

Chapter 1

Implicit Differentiation and Inverses

1.1 Implicit Differentiation

We know:

$$\frac{d}{dx}x^a = ax^{a-1} \quad | \quad a \in \mathbb{Z}$$

We will now extend this formula to cover \mathbb{Q} as well:

$$a = \frac{m}{n} \rightarrow y = x^{\frac{m}{n}} \quad | \quad m, n \in \mathbb{Z}$$

We can start computing the derivative using the chain rule:

$$\begin{aligned} y^n &= x^m \\ \frac{d}{dx}(y^n) &= \frac{d}{dx}(x^m) \\ \frac{d}{dy}(y^n) \cdot \frac{dy}{dx} &= mx^{m-1} \\ \frac{dy}{dx}(ny^{n-1}) &= mx^{m-1} \end{aligned}$$

We finally have an expression for y' :

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{mx^{m-1}}{ny^{n-1}} \\
 &= \frac{m}{n} \cdot \frac{x^{m-1}}{y^{n-1}} \\
 &= \frac{m}{n} \cdot \frac{x^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1}} \\
 &= ax^{m-1-\frac{m}{n}(n-1)} \\
 &= ax^{a-1} \\
 \therefore \frac{d}{dx}x^a &= ax^{a-1} \quad | \quad a \in \mathbb{Q}
 \end{aligned}$$

Example 1.1. The equation of a unit circle is:

$$y^2 = 1 - x^2$$

This can be rewritten as:

$$y = (1 - x^2)^{\frac{1}{2}}$$

We can compute the derivative using the chain rule:

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{2} (1 - x^2)^{\frac{1}{2}-1} \cdot (-2x) \\
 &= \frac{-x}{(1 - x^2)^{\frac{1}{2}}} \\
 &= -\frac{x}{y}
 \end{aligned}$$

However, we can do the same thing using *implicit differentiation*:

$$\begin{aligned}
 x^2 + y^2 &= 1 \\
 \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\
 \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \\
 2x + \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} &= 0 \\
 2x + 2yy' &= 0 \\
 y' &= -\frac{x}{y}
 \end{aligned}$$

Example 1.2. In the following case, it is not so easy to solve for y :

$$y^3 + xy^2 + 1 = 0$$

We will need to use implicit differentiation to find the derivative:

$$\begin{aligned} 3y^2y' + y^2 + 2xyy' + 0 &= 0 \\ y'(3y^2 + 2xy) &= -y^2 \\ y' &= -\frac{y^2}{3y^2 + 2xy} \end{aligned}$$

1.2 Inverses

If $y = f(x)$ and $g(y) = x$, we call g the *inverse* of f , denoted f^{-1} :

$$x = g(y) = f^{-1}(y)$$

Now, we will use implicit differentiation to find the derivative of the inverse function:

$$\begin{aligned} y &= f(x) \\ f^{-1}(y) &= x \\ \frac{d}{dx}(f^{-1}(y)) &= \frac{d}{dx}(x) \\ \frac{d}{dy}(f^{-1}(y)) \cdot \frac{dy}{dx} &= 1 \\ \frac{d}{dy}(f^{-1}(y)) &= \frac{1}{\frac{dy}{dx}} \end{aligned}$$

Example 1.3. The derivative of $y = \tan^{-1}(x)$:

$$\begin{aligned} \tan y &= x \\ \frac{d}{dx}(\tan y) &= \frac{d}{dx}(x) \\ \frac{d}{dy}(\tan y) \cdot \frac{dy}{dx} &= 1 \\ (\csc^2 y) \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2 y \\ &= \cos^2(\tan^{-1}(x)) \end{aligned}$$

This form is messy but we can use geometry to simplify.

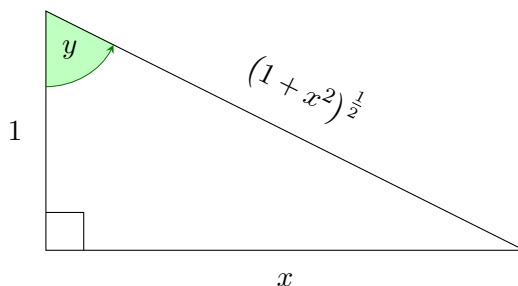


Figure 1.1: Triangle with angles and lengths corresponding to those in the example illustrating differentiation using the inverse function

In the triangle in Figure 1.1, $\tan y = x \Rightarrow y = \tan^{-1}(x)$. From this, we can find:

$$\begin{aligned}\cos y &= \frac{1}{\sqrt{1+x^2}} \\ (\cos y)^2 &= \left(\frac{1}{\sqrt{1+x^2}} \right)^2 \\ \cos^2 y &= \frac{1}{1+x^2} \\ \frac{dy}{dx} &= \frac{1}{1+x^2} \\ \frac{d}{dx} (\tan^{-1}(x)) &= \frac{1}{1+x^2}\end{aligned}$$

1.2.1 Graphing

Suppose $y = f(x)$ and $g(y) = f^{-1}(y) = x$.

To graph f and g together, we need to write g as a function of x . If $g(x) = y$, then $x = f(y)$. What we have done is trade the variables x and y . This is illustrated in Figure 1.2:

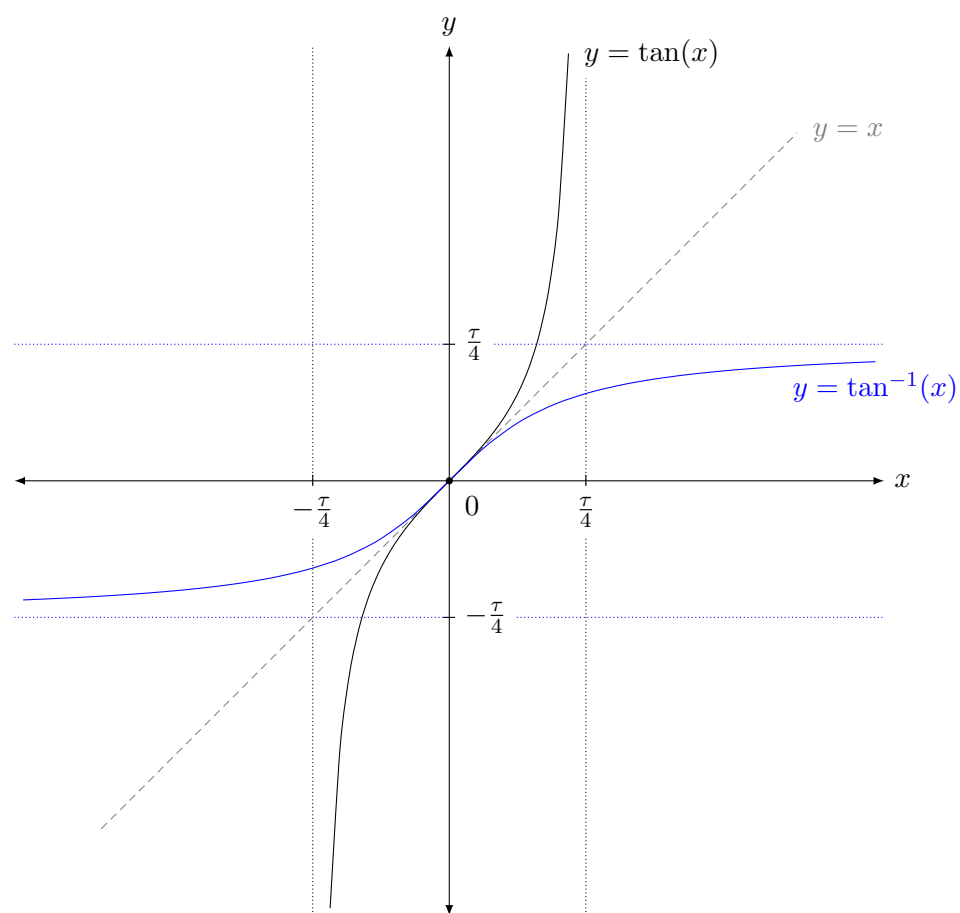


Figure 1.2: You can think about f^{-1} as the graph of f reflected about the line $y = x$