Chapter 1

Implicit Differentiation and Inverses

1.1 Implicit Differentiation

We know:

$$\frac{d}{dx}x^a = ax^{a-1} \quad | \quad a \in \mathbb{Z}$$

We will now extend this formula to cover $\mathbb Q$ as well:

$$a = \frac{m}{n} \to y = x^{\frac{m}{n}} \quad | \quad m, n \in \mathbb{Z}$$

We can start computing the derivative using the chain rule:

$$y^{n} = x^{n}$$

$$\frac{d}{dx}(y^{n}) = \frac{d}{dx}(x^{m})$$

$$\frac{d}{dy}(y^{n}) \cdot \frac{dy}{dx} = mx^{m-1}$$

$$\frac{dy}{dx}(ny^{n-1}) = mx^{m-1}$$

We finally have an expression for y':

$$\frac{dy}{dx} = \frac{mx^{m-1}}{ny^{n-1}}$$

$$= \frac{m}{n} \cdot \frac{x^{m-1}}{y^{n-1}}$$

$$= \frac{m}{n} \cdot \frac{x^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1}}$$

$$= ax^{m-1-\frac{m}{n}(n-1)}$$

$$= ax^{a-1}$$

$$\therefore \frac{d}{dx}x^a = ax^{a-1} \quad | \quad a \in \mathbb{Q}$$

Example 1.1. The equation of a unit circle is:

$$y^2 = 1 - x^2$$

This can be rewritten as:

$$y = (1 - x^2)^{\frac{1}{2}}$$

We can compute the derivative using the chain rule:

$$\frac{dy}{dx} = \frac{1}{2} \left(1 - x^2 \right)^{\frac{1}{2} - 1} \cdot (-2x)$$

$$= \frac{-x}{(1 - x^2)^{\frac{1}{2}}}$$

$$= -\frac{x}{y}$$

However, we can do the same thing using *implicit differentiation*:

$$x^{2} + y^{2} = 1$$

$$\frac{d}{dx}(x^{2} + y^{2}) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^{2}) + \frac{d}{dx}(y^{2}) = 0$$

$$2x + \frac{d}{dy}(y^{2}) \cdot \frac{dy}{dx} = 0$$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

Example 1.2. In the following case, it is not so easy to solve for y:

$$y^3 + xy^2 + 1 = 0$$

We will need to use implicit differentiation to find the derivative:

$$3y^{2}y' + y^{2} + 2xyy' + 0 = 0$$
$$y' (3y^{2} + 2xy) = -y^{2}$$
$$y' = -\frac{y^{2}}{3y^{2} + 2xy}$$

1.2 Inverses

If y = f(x) and g(y) = x, we call g the *inverse* of f, denoted f^{-1} :

$$x = g(y) = f^{-1}(y)$$

Now, we will use implicit differentiation to find the derivative of the inverse function:

$$y = f(x)$$

$$f^{-1}(y) = x$$

$$\frac{d}{dx} (f^{-1}(y)) = \frac{d}{dx}(x)$$

$$\frac{d}{dy} (f^{-1}(y)) \cdot \frac{dy}{dx} = 1$$

$$\frac{d}{dy} (f^{-1}(y)) = \frac{1}{\frac{dy}{dx}}$$

Example 1.3. The derivative of $y = \tan^{-1}(x)$:

$$\tan y = x$$

$$\frac{d}{dx} (\tan y) = \frac{d}{dx} (x)$$

$$\frac{d}{dy} (\tan y) \cdot \frac{dy}{dx} = 1$$

$$(\csc^2 y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \cos^2 y$$

$$= \cos^2 (\tan^{-1}(x))$$

This form is messy but we can use geometry to simplify.

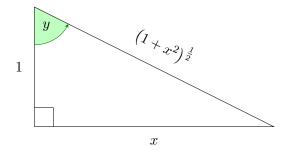


Figure 1.1: Triangle with angles and lengths corresponding to those in the example illustrating differentiation using the inverse function

In the triangle in Figure 1.1, $\tan y = x \Rightarrow y = \tan^{-1}(x)$. From this, we can find:

$$\cos y = \frac{1}{\sqrt{1+x^2}}$$
$$(\cos y)^2 = \left(\frac{1}{\sqrt{1+x^2}}\right)^2$$
$$\cos^2 y = \frac{1}{1+x^2}$$
$$\frac{dy}{dx} = \frac{1}{1+x^2}$$
$$\frac{d}{dx} \left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

1.2.1 Graphing

Suppose y = f(x) and $g(y) = f^{-1}(y) = x$.

To graph f and g together, we need to write g as a function of x. If g(x) = y, then x = f(y). What we have done is trade the variables x and y. This is illustrated in Figure 1.2:

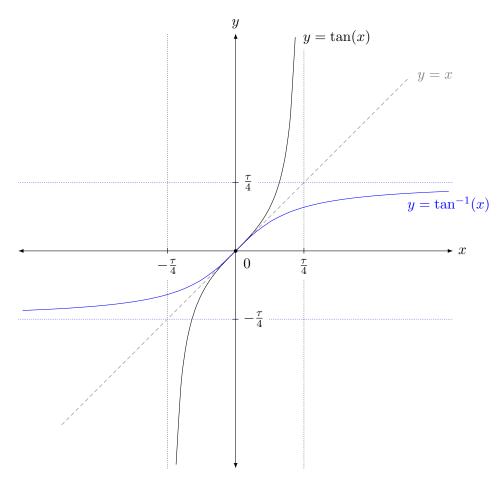


Figure 1.2: You can think about f^{-1} as the graph of f reflected about the line y=x