Chapter 1

Implicit Differentiation and Inverses

1.1 Implicit Differentiation

We know:

$$\frac{d}{dx}x^a = ax^{a-1} \quad | \quad a \in \mathbb{Z}$$

We will now extend this formula to cover $\mathbb{Q}$ as well:

$$a = \frac{m}{n} \rightarrow y = x^{\frac{m}{n}} \quad | \quad m, n \in \mathbb{Z}$$

We can start computing the derivative using the chain rule:

$$y^n = x^n$$
$$\frac{d}{dx}\left(y^n\right) = \frac{d}{dx}\left(x^m\right)$$
$$\frac{d}{dy}\left(y^n\right) \cdot \frac{dy}{dx} = mx^{m-1}$$
$$\frac{dy}{dx}\left(ny^{n-1}\right) = mx^{m-1}$$

We finally have an expression for y':

$$\begin{aligned}
\frac{dy}{dx} &= \frac{mx^{m-1}}{ny^{n-1}} \\
&= \frac{m}{n} \cdot \frac{x^{m-1}}{y^{n-1}} \\
&= \frac{m}{n} \cdot \frac{x^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1}} \\
&= ax^{m-1-\frac{m}{n}(n-1)} \\
&= ax^{a-1}
\end{aligned}$$

$$\therefore \frac{d}{dx}x^a = ax^{a-1} \quad | \quad a \in \mathbb{Q}$$

Example 1.1. The equation of a unit circle is:

$$y^2 = 1 - x^2$$

This can be rewritten as:

$$y = \left(1 - x^2\right)^{\frac{1}{2}}$$

We can compute the derivative using the chain rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\left(1 - x^2\right)^{\frac{1}{2}-1} \cdot (-2x) \\
&= \frac{-x}{\left(1 - x^2\right)^{\frac{1}{2}}} \\
&= -\frac{x}{y}
\end{aligned}$$

However, we can do the same thing using *implicit differentiation*:

$$\begin{aligned}
x^2 + y^2 &= 1 \\
\frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\
\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) &= 0 \\
2x + \frac{d}{dy}\left(y^2\right) \cdot \frac{dy}{dx} &= 0 \\
2x + 2yy' &= 0 \\
y' &= -\frac{x}{y}
\end{aligned}$$

Example 1.2. In the following case, it is not so easy to solve for y:

$$y^3 + xy^2 + 1 = 0$$

We will need to use implicit differentiation to find the derivative:

$$3y^2y' + y^2 + 2xyy' + 0 = 0$$
$$y'\left(3y^2 + 2xy\right) = -y^2$$
$$y' = -\frac{y^2}{3y^2 + 2xy}$$

1.2 Inverses

If $y = f(x)$ and $g(y) = x$, we call g the *inverse* of f, denoted f^{-1}:

$$x = g(y) = f^{-1}(y)$$

Now, we will use implicit differentiation to find the derivative of the inverse function:

$$y = f(x)$$
$$f^{-1}(y) = x$$
$$\frac{d}{dx}\left(f^{-1}(y)\right) = \frac{d}{dx}(x)$$
$$\frac{d}{dy}\left(f^{-1}(y)\right) \cdot \frac{dy}{dx} = 1$$
$$\frac{d}{dy}\left(f^{-1}(y)\right) = \frac{1}{\frac{dy}{dx}}$$

Example 1.3. The derivative of $y = \tan^{-1}(x)$:

$$\tan y = x$$
$$\frac{d}{dx}\left(\tan y\right) = \frac{d}{dx}(x)$$
$$\frac{d}{dy}\left(\tan y\right) \cdot \frac{dy}{dx} = 1$$
$$\left(\csc^2 y\right) \cdot \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \cos^2 y$$
$$= \cos^2\left(\tan^{-1}(x)\right)$$

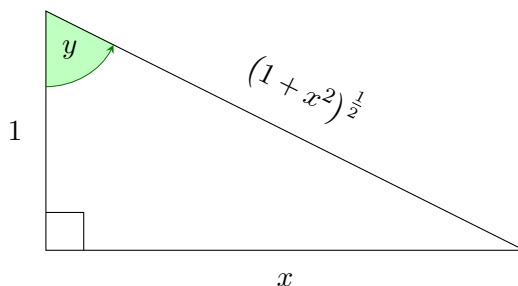This form is messy but we can use geometry to simplify.

Figure 1.1: Triangle with angles and lengths corresponding to those in the example illustrating differentiation using the inverse function

In the triangle in Figure 1.1, $\tan y = x \Rightarrow y = \tan^{-1}(x)$. From this, we can find:

$$\cos y = \frac{1}{\sqrt{1 + x^2}}$$

$$(\cos y)^2 = \left(\frac{1}{\sqrt{1 + x^2}}\right)^2$$

$$\cos^2 y = \frac{1}{1 + x^2}$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1 + x^2}$$

1.2.1 Graphing

Suppose $y = f(x)$ and $g(y) = f^{-1}(y) = x$.

To graph f and g together, we need to write g as a function of x. If $g(x) = y$, then $x = f(y)$. What we have done is trade the variables x and y. This is illustrated in 1.2:
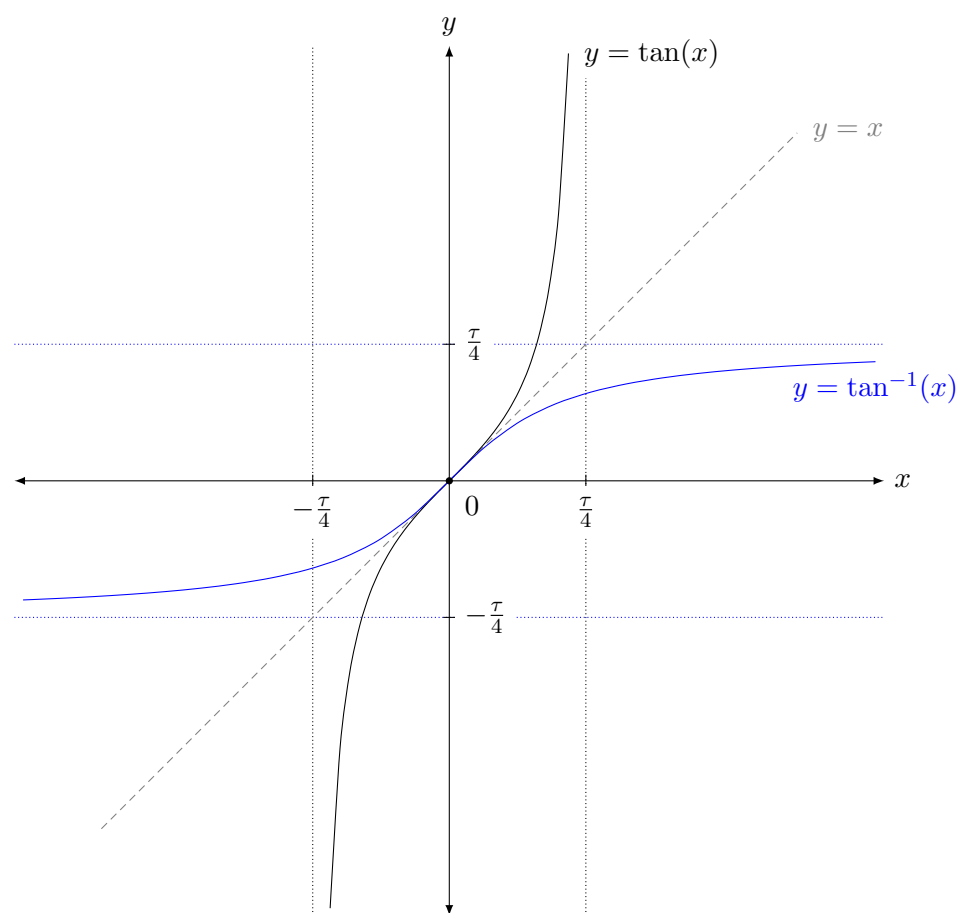
Figure 1.2:   You can think about f^{-1} as the graph of f reflected about the line $y = x$