(x) \\ \frac{d}{dy}(\tan y) \cdot \frac{dy}{dx} &= 1 \\ \left(\csc^2 y\right) \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2 y \\ &= \cos^2\left(\tan^{-1}(x)\right) \end{aligned}$$

This form is messy but we can use geometry to simplify.

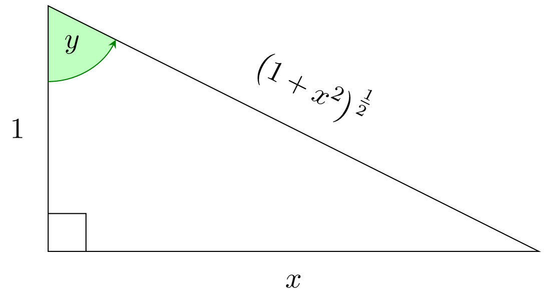

Figure 5.1: Triangle with angles and lengths corresponding to those in the example illustrating differentiation using the inverse function

In the triangle in Figure 5.1, $\tan y = x \Rightarrow y = \tan^{-1}(x)$. From this, we can find:

$$\begin{aligned}
\cos y &= \frac{1}{\sqrt{1+x^2}} \\
(\cos y)^2 &= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 \\
\cos^2 y &= \frac{1}{1+x^2} \\
\frac{dy}{dx} &= \frac{1}{1+x^2} \\
\frac{d}{dx}\left(\tan^{-1}(x)\right) &= \frac{1}{1+x^2}
\end{aligned}$$

Graphing

Suppose $y = f(x)$ and $g(y) = f^{-1}(y) = x$.

To graph f and g together, we need to write g as a function of x. If $g(x) = y$, then $x = f(y)$. What we have done is trade the variables x and y. This is illustrated in Figure 5.2:

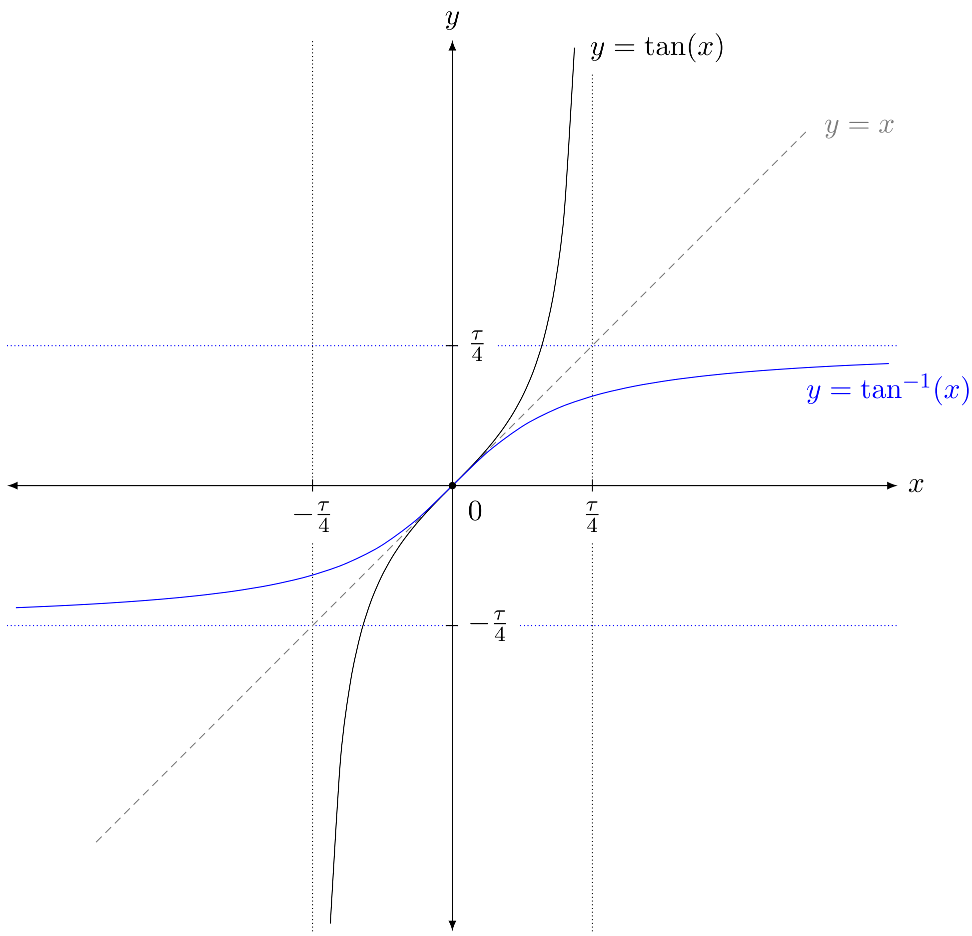

Figure 5.2: We can think about f^{-1} as the graph of f reflected about the line $y = x$

Part II

Applications of Differentiation

Part III

Integration

Part IV

Techniques of Integration

Acknowledgments

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