Taller

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1. Sea

$$\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix} = N_2\left(\boldsymbol{\mu} = \begin{bmatrix} \mu \\ \eta \end{bmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2 & \rho\sigma\tau \\ \rho\sigma\tau & \tau^2 \end{bmatrix}\right)$$

a. **Demostración:** calculemos $|\boldsymbol{\Sigma}|$ y $\boldsymbol{\Sigma}^{-1}$

- $|\boldsymbol{\Sigma}| = \sigma^2\tau^2 - \rho^2\sigma^2\tau^2 = (\sigma\tau)^2(1-\rho^2)$
- $\boldsymbol{\Sigma}^{-1} = \frac{1}{|\boldsymbol{\Sigma}|}\begin{bmatrix} \tau^2 & -\rho\sigma\tau \\ -\rho\sigma\tau & \sigma^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2(1-\rho^2)} & -\frac{\rho}{\sigma\tau(1-\rho^2)} \\ -\frac{\rho}{\sigma\tau(1-\rho^2)} & \frac{1}{\tau^2(1-\rho^2)} \end{bmatrix}$

Y por ende la función de dsitribución del vector aleatorio viene dado por

$$\begin{aligned}
f_{(X,Y)}(x,y) &= \frac{|\boldsymbol{\Sigma}|^{-1/2}}{(2\pi)^{2/2}}\exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} \\
&= \frac{|\boldsymbol{\Sigma}|^{-1/2}}{2\pi}\exp\left\{-\frac{1}{2}\begin{pmatrix} x-\mu \\ y-\eta \end{pmatrix}^T\begin{bmatrix} \frac{1}{\sigma^2(1-\rho^2)} & -\frac{\rho}{\sigma\tau(1-\rho^2)} \\ -\frac{\rho}{\sigma\tau(1-\rho^2)} & \frac{1}{\tau^2(1-\rho^2)} \end{bmatrix}\begin{pmatrix} x-\mu \\ y-\eta \end{pmatrix}\right\} \\
&= \frac{1}{2\pi\sigma\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2}\left[\frac{(x-\mu)^2}{\sigma^2(1-\rho^2)} - 2\frac{\rho(x-\mu)(y-\eta)}{\sigma\tau(1-\rho^2)} + \frac{(y-\eta)^2}{\tau^2(1-\rho^2)}\right]\right\} \\
&= \frac{1}{2\pi\sigma\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2(1-\rho^2)} + \frac{\rho(x-\mu)(y-\eta)}{\sigma\tau(1-\rho^2)} - \frac{(y-\eta)^2}{2\tau^2(1-\rho^2)}\right\}
\end{aligned}$$

Luego la función de densidad de la variable aleatoria $Y|X$ viene dada por:

$$
\begin{aligned}
f_{Y|X=x}(y) &= \frac{(2\pi\sigma\tau\sqrt{1-\rho^2})^{-1}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2(1-\rho^2)}+\frac{\rho(x-\mu)(y-\eta)}{\sigma\tau(1-\rho^2)}-\frac{(y-\eta)^2}{2\tau^2(1-\rho^2)}\right\}}{(2\pi)^{-1/2}\sigma^{-1}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}} \\
&= \frac{(2\pi)^{1/2}\sigma}{2\pi\sigma\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2(1-\rho^2)}+\frac{\rho(x-\mu)(y-\eta)}{\sigma\tau(1-\rho^2)}-\frac{(y-\eta)^2}{2\tau^2(1-\rho^2)}+\cdots\right. \\
\cdots &+ \left.\frac{(x-\mu)^2}{2\sigma^2}\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2(1-\rho^2)}+\frac{\rho(x-\mu)(y-\eta)}{\sigma\tau(1-\rho^2)}-\frac{(y-\eta)^2}{2\tau^2(1-\rho^2)}+\cdots\right. \\
\cdots &+ \left.\frac{(x-\mu)^2}{2\sigma^2}\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu)^2}{\sigma^2}-2\frac{\rho(x-\mu)(y-\eta)}{\sigma\tau}+\frac{(y-\eta)^2}{\tau^2}+\cdots\right.\right. \\
\cdots &- \left.\left.\frac{(1-\rho^2)(x-\mu)^2}{\sigma^2}\right]\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{\rho^2(x-\mu)^2}{\sigma^2}-2\frac{\rho(x-\mu)(y-\eta)}{\sigma\tau}+\frac{(y-\eta)^2}{\tau^2}\right]\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\rho\frac{x-\mu}{\sigma}\right)^2-2\frac{\rho(x-\mu)(y-\eta)}{\sigma\tau}+\left(\frac{y-\eta}{\tau}\right)^2\right]\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\rho\frac{x-\mu}{\sigma}-\frac{y-\eta}{\tau}\right]^2\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{y-\eta}{\tau}-\rho\frac{x-\mu}{\sigma}\right]^2\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{y-\eta}{\tau}-\tau\rho\frac{x-\mu}{\tau\sigma}\right]^2\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{y-\eta}{\tau}-\frac{\tau\rho}{\sigma}\frac{x-\mu}{\tau}\right]^2\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{y-\eta-(\tau\rho)\sigma^{-1}(x-\mu)}{\tau}\right]^2\right\} \\
&= \frac{1}{(2\pi)^{1/2}\tau\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)\tau^2}\left[y-\left(\eta+\frac{\tau\rho}{\sigma}(x-\mu)\right)\right]^2\right\}
\end{aligned}
$$

Con lo que podemos deducir que

$$Y|X=x \sim \mathcal{N}\left(\eta+\frac{\tau\rho}{\sigma}(x-\mu), \tau^2(1-\rho^2)\right)$$

Y por ende, se sigue que:

a. $E\left[Y|X=x\right]=\eta+\frac{\tau\rho}{\sigma}(x-\mu)$

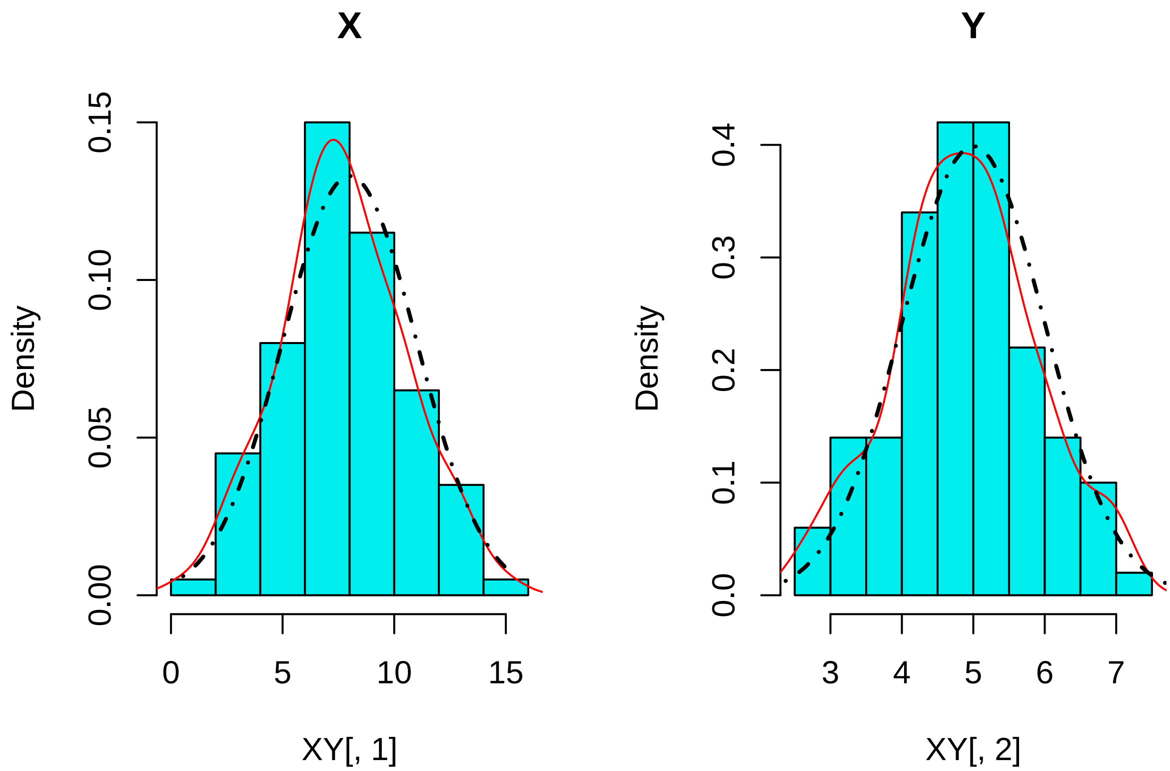
b. $V[Y|X=x]=\tau^2(1-\rho^2)$

c. Usemos la plantilla que nos dan.

```r
rm(list = ls())          # Borramos todos los elementos existentes
library(sm)

set.seed(314159265)      # Fijamos la semilla (reproducibilidad)

# Parámetros de las funciones a usar
mu <- 8          # Media de X
eta <- 5         # Media de Y
sigma <- 3       # Desviaciones estándar de X
tao <- 1         # Desviaciones estándar de Y
rho <- 0.7       # Correlación entre X y Y

# Matriz de varianzas y covarianzas
SIGMA <- c(sigma^2, tao*sigma*rho, tao*sigma*rho, tao^2)
dim(SIGMA) <- c(2, 2)

# Secuencia sobre las que se van a hacer observaciones
x <- seq(from = 5, to = 15, by = 0.1)

# Toma de las muestras
XY <- as.data.frame(MASS::mvrnorm(n = 100,
                                  mu = c(mu, eta),
                                  Sigma = SIGMA))
names(XY) <- c("X", "Y")
```

Veamos unos gráficos de los datos obtenidos

```r
# Configuración del dispositivo de graficación:
par(mfrow = c(1, 2))

# X
hist(XY[, 1], freq = FALSE, col = "cyan2", main = "X")
x1 <- seq(min(XY[, 1]), max(XY[, 1]), by = 0.1)
lines(density(XY[, 1]), col = "red")
lines(x = x1, dnorm(x1, mean = mu, sd = sigma), lty = 4,
      lwd = 2)

# Y
hist(XY[, 2], freq = FALSE, col = "cyan2", main = "Y")
x2 <- seq(min(XY[, 2]), max(XY[, 2]), by = 0.1)
lines(density(XY[, 2]), col = "red")
lines(x = x1, dnorm(x1, mean = eta, sd = tao), lty = 4,
      lwd = 2)
```



d. Los resultados de la estimación del modelo de regresión lineal vienen dados por:

```
# Ajuste del modelo
modelo <- lm(Y ~ X, data = XY)

# Resumen del modelo
summary(modelo)
```

```
##
## Call:
## lm(formula = Y ~ X, data = XY)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.63366 -0.48740 -0.05903  0.47047  1.62302
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.91903    0.20456   14.27   <2e-16 ***
## X            0.25470    0.02493   10.22   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.7006 on 98 degrees of freedom
## Multiple R-squared:  0.5158, Adjusted R-squared:  0.5108
## F-statistic: 104.4 on 1 and 98 DF,  p-value: < 2.2e-16
```

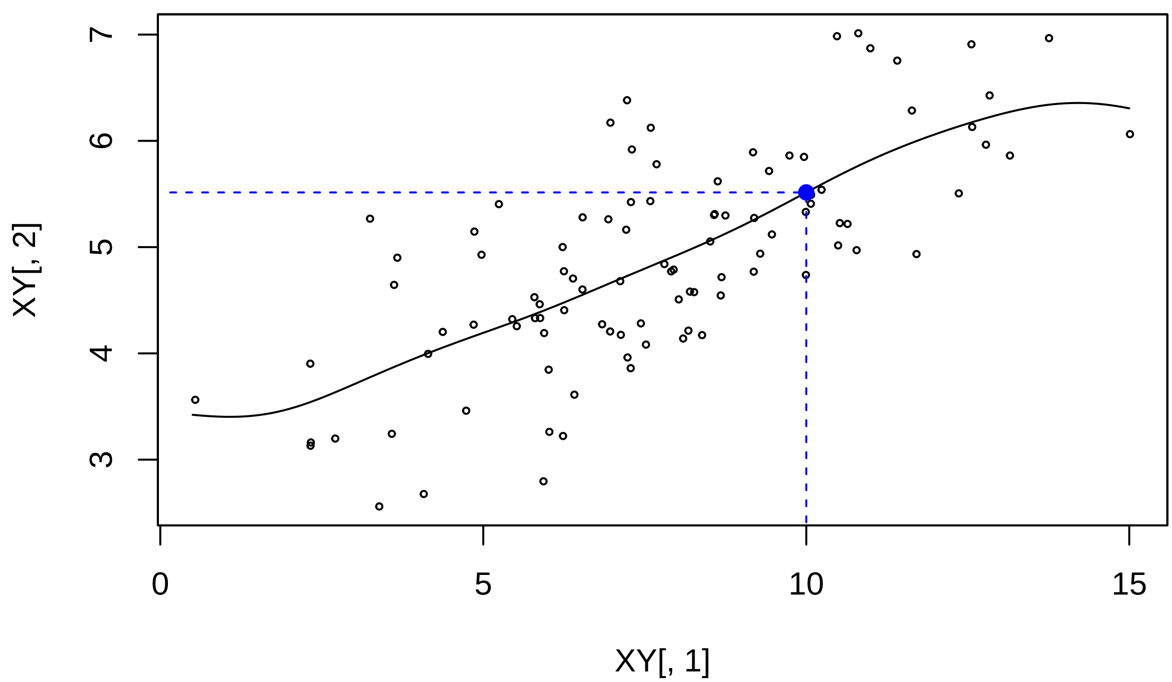
e. Regresión kernel:

```
# Puntos sobre los que vamos a hacer las estimaciones
x_1 <- seq(from = round(min(XY$X), 1), to = max(XY$X), by = 0.1)
```

```r
# Estimación de la regresión kernel
reg_kernel <- sm.regression(x = XY[, 1], y = XY[, 2],
                            eval.points = x_1)

# Índice del valor en donde se encuentra la estimación requerida
pos <- which(reg_kernel$eval.points == 10)

( estimaciOn <- c(reg_kernel$eval.points[pos],
                 reg_kernel$estimate[pos]) )
```

```
## [1] 10.00000  5.51487
```

```r
# Ubicamos el punto en el gráfico
points(x = estimaciOn[1], y = estimaciOn[2],
       pch = 19, col = "blue", cex = 1)
lines(x = c(estimaciOn[c(1, 1)], 0), col = "blue",
      y = c(0, estimaciOn[c(2, 2)]), lty = 2)
```

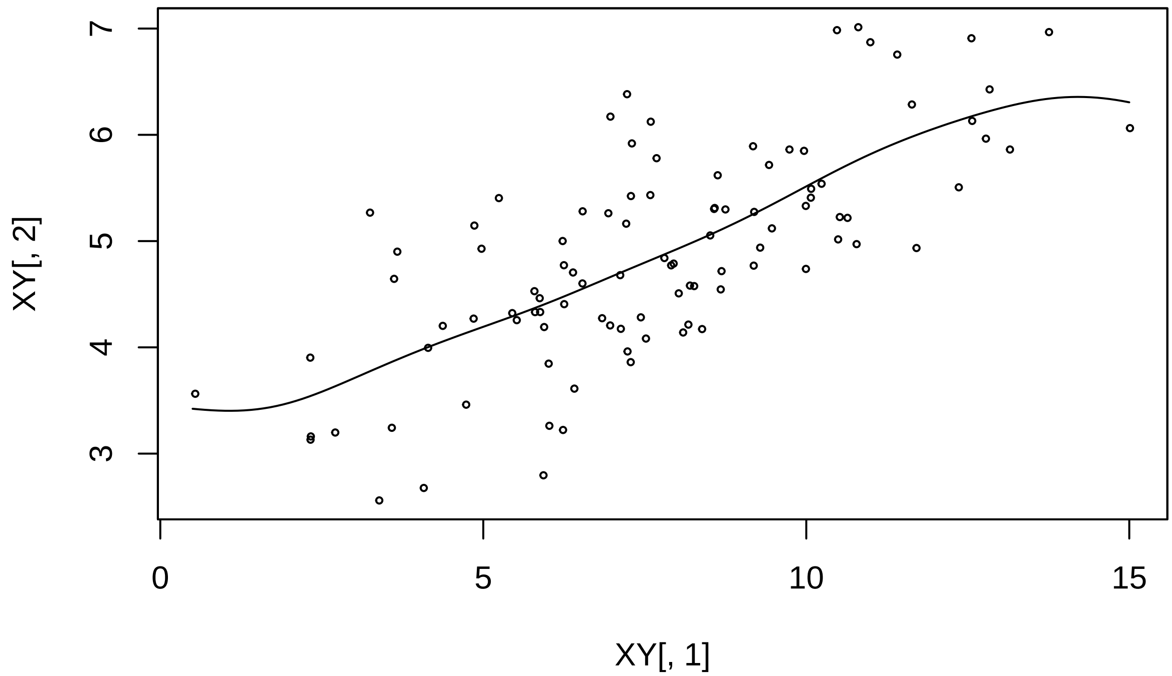


Tenemos que $\hat{E}[Y|X = 10] = 5.5148699$.

f. El valor de $h = 1.4192608$

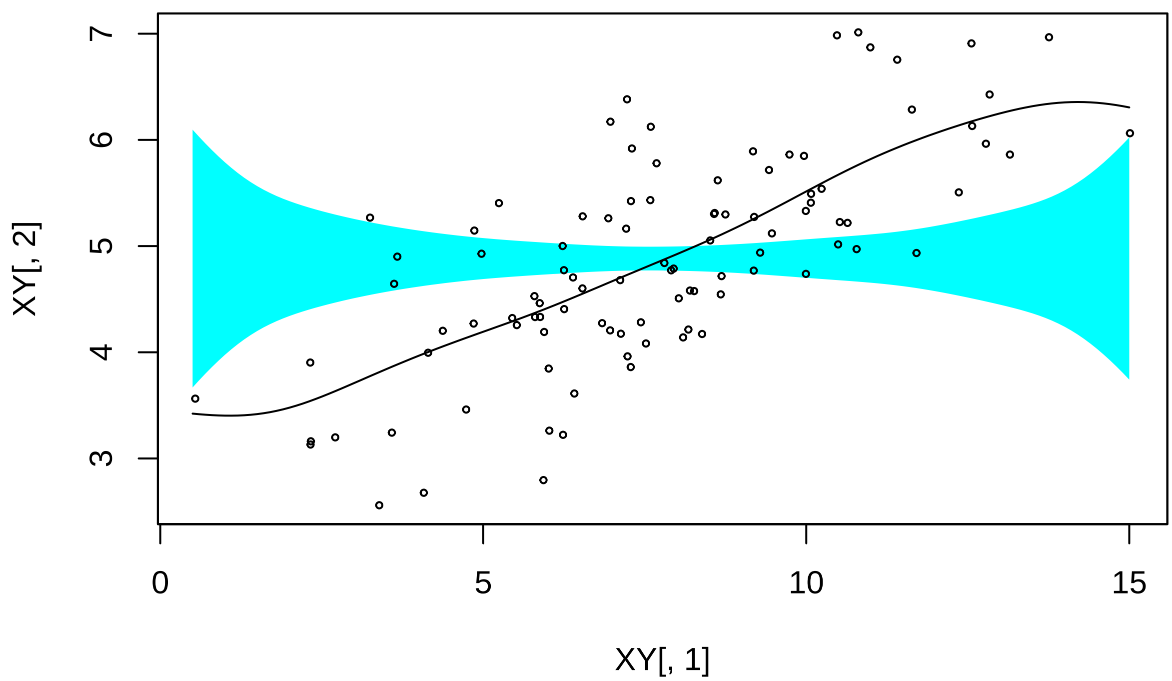
g. Bandas de confianza del 95% (**aún no lo he hecho**)

```r
sm.regression(x = XY[, 1], y = XY[, 2], eval.points = x_1,
              model = "none")
```



h. Bandas de confianza para la hipótesis de no efecto

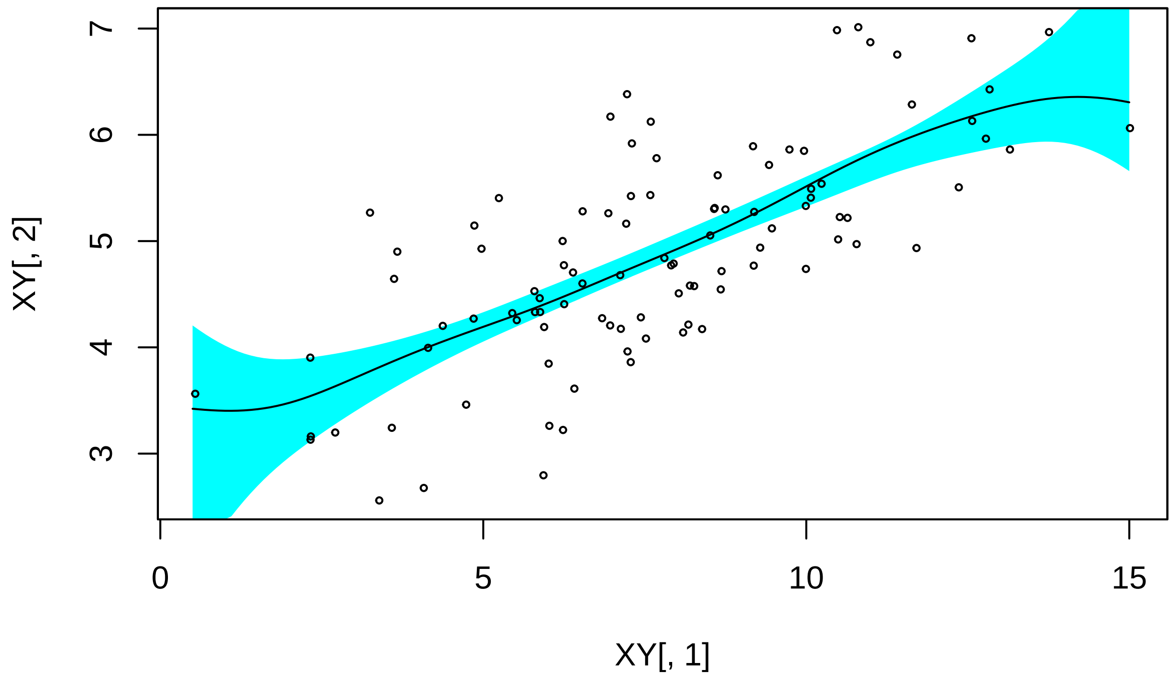
```r
sm.regression(x = XY[, 1], y = XY[, 2], eval.points = x_1,
              model = "no effect")
```




```
## Test of no effect model:  significance =  0
```

i. Bandas de confianza del 95% para la hipótesis de no linealidad

```r
sm.regression(x = XY[, 1], y = XY[, 2], eval.points = x_1,
              model = "linear")
```

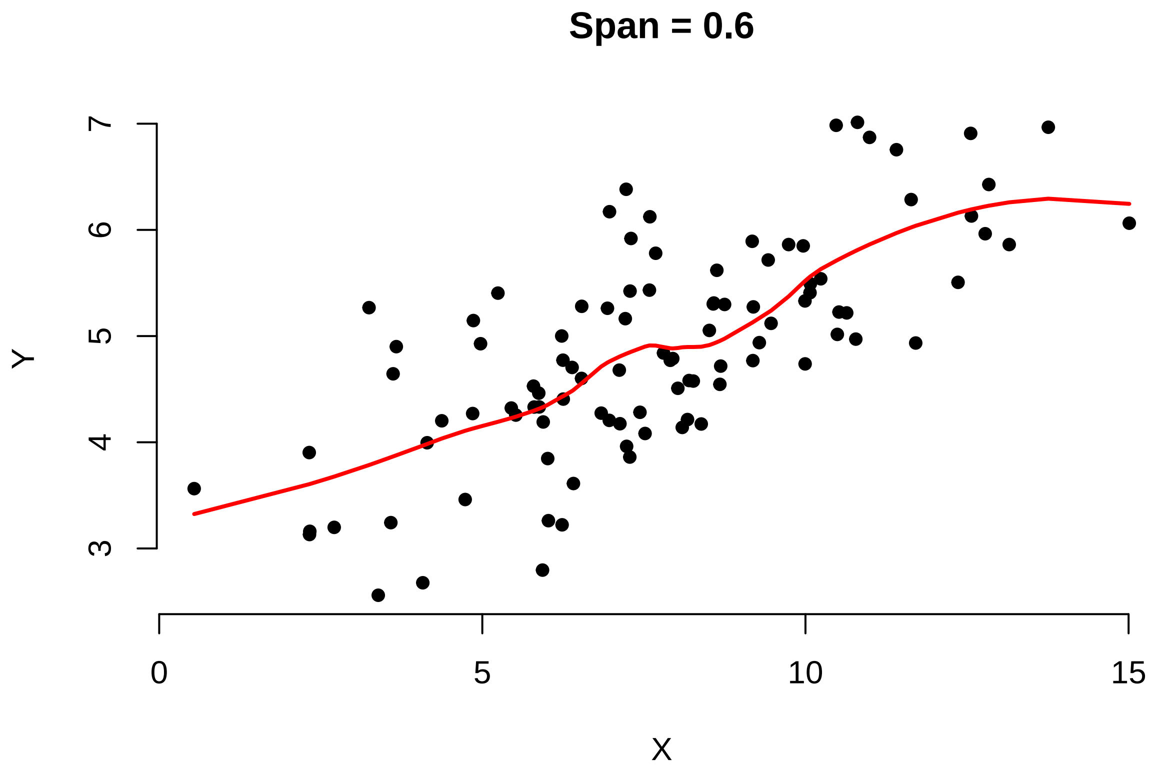



```
## Test of linear model:  significance =  0.913
```

j. Estimación de los diferentes modelos:

- **LOESS:**

```
# Ajuste del modelo
estim_loess <- loess(Y ~ X, data = XY, degree = 2,
                     span = 0.6)

# Valores sobre los que vamos a hacer la estimación
y_pred <- predict(estim_loess)

Indice <- order(XY$X)
with(XY, {plot(x = X, y = Y, main = "Span = 0.6",
               pch = 19, cex = 0.8, frame = FALSE)
          lines(x = X[Indice], y = y_pred[Indice],
                col = "red", lwd = 2)})
```


Span = 0.6
Y
3
4
5
6
7
0
5
10
15
X