V4A2 – ALGEBRAIC GEOMETRY II
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Preliminaries

These notes roughly correspond to the course **V4A2 – Algebraic Geometry II** taught by Prof. Daniel Huybrechts at the Universität Bonn in the Summer 2025 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. Knowledge of commutative algebra, topology, and category theory will be assumed.

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1. Lecture 1 – 7th April 2025

We begin by a consideration of the theory of smoothness, first in the local case. This is done by defining the sheaves of Kähler differentials on schemes – in the local picture, the module of differentials on a ring.

Definition 1.1 (Derivation)**.** Let B be an A-algebra and M a B-module. An morphism of A-modules $D : B \to M$ is an A-derivation if it satisfies the Leibniz rule $\mathrm{d}(xy) = x\mathrm{d}(y) + y\mathrm{d}(x)$ for all $x, y \in B$. Denote the set of A-derivations in M by $\mathrm{Der}_A(B, M)$.

Remark 1.2. It is necessary that M is a B-module, since the Leibniz rule involves elements of B.

Remark 1.3. Observe that the composition $A \to B \to M$is zero since $a = a \cdot 1_B$ and computing we get $\mathrm{d}(a \cdot 1_B) = a\mathrm{d}(1_B)$ by A-linearity, but on the other hand $\mathrm{d}(a \cdot 1_B) = a\mathrm{d}(1_B) + 1_B\mathrm{d}(a)$ by the Leibniz rule, so $a\mathrm{d}(1_B) = 0$ showing $\mathrm{d}(1_B) = 0$ and thus $\mathrm{d}(a) = 0$.

The Kähler differentials of a ring map is the universal recipient of an A-algebra B in the following sense.

Definition 1.4 (Module of Kähler Differentials)**.** Let B be an A-algebra. The module of Kähler differentials of B over A is a B-module $\Omega^1_{B/A}$ with an A-derivation $\mathrm{d} : B \to \Omega^1_{B/A}$ that is initial amongst B-modules recieving an A-derivation from B.

Unwinding the universal property, if M is a B-module recieving an A-derivation from B by $f : B \to M$, there is a unique factorization over $\Omega^1_{B/A}$ as follows.

$$\begin{tikzcd}
\Omega^1_{B/A} \arrow[r] & M \\
B \arrow[u, dashed, "\exists!"] \arrow[ur] &
\end{tikzcd}$$

In particular, there is a bijection $\mathrm{Der}_A(B, M) \leftrightarrow \mathrm{Hom}_{\mathsf{Mod}_B}(\Omega_{B/A}, M)$ functorial in M.

Proposition 1.5. Let B be an A-algebra. The B-module $\Omega^1_{B/A}$ and the A-derivation $\mathrm{d} : B \to \Omega^1_{B/A}$ exist and are unique up to unique isomorphism.

Proof. The module $\Omega^1_{B/A}$ can be constructed as the free B-module on elements $\mathrm{d}x$ for $x \in B$ modulo the relations generated by the Leibniz rule and $\mathrm{d}a = 0$ for $a \in A$. Uniqueness up to unique isomorphism is clear from the universal property and Yoneda's lemma. ■

In special cases, the module of Kähler differentials can be described explicitly.

Example 1.6. Let $A = k, B = k[x_1, \dots, x_n]$. $\Omega^1_{B/A}$ is a free module of rank n with basis $\mathrm{d}x_i$. The map $f \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \mathrm{d}x_i$ is an A-derivation and the map $\mathrm{d}x_i \mapsto x_i$ defines an isomorphism $\Omega^1_{B/A} \to B^{\oplus n}$.

Kähler differentials are also fairly easy to understand in the case of ring localizations and ring quotients. These will be important in understanding the sheaves of Kähler differentials of open and closed immersions in the case of schemes, respectively.

Proposition 1.7. Let A be a ring.

(i) If $B = S^{-1}A$, then $\Omega^1_{B/A} = 0$.

(ii) If $B = A/I$ for $I \subseteq A$ an ideal, then $\Omega^1_{B/A} = 0$.

Proof of (i). We already have that $\mathrm{d}a = 0$ for all $a \in A$. We then observe that writing $a = s \cdot \frac{a}{s}$ we have

$$\begin{aligned} 0 = \mathrm{d}(a) = \mathrm{d}\left(s \cdot \frac{a}{s}\right) &= s\mathrm{d}\left(\frac{a}{s}\right) + \frac{a}{s}\mathrm{d}(s) \\ &= s\mathrm{d}\left(\frac{a}{s}\right) && s \in A \Rightarrow \mathrm{d}s = 0 \end{aligned}$$

so $s\mathrm{d}(\frac{a}{s}) = 0$ and $\mathrm{d}(\frac{a}{s}) = 0$ whence the claim. ■

Proof of (ii). The map $A \to B$ is surjective, so this is precisely the situation Remark 1.3. ■

We can additionally understand sheaves of Kähler differentials in towers. Let $A \to B \to C$ be maps of rings. There is a natural C-linear map $\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$ which is a C-module homomorphism induced by the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\mathrm{d}_{B/A}"'] & C \arrow[r, "\mathrm{d}_{C/A}"] & \Omega^1_{C/A} \\ \Omega^1_{B/A} \arrow[urr, dashed, "\exists!"'] & & \end{tikzcd}$$

where the top row is both A and B-linear inducing a unique B-module map $\Omega^1_{B/A} \to \Omega^1_{C/A}$, considering the latter as a B-module. By the extension-restriction adjunction, however, we have

$$\mathrm{Hom}_{\mathsf{Mod}_B}(\Omega_{B/A}, \Omega_{C/A}|_B) \leftrightarrow \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega_{B/A} \otimes_B C, \Omega_{C/A})$$

hence the data of the dotted map in the diagram above gives rise to a unique map $\Omega_{B/A} \otimes_B C \to \Omega_{C/A}$. Arguing similarly, there is a C-linear map $\Omega^1_{C/A} \to \Omega^1_{C/B}$ induced by

$$\begin{tikzcd} C \arrow[r, "\mathrm{d}_{C/B}"] \arrow[d, "\mathrm{d}_{C/A}"'] & \Omega^1_{C/B} \\ \Omega^1_{C/A} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

where the map is induced by the universal property as any B-derivation is also an A-derivation.

The maps in the preceding discussion assemble to give the following proposition.

Proposition 1.8. Let $A \to B \to C$ be maps of rings. There is an exact sequence

$$\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0.$$

Proof. The above discussion gives the existence of such maps, so it remains to show exactness at $\Omega^1_{C/A}$ and surjectivity of the map $\Omega^1_{C/A} \to \Omega^1_{C/B}$.

We begin with the latter, where by the quotient construction of Proposition 1.5 it suffices to observe that $\Omega^1_{C/B}$ is a quotient of $\Omega^1_{C/A}$.

For the former, we note that for a fixed C-module M we have an exact sequence

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M|_B)$$

since an A-derivation $(\mathrm{d} : C \to M)$ is taken to the composite $B \to C \to M$ which is zero when the map is also a B-derivation. Rewriting this using the universal property, this is

$$0 \to \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega^1_{C/B}, M) \to \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega^1_{C/A}, M) \to \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega^1_{B/A} \otimes_B C, M)$$

which by contravariant exatness of the Hom-functor (see [Stacks, Tag 0582] for the precise statement), is the claim. ∎

As a corollary, we can deduce the following fact about localizations.

Corollary 1.9. Let B be an A-algebra and S a multiplicative subset of B. Then $S^{-1}\Omega^1_{B/A} \cong \Omega^1_{S^{-1}B/A}$.

Proof. Apply Proposition 1.8 to $C = S^{-1}B$ and note that $\Omega^1_{C/B} = 0$ so the map $S^{-1}\Omega^1_{B/A} \to \Omega^1_{S^{-1}B/A}$ is surjective. To prove injectivity, we produce an inverse map which is an A-derivation of $S^{-1}B$ to $S^{-1}\Omega^1_{B/A}$ by $\mathrm{d}(\frac{b}{s}) \mapsto \frac{1}{s}\mathrm{d}(b) - \frac{1}{s^2}b\mathrm{d}(s)$ which by the universal property can be seen to be the inverse. ∎

Note that in general $\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$ is rarely injective.

Example 1.10. Let $A = k, B = k[x], C = k[x]/(x)$. So $\Omega^1_{B/A} \cong B\mathrm{d}x$ but $\Omega_{C/A} = \Omega_{k/k} = 0$.

On the other hand, there are situations in which the exact sequence of Proposition 1.8 extends to a short exact sequence.

Example 1.11. Let B be an A-algebra and $C = B[x_1, \ldots, x_n]$. We then have a split short exact sequence

$$0 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

where denoting the map $\Omega^1_{C/A} \to \Omega^1_{C/B}$ by φ, we have the splitting $\Omega^1_{C/A} \to (\Omega^1_{B/A} \otimes_B C) \oplus \Omega^1_{C/B}$ prescribed by the C-derivation $f \mapsto \mathrm{d}_{B/A}(f) + \varphi(f)$ under the bijection

$$\mathrm{Hom}_{\mathsf{Mod}_C}(\Omega_{C/A}, (\Omega_{B/A} \otimes_B C) \oplus \Omega_{C/B}) \leftrightarrow \mathrm{Der}_A(C, (\Omega_{B/A} \otimes_B C) \oplus \Omega_{C/B}).$$

The following proposition describes the behavior of the module of Kähler differentials with respect to tensor products.

Proposition 1.12. Let B, A' be A-algebras. Then there is an isomorphism of B-modules $\Omega^1_{B/A} \otimes_B (B \otimes_A A') \cong \Omega^1_{(B\otimes_A A')/A'}$.

Proof. We contemplate the diagram

$$\begin{tikzcd} B \otimes_A A' \arrow[rr] \arrow[d] & & \Omega^1_{B/A} \otimes_A A' \\ \Omega^1_{(B\otimes_A A')/A'} \arrow[urr, dashed, "\exists!"'] & & \end{tikzcd}$$

where the solid arrows are $B \otimes_A A'$-linear with $\Omega^1_{B/A} \otimes_A A' \cong (\Omega^1_{B/A} \otimes_B (B \otimes_A A'))$ and the dotted arrow induced by the universal property of $\Omega^1_{(B\otimes_A A')/A'}$. By applying the tensor-hom adjunction and the universal property of derivations, prescribing an inverse map to the dotted arrow is equivalent to producing an A-derivation of B in $\Omega^1_{(B\otimes_A A')/A'}$ and one observes that the map $b \mapsto \mathrm{d}_{(B\otimes_A A')/A'}(b \otimes 1)$ gives an inverse, whence the claim. ■

We now treat the case of quotients.

Proposition 1.13. Let $A \to B \to C$ be maps of rings where $C \cong B/\mathfrak{b}$ for some ideal $\mathfrak{b} \subseteq B$. There is an exact sequence

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to 0.$$

Proof. We first observe that $\Omega^1_{C/B} = 0$ by Proposition 1.7 and $\Omega^1_{B/A} \otimes_B C \cong \Omega^1_{B/A}/\mathfrak{b}\Omega^1_{B/A}$.

We denote $\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B/A} \otimes_B C$ by δ, $b \mapsto \mathrm{d}b \otimes 1$. We first show δ is well-defined. For this, we want to show that $\mathrm{d}(b_1 b_2) \otimes 1$ is zero for $b_1, b_2 \in \mathfrak{b}$. Indeed, using the Leibniz rule, we have

$$\mathrm{d}(b_1 b_2) \otimes 1 = \mathrm{d}(f_2) \otimes f_1 + \mathrm{d}(f_1) \otimes f_2 \in \mathfrak{b}\Omega_{B/A}$$

hence zero in the quotient, showing the map is well-defined.

The diagram is a complex as db maps to zero in $\Omega^1_{C/A}$. The kernel of $\Omega^1_{B/A} \to \Omega^1_{C/A}$ is generated by the B-submodule $\mathfrak{b}\Omega^1_{B/A}$ and the elements $\mathrm{d}b$ for $b \in \mathfrak{b}$, showing exactness of the complex in the middle. ■

This specializes to finite type algebras.

Corollary 1.14. Let C be a finite type A-algebra – that is, the quotient of $B = A[x_1, \ldots, x_n]$. Then $\Omega^1_{C/A}$ is a finitely generated C-module.

Proof. Set $B = A[x_1, \ldots, x_n]$ for which $C = B/\mathfrak{b}$. Exactness of the sequence in Proposition 1.13 gives a surjection $\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$, and observing that $\Omega^1_{B/A} \otimes_B C \cong B^{\oplus n} \otimes_B C \cong C^{\oplus n}$ gives a surjection $C^{\oplus n} \to \Omega^1_{C/A}$, showing that it is finitely generated. ■

Let us consider the case of quotients of multivariate polynomial rings by a single polynomial.

Example 1.15. Let A be a ring, $B = A[x_1, \ldots, x_n]$, and $C = B/(f)$ for $f \in B$. By Proposition 1.13 and Corollary 1.14, we have that $\Omega^1_{C/A}$ is the cokernel of the map $\delta : (f)/(f)^2 \to \Omega^1_{B/A} \otimes_B C \cong C^{\oplus n}$ of Proposition 1.13, so is the quotient $\left(\bigoplus_{i=1}^n C\mathrm{d}x_i\right)/\mathrm{d}f$.

We can also consider the case of k-algebras.

Corollary 1.16. Let A be a k-algebra and $\mathfrak{m}$ a maximal ideal in A such that $\kappa(\mathfrak{m}) = A/\mathfrak{m} \cong k$. Then $\Omega^1_{A/k} \otimes_k \kappa(\mathfrak{m}) \cong \mathfrak{m}/\mathfrak{m}^2$.

Proof. This is precisely Proposition 1.13 for $k \to k[x_1, \ldots, x_n] \to A$, and the map $\mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{A/k} \otimes_k \kappa(\mathfrak{m})$ is a surjection between vector spaces of the same dimension, hence an isomorphism. ■

Note that this is the dual of the Zariski tangent space $\mathrm{Hom}_{\mathsf{Vec}_{\kappa(\mathfrak{m})}}(\mathfrak{m}/\mathfrak{m}^2, \kappa(\mathfrak{m}))$, motivating the connection to schemes.

2. Lecture 2 – 10th April 2025

We begin with an example.

Example 2.1. Let $A = k, B = k[x,y], C = B/\mathfrak{b}$ where $\mathfrak{b} = (xy)$. We have that $\mathrm{Spec}(B)$ is the affine plane $\mathbb{A}^2_k$ and $\mathrm{Spec}(C)$ is the union of the two coordinate axes. The exact sequence of Proposition 1.13 gives

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{k[x,y]/k} \otimes_{k[x,y]} C \to \Omega^1_{C/k} \to 0.$$

Explicitly identifying $\mathfrak{b}/\mathfrak{b}^2$ with the C-module $(xy)/(x^2y^2)$ module-isomorphic to C by $1 \mapsto xy$ and $\Omega^1_{k[x,y]/k}$ with the free $k[x,y]$-module $B\mathrm{d}x \oplus B\mathrm{d}y$, we observe that the map $\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{k[x,y]/k} \otimes_{k[x,y]} C$ is given by $\overline{xy} \mapsto \mathrm{d}(xy) \otimes 1 = (x\mathrm{d}y + y\mathrm{d}x) \otimes 1$. This yields a map $C \to C \oplus C$ by $1 \mapsto (y,x)$ and the cokernel of this map is the kernel of the map $C \oplus C \to C$ by $(a,b) \mapsto ax - by$ so by exactness the image of $C \oplus C \to C$ is the ideal $(x,y) \subseteq C$ showing $\Omega^1_{C/k} \cong (x,y) \subseteq C$. Thus $\Omega_{C/k} \otimes_C \frac{k[x,y]}{(x,y)} \cong kx \oplus ky$ and in particular $\Omega^1_{C/k} \otimes_C k$ is of k-dimension 2. For all points $\mathfrak{p} \in \mathrm{Spec}(C) \setminus \{(x,y)\}$, we have $\Omega_{C/k} \otimes_C \kappa(\mathfrak{p}) \cong k$ extending the exact sequence above to a short exact sequence.

In what follows, we will use the following lemma for Kähler differentials of field extensions, the proof of which we omit.

Lemma 2.2. Let k be a field and K/k a separable extension. Then $\Omega^1_{K/k} = 0$.

This lemma, in conjunction with Proposition 1.13, shows that for maximal ideals of k-algebras A with separable residue field, the base change of the sheaf of Kähler differentials to k is isomorphic to the Zariski tangent space $\mathfrak{m}/\mathfrak{m}^2$.

Proposition 2.3. Let A be a finite type k-algebra and $\mathfrak{m} \subseteq A$ maximal with residue field $\kappa(\mathfrak{m})$ separable over k. There is an isomorphism $\frac{\mathfrak{m}}{\mathfrak{m}^2} \to \Omega^1_{A/k} \otimes_A k$.

Proof. Let $\varphi : k[x_1, \ldots, x_n] \to A$ with kernel $\ker(\varphi) = \mathfrak{a}$ and $\widetilde{\mathfrak{m}} = \varphi^{-1}(\mathfrak{m})$. This yields a surjective map $\frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2} \to \frac{\mathfrak{m}}{\mathfrak{m}^2}$ with kernel $\mathfrak{a}$. By Proposition 1.13 we have the following diagram with bottom row exact

$$\begin{tikzcd}
\mathfrak{a} \arrow[r] \arrow[d] & \frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2} \arrow[r] \arrow[d] & \frac{\mathfrak{m}}{\mathfrak{m}^2} \arrow[r] \arrow[d] & 0 \\
\frac{\mathfrak{a}}{\mathfrak{a}^2} \arrow[r] & \Omega^1_{k[x_1,\ldots,x_n]/k} \otimes_{k[x_1,\ldots,x_n]} k \arrow[r] & \Omega^1_{A/k} \otimes_A k \arrow[r] & 0
\end{tikzcd}$$

and noting the top row is the $- \otimes_{k[x_1,\ldots,x_n]} k$ of the bottom, a right-exact operation, we get the claim. ■

We deduce the following result which will be required for defining the sheaf of Kähler differentials on schemes more generally.

Corollary 2.4. Let B be an A-algebra and I the kernel of the map $B \otimes_A B \to B$ by $b_1 \otimes b_2 \mapsto b_1 b_2$. Then $\Omega^1_{B/A} \cong I/I^2$ as B-modules by $\mathrm{d}b \mapsto 1 \otimes b - b \otimes 1$, and where I/I^2 has the structure of a B-module by $b(b_1 \otimes b_2) = bb_1 \otimes b_2$.

Proof. We use the universal property of Kähler differentials. Defining δ by $b \mapsto 1 \otimes b - b \otimes 1$ we get the diagram

$$\begin{tikzcd} B \arrow[r, "\delta"] \arrow[d] & I/I^2 \\ \Omega^1_{B/A}. \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Note for $b_1, b_2 \in B$, we compute

$$\delta(b_1 \cdot b_2) = 1 \otimes b_1 b_2 - b_1 b_2 \otimes 1 = b_1 \delta(b_2) + b_2 \delta(b_1) = b_1(1 \otimes b_2 - b_2 \otimes 1) + b_2(1 \otimes b_1 - b_1 \otimes 1).$$

The difference of the two expresssions is $(1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1)$ the product of two elements of I, hence in I^2, hence vanishes in the quotient. Since this is a derivation, there exists an extension $\Omega^1_{B/A} \to I/I^2$.

To show surjectivity, we consider an element $\sum b_i \otimes b_i' \in I$ and compute

$$\begin{aligned} \sum b_i \otimes b_i' &= \sum b_i(1 \otimes b_i') \\ &= \sum b_i(1 \otimes b_i') - \underbrace{\left(\sum b_i b_i'\right)}_{=0} \otimes 1 \\ &= \sum b_i(1 \otimes b_i' - b_i' \otimes 1) \end{aligned}$$

showing it is surjective.

To show injectivity, we consider the diagram

$$\begin{tikzcd} B \arrow[r, "\delta"] \arrow[d, "\mathrm{d}"'] & M \\ \Omega^1_{B/A} \arrow[ur] \arrow[d] & \\ I/I^2 \arrow[uur, dashed] & \end{tikzcd}$$

The existence of a dotted arrow rendering the entire diagram commutative would imply the injectivity of δ for $M = \Omega^1_{B/A}$. Note that the B-module $B \oplus M$ can be given the structure of a free algebra by the map $b \mapsto b \oplus 0$ and multiplication $(b_1, m_1) \cdot (b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1)$, which defines a B-algebra in which M is an ideal with square zero. We can define a map $\varphi : B \otimes_A B \to B \oplus M$ by $b_1 \otimes b_2 \mapsto (b_1 b_2, b_1 \delta(b_2))$ which is a homomorphism of A-algebras and where the image of the ideal I is zero since M^2 is zero. Thus an extension $\psi : I/I^2 \to M$ and the diagram commutes, yielding injectivity, and hence the claim. ∎

We now seek to define the sheaf of Kähler differentials on a scheme.

Definition 2.5 (Relative Kähler Differentials of a Scheme)**.** Let $f : X \to Y$ be a morphism of schemes. The sheaf of Kähler differentials of X is locally given by the $\mathcal{I}/\mathcal{I}^2$ of the locally closed embedding $X \to X \times_Y X$.

We will most often be interested in the situation where $Y = \mathrm{Spec}(k)$ and f is the structure map of f as a k-scheme.

Example 2.6. Let $X = \mathbb{A}_k^n$ over $\mathrm{Spec}(A)$. $\Omega^1_{X/A}$ is free of rank n as in Example 1.6.

Example 2.7. Let $X = \mathbb{P}_A^n$ over $\mathrm{Spec}(A)$. $\Omega^1_{X/A}$ is the locally free sheaf of rank n obtained by gluing the free sheaves of Example 2.6 on the distinguished affine opens $D_+(x_i)$ of $\mathbb{P}_A^n$.

The sheaf of Kähler differentials is another example of an interesting sheaf on schemes which is not the structure sheaf. In the case of projective space, the relationship between the sheaf of Kähler differentials is related to the structure sheaf by the Euler sequence.

Theorem 2.8 (Euler Sequence)**.** Let A be a ring. There is a short exact sequence of sheaves

$$0 \longrightarrow \Omega^1_{\mathbb{P}_A^n/A} \longrightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1)^{\oplus n+1} \xrightarrow{(x_0,\dots,x_n)} \mathcal{O}_{\mathbb{P}_A^n} \longrightarrow 0$$

on $\mathbb{P}_A^n$.

Proof. We have $X = \mathrm{Proj}(B)$ with $B = A[x_0, \dots, x_n]$ and $\mathcal{O}_{\mathbb{P}_A^n}(-1) = \widetilde{B(1)}$. The map $\mathcal{O}_{\mathbb{P}_A^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}_A^n}$ is given by the module homomorphism given by the dot product map, and is surjective since $\bigcap_{i=0}^n V_+(x_i) = \emptyset$ and $- \otimes \mathcal{O}_{\mathbb{P}_A^n}(1)$ being right-exact. We show that the kernel of this map is $\Omega^1_{\mathbb{P}_A^n/A}$ affine-locally.

On $D_+(x_i)$, the map is given by localizations $B(-1)^{\oplus n+1}_{(x_i)} \to B_{(x_i)}$ and the kernel is free of rank n generated by $e_j - \frac{x_j}{x_i} e_i$ for $j \neq i$. In particular, the kernel is a free $\mathcal{O}_{D_+(x_i)}$-module generated by $\frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i$ for $j \neq i$. Recall that $\Omega^1_{\mathbb{P}_A^n/A}$ is the free $\mathcal{O}_{D_+(x_i)}$-module spanned by $\mathrm{d}(\frac{x_0}{x_i}), \dots, \mathrm{d}(\frac{x_n}{x_i})$ and the isomorphism to the kernel of the map is given by $\mathrm{d}(\frac{x_j}{x_i}) \mapsto \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i$. The isomorphisms glue by inspection, giving the isomorphism of sheaves. ■

Example 2.9. Let $n = 1$. The Euler sequence gives $0 \to \Omega^1_{\mathbb{P}_A^1/A} \to \mathcal{O}_{\mathbb{P}_A^1}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}_A^1} \to 0$ where passing to determinants gives $\det(\mathcal{O}_{\mathbb{P}_A^1}(-1)^{\oplus 2}) \cong \det(\mathcal{O}_{\mathbb{P}_A^1}) \otimes \det(\Omega^1_{\mathbb{P}_A^1})$ showing $\Omega^1_{\mathbb{P}_A^1/A} \cong \mathcal{O}_{\mathbb{P}_A^1}(-2)$.

Example 2.10. For $n > 1$m $\Omega^1_{\mathbb{P}_A^n/A}$ is never a direct sum of line bundles. We have $\det(\Omega^1_{\mathbb{P}_A^n/A}) \cong \mathcal{O}_{\mathbb{P}_A^n}(-n-1)$. Twisting by $\mathcal{O}_{\mathbb{P}_A^n}(1)$, we get a short exacts sequence

$$0 \to H^0(\mathbb{P}_A^n, \Omega^1_{\mathbb{P}_A^n/1}) \to H^0(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n}^{\oplus n+1}) \to H^0(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n}(1)) \to 0$$

since H^1 of all the sheaves in the Euler sequence vanishes. If $\Omega^1_{\mathbb{P}_A^n/A} \cong \bigoplus \mathcal{O}_{\mathbb{P}_A^n}(a_i)$ then $a_i \leq -2$ and $\sum a_i \leq -2n$ which has global sections, a contradiction for $n \geq 2$.

3. Lecture 3 – 14th April 2025

Recall that the cotangent sheaf on $\mathbb{A}^n_A, \mathbb{P}^n_A$ over $\mathrm{Spec}(A)$ are locally free sheaves by Examples 2.6 and 2.7. One is then led to consider for what other S-schemes X is $\Omega^1_{X/S}$ locally free. This is roughly captured by smoothness. Moreover, as suggested by Proposition 2.3, the notion of smoothness is connected to the Zariski tangent space, which in the case of algebraic geometry – unlike differential geometry – need not coincide with the geometric tangent space, especially in characteristic p situations.

We recall the definition of the Zariski tangent space.

Definition 3.1 (Zariski Tangent Space of a Ring)**.** Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = A/\mathfrak{m}$. The Zariski tangent space of A is the κ-vector space $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^\vee = \mathrm{Hom}_{\mathsf{Vect}_\kappa}(\frac{\mathfrak{m}}{\mathfrak{m}^2}, \kappa)$.

Definition 3.2 (Zariski Tangent Space of a Scheme)**.** Let X be a scheme and $x \in X$ a point. The Zariski tangent space $T_{X,x}$ is the Zariski tangent space of the local ring $\mathcal{O}_{X,x}$.

Example 3.3. Let A be a ring and $\mathfrak{p} \subseteq \mathrm{Spec}(A)$. The Zariski tangent space $T_{\mathrm{Spec}(A),\mathfrak{p}}$ is given by $(\frac{\mathfrak{p}A_\mathfrak{p}}{(\mathfrak{p}A_\mathfrak{p})^2})^\vee$. This is a vector space over the field $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.

Example 3.4. Let k be a field and $x \in \mathbb{A}^n_k(k)$ a closed k-rational point hence of the form $(x_1 - a_1, \dots, x_n - a_n)$. Define a map $D_x : k[x_1, \dots, x_n] \to \mathrm{Hom}_{\mathsf{Vect}_k}(k^n, k)$ by

$$f \mapsto \left[(\alpha_i)_{i=1}^n \mapsto \sum_{i=1}^n \alpha_i \frac{\partial f}{\partial x_i}(x)\right].$$

The map is k-linear and satisfies the Leibniz rule, hence defines a k-linear derivation which is a k-vector space. This defines an isomorphism between $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^2$ and $\mathrm{Hom}_{\mathsf{Vect}_k}(k^n, k)$ by considering the Taylor expansion of a polynomial

$$f = f(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)(x_i - a_i) + \underbrace{O(x^2)}_{\in \mathfrak{m}^2}$$

hence the map is zero on $f \in \mathfrak{m}^2$ showing that $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^\vee \to \mathrm{Hom}_{\mathsf{Vect}_k}(k^n, k)$ by $(x_i - a_i) \mapsto e_i^\vee$ is an injection between vector spaces of the same dimension and hence an isomorphism.

Example 3.5. In general, one can still define a map on non-rational points with target $\mathrm{Hom}_{\mathsf{Vect}_{\kappa(\mathfrak{p})}}(\kappa(\mathfrak{p})^n, \kappa(\mathfrak{p}))$ which may fail to be injective. Let k be a field of characteristic p and consider $(x^p - a) \subseteq k[x]$ which is maximal when $a^{1/p} \notin k$. We have $\frac{\mathfrak{m}}{\mathfrak{m}^2} = \frac{(x^p-a)}{(x^p-a)^2} \cong k$ which defines a map $\mathrm{Hom}_{\mathsf{Vect}_k}(k, k)$ by Example 3.4 which is the zero map as $px^{p-1} = 0$.

In what follows, we will use the following result for closed subschemes of affine spaces.

Proposition 3.6. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be an ideal defining $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$. Let $x \in X(k) \subseteq \mathbb{A}_k^n(k)$. Then $T_{X,x}$ is the annihilator of the image of $\mathfrak{a}$ under D_x

Proof. We have a short exact sequence

$$0 \to \mathfrak{a} \to \widetilde{\mathfrak{m}} \to \mathfrak{m} \to 0$$

inducing

$$0 \to \frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2} \to \frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2} \to \frac{\mathfrak{m}}{\mathfrak{m}^2} \to 0. \tag{3.1}$$

Applying the right-exact functor $\operatorname{Hom}_{\mathsf{Vect}_k}(-, k)$ we get

$$\left(\frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2}\right)^\vee \to T^\vee_{\mathbb{A}_k^n, x} \to T^\vee_{X,x} \to 0$$

where the map $\left(\frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2}\right)^\vee \to \operatorname{Hom}_{\mathsf{Vect}_k}(k^n, k)$ by taking $\mathfrak{a}$-derivations as in Example 3.4. ∎

An analogous proof can be used to show that the Zariski tangent space is the cokernel of the Jacobian matrix.

Corollary 3.7. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be an ideal defining $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$. Let $x \in X(k) \subseteq \mathbb{A}_k^n(k)$. Then $T^\vee_{X,x} \cong \operatorname{coker}(J_x)$ where J_x is the Jacobian at x.

Proof. We use the short exact sequence (3.1) and observe that the map $\frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2} \to \frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2}$ is given by multiplication by the Jacobian, giving the claim. ∎

Having related this to the Zariski tangent space, we want to relate the Jacobian matrix to the sheaf/module of Kähler differentials, an analogy suggested by Proposition 2.3.

Proposition 3.8. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be an ideal defining $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$. Let $x \in X(k) \subseteq \mathbb{A}_k^n(k)$. Then the corank of the Jacobian J_x is equal to $\dim_{\kappa(x)} \Omega^1_{X/k} \otimes \kappa(x)$.

Proof. Applying $- \otimes \kappa(x)$ to the short exact sequence of Proposition 1.13, we have

$$\frac{\mathfrak{a}}{\mathfrak{a}^2} \otimes \kappa(x) \to \Omega^1_{\mathbb{A}_k^n/k} \otimes \kappa(x) \to \Omega^1_{X/k} \otimes \kappa(x) \to 0$$

which factors over the image of the Jacobian J_x. As such, we get that $\dim_{\kappa(x)} \Omega^1_{X/k} = n - \dim(\operatorname{im}(J_x)) = n - \operatorname{rank}(J_x)$ which is precisely the corank. ∎

Moreover, for a general point x, the property of the Zariski tangent space being isomorphic to the scalar extension of the sheaf of Kähler differentials.

Proposition 3.9. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be an ideal defining $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$. Let $x \in X$. $\kappa(x)$ is a separable extension of k if and only if $T^\vee_{X,x} \cong \Omega^1_{X/k} \otimes \kappa(x)$.

Proof. ($\Rightarrow$) If $\kappa(x)$ is separable over k, then $\Omega^1_{\kappa(x)/k} = 0$ by Lemma 2.2 so $\frac{\mathfrak{m}}{\mathfrak{m}^2} \to \Omega^1_{X,k} \otimes \kappa(x)$ is surjective, but this shows that we have a surjection of $\kappa(x)$-vector spaces of the same dimension, hence an isomorphism.

($\Leftarrow$) If the Zariski cotangent space is isomorphic to teh sheaf of differentials, then the cokernel $\Omega^1_{\kappa(x)/k}$ of the exact sequence Proposition 1.13 is zero, showing that $\kappa(x)/k$ is separable. ∎

Note that the equality $\dim(T_{X,x}) = \dim_{\kappa(x)} \Omega^1_{X/k} \otimes \kappa(x)$ does not imply the natural map is an isomorphism when $\kappa(x)$ is not separable over k.

Example 3.10. Let k be a field of characteristic p and consider $(x^p - a) \subseteq k[x]$ for $a^{1/p} \notin k$. Denoting $X = V(x^p - a) \subseteq \mathbb{A}^1_k$, we have $\dim(T_{X,x}) = 1 = \dim_{\kappa(x)} \Omega^1_{X/k} \otimes \kappa(x)$ but $\Omega^1_{\kappa(x)/k}$ is nonzero as $\kappa(x)$ is not a separable extension of k.

We arrive at the notion of smoothness for schemes.

Definition 3.11 (Smooth Scheme)**.** Let X be a scheme of finite type over a field k. X is smooth of pure dimension d if

(i) each of the finitely many irreducible components of X are of dimension d, and
(ii) every point $x \in X$ is contained in an affine open neighborhood where the Jacobian matrix is of corank d.

Remark 3.12. Smoothness is a relative notion, determined by the structure map to $\mathrm{Spec}(k)$.

Remark 3.13. By Example 3.3, this construction is independent of the choice of chart.

Remark 3.14. It suffices to verify this condition on closed points, as if the Jacobian is rank-deficient at some non-closed point, then it is rank-deficient at any specialization.

Intuitively, we can view X locally as the fiber of a map $\mathbb{A}^n_k \to \mathbb{A}^r_k$ defined by the r polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, and where x being in the fiber over zero implies that the tangent space $T_{X,x}$ is the kernel of the map $(f_1, \ldots, f_r)$ hence equal to the dimension of the fiber. We introduce the notion of being geometrically smooth.

Definition 3.15 (Geometrically Smooth Scheme)**.** Let X be a scheme of finite type over a field k. X is geometrically smooth if the base change $X_{\overline{k}}$ to the algebraic closure is smooth over $\overline{k}$.

This is in fact equivalent to the condition of being smooth.

Lemma 3.16. Let X be a scheme of finite type over a field k. X is a smooth k-scheme if and only if it is geometrically smooth.

Proof. We use the Cartesian square

$$\begin{tikzcd} X_{\overline{k}} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\overline{k}) \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

where using Proposition 1.8, we have $\Omega^1_{X/k} \otimes \kappa(x) \cong \Omega^1_{X_{\overline{k}}/\overline{k}} \otimes \kappa(y)$ where y is the closed point corresponding to x in $X_{\overline{k}}$. This isomorphism of sheaves characterizes the Jacobian being full rank at x, y, hence the smoothness conditions are equivalent. ■

While smoothness depends on the structure of X as a k-scheme, it is closely related to the absolute notion of regularity.

We recall the relevant definitons from commutative algebra.

Definition 3.17 (Regular Local Ring)**.** Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field $\kappa = A/\mathfrak{m}$. A is a regular local ring if $\dim(A) = \dim_\kappa(\frac{\mathfrak{m}}{\mathfrak{m}^2})$.

Definition 3.18 (Regular Ring)**.** Let A be a Noetherian ring. A is ar egular ring if for all primes $\mathfrak{p} \subseteq A$, the localization $A_\mathfrak{p}$ is a regular local ring.

Remark 3.19. Checking regularity of an arbitrary Noetherian ring can be done on maximal ideals by reasoning analogous to that of Remark 3.14.

This allows us to define regularity of schemes.

Definition 3.20 (Regular Scheme)**.** Let X be a locally Noetherian scheme. X is regular if for all closed points $x \in X$, the local ring $\mathcal{O}_{X,x}$ is regular.

Remark 3.21. By Definition 3.18, this is equivalent to each point admitting an affine neighborhood given by the Zariski spectrum of a regular ring.

Remark 3.22. In contrast to Remark 3.12, regularity is absolute and does not depend on any structure map of X.

The notions of regularity and smoothness are connected by the following proposition.

Proposition 3.23. Let X be a scheme of finite type over an algebraically closed field k. X is k-smooth if and only if X is regular.

Proof. Both conditions can be checked affine-locally, so without loss of generality, we can take $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n_k$. By Proposition 3.6 the dimension of the Zariski tangent space of any $x \in X$ is dimension of the image of the map D_x defined in Example 3.4, which is equal to the rank of the Jacobian J_x. This is of rank equal to the Zariski tangent space (ie. X is regular) if and only if the corank of the Jacobian is $\dim(X)$ (ie. X is smooth). ■

Over general fields, smoothness implies regularity, but not the converse.

Corollary 3.24. Let X be a scheme of finite type over a field k. If X is k-smooth, then X is regular.

Proof. By locality, we reduce once more to $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n_k$. By smoothness, $D_x(\mathfrak{a}) = \operatorname{rank}(J_x)$ and by the short exact sequence (3.1) we have

$$\dim_{\kappa(\mathfrak{m})}\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) = \dim_{\kappa(\mathfrak{m})}\left(\frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2}\right) - \dim_{\mathfrak{m}}\left(\frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2}\right)$$

showing that the dimension of the Zariski tangent space at x is equal to the dimension of X, hence X is regular. ■

We now see an example of a regular non-smooth scheme.

Example 3.25. Let k be a field of characteristic p and consider $X = V(x^p - a) \subseteq \mathbb{A}^1_k$ and where $a^{1/p} \notin k$. $\mathrm{Spec}(\frac{k[x]}{(x^p-a)})$ is the Zariski spectrum of a field, hence regular, but X is not geometrically smooth and hence not smooth.

References

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