Rankine Vortex Equations

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https://github.com/Daniel-Butt/Rankine_Tornado_Vortex_Solver

By Definition:

V_r = Maximum radial velocity, $V_r \in \mathbb{R}^+$
V_t = Maximum tangential velocity, $V_t \in \mathbb{R}^+$
V_s = Displacement velocity, $V_s \in \mathbb{R}^+$
V_c = Tree critical failure velocity, $V_c \in \mathbb{R}^+$
R_{max} = Radius of maximum tangential velocity, $R_{max} \in \mathbb{R}^+$

V_r, V_t, V_s, V_c, and R_{max} are all input parameters for a given Rankine Vortex.

r = Radial distance from origin, $r \in \mathbb{R}^+$

Rankine Vortex Equation

$$|\vec{V}_{tan}| = \begin{cases} V_t r R_{max}^{-1} & \text{if } r \leqslant R_{max} \\ V_t r^{-1} R_{max} & \text{if } r > R_{max} \end{cases} \qquad |\vec{V}_{rad}| = \begin{cases} V_r r R_{max}^{-1} & \text{if } r \leqslant R_{max} \\ V_r r^{-1} R_{max} & \text{if } r > R_{max} \end{cases}$$

For simplification purposes, V_s is assumed to always be in the positive y direction. Though it should be noted that the results can simply be rotated should V_s be in a different direction.

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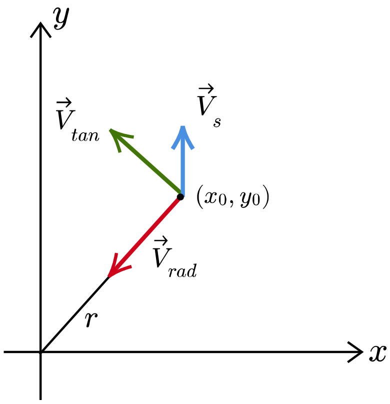
1. Calculating Velocity Vector At Point (x, y)



$$r = \sqrt{x^2 + y^2} \quad \widehat{V}_{rad} = \left[\frac{-x}{r}, \frac{-y}{r} \right], \widehat{V}_{tan} = \left[\frac{-y}{r}, \frac{x}{r} \right]$$

<u>if $r \leqslant R_{max}$ (inner solution)</u>

$$\vec{V}_{rad} = \left[-xV_r R_{max}^{-1}, \ -yV_r R_{max}^{-1} \right]$$
$$\vec{V}_{tan} = \left[-yV_t R_{max}^{-1}, \ xV_t R_{max}^{-1} \right]$$
$$\vec{V}_s = [\,0, \ V_s]$$

$$\boldsymbol{\vec{V} = \left[-R_{max}^{-1}(xV_r + yV_t), \ R_{max}^{-1}(xV_t - yV_r) + V_s \right]}$$

<u>if $r > R_{max}$ (outer solution)</u>

$$\vec{V}_{rad} = \left[-xV_r r^{-2} R_{max}, \ -yV_r r^{-2} R_{max} \right]$$
$$\vec{V}_{tan} = \left[-yV_t r^{-2} R_{max}, \ xV_t r^{-2} R_{max} \right]$$
$$\vec{V}_s = [\,0, \ V_s]$$

$$\boldsymbol{\vec{V} = \left[-r^{-2} R_{max}(xV_r + yV_t), \ r^{-2} R_{max}(xV_t - yV_r) + V_s \right]}$$

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2. Solving For The Point (x, y) Of Maximum Velocity

By definition, $\vec{V}_{max}$ occurs when $r = R_{max}$ or $r^2 = x^2 + y^2 = R_{max}^2$

Since $r = R_{max}$, the inner solution for $\vec{V}$ is required.

In order to find position of maximum velocity we must solve for the maximum of the following system of equations.

$$g(x, y) = x^2 + y^2 - R_{max}^2 = 0,$$

$$f(x, y) = |\overrightarrow{V}|^2 = \left[R_{max}^{-1}(xV_r + yV_t)\right]^2 + \left[R_{max}^{-1}(xV_t - yV_r) + V_s\right]^2$$

This problem can be solved various ways, however, this is a pretty typical Lagrangian Multiplier Optimization problem,

i.e. $\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y),\ \nabla\mathcal{L} = 0$

Expanding,

$$f(x, y) = R_{max}^{-2}\left(V_r^2 + V_t^2\right)x^2 + R_{max}^{-2}\left(V_r^2 + V_t^2\right)y^2 + 2R_{max}^{-1}V_sV_tx - 2R_{max}^{-1}V_sV_ry + V_s^2$$

For simplification, let

$$a = R_{max}^{-2}\left(V_r^2 + V_t^2\right),\ b = 2R_{max}^{-1}V_sV_t,\ c = 2R_{max}^{-1}V_sV_r,\ d = V_s^2$$

$$\Longrightarrow f(x, y) = ax^2 + ay^2 + bx - cy + d$$

Solving the Lagrangian multiplier,

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y) = ax^2 + ay^2 + bx - cy + d\ +\ \lambda(x^2 + y^2 - R_{max}^2)$$

$$\nabla\mathcal{L} = 0 \Longrightarrow \frac{\partial\mathcal{L}}{\partial x} = \frac{\partial\mathcal{L}}{\partial y} = \frac{\partial\mathcal{L}}{\partial\lambda} = 0$$

$$\frac{\partial\mathcal{L}}{\partial x} = (2a + 2\lambda)x + b = 0 \Longrightarrow x = \frac{-b}{2a + 2\lambda}$$

$$\frac{\partial\mathcal{L}}{\partial y} = (2a + 2\lambda)y - c = 0 \Longrightarrow y = \frac{c}{2a + 2\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - R_{max}^2 = 0$$

Solving for λ yields,

$$\lambda = -a \pm a\sqrt{1 - \frac{b^2 + c^2}{4R_{max}^2}}$$

Subbing in λ yields,

$$x = \mp \frac{bR_{max}}{\sqrt{b^2 + c^2}} \qquad y = \pm \frac{cR_{max}}{\sqrt{b^2 + c^2}}$$

By visual inspection of graphed Rankine Vortexes it is clear the maximum occurs at,

$$x = \frac{bR_{max}}{\sqrt{b^2 + c^2}} \qquad y = -\frac{cR_{max}}{\sqrt{b^2 + c^2}}$$

$b = 2R_{max}^{-1} V_s V_t, \;\; c = 2R_{max}^{-1} V_s V_r$

$$\implies \mathbf{x} = \frac{\mathbf{V_t R_{max}}}{\sqrt{\mathbf{V_t}^2 + \mathbf{V_r}^2}} \qquad \mathbf{y} = -\frac{\mathbf{V_r R_{max}}}{\sqrt{\mathbf{V_t}^2 + \mathbf{V_r}^2}}$$

*other point is a minimum.

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3. Solving For A Tree Fall Pattern Vector Given An X-Coordinate

For this relatively simple tornado tree fall model, it is assumed that a tree will fail (snap/uproot), the instant that the wind velocity around the tree exceeds V_c, the tree critical failure velocity, and that a tree will not fail otherwise.

Given that the tornado is assumed to be moving in the positive y direction (V_s), solving for a tree fall pattern vector given an x-coordinate entails finding the greatest y-coordinate where the magnitude of the wind velocity, $|\vec{V}|$, at the point (x, y) exceeds the tree critical failure velocity, V_c. The greatest y-coordinate is needed as this will be the first occurrence where $|\vec{V}| \geqslant V_c$ for a tornado moving in the positive y direction (V_s). See the simulation on the GitHub page for a visual demo.

If $r \leqslant R_{max}$ (inner solution)

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From 2.

$$|\overrightarrow{V}|^2 = V_c^2 = R_{max}^{-2}\left(V_r^2 + V_t^2\right)x^2 + R_{max}^{-2}\left(V_r^2 + V_t^2\right)y^2 + 2R_{max}^{-1}V_sV_tx - 2R_{max}^{-1}V_sV_ry + V_s^2$$

$$\implies 0 = R_{max}^{-2}\left(V_r^2 + V_t^2\right)y^2 - 2R_{max}^{-1}V_sV_ry + R_{max}^{-2}\left(V_r^2 + V_t^2\right)x^2 + 2R_{max}^{-1}V_sV_tx + V_s^2 - V_c^2$$

For simplification, let

$$a = R_{max}^{-2}\left(V_r^2 + V_t^2\right), b = \ -2R_{max}^{-1}V_sV_r,\ \ c = R_{max}^{-2}\left(V_r^2 + V_t^2\right)x^2 + 2R_{max}^{-1}V_sV_tx + V_s^2 - V_c^2$$

$$\implies 0 = ay^2 + by + c$$

Solving the quadratic yields,

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Subbing in an simplifying,

$$\mathbf{y} = \frac{\mathbf{V_sV_r \pm \sqrt{{V_s}^2{V_r}^2 - \left({V_r}^2 + {V_t}^2\right)\left(R_{max}^{-2}\left({V_r}^2 + {V_t}^2\right)x^2 + 2R_{max}^{-1}V_sV_tx + {V_s}^2 - V_c^2\right)}}}{\mathbf{R_{max}^{-1}\left({V_r}^2 + {V_t}^2\right)}}$$

If $r > R_{max}$ (outer solution)

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From 1. plus a similar process as in 2.

$$0 = R_{max}^2\left(V_r^2 + V_t^2\right)y^2 - 2r^2 R_{max}V_sV_ry + R_{max}^2\left(V_r^2 + V_t^2\right)x^2 + 2r^2R_{max}V_sV_tx + r^4\left(V_s^2 - V_c^2\right)$$

$$r^2 = x^2 + y^2$$

For simplification, let

$$a = R_{max}^2\left(V_r^2 + V_t^2\right), b = 2R_{max}V_sV_r,\ \ c = 2R_{max}V_sV_t,\ \ d = V_s^2 - V_c^2$$

$$\implies 0 = ay^2 - b\left(x^2 + y^2\right)y + ax^2 + c\left(x^2 + y^2\right)x + d\left(x^2 + y^2\right)^2$$

Solving for y yields,

$y = \dfrac{b \pm \sqrt{b^2 - 4d(a + x(c + dx))}}{2d}$ or if $d = 0$, $y = \dfrac{a + cx}{b}$

Subbing in an simplifying,

$$\mathbf{y} = \frac{\mathbf{R_{max}V_sV_r} \pm \sqrt{\mathbf{R_{max}^2V_s^2V_r^2} - \left(\mathbf{V_s^2 - V_c^2}\right)\left(\mathbf{R_{max}^2}\left(\mathbf{V_r^2 + V_t^2}\right) + \mathbf{2R_{max}V_sV_tx} + \left(\mathbf{V_s^2 - V_c^2}\right)\mathbf{x^2}\right)}}{\left(\mathbf{V_s^2 - V_c^2}\right)}$$

or if $d = 0$,

$$\mathbf{y} = \frac{\mathbf{R_{max}}\left(\mathbf{V_r^2 + V_t^2}\right) + \mathbf{2V_sV_tx}}{\mathbf{2V_sV_r}}$$

*maybe this can be simplified more?

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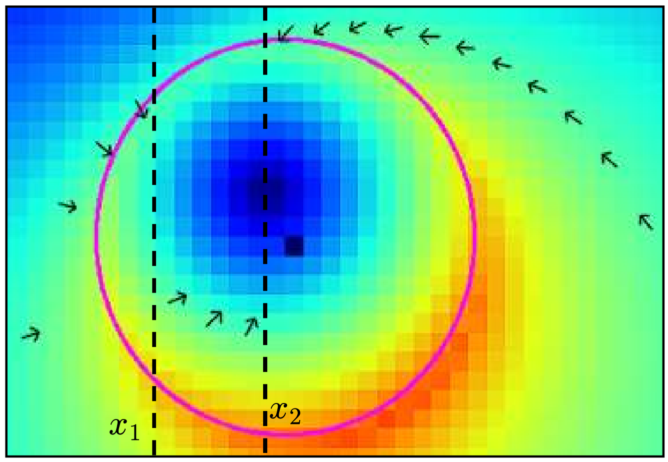
4. Solving For Asymptotes

When solving for the a Tree Fall Pattern (described in 4.), it is common to get both inner ($r \leqslant R_{max}$) and outer solutions ($r > R_{max}$), I'm defining the asymptotes as the dividing lines when solutions switch from inner to outer or vice versa.



This can be seen best by the dashed lines in the above images.

When finding solutions where $x_2 < x < x_1$, only outer solutions need be considered and conversely, finding solutions where $x_1 < x < x_2$, only inner solutions need be considered. Furthermore, if for a given set of input parameter no asymptotes exist, then only outer solutions need be considered.

<u>To find Asymptotes</u>

As can be seen in the figure above, the asymptotes occur as the y-coordinate of the outer solutions approach the top half of the circle $x^2 + y^2 = R_{max}^2 \implies y = \sqrt{R_{max}^2 - x^2}$.

Subbing this y value into the outer pattern solution equation from 3. gives,

$$\sqrt{R_{max}^2 - x^2} = \frac{b \pm \sqrt{b^2 - 4d(a + x(c + dx))}}{2d},$$

where $a = R_{max}^2\left(V_r^2 + V_t^2\right), b = 2R_{max}V_sV_r,\ c = 2R_{max}V_sV_t,\ d = V_s^2 - V_c^2$

Solving for x yields,

$$x = \frac{-\left(ac + cdR_{max}^2\right) \pm b\sqrt{R_{max}^2\left(b^2 + c^2 - 2ad - d^2R_{max}^2\right) - a^2}}{b^2 + c^2}$$

Subbing in an simplifying,

$$\boldsymbol{x = \frac{-R_{max}V_t\left(V_r^2 + V_t^2 + V_s^2 - V_c^2\right) \pm R_{max}V_r\sqrt{D}}{2V_s\left(V_r^2 + V_t^2\right)}},$$

where $\mathbf{D = 2\left({V_r}^2 + {V_t}^2\right)\left({V_s}^2 + {V_c}^2\right) - \left({V_s}^2 - {V_c}^2\right)^2 - \left({V_r}^2 + {V_t}^2\right)^2}$, is the discriminant.

if the discriminant is negative, no asymptotes exist (imaginary solutions)
$0 > 2\left(V_r^2 + V_t^2\right)\left(V_s^2 + V_c^2\right) - \left(V_s^2 - V_c^2\right)^2 - \left(V_r^2 + V_t^2\right)^2$

For our purposes it is very unlikely, but if $D = 0$ then one asymptotes exists.

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5. Solving For Axis Of Convergence

First solve for the asymptotes between the inner and outer solutions (see 4.). If asymptotes exists, take the asymptotes which is closer to the point of maximum velocity (see 2.), in effectively all cases, this is the asymptote with the greatest x-coordinate. If asymptotes don't exist, then find the x-coordinate which produces an outer solution whose direction is parallel to the displacement vector of the tornado.

Discriminant from 4.

$$D = 2\left(V_r^2 + V_t^2\right)\left(V_s^2 + V_c^2\right) - \left(V_s^2 - V_c^2\right)^2 - \left(V_r^2 + V_t^2\right)^2$$

if $D \geqslant 0$:

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$$\widetilde{\boldsymbol{x}} = \max\left(\frac{-R_{max}V_t\left(V_r^2 + V_t^2 + V_s^2 - V_c^2\right) \pm R_{max}V_r\sqrt{D}}{2V_s\left(V_r^2 + V_t^2\right)}\right)$$

else

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Solve for an outer solution vector whose direction is parallel to the displacement vector of the tornado (in this case vertical).

For outer vectors, from 1.

$$\vec{V} = \left[-r^{-2}R_{max}(xV_r + yV_t),\ r^{-2}R_{max}(xV_t - yV_r) + V_s\right]$$

For this vector to be vertical,

$$0 = -r^{-2}R_{max}(xV_r + yV_t), \quad 0 \neq\ r^{-2}R_{max}(xV_t - yV_r) + V_s$$

$$\Longrightarrow x = -\frac{V_t}{V_r}y,\ x \neq \frac{V_r}{V_t}y$$

For outer pattern solutions,

$$y = \frac{b \pm \sqrt{b^2 - 4d(a + x(c + dx))}}{2d} = -\frac{V_r}{V_t}x,$$

let $\lambda = \dfrac{V_t}{V_r},\ a = R_{max}^2\left(V_r^2 + V_t^2\right), b = 2R_{max}V_sV_r,\ c = 2R_{max}V_sV_t,\ d = V_s^2 - V_c^2$

$$\implies x = -\lambda \frac{b \pm \sqrt{b^2 - 4d(a + x(c + dx))}}{2d}$$

Solving for x yields,

$$x = \frac{-\frac{\lambda(c\lambda+b)}{d} \pm \sqrt{\frac{\lambda^2(c\lambda + b)^2}{d^2} - \frac{4a\lambda^2\left(\lambda^2 + 1\right)}{d}}}{2\left(\lambda^2 + 1\right)}$$

Simplifying gives,

$$\boldsymbol{\tilde{x}} = \frac{-2\mathrm{R}_{\mathrm{max}}\mathrm{V}_{\mathrm{s}}\left(\mathrm{V}_{\mathrm{t}}^{3} + \mathrm{V}_{\mathrm{t}}\mathrm{V}_{\mathrm{r}}^{2}\right) \pm \mathrm{V}_{\mathrm{t}}\mathrm{R}_{\mathrm{max}}\sqrt{2\mathrm{V}_{\mathrm{s}}\left(\mathrm{V}_{\mathrm{t}}^{2}\mathrm{V}_{\mathrm{r}} + \mathrm{V}_{\mathrm{r}}^{3}\right)^{2} - 4\mathrm{R}_{\mathrm{max}}\left(\mathrm{V}_{\mathrm{r}}^{2} + \mathrm{V}_{\mathrm{t}}^{2}\right)^{2}\left(\mathrm{V}_{\mathrm{s}}^{2} - \mathrm{V}_{\mathrm{c}}^{2}\right)}}{2\left(\mathrm{V}_{\mathrm{t}}^{2} + 1\right)\left(\mathrm{V}_{\mathrm{s}}^{2} - \mathrm{V}_{\mathrm{c}}^{2}\right)}$$

Where one solution is invalid some how...
or if $d = 0$,

$$y = \frac{a + cx}{b} \implies x = -\lambda \frac{a + cx}{b}$$

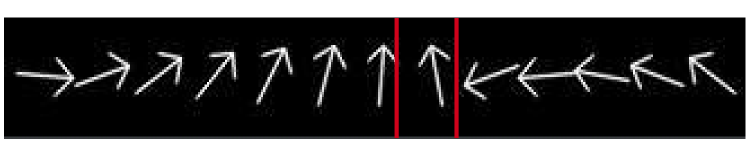
Solving for x yields,

$$x = \frac{-\lambda a}{b + c\lambda}$$

Simplifying gives,

$$\boldsymbol{\tilde{x}} = \frac{-V_t R_{max}}{2V_s}$$

<u>The Ambiguous Case</u>
Consider the following tree fall pattern,



x_1 x_2

At x_1 the vector is perfectly aligned with the displacement of the tornado, but x_2 is an asymptote and the vectors on either side of x_2 have the biggest difference in angle.
So, should the axis of convergence occur at x_1 or x_2? I say x_2 but this is clearly debatable.

6. Solving For Pattern Width

Inner Solution

from 3.

$$y = \frac{V_s V_r \pm \sqrt{V_s^2 V_r^2 - \left(V_r^2 + V_t^2\right)\left(R_{max}^{-2}\left(V_r^2 + V_t^2\right)x^2 + 2R_{max}^{-1} V_s V_t x + V_s^2 - V_c^2\right)}}{R_{max}^{-1}\left(V_r^2 + V_t^2\right)}$$

Inner solution occur so long as the value under the square root is not negative

$$0 = R_{max}^{-2}\left(V_r^2 + V_t^2\right)^2 x^2 + 2R_{max}^{-1} V_s V_t \left(V_r^2 + V_t^2\right)x + \left(V_s^2 - V_c^2\right)\left(V_r^2 + V_t^2\right) - V_s^2 V_r^2$$

$$\implies x = \frac{-V_s V_t R_{max} \pm V_c R_{max}\sqrt{V_r^2 + V_t^2}}{\left(V_r^2 + V_t^2\right)} = x_i$$

Outer Solution

$$y = \frac{R_{max} V_s V_r \pm \sqrt{R_{max}^2 V_s^2 V_r^2 - \left(V_s^2 - V_c^2\right)\left(R_{max}^2\left(V_r^2 + V_t^2\right) + 2R_{max} V_s V_t x + \left(V_s^2 - V_c^2\right)x^2\right)}}{\left(V_s^2 - V_c^2\right)}$$

$$0 = \left(V_s^2 - V_c^2\right)^2 x^2 + 2R_{max} V_s V_t \left(V_s^2 - V_c^2\right)x + R_{max}^2\left(V_s^2 - V_c^2\right)\left(V_r^2 + V_t^2\right) - R_{max}^2 V_s^2 V_r^2$$

$$\implies x = \frac{-V_s V_t R_{max} \pm V_c R_{max}\sqrt{V_r^2 + V_t^2}}{\left(V_s^2 - V_c^2\right)} = x_o$$

Thus,

$\boldsymbol{(x_{min}, x_{max}) = (\min\{x_i, x_o\}, \max\{x_i, x_o\})}$