Problem set 8

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Exercises

Exercise 8.1. Let $(C^i, \partial^i)_{i\in I}$ be a family of chain complexes. Show that

$$H_n\left(\bigoplus_{i\in I} C^i\right) \cong \bigoplus_{i\in I} H_n(C^i).$$

Solution

Notice that by the definitions, the functions ∂ are defined by the components, so we have

$$\operatorname{im}\partial = \bigoplus_{i\in I} \operatorname{im}\partial^i \quad \text{and} \quad \ker\partial = \bigoplus_{i\in I} \ker\partial^i.$$

Therefore

$$H_n\left(\bigoplus_{i\in I} C^i\right) = \ker\partial_n / \operatorname{im}\partial_{n+1} = \bigoplus_{i\in I} (\ker\partial_n^i / \operatorname{im}\partial_{n+1}^i) = \bigoplus_{i\in I} H_n^i.$$

□

Exercise 8.2. Let X be a topological space.

(1) If f is a (not necessarily closed) path in X, prove that f is homologous to $-f^{-1}$.

(2) Let a, b, c be (not necessarily closed) paths in X, such that $a * b * c$ is defined, and a closed path. Prove that in $H_1(X)$,

$$[a * b * c]_H = [a]_H + [b]_H + [c]_H.$$

Solution

(1) I am not sure what homologous means when f and $-f^{-1}$ aren't in $Z_1(X)$. We prove however that $f - (-f^{-1}) \in B_1(X)$, because I assume that that is what is wanted. Let $\sigma_f : \Delta^2 \to X$ be the 2-simplex defined by $\sigma_f(x, y, z) = f(y)$. Notice that

$$\partial\sigma_f = \sigma_f \circ d^0 - \sigma_f \circ d^1 + \sigma_f \circ d^2 = \sigma_f(0, x, y) - \sigma_f(x, 0, y) + \sigma_f(x, y, 0)$$

But $\sigma_f(0, x, y)$ is just f and $f(x, y, 0)$ is just f^{-1} and $\sigma_f(x, 0, y)$ is just the constant 1-simplex at $f(0)$. But every constant 1-simplex is the image of the constant 2-simplex at that point, so we can subtract $f(0)$ from it, and still stay in $B_1(x)$.

(1) We prove that

$$[a * b * c]_H - [a]_H - [b]_H - c_[H] \in B_1(X).$$

Let $\sigma_{abc} : \Delta^2 \to X$ be the 2-chain which maps into the degenerate triangle with sides a, $a * b * c$, and $b * c$. Let $\sigma_{bc} : \Delta^2 \to X$ map into the degenerate triangle with sides b, bc, and c. Notice that

$$\partial\sigma_{abc} = [a]_H - [a * b * c]_H + [b * c]_H,$$

and

$$\partial\sigma_{bc} = [b]_H - [b * c]_H + [c]_H.$$

So $\partial(-\sigma_{abc} - \sigma_{bc}) = [a*b*c]_H - [a]_H - [b]_H - [c]_H$, so they are homologous. □

Exercise 8.3. Show that if X is a deformation retract of Y, then $H_n(X) \cong H_n(Y)$ for all $n \geq 0$.

Solution

Is this not trivial because deformation retracts are homotopies, and homology is homotopy equivalent? Is there something which I have misunderstood? □

Exercise 8.4. Compute the singular homology groups of the topologist's sine curve

$$X = \{(x, \sin\frac{1}{x})|0 < x \geq 1\} \cup \{(0, 0)\}.$$

Solution

Notice that the path-components of X are $X_0 = \{(x, \sin\frac{1}{x})|0 < x \leq 1\}$ and $X_1 = (0, 0)$, so we have

$$H_n(X) = H_n(X_0) \oplus H_n(X_1).$$

Because X_1 is a singleton set we have $H_0(X_1) = \mathbb{Z}$, and $H_n(X_1) = 0$ for positive n. Notice that function $(x, \sin\frac{1}{x}) \mapsto x$ is continuous, and has a continuous inverse $x \mapsto (x, \sin\frac{1}{x})$ so X_0 is homeomorphic to the half-open interval $(0, 1]$. But the homology groups of the half-open interval are just 0, except when $n = 0$, when it is $\mathbb{Z}$. So we have $H_0(X) = \mathbb{Z}^2$, and $H_n(X) = 0$ for positive n. $\square$

Exercise 8.5. Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian, and find a basis.

Solution

The relative homology group being free abelian can be done with the long exact sequence in relative singular homology, but I don't know how to prove that it has a basis $\mathbb{Q} \setminus \{x_0\}$ for some $x_0 \in \mathbb{Q}$. $\square$