

## Chapter 1

# Implicit Differentiation and Inverses

### 1.1 Implicit Differentiation

We know:

$$\frac{d}{dx}x^a = ax^{a-1} \quad | \quad a \in \mathbb{Z}$$

We will now extend this formula to cover  $\mathbb{Q}$  as well:

$$a = \frac{m}{n} \rightarrow y = x^{\frac{m}{n}} \quad | \quad m, n \in \mathbb{Z}$$

We can start computing the derivative using the chain rule:

$$\begin{aligned} y^n &= x^m \\ \frac{d}{dx}(y^n) &= \frac{d}{dx}(x^m) \\ \frac{d}{dy}(y^n) \cdot \frac{dy}{dx} &= mx^{m-1} \\ \frac{dy}{dx}(ny^{n-1}) &= mx^{m-1} \end{aligned}$$

We finally have an expression for  $y'$ :

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{mx^{m-1}}{ny^{n-1}} \\
 &= \frac{m}{n} \cdot \frac{x^{m-1}}{y^{n-1}} \\
 &= \frac{m}{n} \cdot \frac{x^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1}} \\
 &= ax^{m-1-\frac{m}{n}(n-1)} \\
 &= ax^{a-1} \\
 \therefore \frac{d}{dx}x^a &= ax^{a-1} \quad | \quad a \in \mathbb{Q}
 \end{aligned}$$

**Example 1.1.1.** The equation of a unit circle is:

$$y^2 = 1 - x^2$$

This can be rewritten as:

$$y = (1 - x^2)^{\frac{1}{2}}$$

We can compute the derivative using the chain rule:

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{2} (1 - x^2)^{\frac{1}{2}-1} \cdot (-2x) \\
 &= \frac{-x}{(1 - x^2)^{\frac{1}{2}}} \\
 &= -\frac{x}{y}
 \end{aligned}$$

However, we can do the same thing using *implicit differentiation*:

$$\begin{aligned}
 x^2 + y^2 &= 1 \\
 \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\
 \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0 \\
 2x + \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} &= 0 \\
 2x + 2yy' &= 0 \\
 y' &= -\frac{x}{y}
 \end{aligned}$$

**Example 1.1.2.** In the following case, it is not so easy to solve for  $y$ :

$$y^3 + xy^2 + 1 = 0$$

We will need to use implicit differentiation to find the derivative:

$$\begin{aligned} 3y^2y' + y^2 + 2xyy' + 0 &= 0 \\ y'(3y^2 + 2xy) &= -y^2 \\ y' &= -\frac{y^2}{3y^2 + 2xy} \end{aligned}$$

## 1.2 Inverses

If  $y = f(x)$  and  $g(y) = x$ , we call  $g$  the *inverse* of  $f$ , denoted  $f^{-1}$ :

$$x = g(y) = f^{-1}(y)$$

Now, we will use implicit differentiation to find the derivative of the inverse function:

$$\begin{aligned} y &= f(x) \\ f^{-1}(y) &= x \\ \frac{d}{dx}(f^{-1}(y)) &= \frac{d}{dx}(x) \\ \frac{d}{dy}(f^{-1}(y)) \cdot \frac{dy}{dx} &= 1 \\ \frac{d}{dy}(f^{-1}(y)) &= \frac{1}{\frac{dy}{dx}} \end{aligned}$$

**Example 1.2.1.** The derivative of  $y = \tan^{-1}(x)$ :

$$\begin{aligned} \tan y &= x \\ \frac{d}{dx}(\tan y) &= \frac{d}{dx}(x) \\ \frac{d}{dy}(\tan y) \cdot \frac{dy}{dx} &= 1 \\ (\csc^2 y) \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2 y \\ &= \cos^2(\tan^{-1}(x)) \end{aligned}$$

This form is messy. We will use geometry to simplify: