V4A2 – ALGEBRAIC GEOMETRY II
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Preliminaries

These notes roughly correspond to the course **V4A2 – Algebraic Geometry II** taught by Prof. Daniel Huybrechts at the Universität Bonn in the Summer 2025 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. Knowledge of commutative algebra, topology, and category theory will be assumed.

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1. Lecture 1 – 7th April 2025

We begin by a consideration of the theory of smoothness, first in the local case. This is done by defining the sheaves of Kähler differentials on schemes – in the local picture, the module of differentials on a ring.

Definition 1.1 (Derivation)**.** Let B be an A-algebra and M a B-module. An morphism of A-modules $D : B \to M$ is an A-derivation if it satisfies the Leibniz rule $\mathrm{d}(xy) = x\mathrm{d}(y) + y\mathrm{d}(x)$ for all $x, y \in B$. Denote the set of A-derivations in M by $\mathrm{Der}_A(B, M)$.

Remark 1.2. It is necessary that M is a B-module, since the Leibniz rule involves elements of B.

Remark 1.3. Observe that the composition $A \to B \to M$is zero since $a = a \cdot 1_B$ and computing we get $\mathrm{d}(a \cdot 1_B) = a\mathrm{d}(1_B)$ by A-linearity, but on the other hand $\mathrm{d}(a \cdot 1_B) = a\mathrm{d}(1_B) + 1_B\mathrm{d}(a)$ by the Leibniz rule, so $a\mathrm{d}(1_B) = 0$ showing $\mathrm{d}(1_B) = 0$ and thus $\mathrm{d}(a) = 0$.

The Kähler differentials of a ring map is the universal recipient of an A-algebra B in the following sense.

Definition 1.4 (Module of Kähler Differentials)**.** Let B be an A-algebra. The module of Kähler differentials of B over A is a B-module $\Omega^1_{B/A}$ with an A-derivation $\mathrm{d} : B \to \Omega^1_{B/A}$ that is initial amongst B-modules recieving an A-derivation from B.

Unwinding the universal property, if M is a B-module recieving an A-derivation from B by $f : B \to M$, there is a unique factorization over $\Omega^1_{B/A}$ as follows.

$$\begin{tikzcd}
\Omega^1_{B/A} \arrow[r] & M \\
B \arrow[u, dashed, "\exists!"] \arrow[ur] &
\end{tikzcd}$$

In particular, there is a bijection $\mathrm{Der}_A(B, M) \leftrightarrow \mathrm{Hom}_{\mathsf{Mod}_B}(\Omega_{B/A}, M)$ functorial in M.

Proposition 1.5. Let B be an A-algebra. The B-module $\Omega^1_{B/A}$ and the A-derivation $\mathrm{d} : B \to \Omega^1_{B/A}$ exist and are unique up to unique isomorphism.

Proof. The module $\Omega^1_{B/A}$ can be constructed as the free B-module on elements $\mathrm{d}x$ for $x \in B$ modulo the relations generated by the Leibniz rule and $\mathrm{d}a = 0$ for $a \in A$. Uniqueness up to unique isomorphism is clear from the universal property and Yoneda's lemma. ■

In special cases, the module of Kähler differentials can be described explicitly.

Example 1.6. Let $A = k, B = k[x_1, \ldots, x_n]$. $\Omega^1_{B/A}$ is a free module of rank n with basis $\mathrm{d}x_i$. The map $f \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \mathrm{d}x_i$ is an A-derivation and the map $\mathrm{d}x_i \mapsto x_i$ defines an isomorphism $\Omega^1_{B/A} \to B^{\oplus n}$.

Kähler differentials are also fairly easy to understand in the case of ring localizations and ring quotients. These will be important in understanding the sheaves of Kähler differentials of open and closed immersions in the case of schemes, respectively.

Proposition 1.7. Let A be a ring.

(i) If $B = S^{-1}A$, then $\Omega^1_{B/A} = 0$.

(ii) If $B = A/I$ for $I \subseteq A$ an ideal, then $\Omega^1_{B/A} = 0$.

Proof of (i). We already have that $\mathrm{d}a = 0$ for all $a \in A$. We then observe that writing $a = s \cdot \frac{a}{s}$ we have

$$\begin{aligned} 0 = \mathrm{d}(a) = \mathrm{d}\left(s \cdot \frac{a}{s}\right) &= s\mathrm{d}\left(\frac{a}{s}\right) + \frac{a}{s}\mathrm{d}(s) \\ &= s\mathrm{d}\left(\frac{a}{s}\right) && s \in A \Rightarrow \mathrm{d}s = 0 \end{aligned}$$

so $s\mathrm{d}(\frac{a}{s}) = 0$ and $\mathrm{d}(\frac{a}{s}) = 0$ whence the claim. ■

Proof of (ii). The map $A \to B$ is surjective, so this is precisely the situation Remark 1.3. ■

We can additionally understand sheaves of Kähler differentials in towers. Let $A \to B \to C$ be maps of rings. There is a natural C-linear map $\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$ which is a C-module homomorphism induced by the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\mathrm{d}_{B/A}"'] & C \arrow[r, "\mathrm{d}_{C/A}"] & \Omega^1_{C/A} \\ \Omega^1_{B/A} \arrow[urr, dashed, "\exists!"'] & & \end{tikzcd}$$

where the top row is both A and B-linear inducing a unique B-module map $\Omega^1_{B/A} \to \Omega^1_{C/A}$, considering the latter as a B-module. By the extension-restriction adjunction, however, we have

$$\mathrm{Hom}_{\mathsf{Mod}_B}(\Omega_{B/A}, \Omega_{C/A}|_B) \leftrightarrow \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega_{B/A} \otimes_B C, \Omega_{C/A})$$

hence the data of the dotted map in the diagram above gives rise to a unique map $\Omega_{B/A} \otimes_B C \to \Omega_{C/A}$. Arguing similarly, there is a C-linear map $\Omega^1_{C/A} \to \Omega^1_{C/B}$ induced by

$$\begin{tikzcd} C \arrow[r, "\mathrm{d}_{C/B}"] \arrow[d, "\mathrm{d}_{C/A}"'] & \Omega^1_{C/B} \\ \Omega^1_{C/A} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

where the map is induced by the universal property as any B-derivation is also an A-derivation.

The maps in the preceding discussion assemble to give the following proposition.

Proposition 1.8. Let $A \to B \to C$ be maps of rings. There is an exact sequence

$$\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0.$$

Proof. The above discussion gives the existence of such maps, so it remains to show exactness at $\Omega^1_{C/A}$ and surjectivity of the map $\Omega^1_{C/A} \to \Omega^1_{C/B}$.

We begin with the latter, where by the quotient construction of Proposition 1.5 it suffices to observe that $\Omega^1_{C/B}$ is a quotient of $\Omega^1_{C/A}$.

For the former, we note that for a fixed C-module M we have an exact sequence

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M|_B)$$

since an A-derivation $(\mathrm{d} : C \to M)$ is taken to the composite $B \to C \to M$ which is zero when the map is also a B-derivation. Rewriting this using the universal property, this is

$$0 \to \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega^1_{C/B}, M) \to \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega^1_{C/A}, M) \to \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega^1_{B/A} \otimes_B C, M)$$

which by contravariant exatness of the Hom-functor (see [Stacks, Tag 0582] for the precise statement), is the claim. ∎

As a corollary, we can deduce the following fact about localizations.

Corollary 1.9. Let B be an A-algebra and S a multiplicative subset of B. Then $S^{-1}\Omega^1_{B/A} \cong \Omega^1_{S^{-1}B/A}$.

Proof. Apply Proposition 1.8 to $C = S^{-1}B$ and note that $\Omega^1_{C/B} = 0$ so the map $S^{-1}\Omega^1_{B/A} \to \Omega^1_{S^{-1}B/A}$ is surjective. To prove injectivity, we produce an inverse map which is an A-derivation of $S^{-1}B$ to $S^{-1}\Omega^1_{B/A}$ by $\mathrm{d}(\frac{b}{s}) \mapsto \frac{1}{s}\mathrm{d}(b) - \frac{1}{s^2}b\mathrm{d}(s)$ which by the universal property can be seen to be the inverse. ∎

Note that in general $\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$ is rarely injective.

Example 1.10. Let $A = k, B = k[x], C = k[x]/(x)$. So $\Omega^1_{B/A} \cong B\mathrm{d}x$ but $\Omega_{C/A} = \Omega_{k/k} = 0$.

On the other hand, there are situations in which the exact sequence of Proposition 1.8 extends to a short exact sequence.

Example 1.11. Let B be an A-algebra and $C = B[x_1, \ldots, x_n]$. We then have a split short exact sequence

$$0 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

where denoting the map $\Omega^1_{C/A} \to \Omega^1_{C/B}$ by φ, we have the splitting $\Omega^1_{C/A} \to (\Omega^1_{B/A} \otimes_B C) \oplus \Omega^1_{C/B}$ prescribed by the C-derivation $f \mapsto \mathrm{d}_{B/A}(f) + \varphi(f)$ under the bijection

$$\mathrm{Hom}_{\mathsf{Mod}_C}(\Omega_{C/A}, (\Omega_{B/A} \otimes_B C) \oplus \Omega_{C/B}) \leftrightarrow \mathrm{Der}_A(C, (\Omega_{B/A} \otimes_B C) \oplus \Omega_{C/B}).$$

The following proposition describes the behavior of the module of Kähler differentials with respect to tensor products.

Proposition 1.12. Let B, A' be A-algebras. Then there is an isomorphism of B-modules $\Omega^1_{B/A} \otimes_B (B \otimes_A A') \cong \Omega^1_{(B \otimes_A A')/A'}$.

Proof. We contemplate the diagram

$$\begin{tikzcd} B \otimes_A A' \arrow[r] \arrow[d] & \Omega^1_{B/A} \otimes_A A' \\ \Omega^1_{(B\otimes_A A')/A'} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

where the solid arrows are $B \otimes_A A'$-linear with $\Omega^1_{B/A} \otimes_A A' \cong (\Omega^1_{B/A} \otimes_B (B \otimes_A A'))$ and the dotted arrow induced by the universal property of $\Omega^1_{(B \otimes_A A')/A'}$. By applying the tensor-hom adjunction and the universal property of derivations, prescribing an inverse map to the dotted arrow is equivalent to producing an A-derivation of B in $\Omega^1_{(B \otimes_A A')/A'}$ and one observes that the map $b \mapsto \mathrm{d}_{(B \otimes_A A')/A'}(b \otimes 1)$ gives an inverse, whence the claim. ■

We now treat the case of quotients.

Proposition 1.13. Let $A \to B \to C$ be maps of rings where $C \cong B/\mathfrak{b}$ for some ideal $\mathfrak{b} \subseteq B$. There is an exact sequence

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to 0.$$

Proof. We first observe that $\Omega^1_{C/B} = 0$ by Proposition 1.7 and $\Omega^1_{B/A} \otimes_B C \cong \Omega^1_{B/A}/\mathfrak{b}\Omega^1_{B/A}$.

We denote $\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B/A} \otimes_B C$ by δ, $b \mapsto \mathrm{d}b \otimes 1$. We first show δ is well-defined. For this, we want to show that $\mathrm{d}(b_1 b_2) \otimes 1$ is zero for $b_1, b_2 \in \mathfrak{b}$. Indeed, using the Leibniz rule, we have

$$\mathrm{d}(b_1 b_2) \otimes 1 = \mathrm{d}(f_2) \otimes f_1 + \mathrm{d}(f_1) \otimes f_2 \in \mathfrak{b}\Omega_{B/A}$$

hence zero in the quotient, showing the map is well-defined.

The diagram is a complex as db maps to zero in $\Omega^1_{C/A}$. The kernel of $\Omega^1_{B/A} \to \Omega^1_{C/A}$ is generated by the B-submodule $\mathfrak{b}\Omega^1_{B/A}$ and the elements $\mathrm{d}b$ for $b \in \mathfrak{b}$, showing exactness of the complex in the middle. ■

This specializes to finite type algebras.

Corollary 1.14. Let C be a finite type A-algebra – that is, the quotient of $B = A[x_1, \ldots, x_n]$. Then $\Omega^1_{C/A}$ is a finitely generated C-module.

Proof. Set $B = A[x_1, \ldots, x_n]$ for which $C = B/\mathfrak{b}$. Exactness of the sequence in Proposition 1.13 gives a surjection $\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$, and observing that $\Omega^1_{B/A} \otimes_B C \cong B^{\oplus n} \otimes_B C \cong C^{\oplus n}$ gives a surjection $C^{\oplus n} \to \Omega^1_{C/A}$, showing that it is finitely generated. ■

Let us consider the case of quotients of multivariate polynomial rings by a single polynomial.

Example 1.15. Let A be a ring, $B = A[x_1, \ldots, x_n]$, and $C = B/(f)$ for $f \in B$. By Proposition 1.13 and Corollary 1.14, we have that $\Omega^1_{C/A}$ is the cokernel of the map $\delta : (f)/(f)^2 \to \Omega^1_{B/A} \otimes_B C \cong C^{\oplus n}$ of Proposition 1.13, so is the quotient $\left(\bigoplus_{i=1}^n C\mathrm{d}x_i\right)/\mathrm{d}f$.

We can also consider the case of k-algebras.

Corollary 1.16. Let A be a k-algebra and $\mathfrak{m}$ a maximal ideal in A such that $\kappa(\mathfrak{m}) = A/\mathfrak{m} \cong k$. Then $\Omega^1_{A/k} \otimes_k \kappa(\mathfrak{m}) \cong \mathfrak{m}/\mathfrak{m}^2$.

Proof. This is precisely Proposition 1.13 for $k \to k[x_1, \ldots, x_n] \to A$, and the map $\mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{A/k} \otimes_k \kappa(\mathfrak{m})$ is a surjection between vector spaces of the same dimension, hence an isomorphism. ■

Note that this is the dual of the Zariski tangent space $\mathrm{Hom}_{\mathsf{Vec}_{\kappa(\mathfrak{m})}}(\mathfrak{m}/\mathfrak{m}^2, \kappa(\mathfrak{m}))$, motivating the connection to schemes.

2. Lecture 2 – 10th April 2025

We begin with an example.

Example 2.1. Let $A = k, B = k[x,y], C = B/\mathfrak{b}$ where $\mathfrak{b} = (xy)$. We have that $\mathrm{Spec}(B)$ is the affine plane $\mathbb{A}^2_k$ and $\mathrm{Spec}(C)$ is the union of the two coordinate axes. The exact sequence of Proposition 1.13 gives

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{k[x,y]/k} \otimes_{k[x,y]} C \to \Omega^1_{C/k} \to 0.$$

Explicitly identifying $\mathfrak{b}/\mathfrak{b}^2$ with the C-module $(xy)/(x^2y^2)$ module-isomorphic to C by $1 \mapsto xy$ and $\Omega^1_{k[x,y]/k}$ with the free $k[x,y]$-module $B\mathrm{d}x \oplus B\mathrm{d}y$, we observe that the map $\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{k[x,y]/k} \otimes_{k[x,y]} C$ is given by $\overline{xy} \mapsto \mathrm{d}(xy) \otimes 1 = (x\mathrm{d}y + y\mathrm{d}x) \otimes 1$. This yields a map $C \to C \oplus C$ by $1 \mapsto (y,x)$ and the cokernel of this map is the kernel of the map $C \oplus C \to C$ by $(a,b) \mapsto ax - by$ so by exactness the image of $C \oplus C \to C$ is the ideal $(x,y) \subseteq C$ showing $\Omega^1_{C/k} \cong (x,y) \subseteq C$. Thus $\Omega_{C/k} \otimes_C \frac{k[x,y]}{(x,y)} \cong kx \oplus ky$ and in particular $\Omega^1_{C/k} \otimes_C k$ is of k-dimension 2. For all points $\mathfrak{p} \in \mathrm{Spec}(C) \setminus \{(x,y)\}$, we have $\Omega_{C/k} \otimes_C \kappa(\mathfrak{p}) \cong k$ extending the exact sequence above to a short exact sequence.

In what follows, we will use the following lemma for Kähler differentials of field extensions, the proof of which we omit.

Lemma 2.2. Let k be a field and K/k a separable extension. Then $\Omega^1_{K/k} = 0$.

This lemma, in conjunction with Proposition 1.13, shows that

References

[Stacks] The Stacks project authors. *The Stacks project.* `https://stacks.math.columbia.edu`. 2024.

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