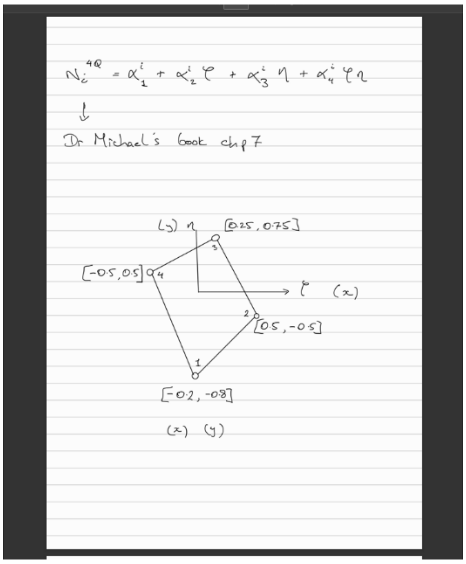
Problem 3 – Mathematics of Element Formulations showing analytical calculations

Part A) Derivation of Shape Functions for N_i where $i = 1,2,3,4$

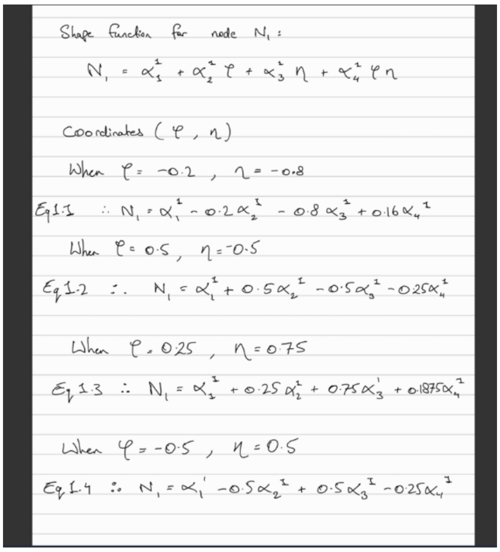


$N_i^{4Q} = \alpha_1^i + \alpha_2^i \xi + \alpha_3^i \eta + \alpha_4^i \xi\eta$

↓

Dr Michael's book chp 7

Figure 3.01: Derivation of Shape functions (page 1 of 9)



Shape function for node N_1:

$N_1 = \alpha_1^1 + \alpha_2^1 \xi + \alpha_3^1 \eta + \alpha_4^1 \xi\eta$

Coordinates (ξ, η)

When $\xi = -0.2$, $\eta = -0.8$

Eq 1.1 $\therefore N_1 = \alpha_1^1 - 0.2\alpha_2^1 - 0.8\alpha_3^1 + 0.16\alpha_4^1$

When $\xi = 0.5$, $\eta = -0.5$

Eq 1.2 $\therefore N_1 = \alpha_1^1 + 0.5\alpha_2^1 - 0.5\alpha_3^1 - 0.25\alpha_4^1$

When $\xi = 0.25$, $\eta = 0.75$

Eq 1.3 $\therefore N_1 = \alpha_1^1 + 0.25\alpha_2^1 + 0.75\alpha_3^1 + 0.1875\alpha_4^1$

When $\xi = -0.5$, $\eta = 0.5$

Eq 1.4 $\therefore N_1 = \alpha_1^1 - 0.5\alpha_2^1 + 0.5\alpha_3^1 - 0.25\alpha_4^1$

Figure 3.02: Derivation of Shape functions (page 2 of 9)

Eq 1.1 $1 = \alpha_1^1 - 0.2\alpha_2^1 - 0.8\alpha_3^1 + 0.16\alpha_4^1$

Eq 1.2 $0 = \alpha_1^1 + 0.5\alpha_2^1 - 0.5\alpha_3^1 - 0.25\alpha_4^1$

Eq 1.3 $0 = \alpha_1^1 + 0.25\alpha_2^1 + 0.75\alpha_3^1 + 0.1875\alpha_4^1$

Eq 1.4 $0 = \alpha_1^1 - 0.5\alpha_2^1 + 0.5\alpha_3^1 - 0.25\alpha_4^1$

Solved Simultaneously w/ Calculator

@ Node 1, $N_1 = 1$

Node 1 = 0 @ Nodes (2, 3, 4)

$\alpha_1^1 = 0.294985$, $\alpha_2^1 = -0.5162242$

$\alpha_3^1 = -0.5162242$, $\alpha_4^1 = 1.179941$

Figure 3.03: Derivation of Shape Functions (page 3 of 9)

Shape function for Node N_2

$$N_2 = \alpha_1^2 + \alpha_2^2 \varphi + \alpha_3^2 \eta + \alpha_4^2 \varphi \eta$$

When $\varphi = -0.2$, $\eta = -0.8$

Eq 2.1 $\therefore N_2 = \alpha_1^2 - 0.2\alpha_2^2 - 0.8\alpha_3^2 + 0.16\alpha_4^2$

When $\varphi = 0.5$, $\eta = -0.5$

Eq 2.2 $\therefore N_2 = \alpha_1^2 + 0.5\alpha_2^2 - 0.5\alpha_3^2 - 0.25\alpha_4^2$

When $\varphi = 0.25$, $\eta = 0.75$

Eq 2.3 $\therefore N_2 = \alpha_2^1 + 0.25\alpha_2^2 + 0.75\alpha_3^2 + 0.1875\alpha_4^2$

When $\varphi = -0.5$, $\eta = 0.5$

Eq 2.4 $\therefore N_2 = \alpha_2^1 - 0.5\alpha_2^2 + 0.5\alpha_3^2 - 0.25\alpha_4^2$

Figure 3.04: Derivation of Shape Functions (page 4 of 9)

Similar to Node 1 ... (Solving Simultaneously

@ Node 2, $N_2 = 0$

Node 2 = 0, @ Nodes (1, 3, 4)

$\alpha_2^1 = 0.5$, $\alpha_2^2 = -0.278523$

$\alpha_2^3 = 0.130872$, $\alpha_2^4 = -2.818792$

Figure 3.05: Derivation of Shape Functions (page 5 of 9)

Shape function for Node N_3

$N_3 = \alpha_1^3 + \alpha_2^3 \xi + \alpha_3^3 \eta + \alpha_4^3 \xi \eta$

When $\xi = -0.2$, $\eta = -0.8$

Eq 3.1 $\therefore N_3 = \alpha_1^3 - 0.2\alpha_2^3 - 0.8\alpha_3^3 + 0.16\alpha_4^3$

When $\xi = 0.5$, $\eta = -0.5$

Eq 3.2 $\therefore N_3 = \alpha_1^3 + 0.5\alpha_2^3 - 0.5\alpha_3^3 - 0.25\alpha_4^3$

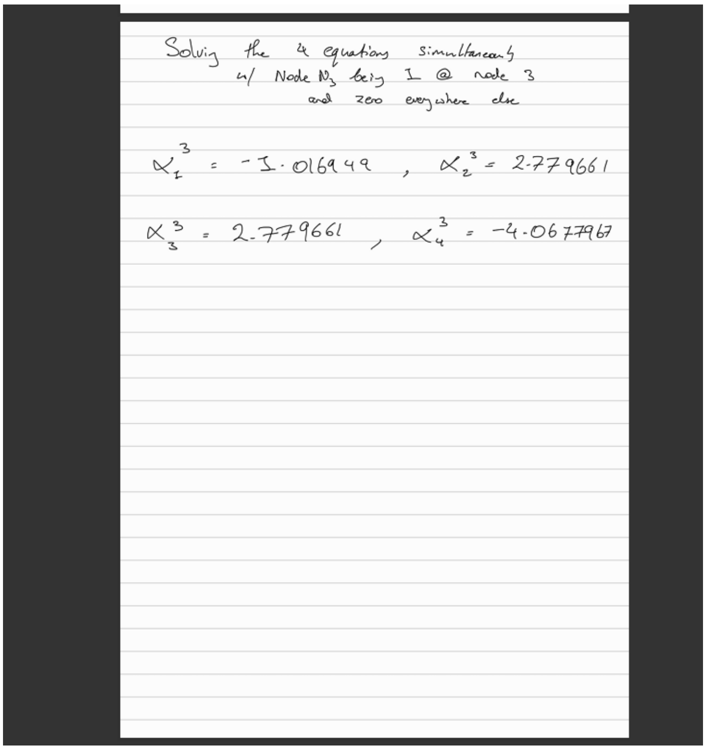
When $\xi = 0.25$, $\eta = 0.75$

Eq 3.3 $\therefore N_3 = \alpha_1^3 + 0.25\alpha_2^3 + 0.75\alpha_3^3 + 0.1875\alpha_4^3$

When $\xi = -0.5$, $\eta = 0.5$

Eq 3.4 $\therefore N_3 = \alpha_1^3 - 0.5\alpha_2^3 + 0.5\alpha_3^3 - 0.25\alpha_4^3$

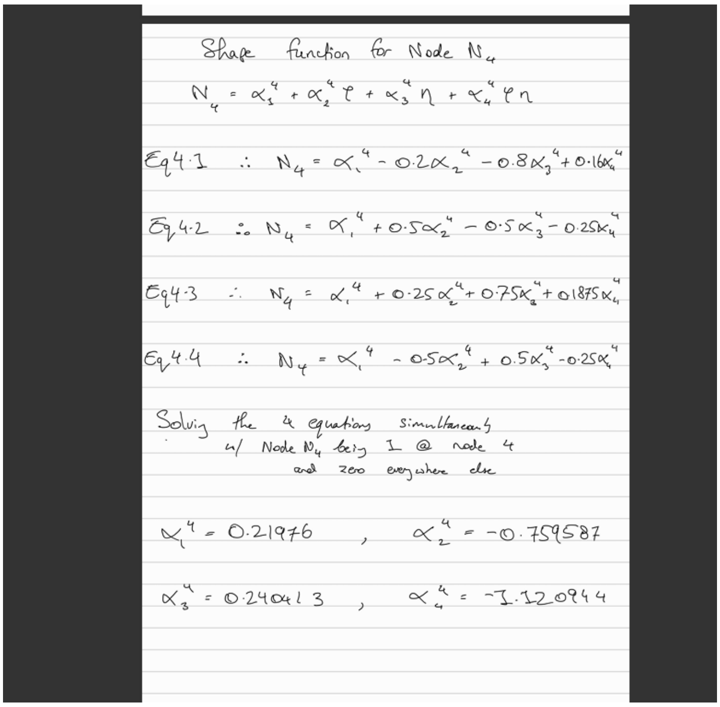
Figure 3.06: Derivation of Shape Functions (page 6 of 9)


Solving the 4 equations simultaneously
w/ Node N_3 being 1 @ node 3
and zero everywhere else

$\alpha_1^3 = -1.016949$, $\alpha_2^3 = 2.779661$

$\alpha_3^3 = 2.779661$, $\alpha_4^3 = -4.0677967$

Figure 3.07: Derivation of Shape Functions (page 7 of 9)


Shape function for Node N_4

$N_4 = \alpha_1^4 + \alpha_2^4 \xi + \alpha_3^4 \eta + \alpha_4^4 \xi \eta$

Eq4.1 $\therefore N_4 = \alpha_1^4 - 0.2\alpha_2^4 - 0.8\alpha_3^4 + 0.16\alpha_4^4$

Eq4.2 $\therefore N_4 = \alpha_1^4 + 0.5\alpha_2^4 - 0.5\alpha_3^4 - 0.25\alpha_4^4$

Eq4.3 $\therefore N_4 = \alpha_1^4 + 0.25\alpha_2^4 + 0.75\alpha_3^4 + 0.1875\alpha_4^4$

Eq4.4 $\therefore N_4 = \alpha_1^4 - 0.5\alpha_2^4 + 0.5\alpha_3^4 - 0.25\alpha_4^4$

Solving the 4 equations simultaneously
w/ Node N_4 being 1 @ node 4
and zero everywhere else

$\alpha_1^4 = 0.21976$, $\alpha_2^4 = -0.759587$

$\alpha_3^4 = 0.240413$, $\alpha_4^4 = -1.120944$

Figure 3.08: Derivation of Shape Functions (page 8 of 9)

The final equation (Shape Function)

$$N_i^{4Q} = \alpha_1^i + \alpha_2^i \varphi + \alpha_3^i \eta + \alpha_4^i \varphi\eta$$

$$N_1 = 0.295 - 0.516\varphi - 0.516\eta + 1.180\varphi\eta$$

$$N_2 = 0.5 - 0.279\varphi + 0.131\eta - 2.819\varphi\eta$$

$$N_3 = -1.017 + 2.780\varphi + 2.780\eta - 4.068\varphi\eta$$

$$N_4 = 0.220 - 0.760\varphi + 0.240\eta - 1.121\varphi\eta$$

Figure 3.09: Derivation of Shape functions (page 9 of 9)

Figures 3.01 to 3.09 show the step-by-step derivation of the shape functions.

Part B) Shape function plot

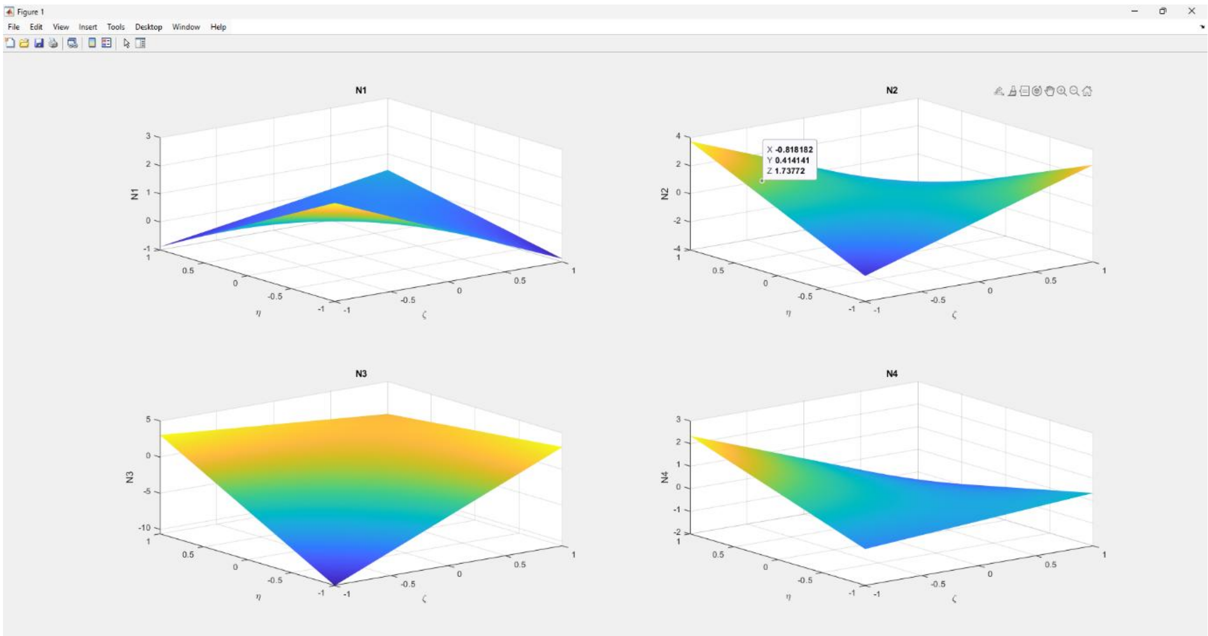


Figure 3.10: Shape Function plots

Part C) Derivation of Resulting Strain-Displacement Matrix, **B**

Strain-Displacement, B^e Matrix

$$e^e = \begin{bmatrix} \dfrac{\partial u^e}{\partial \xi} \\[2ex] \dfrac{\partial u^e}{\partial \eta} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dfrac{\partial N_1^e}{\partial \xi} u_1^e + \dfrac{\partial N_2^e}{\partial \xi} u_2^e + \dfrac{\partial N_3^e}{\partial \xi} + \dfrac{\partial N_4^e}{\partial \xi} u_4^e \\[2ex] \dfrac{\partial N_1^e}{\partial \eta} u_1^e + \dfrac{\partial N_2^e}{\partial \eta} u_2^e + \dfrac{\partial N_3^e}{\partial \eta} u_3^e + \dfrac{\partial N_4^e}{\partial \xi} u_4^e \end{bmatrix}$$

$$\Rightarrow e^e = \underbrace{\begin{bmatrix} \dfrac{\partial N_1^e}{\partial \xi} & \dfrac{\partial N_2^e}{\partial \xi} & \dfrac{\partial N_3^e}{\partial \xi} & \dfrac{\partial N_4^e}{\partial \xi} \\[2ex] \dfrac{\partial N_1^e}{\partial \eta} & \dfrac{\partial N_2^e}{\partial \eta} & \dfrac{\partial N_3^e}{\partial \eta} & \dfrac{\partial N_4^e}{\partial \eta} \end{bmatrix}}_{B^e} \underbrace{\begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \end{bmatrix}}_{u^e}$$

Figure 3.11: Derivation of B Matrix (page 1 of 4)

$$N_1 = 0.295 - 0.516\xi - 0.516\eta + 1.180\xi\eta$$

$$N_2 = 0.5 - 0.279\xi + 0.131\eta - 2.819\xi\eta$$

$$N_3 = -1.017 + 2.780\xi + 2.780\eta - 4.068\xi\eta$$

$$N_4 = 0.220 - 0.760\xi + 0.240\eta - 1.121\xi\eta$$

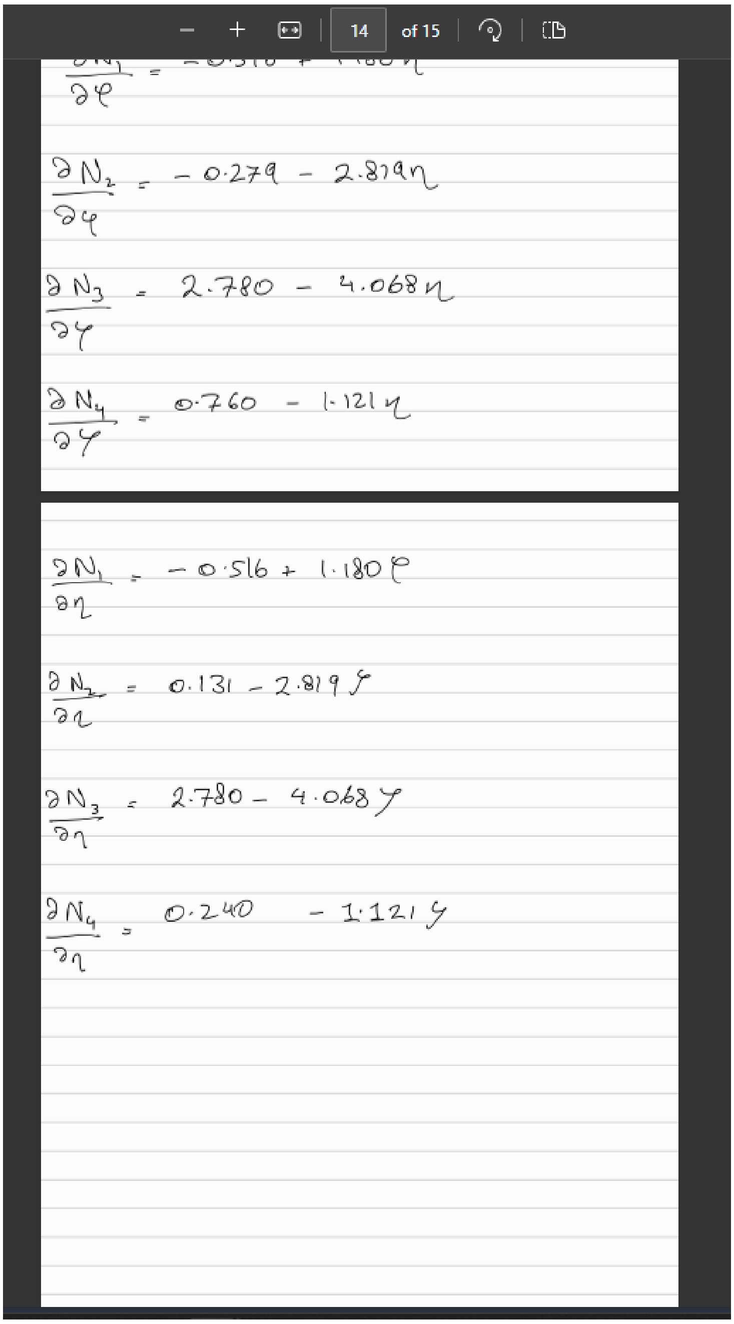
$$\frac{\partial N_1}{\partial \xi} = -0.516 + 1.180\eta$$

$$\frac{\partial N_2}{\partial \xi} = -0.279 - 2.819\eta$$

$$\frac{\partial N_3}{\partial \xi} = 2.780 - 4.068\eta$$

$$\frac{\partial N_4}{\partial \xi} = 0.760 - 1.121\eta$$

Figure3.12: Derivation of B Matrix (page 2 of 4)



$$\frac{\partial N_2}{\partial \xi} = -0.279 - 2.819\eta$$

$$\frac{\partial N_3}{\partial \xi} = 2.780 - 4.068\eta$$

$$\frac{\partial N_4}{\partial \xi} = 0.760 - 1.121\eta$$

$$\frac{\partial N_1}{\partial \eta} = -0.516 + 1.180\xi$$

$$\frac{\partial N_2}{\partial \eta} = 0.131 - 2.819\xi$$

$$\frac{\partial N_3}{\partial \eta} = 2.780 - 4.068\xi$$

$$\frac{\partial N_4}{\partial \eta} = 0.240 - 1.121\xi$$

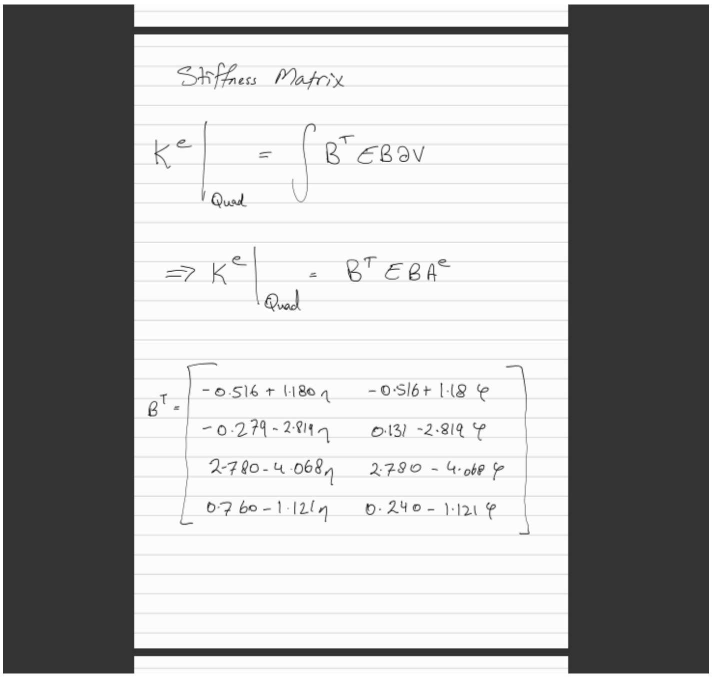
Figure 3.13: Derivation of B Matrix (page 3 of 4)

$$
\begin{bmatrix}
-0.516 + 1.180\eta & -0.279 - 2.819\eta & 2.780 - 4.068\eta & 0.760 - 1.121\eta \\
-0.516 + 1.180\varphi & 0.131 - 2.819\varphi & 2.780 - 4.068\varphi & 0.240 - 1.121\varphi
\end{bmatrix}
$$

B Matrix

Figure 3.14: Derivation of B Matrix (page 4 of 4)

Part D) Stiffness Matrix, K



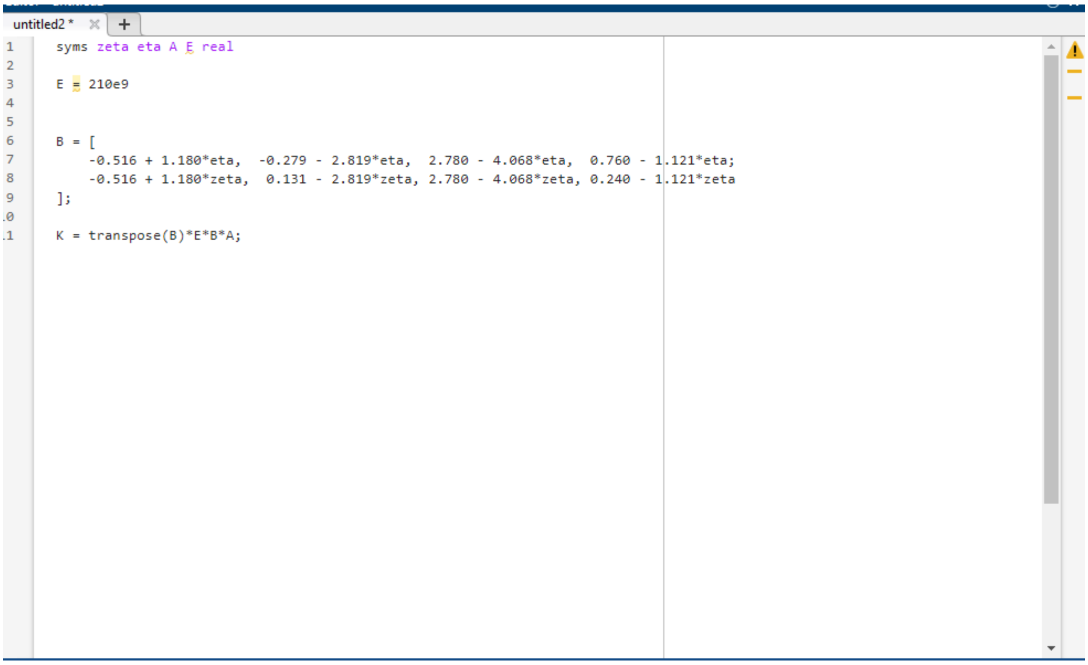
Stiffness Matrix

$$K^e\Big|_{Quad} = \int B^T E B \, \partial V$$

$$\Rightarrow K^e\Big|_{Quad} = B^T E B A^e$$

$$B^T = \begin{bmatrix} -0.516 + 1.180\eta & -0.516 + 1.18\varphi \\ -0.279 - 2.819\eta & 0.131 - 2.819\varphi \\ 2.780 - 4.068\eta & 2.780 - 4.068\varphi \\ 0.760 - 1.121\eta & 0.240 - 1.121\varphi \end{bmatrix}$$

Figure 3.15: derivation of stiffness Matrix (page 1 of 1)



untitled2 *

```
syms zeta eta A E real

E = 210e9


B = [
    -0.516 + 1.180*eta,  -0.279 - 2.819*eta,  2.780 - 4.068*eta,  0.760 - 1.121*eta;
    -0.516 + 1.180*zeta,  0.131 - 2.819*zeta, 2.780 - 4.068*zeta, 0.240 - 1.121*zeta
];

K = transpose(B)*E*B*A;
```

Figure 3.16: Matlab formulation of stiffness matrix, K