Advanced Applications In Inertial Navigation Systems

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Navigation Frame EOM 1

- Navigation frame equations:

Position	$\dot{r}^n = Dv^n$
Velocity	$\dot{v}^n = T_b^n f^b + g^n - \left(\Omega_{en}^n + 2\Omega_{ie}^n\right) v^n$
Orientation	$\dot{T}_b^n = T_b^n \Omega_{ib}^b - \left(\Omega_{ie}^n + \Omega_{en}^n\right) T_b^n$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_M + h} & 0 & 0 \\ 0 & \frac{1}{\cos(\phi)\left[R_N + h\right]} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_n \\ v_e \\ v_d \end{bmatrix} = Dv^n$$

Meridian radius

$$R_M = \frac{R_e(1 - e^2)}{(1 - e^2 \sin^2(\phi))^{3/2}}$$

$$R_N = \frac{R_e}{(1 - e^2 \sin^2(\phi))^{1/2}}$$

Normal radius

Navigation Frame EOM 2

- Summarize navigation frame equations:

Position	$\dot{r}^n = Dv^n$
Velocity	$\dot{v}^n = T_b^n f_{ib}^b + g^n - \left(\Omega_{en}^n + 2\Omega_{ie}^n\right)v^n$
Orientation	$\dot{T}_b^n = T_b^n \Omega_{ib}^b - \left(\Omega_{ie}^n + \Omega_{en}^n\right)T_b^n$

$$\omega_{en}^n = \begin{bmatrix} \dfrac{v_e}{R_N + h} \\ -\dfrac{v_n}{R_M + h} \\ -\dfrac{v_e \tan(\phi)}{R_N + h} \end{bmatrix} \quad \omega_{ie}^n = \begin{bmatrix} \omega_{ie}\cos(\phi) \\ 0 \\ -\omega_{ie}\sin(\phi) \end{bmatrix} \quad g^n = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \quad f_{ib}^b = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

Navigation Frame EOM 3

- Summarize navigation frame equations:

Position	$\dot{r}^n = Dv^n$
Velocity	$\dot{v}^n = T_b^n f_{ib}^b + g^n - \left(\Omega_{en}^n + 2\Omega_{ie}^n\right)v^n$
Orientation	$\dot{T}_b^n = T_b^n \Omega_{ib}^b - \left(\Omega_{ie}^n + \Omega_{en}^n\right)T_b^n$

$$\omega_{ib}^b = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Error State Dynamic Equations

- Navigation error are due:
  - Initial condition errors
  - Accumulation of instrumentation errors the integration errors.
- Derive a state-space model for the navigation error vector:

$$\begin{bmatrix} \delta \dot{p}^n \\ \delta \dot{v}^n \\ \dot{\varepsilon}^n \end{bmatrix} = \begin{bmatrix} F_{rr} & F_{rv} & F_{re} \\ F_{vr} & F_{vv} & F_{ve} \\ F_{er} & F_{ev} & F_{ee} \end{bmatrix} \begin{bmatrix} \delta p^n \\ \delta v^n \\ \varepsilon^n \end{bmatrix} + \begin{bmatrix} 0_{3\times3} & 0_{3\times3} \\ T^{b\rightarrow n} & 0_{3\times3} \\ 0_{3\times3} & T^{b\rightarrow n} \end{bmatrix} \begin{bmatrix} \delta f \\ \delta\omega \end{bmatrix}$$

- where $\delta f, \delta\omega$ are sensor measurement residuals which are represented as white noise.

Position Error States

$$F_{rr} = \begin{bmatrix} 0 & 0 & \dfrac{-v_{n\,ins}}{\left(R_M + h_{ins}\right)^2} \\ \dfrac{v_{e\,ins}\sin\left(\phi_{ins}\right)}{\cos^2\left(\phi_{ins}\right)\left[R_N + h_{ins}\right]} & 0 & \dfrac{-v_{e\,ins}}{\cos\left(\phi_{ins}\right)\left[R_N + h_{ins}\right]^2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_{rv} = \begin{bmatrix} \dfrac{1}{\left(R_M + h_{ins}\right)} & 0 & 0 \\ 0 & \dfrac{1}{\cos\left(\phi_{ins}\right)\left[R_N + h_{ins}\right]} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$F_{r\varepsilon} = 0_{3\times3}$$

Velocity Error States

$$F_{vr}=\begin{bmatrix} -2v_e\omega_{ie}\cos(\phi)-\dfrac{v_e^2}{(R_N+h)\cos^2(\phi)} & 0 & -\dfrac{v_d v_n}{(R_M+h)^2}+\dfrac{v_e^2\tan(\phi)}{(R_N+h)^2} \\ 2\omega_{ie}\left(-v_d\sin(\phi)+v_n\cos(\phi)\right)+\left[\dfrac{v_n v_e}{(R_N+h)\cos^2(\phi)}\right] & 0 & -\dfrac{v_d v_e}{(R_N+h)^2}-\dfrac{v_n v_e\tan(\phi)}{(R_N+h)^2} \\ 2v_e\omega_{ie}\sin(\phi) & 0 & \dfrac{v_n^2}{(R_M+h)^2}+\dfrac{v_e^2}{(R_N+h)^2}-\dfrac{2g}{R_e+h} \end{bmatrix}$$

$$F_{vv}=\begin{bmatrix} \dfrac{v_d}{(R_M+h)} & -\dfrac{2v_e\tan(\phi)}{(R_N+h)}-2\omega_{ie}\sin(\phi) & \dfrac{v_n}{R_M+h} \\ \left(2\omega_{ie}\sin(\phi)+\dfrac{v_e\tan(\phi)}{R_N+h}\right) & \dfrac{v_d}{(R_N+h)}+\dfrac{v_n\tan(\phi)}{(R_N+h)} & \left(2\omega_{ie}\cos(\phi)+\dfrac{v_e\tan(\phi)}{R_N+h}\right) \\ -\dfrac{2v_n}{R_M+h} & \left(-2\omega_{ie}\cos(\phi)-\dfrac{2v_e}{R_N+h}\right) & 0 \end{bmatrix}$$

$$F_{v\varepsilon}=-\left[f_{ins}^{n}\times\right]$$

Misalignment Error States

$$F_{\varepsilon r} = \begin{bmatrix} \omega_{ie}\sin(\phi) & 0 & \dfrac{v_e}{(R_N+h)^2} \\ 0 & 0 & -\dfrac{v_n}{(R_M+h)^2} \\ \omega_{ie}\cos(\phi) + \dfrac{v_e}{(R_N+h)\cos^2(\phi)} & 0 & -\dfrac{v_e\tan(\phi)}{(R_N+h)^2} \end{bmatrix}$$

$$F_{\varepsilon v} = \begin{bmatrix} 0 & -\dfrac{1}{(R_N+h)} & 0 \\ \dfrac{1}{(R_M+h)} & 0 & 0 \\ 0 & \dfrac{\tan(\phi)}{(R_N+h)} & 0 \end{bmatrix}$$

$$F_{\varepsilon\varepsilon} = -\left[\omega_{in}^{n} \times\right]$$

Augmented Error State

$$\delta f = b_a(t) + w_a, \dot{b}_a(t) = -\frac{1}{\tau_a} b_a(t) + w_{agm}$$

$$\delta\omega = b_g(t) + w_g, \dot{b}_g(t) = -\frac{1}{\tau_g} b_g(t) + w_{ggm}$$

$$\begin{bmatrix} \dot{b}_a \\ \dot{b}_g \end{bmatrix} = \begin{bmatrix} F_a & 0 \\ 0 & F_g \end{bmatrix} \begin{bmatrix} b_a \\ b_g \end{bmatrix} + \begin{bmatrix} I_3 & 0 \\ 0 & I_3 \end{bmatrix} \begin{bmatrix} w_{agm} \\ w_{ggm} \end{bmatrix}$$

$$\begin{bmatrix} \delta\dot{p}^n \\ \delta\dot{v}^n \\ \dot{\varepsilon}^n \\ \dot{b}_a \\ \dot{b}_g \end{bmatrix} = \begin{bmatrix} F_{rr} & F_{rv} & F_{re} & 0_{3\times3} & 0_{3\times3} \\ F_{vr} & F_{vv} & F_{ve} & T^{b\to n} & 0_{3\times3} \\ F_{er} & F_{ev} & F_{ee} & 0_{3\times3} & T^{b\to n} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & F_a & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & F_g \end{bmatrix} \begin{bmatrix} \delta p^n \\ \delta v^n \\ \varepsilon^n \\ b_a \\ b_g \end{bmatrix} + \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ T^{b\to n} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & T^{b\to n} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_3 & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_3 \end{bmatrix} \begin{bmatrix} w_a \\ w_g \\ w_{agm} \\ w_{ggm} \end{bmatrix}$$

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Thanks!

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