Chapter 1

Limits and Continuity

1.1 Easy Limits

With easy limits, you can get the answer simply by plugging in the limiting value:

$$\lim_{x \to 3} \frac{x^2 + x}{x + 1} = \frac{(3)^2 + (3)}{(3) + 1} = 3$$

1.2 Continuity

Definition 1.1. We say $f(x)$ is continuous at x_0 when:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Example 1.1. Consider the following *piecewise* function:

$$f(x) = \begin{cases} x + 1 & : \ x > 0 \\ -x & : \ x \leq 0 \end{cases}$$

It is said to be *discontinuous* (see Figure 1.1) at $x = 0$ since we know $f(0) = 0$, but for $x > 0$:

$$\lim_{x \to 0} f(x) = 1$$

We can say f is continuous from the left at $x = 0$, but not the right.

Definition 1.2 (Right-Hand Limit)**.**

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} f(x) \ : \ x > 0$$

Definition 1.3 (Left-Hand Limit)**.**

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) \ : \ x < 0$$

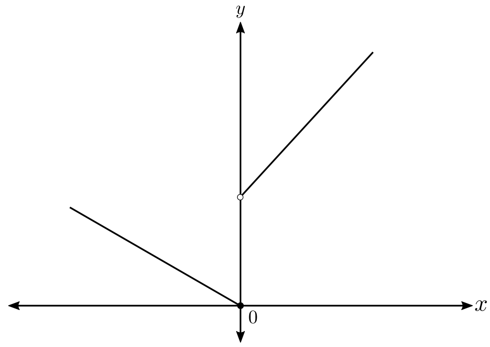

Figure 1.1: Graph of a discontinuous function

1.2.1 Discontinuities

Definition 1.4. A discontinuity is removable if:

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) \neq f(x_0)$$

Case. The function $\frac{\sin(x)}{x}$ is undefined for $x = 0$. However, its limit as $x \to 0$ still exists:

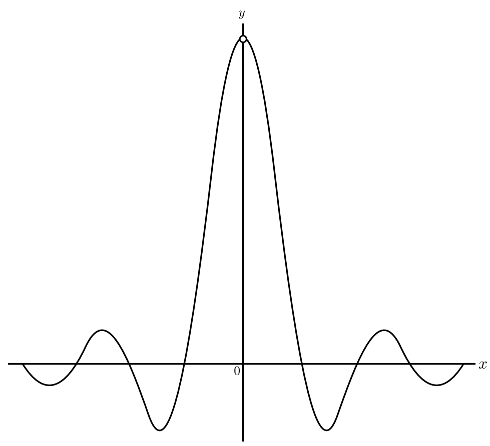

Figure 1.2: A removable discontinuity — the function is continuous everywhere except one point

Definition 1.5. A jump discontinuity is when $\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$ even if they both exist.

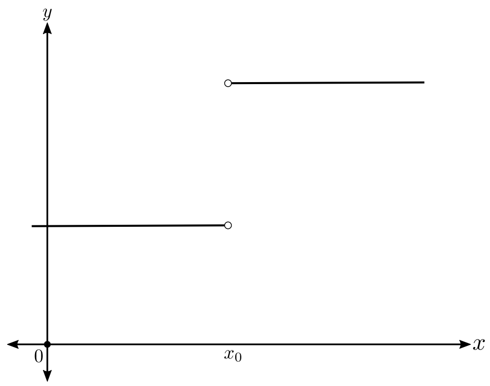

Figure 1.3: An example of a jump discontinuity

Definition 1.6. There is an infinite discontinuity when the right- and left-hand limits are both infinite, but in opposite directions.

Case. $f(x) = \frac{1}{x}$ has an infinite discontinuity (*singularity*) at $x = 0$:

$$\lim_{x \to x_0^+} \frac{1}{x} = -\infty \quad , \quad \lim_{x \to x_0^+} \frac{1}{x} = \infty$$

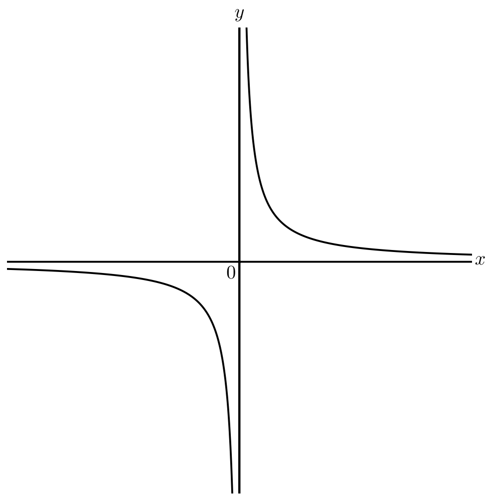

Figure 1.4: $f(x) = \frac{1}{x}$ is an example of an infinite discontinuity

Ugly Discontinuities

The function shown in Figure 1.5 doesn't even go to $\pm\infty$ — it doesn't make sense to say it "goes to" anything. For something like this, we say the limit does not exist (DNE).

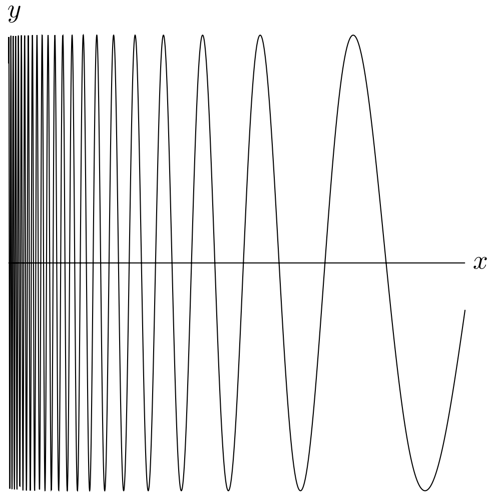

Figure 1.5: An example of an ugly discontinuity: a function that oscillates a lot as it approaches the origin

Theorem 1.1. *If f is differentiable at x_0, then f is continuous at x_0*

Proof. From Definition 1.1, we need to show:

$$\begin{aligned} &\lim_{x \to x_0} f(x) = f(x_0) \\ \Rightarrow &\lim_{x \to x_0} f(x) - f(x_0) = 0 \end{aligned}$$

The LHS can be rewritten as:

$$\begin{aligned} \lim_{x \to x_0} f(x) - f(x_0) &= \lim_{x \to x_0} f(x) - f(x_0) \cdot \frac{x - x_0}{x - x_0} \quad \because \quad \frac{x - x_0}{x - x_0} = 1 \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) \end{aligned}$$

If f is differentiable, we can use Equation **??** and rearrange to get:

$$\begin{aligned}\lim_{x\to x_0} f(x) - f(x_0) &= \lim_{x\to x_0} (x - x_0)\cdot f'(x_0)\\ &= (x_0 - x_0)\cdot f'(x_0)\\ &= 0\cdot f'(x_0)\\ &= 0\end{aligned}$$

□

Note. You can never divide by zero! The first step was to multiply by $\frac{x-x_0}{x-x_0}$. It looks as if this is illegal because we are multiplying by $\frac{0}{0}$ when $x = x_0$. But, when computing the limit as $x \to x_0$, we always assume $x \neq x_0$. In other words, $x - x_0 \neq 0$. Therefore, the proof is valid.