Introduction to Symplectic Geometry
Notes from Summer Semester 2025

Daniel Arone

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These notes are on the course V5D6 - Introduction to Symplectic Geometry[1], lectured jointly by Nathaniel Bottman and Laurent Côté. They are entirely my own, as are any potential mistakes in what follows. Proceed with caution. These notes are available on GitHub and Overleaf.

This course covers the basic topics in symplectic geometry. We will not discuss Floer theory or J-holomorphic curves. The main reference for this course is [MS17] by McDuff and Salamon.

Contents

[1] https://www.math.uni-bonn.de/~lcote/V5D6_2025.html

1 Introduction to symplectic topology

LECTURE 1 (10.04.25).

In this lecture we covered linear symplectic geometry and the cotangent bundle as the first example of a symplectic manifold. References for this lecture are sections 2.1-2.3 and 3.1 of Mc-Duff and Salamon's *Introduction to Symplectic Topology*.

Definition. A *symplectic manifold* is a pair (M, ω) of an even-dimensional smooth manifold without boundary M, and a closed non-degenerate 2-form ω on M.

An equivalent condition to closedness and non-degeneracy would be that the highest wedge power of ω is a volume form. So in particular it follows that M is orientable.

Linear symplectic geometry

We now turn our attention to the simpler linear setting of symplectic geometry.

Definition. A symplectic vector space is a pair (V, ω) of a finite-dimensional vector space V over $\mathbb{R}$, and a non-degenerate skew-symmetric bilinear form ω on V.

Proposition 1.1. V is even-dimensional.

Proof. This follows immediately from the existence of a non-degenerate skew-symmetric bilinear form. □

Example. Let $V = \mathbb{R}^{2n}$ and ω_0 be given by

$$\omega_0(v, w) = v^{\mathsf{T}} \begin{bmatrix} 0 & \mathrm{Id}_{\mathbb{R}^n} \\ -\mathrm{Id}_{\mathbb{R}^n} & 0 \end{bmatrix} w.$$

This shall be the canonical symplectic form on Euclidean space, and we call the specified matrix $-J_0$. If we denote by $x_1, \ldots, x_n, y_1, \ldots, y_n$ the standard basis on $\mathbb{R}^{2n}$, then we have

$$\omega_0(x_i, x_j) = \omega(y_i, y_j) = 0$$

for all i, j and

$$\omega(x_i, y_j) = \delta_{ij}.$$

Definition. A *linear symplectomorphism* is a linear isomorphism $\psi : (V, \omega) \to (V', \omega')$ such that $\psi^* \omega' = \omega$. Explicitly this means that

$$\omega(v, w) = \omega'(\psi v, \psi w).$$

> **Remark.** We may also consider "symplectic linear maps" in general, but because the symplectic form is non-degenerate the pull-back condition requires these to be injective. Thus a symplectic linear map is just a symplectomorphism onto its image.

A linear symplectomorphism of $(\mathbb{R}^{2n}, \omega_0)$ is then represented by a matrix $A \in \mathrm{GL}_{2n}(\mathbb{R})$ such that $A^\intercal (J_0) A = J_0$.

Definition. A matrix $A \in \mathrm{GL}_{2n}(\mathbb{R})$ satisfying this equation is called symplectic. Denote the group of symplectic matrices by $\mathrm{Sp}(2n)$ or $\mathrm{Sp}(2n, \mathbb{R})$. A computation shows that $\det A = \pm 1$ for symplectic A. In fact, the Pfaffian may be used to show that $\det A = 1$.

compute this

Let (V, ω) be a symplectic vector space and $W \subset V$ a linear subspace.

Definition. The *symplectic complement* of W is

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

We then say that $W \subset V$ is

- *isotropic* if $W \subset W^\omega$,
- *coisotropic* if $W^\omega \subset W$,
- *symplectic* if $W \cap W^\omega = \varnothing$, and
- *Lagrangian* if $W = W^\omega$.

If W is symplectic, then $(W, \omega|_W)$ is a symplectic vector space.

Lemma 1.2. We have the following two results,

$$\dim W + \dim W^\omega = \dim V, \quad \text{and} \quad (W^\omega)^\omega = W.$$

Proof. Consider the map

$$i_\omega : V \to V^*, \quad i_\omega(v)(w) = \omega(v, w).$$

Then this is an isomorphism, as ω is non-degenerate. Now see that

$$i_\omega(W^\omega) = \{\varphi \in V^* | \varphi(w) = 0 \text{ for all } w \in W\} = W^\perp.$$

Because i_ω is an isomorphism, this means that

$$\dim W + \dim W^\omega = \dim W + \dim W^\perp = \dim V.$$

See that by definition $W \subset (W^\omega)^\omega$ and as they have the same dimension they must then be equal. $\square$

Corollary. The following 3 results follow immediately,

- W is symplectic $\iff$ W^ω is symplectic,
- W is isotropic $\iff$ W^ω is coisotropic, and
- W is Lagrangian $\implies$ $\dim W = \frac{1}{2}\dim V$.

Theorem 1.3. Any symplectic vector space (V, ω) of dimension $2n$ is isomorphic $(\mathbb{R}^{2n}, \omega_0)$.

Proof. It suffices to construct a *symplectic basis*, i.e. a basis $u_1, \ldots, u_n, v_1, \ldots, v_n \in V$ such that

prove this

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0,$$

for all i, j and

$$\omega(u_i, v_j) = \delta_{ij}.$$

To construct such a basis first choose any non-zero $u_1 \in V$, and because ω is non-degenerate, we can find some $v_1 \in V$ such that $\omega(u_1, v_1) = 1$. See that u_1, v_1 span a symplectic subspace. Repeating this process on the symplectic complement of their span then gets us a symplectic basis. □

Symplectic manifolds

Definition. A symplectomorphism $(M, \omega_M) \to (N, \omega_N)$ is a diffeomorphism φ such that $\varphi^*\omega_N = \omega_M$. We denote the symplectomorphisms of (M, ω_M) by $\mathrm{Symp}(M, \omega_M)$. This is a lie group. We say that a vector field $X \in \mathfrak{X}(M)$ is symplectic if $i_X\omega$ is closed. We denote these by $\chi(M, \omega)$.

> **Remark.** By i_X we denote the interior derivative or contraction, i.e. for a k-form μ,
>
> $$i_X\mu(X_1, \ldots, X_{k-1}) = \mu(X, X_1, \ldots, X_{k-1}).$$
>
> Sometimes this is also denoted by $X \lrcorner \mu$.

Proposition 1.4. Integrating symplectic vector fields results in a symplectomorphism. Specifically if we have a smooth families (φ_t) of smooth maps and (X_t) of smooth vector fields such that

$$\varphi_0 = \mathrm{id} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = X_t \circ \varphi_t.$$

Then φ_t is symplectic for all t if and only if X_t is symplectic for all t.

Proof. □

prove this

A bit on the cotangent bundle

We consider the cotangent bundle T^*L of a closed manifold L.

Definition (Canonical 1-form)**.** We denote by $\pi : T^*L \to L$ the projection of the fibers to the base space. Then we can take the pullback along the projection as follows. For $p \in L, \xi \in T_p^*(L)$ and a tangent vector $v \in T_{(p,\xi)}T^*L$, denote

$$\lambda_{\mathrm{can}}(p, \xi)(v) := (\xi \circ \mathrm{d}_p\pi)(v).$$

We call $\lambda_{\mathrm{can}} \in \Omega^1(T^*L)$ the *canonical* 1*-form*.

Proposition 1.5. Let L be a smooth manifold, and define $\omega_{\mathrm{can}} := -\,\mathrm{d}\lambda_{\mathrm{can}} \in \Omega^2(T^*L)$ the canonical symplectic form. Then $(T^*L, \omega_{\mathrm{can}})$ is a symplectic manifold.

Proof. It is clear that ω_{can} is closed because it is exact. Verifying that ω_{can} is non-degenerate is left to the reader. □

Remark. Let $x : U \to \mathbb{R}^n$ be a chart of L with coordinates $x_1, \ldots, x_n$. At a point $p \in L$ see that the differentials $(\,\mathrm{d}x_i)_{i=1}^n$ form a basis of the cotangent space T_q^*L and denoting $y_i = \mathrm{d}x_i$ we get local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ for T^*U. Then one can define the canonical 1-form in local coordinates to be

$$\lambda_{\mathrm{can}} = y\,\mathrm{d}x.$$

Then $\omega_{\mathrm{can}} = \mathrm{d}x \wedge \mathrm{d}y$ in local coordinates.

Proposition 1.6. Let $\sigma \in \Omega^1(L)$ be a 1-form. Consider it as a map $\sigma : L \to T^*L$. Then λ_{can} is described by the universal property that the pullback satisfyies

$$\sigma^*\lambda_{\mathrm{can}} = \sigma,$$

for any σ.

Proof. This is done in local coordinates on some chart $x : U \to \mathbb{R}^n$. We write

$$\sigma = \sum_{i=1}^n \sigma_i(x_1, \ldots, x_n)\,\mathrm{d}x_i.$$

Then this is a map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \sigma_1, \ldots, \sigma_n)$. Thus

$$\sigma^*\lambda_{\mathrm{can}} = \sigma^*\left(\sum_{i=1}^n y_i\,\mathrm{d}x_i\right) = \sum_{i=1}^n \sigma_i(x_1, \ldots, x_n)\,\mathrm{d}x_i = \sigma.$$

To verify that this uniquely determines λ is suffices to note that if $\sigma^*\mu = 0$ for all σ, then $\mu = 0$. □

Lecture 2 (17.04.25).

The references for this lecture are sections 3.2 and 3.3 of McDuff-Salamon.

Recap

write this

Theorem (Reynolds transport theorem)**.** Let $\varphi_t : M \to M$ be a smooth family of diffeomorphism generated by vector fields X_t, and η_t a family of symplectic forms. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_t^* \eta_t\right) = \varphi_t^* \left(\frac{\mathrm{d}\eta_t}{\mathrm{d}t} + L_{X_t}\eta_t \right)$$

Moser's trick and its consequences

Theorem 1.7 (Moser's trick)**.** Let M be a closed smooth even-dimensional manifold. Let $(\omega_t)_{[t\in[0,1]}$ a family of symplectic forms on M and $(\sigma_t)_{t\in[0,1]}$ a family of smooth 1-forms such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \mathrm{d}\sigma_t.$$

Then there exists a family $(\psi_t)_{t\in[0,1]}$ of diffeomorphisms on M satisfying

$$\psi_t^*\omega_t = \omega_0.$$

To prove this we require the following lemma.

Lemma 1.8. Let M be a smooth $2n$-manifold, $Q \subset M$ a closed submanifold, and $\omega_1 0, \omega_1 \in \Omega^2(M)$ closed forms such that for all points $q \in Q$, $\omega_{0,q}$ and $\omega_{1,q}$ are non-degenerate and agree on all of T_qM.

Proof. □

prove this

We are now equipped to prove Moser's trick.

Proof. □

prove this

Consequences of Moser's trick

short bit on Darboux's theorem

Theorem (Darboux)**.** Let (M, ω) be a symplectic $2n$-manifold. Then for all points $p \in M$, there exists some neighborhood $U \ni p$ such that $(U, \omega|_U)$ is symplectomorphic an open $V \subset \mathbb{R}^{2n}$ equipped with the canonical symplectic form.

Proof. □

prove this

Theorem (Moser's stability theorem)**.** Let M be a closed manifold, and $(\omega_t)_{t\in[0,1]}$ a smooth family of symplectic forms such that $[\omega_t] \in H^2_{\mathrm{dR}}(M, \mathbb{R})$ is fixed independently of t. Then there exists a family of diffeomorphism $(\psi_t)_{t\in[0,1]}$ such that

$$\psi_t^*\omega_t = \omega_0.$$

Proof. □

prove this

Definition. We say that a smooth submanifold $Q \subset (M, \omega)$ is symplectic, (co)isotropic, or Lagrangian if for all $q \in Q$ the tangent space $T_qQ \subset T_qM$ satisfies the corresponding property with respect to ω.

Recall specifically that $T_qQ \subset T_qM$ is Lagrangian if $(T_qQ)^\omega = T_qQ$ or equivalently if $\dim Q = \dim M/2$ and $\omega_q|_{T_qQ} = 0$.

Example $(T^*L, \omega_{\text{can}})$**.** As previously discussed, for a manifold L the cotangent bundle and canonical form give a symplectic manifold $(T^*L, \omega_{\text{can}}$. Now the zero section $L \subset T^*L$ and the cotangent space $T_q^*L \subset T^*L$ are Lagrangian submanifolds.

tikz this

In fact we will next show that any closed Lagrangian submanifold is locally like T^*L.

Theorem (Weinstein neighborhood theorem)**.** Let $L^n \subset (M^{2n}, \omega)$ be a Lagrangian submanifold. Then there exist open sets

$$L \subset U \subset M \quad \text{and} \quad L \subset V \subset T^*L,$$

and a symplectomorphism $\varphi : (V, \omega_{\text{can}}) \to (U, \omega|_U)$ such that $\varphi|_L = \mathrm{id}_L$.

Proof. $\square$

prove this

2 Dynamics and integrable systems

Lecture 3 (24.04.25).

In this lecture we covered Hamiltonian mechanics and vector fields, as well as the Poisson bracket. The goal of this lecture was to introduce some the historical and physical motivation for symplectic geometry. In the notes for this lecture we omit some of the physical calculations for brevity as they will not be relevant going further. We attempt to retain the physical and historical motivation behind symplectic geometry.

References for this lecture are sections 1.1 and 3.1 of [MS17]. *(Note: From this point onward the references are specifically for the third edition of the book, I am unsure which edition the previous ones refer to.)*

Hamiltonian mechanics

Consider a physical system, the configurations of which may be described by a point $x \in \mathbb{R}^n$. For example, the positions of three celestial objects may be described by a point in $\mathbb{R}^9$. We are interested in trajectories in this space, denoted by $t \mapsto x(t)$. Now suppose that we have a function

$$L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad L = L(t, x, v)$$

such that trajectories are the critical points of the "action functional":

$$I(x) := \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, \mathrm{d}t.$$

In fact, such a function models the difference between the kinetic and potential energies of the system. *(Note: In the tradition of the physicists we will abuse notation and denote $L(t, x(t), v(t))$ by $L(t, x, v)$.)*

Lemma 2.1. A minimal path $x : [t_0, t_1] \to \mathbb{R}^n$ satisfies the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial x},$$

where by $\partial L/\partial v$ we refer to $(\partial L/\partial v_1, \ldots, \partial L/\partial v_n)$, and similarly for x.

Proof. See Lemma 1.1.1 in [MS17]. □

This is the Lagrangian formulation of mechanics. To transform this into the Hamiltonian formulation of mechanics we apply the Legendre transformation to our coordinates. We replace v_i by y_i where

$$y_i = \frac{\partial L}{\partial v_i}(x, v).$$

This is a valid coordinate transformation as long as the Legendre condition,

$$\det\left(\frac{\partial^2 L}{\partial v_i \partial v_j}\right)_{ij} \neq 0,$$

holds. For clarity of notation we denote $G_i(t, x, y) = v_i$, and avoid reference to v so that we may take y as a given.

Definition (Hamiltonian)**.** We define the Hamiltonian to be

$$H(t, x, y) = \sum_{i=1}^{n} y_i \cdot G_i(t, x, y) - L(t, x, G(t, x, y)).$$

This represents the total energy within the system.

We omit the computation but one may see that

$$\frac{\partial H}{\partial x_i} = -\dot{y}_i \quad \text{and} \quad \frac{\partial H}{\partial y_i} = \dot{x}_i.$$

These are called Hamilton's equations. Note that while the Lagrangian formalism is concerned with vectors in space and the tangent bundle, in Hamilton's reformulation we focus on the cotangent bundle.

Write $z(t) = (x(t), y(t))$. Then we may rewrite Hamilton's equations as

$$\dot{z} = -J_0 \nabla H(z), \quad J_0 = \begin{bmatrix} 0 & -\mathrm{id}_n \\ \mathrm{id}_n & 0 \end{bmatrix}.$$

The Poisson bracket

In what follows we assume that H is independent of t, so $H(t, x(t), y(t)) = H(x(t), y(t))$.

finish this lecture

Lecture 4 (08.05.25).

The references for this lecture are sections 1.3-1.6 of [Eva23].

3 Lagrangian submanifolds and toric symplectic manifolds

Lecture 5

Lecture 6

4 Symplectic capacities and embeddings

Lecture 7

Lecture 8

Lecture 9

Lecture 10

5 Hamiltonian dynamics

Lecture 11

References

[Eva23] Jonny Evans. *Lectures on Lagrangian Torus Fibrations*. London Mathematical Society Student Texts. Cambridge University Press, 2023.

[MS17] Dusa McDuff and Dietmar Salamon. *Introduction to Symplectic Topology*. Oxford University Press, Mar. 2017. isbn: 9780198794899. doi: 10.1093/oso/9780198794899.001.0001. url: https://doi.org/10.1093/oso/9780198794899.001.0001.