V5A4 – HABIRO COHOMOLOGY
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Preliminaries

These notes roughly correspond to the course **V5A4 – Habiro Cohomology** taught by Prof. Peter Scholze at the Universität Bonn in the Summer 2025 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. Recordings of the lecture are availible at the following link:

`archive.mpim-bonn.mpg.de/id/eprint/5155/`

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1. Lecture 1 – 11th April 2025

Recall that the construction of the Habiro ring of a number field [GS+24, Sch24] was motivated by an expectation of the instructor, circa 2017, that there exists some form of "Habiro cohomology." Within this larger aspriational framework, the Habiro ring of a number field serves as the zero-dimensional case where the variety is a discrete collection of points. More precisely, in the case of the Habiro ring of a number field, there are certian q-series related to pertubative Chern-Simons theory giving rise to an explicit approach to Habiro rings of number fields. In particular, these q-series from pertubative Chern-Simons theory as computed by Garoufalidis and Zagier arise as elements of the abstract Habiro ring of a number field.

The goal of this course, then, is to explicate this aspirational framework of Habiro cohomology that synthesizes the concrete approach of Garoufalidis-Zagier with the instructor's abstract approach. In particular, we will define a new explicit cohomology theory for algebraic varieties that has specializations to clasical cohomology theories: de Rham cohomology as well as p-adic étale cohomology, crystalline cohomology, and prismatic cohomology for all primes p. Moreover, this cohomology theory will extend to the rigid-analytic setting of Berkovich spaces.

Let recall a modern definiton of Weil-type cohomology theories for algebraic varieties: functors

$$\mathsf{Sch}_k^{\mathsf{sft}} \longrightarrow \mathsf{Pr}_A^{\mathsf{L}}$$

where $\mathsf{Sch}_k^{\mathsf{sft}}$ is the category of separated finite type schemes over k and $\mathsf{Pr}_A^{\mathsf{L}}$ the category of presentable A-linear categories with a six-functor formalism and satisfying the Künneth formula. In particular this exculedes some cohomology theories such as motivic cohomology.

The state of the art of Weil-type cohomology theories for algebraic varieties can be summarized in the following diagram.

The instructor remarks that this is his favorite diagram.

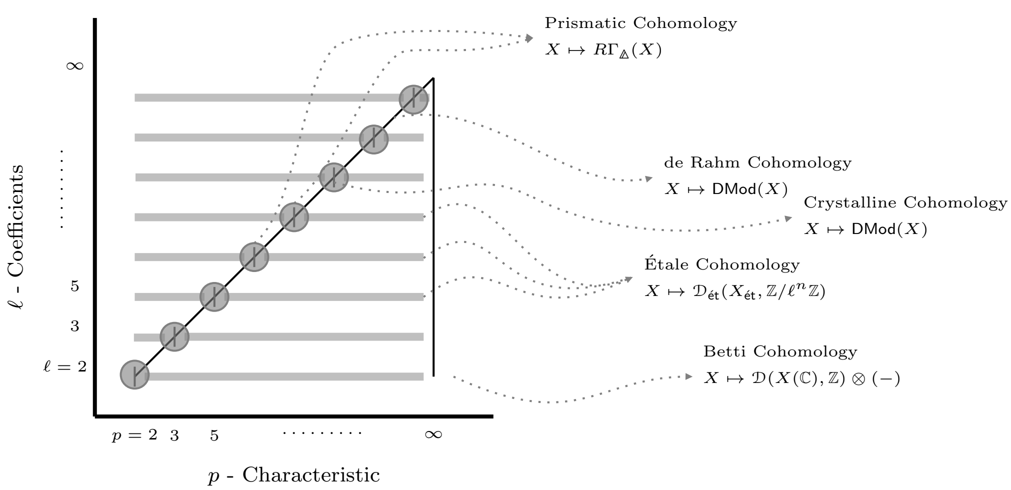

Figure 1. Cohomology theories for algebraic varieties. Or: the instructor's favorite diagram.

- Betti cohomology $X \mapsto \mathcal{D}(X(\mathbb{C}), \mathbb{Z}) \otimes (-)$ produces a cohomology theory for complex schemes. But coefficients can be taken in any field by base change.
- de Rham cohomology $X \mapsto \mathsf{DMod}(X)$ associating to a scheme its category of D-modules produces a cohomology theory for k-schemes (modulo technicalities). This produces a k-vector space for a k-scheme, hence has coefficients equal to the characteristic of the scheme.
- Étale cohomology as defined by Grothendieck $X \mapsto \mathcal{D}_{\text{ét}}(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$ produces for a k-scheme X, a cohomology theory with $\mathbb{Z}/\ell^n\mathbb{Z}$-coeffients with ℓ of characteristic distinct from that of k. That étale cohomology is able to produce cohomology in coefficients modulo powers of ℓ is represented by the thickening of the horizontal. Note that étale cohomology satisfies the Künneth formula, but not its categorical variant.
- Crystalline cohomology after Grothendieck, Berthelot, Caro, et. al. that associates to a k-scheme where k is of positive characteristic a cohomology theory $X \mapsto \mathsf{DMod}(X)$ that associates to X its category of arithmetic D-modules and which satisfies the categorical Künneth formula. This produces a module over the Witt vectors $W(k)$ of k for a k-scheme, and is represented by vertical thickenings at the characteristic.
- Prismatic cohomology was defined by Bhatt-Scholze [BS22] as a universal cohomology theory at the (p,p)-point by computing the structure sheaf cohomology of the prismatic site $X \mapsto R\Gamma_{\Delta}(X)$ where X is a scheme over $\mathcal{O}_K$ where K is a mixed characteristic local field which has coefficients valued in prisms.[1]

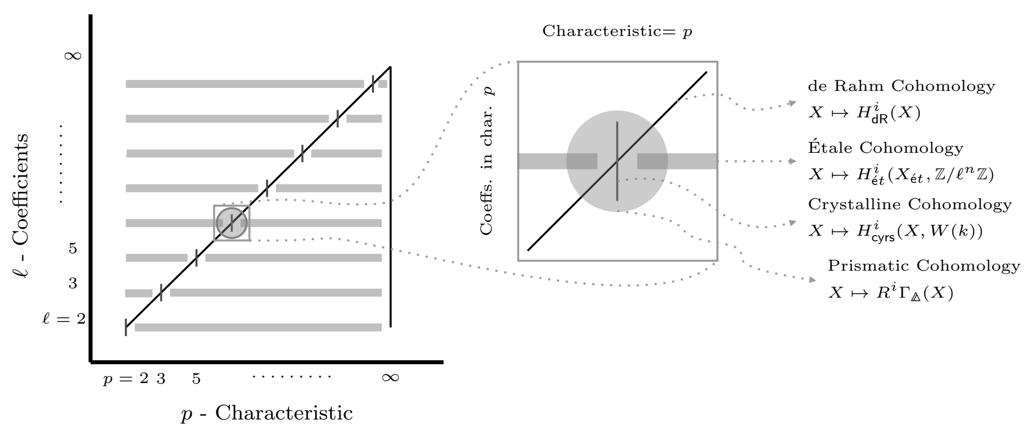

FIGURE 2. Prismatic cohomology at the (p,p)-point.

Moreover, the diagram reflects several important comparison phenomena between the abovementioned cohomology theories:

[1]It would be more precise to state this using "derived category of sheaves" associated to prismatic cohomology, namely the category of F-gauges a là Bhatt-Lurie [Bha22], but we do not comment on this further.

- The intersection of the lines corresponding to Betti and de Rham cohomology at the (∞, ∞)-point is substantiated by the comparison isomorphism between singular cohomology with $\mathbb{C}$-coefficients and de Rham cohomology via the Riemann-Hilbert correspondence.
- The intersection of the lines corresponding to étale and Betti cohomology at the (∞, p)-points are substantiated by the Artin's comparison isomorphism between étale and Betti cohomology.
- The intersection of the thickenings of crystalline cohomology meeting de Rham cohomology along the diagonal at the (p, p)-point is substantiated by the isomorphism between crystaline cohomology reduced modulo p and de Rham cohomology.
- Prismatic cohomology as depicted in Figure 1 admits specializations to de Rham, crystalline, and étale cohomology. Prismatic cohomology is additionally compatible with the structures of the various cohomology theories around the (p, p)-point, specializing to the action of the Frobenius in crystalline cohomology, the Hodge-Tate filtration in the case of de Rham cohomology, and the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ in the case of étale cohomology.
- The "prismatization" at the (∞, ∞)-point is the content of classical complex Hodge theory, which considers Hodge filtrations on de Rham cohomology and associated objects.

Observe, then, that de Rham cohomology is the unifying cohomology theory on the diagonal, while prismatic cohomology only exists at a fixed prime. One then wonders if there is a way to unify the cohomology theories along the diagonal. This is provided by Habiro cohomology, at least in the positive characteristic case.

The instructor remarks that he is unsure how to unify Habiro cohomology with classical Hodge theory.

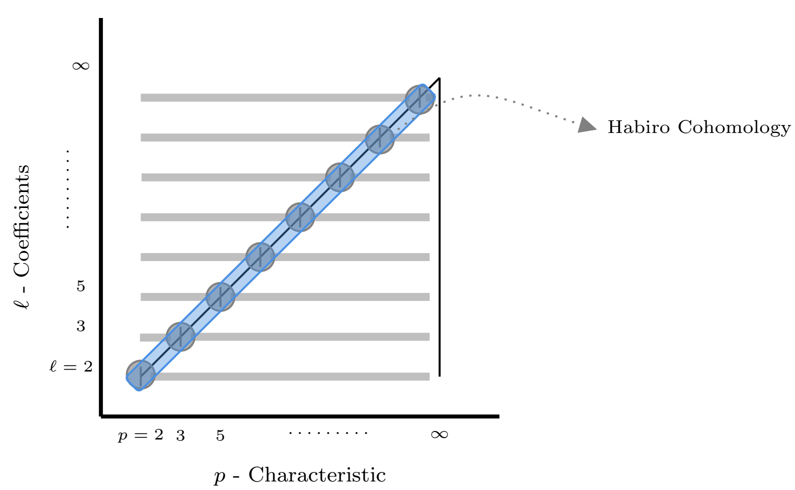

FIGURE 3. The role of Habiro cohomology highlighted in blue generalizing prismatic cohomology at all primes. Compare Figure 1.

That is to say that Habiro cohomology, covering a neighborhood of the de Rham diagonal, specializes to prismatic cohomology at each prime, and spreads out further than prismatic cohomology along the horizontal étale branches in an appropriate sense.

The starting point of Habiro cohomology is the example of the q-de Rham prism, the definition of which we now recall.

Example 1.1. The q-de Rham prism is the prism $(\mathbb{Z}_p[[q-1]], [p]_q)$ where $[p]_q = \frac{1-p^n}{1-q}$ is the q-deformation of p with a Frobenius action by $q \mapsto q^p$. The quotient $\mathbb{Z}_p[[q-1]]/([p]_q)$ is precisely the quotient by the p-th cyclotomic polynomial and hence isomorphic to the cyclotomic extension $\mathbb{Z}_p[\zeta_p]$.

Computing the prismatic cohomology of $\mathbb{A}^1_{\mathbb{Z}_p[\zeta_p]}$ relative to the q-de Rham prism, one finds that this is computed by an obvious q-deformation of the de Rham complex. The cohomological comparisons of the preceding discussion suggest that there is a deformation of the de Rham complex given by

$$\nabla_q : \mathbb{Z}_p[\zeta_p][x][[q-1]] \longrightarrow \mathbb{Z}_p[\zeta_p][x][[q-1]]$$

by $x^n \mapsto [n]_q x^{n-1}$. It is not *a priori* clear why q-deformations appear in this setting. Moreover, the construction of prismatic cohomology over the q-de Rham prism is expected to be functorial in automorphisms of $\mathbb{A}^1_{\mathbb{Z}_p[\zeta_p]}$ but it is unclear if (and how) this construction is invariant under change of coordinates. Additionally, the q-deformation suggests that by removing p everywhere, one can find a construction independent that works for all primes p. In particular, the instructor conjectures in [Sch17] the following:

Conjecture 1.2 (Scholze; [Sch17, Conj. 1.1])**.** If R is a smooth $\mathbb{Z}$-algebra equipped with an étale map $\operatorname{Spec}(R) \to \mathbb{A}^d_{\mathbb{Z}}$, there is a cohomology theory for smooth proper varieties over R valued in finitely generated $R[[q-1]]$-modules with a q-connection.

The q-connection captures precisely the difficulties with coordinate transformations articulated above, and the specialization at $q = 1$ recovers the de Rham cohomology of X with a Gauss-Manin connection. This suggests that algebraic varieties have a canonical q-deformation with connection compatible with the Gauss-Manin connection on classical de Rham cohomology, and was proven after p-adic completion in [BS22] and in general by Ferdinand Wagner in [Wag24] using the machinery of adelic gluing.

Theorem 1.3 (Wagner; [Wag24, Thm. 1.7])**.** Let R be a smooth framed $\mathbb{Z}$-algebra. There is an isomorphism between the $(q-1)$-completed q-de Rham–Witt complex and the cohomology of the quotient of the q-Hodge complex by $(q^m - 1)$.

Let us consider an example of this phenomenon.

Example 1.4. Consider the Legendre family of elliptic curves X with affine model $y^2 = x(x-1)(x-\lambda)$ over $R = \mathbb{Z}[\frac{1}{2}, \lambda, \frac{1}{\lambda(1-\lambda)}]$. We have $H^1_{\mathsf{dR}}(X)$ free of rank 2, containing the Hodge filtration $\mathrm{Fil}^1_{\mathsf{Hdg}} = H^0(X, \Omega^1_{X/R})$ with canonical differential

$\omega = \frac{\mathrm{d}x}{y}$. Denoting ∇ the connection on $H^1_{\mathsf{dR}}(X)$, we have $\omega, \nabla(\omega)$ a basis of $H^1_{\mathsf{dR}}(X)$ and

$$\nabla^2(\omega) = \frac{1}{4\lambda(1-\lambda)} + \frac{2\lambda - 1}{\lambda(1-\lambda)}\nabla(\omega).$$

A horizontal section is $f(\lambda)\cdot\lambda(1-\lambda) - f'(\lambda)\lambda(1-\lambda)\nabla(\omega)$ for a certain hypergeometric function $f(\lambda) = \sum_{n\geq 0}\prod_{i=0}^{n-1}\left(\frac{i+\frac{1}{2}}{i+1}\right)^2\lambda^n$.

There is a q-analogue of hypergeometric functions.

Example 1.5. The q-hypergeometric function

$$\sum_{n\geq 0}\prod_{i=0}^{n-1}\left(\frac{[i+\frac{1}{2}]_q}{[i+1]_q}\right)^2\lambda^n$$

satisfies a second order q-difference equation that deforms the Picard-Fuchs equation whose solutions describe periods of elliptic curves [nLab-a].

The example suggests that there is a possible connection between q-hypergeometric functions – the q-analogue of hypergeometric functions – and q-deformations of de Rham cohomology.

In the case of de Rham cohomology as in Example 1.4, there is not only a connection ∇, but also a choice of canonical vector $\omega = \frac{\mathrm{d}x}{y}$ obtained by the filtration. Then considering the differential equation the class satisfies produces the desired differential equation – the module and connection alone are insufficient to produce the differential equation. The main barrier to considering the q-analogue, then, was the lack of choice of such a class.

Recent computations of Shirai [Shi20] and work of Garoufalidis-Wheeler remedy this by producing explicit classes in q-de Rham cohomology, allowing the procedure above to be repeated.

This course will consider what happens to these q-deformations when q approaches a root of unity ζ_m, knowing that it recovers the classical construction at $q = 1$. Working over the Habiro ring

$$\mathcal{H} = \lim_{m,n\geq 1}\mathbb{Z}[q]/(1-q^n)^m = \lim_n \mathbb{Z}[q]/(q;q)_n$$

allows us to consider specializations at different roots of unity.

One issue that arises in trying to naïvely generalize Habiro cohomology to schemes of higher dimension is that the specialization of prismatic cohomology over the q-de Rham prism at $q = 1$ recovers de Rham cohomology, but at other roots of unity recovers only Hodge cohomology – this does not put all roots of unity on equal footing. But if the q-de Rham cohomology could be modified to be Hodge cohomology in an appropriate manner. This was shown by Meyer-Wagner in [MW24].

Theorem 1.6 (Meyer-Wagner; [MW24, Thm. 1.7])**.** Let R be a p-torsion free p-complete ring which is a quasiregular quotient over $\mathbb{Z}_p$ and such that the Frobenius on R/p is semiperfect. If R admits a lift to a p-complete $\mathbb{E}_1$ ring spectrum $\mathbb{S}_R$ such that $R \simeq \mathbb{S}_R\widehat{\otimes}_{\mathbb{S}_p}\mathbb{Z}_p$ then the q-Hodge filtration on the p-complete derived

q-de Rham complex is a q-deformation of the Hodge filtration on the (ordinary) p-complete derived de Rham complex.

The proof of Meyer-Wagner once again leverages highly technical machinery, in particular the relationship bewteen prismatic cohomology and topological cyclic homology. However, there is a more computational way of achieving the same goal.

Here "ring stack" and "analytic" are to be taken in the sense of condensed mathematics [CS23].

Theorem 1.7 (Scholze)**.** There is an explicit ring stack over an analytic version of the Habiro ring yielding a full six-functor formalism.

Remark 1.8. This in particular yields a sheaf theory.

These are related to the constructions of the ring stacks for prismatic cohomology following Drinfeld [Dri20] and Bhatt-Lurie [BL22].

While multiplication is easy to define in this ring, addition is not: in particular, the instructor remarks that he spent a whole day computing what $1 + 1$ is in this ring.

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