Our physics model is deterministic:

is the simulation. is a particular configuration of the initial condition. is the initial condition space containing all possible . is the outcome, the final number given by the roll of a dice.

If we color the initial condition space according to outcome :

[img] plotted by matlab

We can see cells separated by borders and vertices. Each cell is colored according to the outcome .

The above scatter plot is 2D just for visualization. In this project, we consider to have 6 dimensions: 3 from initial linear velocity and 3 from initial angular momentum . Why not consider initial height ? If was not a constant, the research question would have many trivial solutions: set and let go of the dice.

Now, consider this hypothesis:

Where is a 6-dimensional vector. intuitively describes that you can move and stay in the same cell as long as your step is arbitrarily small. That makes sense because physics laws are mostly continuous. However, if is true for all points, then apparently the entire space can only have one color! That is counterfactual. More specifically, fails to describe the borders in the cell structure of .

The borders in , as the below video shows, is when the dice ends up standing on its edge.

[video]

There exists exact solution of such that the dice will stand on its edge forever. These points in are *undefined*, because they will never give us a valid face result. We are ready to improve :

where has definition

is what we believe to be true. Sadly, we have not been able to prove it. The rest of this writeup will treat as an assumption. Again, please note that is an unproved assumption.

What describes is what we call a “wedged continuous space”. To understand we need to understand what we mean by “wedge”. Let us take a simpler example. Consider a ball climbing a hill.

[img]

The hill is a frictionless arc whose top is higher relative to the ball. The ball has mass and starts out with a linear velocity tangent to the hill whose mode is . The outcome of the experiment, , denotes whether the ball will roll pass the hill or return to where it starts. [footnote: For more careful readers, please consider the ball and the hill to be a ring and a rail, because we are assuming the ball never leaves the surface.]

When the initial kinetic energy of the ball equals to the potential energy the ball would have at the top, , what will the outcome be?

Will the ball roll pass the hill? If so, then the ball must pass the top of the hill with some non-zero velocity. This breaks the conservation of energy, hence impossible. Will the ball return to where it starts? If so, then the ball’s velocity along x axis must go from positive to negative, passing zero velocity somewhere, either at the top or before the top. Can the ball reach zero velocity at the top? If we rewind time from the situation where the ball sits still at the top, we see that the ball will never move, unable to reach its initial condition in finite time, hence impossible. Can the ball reach zero velocity before reaching the top? Again, the conservation of energy would be broken. In conclusion, the ball will neither roll pass the hill nor return to where it starts. Outcome is undefined.

The ball will approach the top of the hill as , but never reaching the top.

We think “physics is continuous”, but has an undefined point separating two distinct values! This may feel paradoxical. To gain insight, let us borrow a powerful visualization tool, phase space. Let x axis represent , and y axis represent angular velocity of the ball in regard to the center of the arc.

[img ball-hill annotated] [phase space]

Each point in the phase space represent a state of the system. The evolution of the system can be represented by a path in the phase space. For a path to be valid, it must follow the vector field in the image, which is given by:

(See the appendix for how this is derived.)

Now we can translate our vague feeling “physics is continuous” into something tangible: *the vector field is continuous and differentiable*. Our “paradox”, in the context of phase space, thus translates to: two arbitrarily close initial conditions can lead to separated paths in the continuous and differentiable vector field.

[img]

Normalizing the vector field has the effect of ignoring time in the phase space. In the normalized vector field, point evaluates to , which is undefined! This point corresponds to an unstable equilibrium of the system. Four paths that connect to represents the scenarios when the ball approaches the top of the hill as . The four paths separate the phase space into four sections. Within each section, two arbitrarily close initial conditions always lead to arbitrarily close paths. Only when two initial conditions are in neighboring sections, separated just by the border, can the anomaly appear.

[img]

In a “wedged continuous space”, points like are the “wedges”. They divide the phase space into sections according to the topology of the evolution of the system. Similarly, the initial condition space can be divided into discrete cells.

The dice simulation is largely similar to the ball-hill experiment, since the relevant trajectory of the center of mass of the dice when it rotates about its edge is an arc. [footnote: it is still different from the ball-hill experiment because a) the dice has moment of inertia, and b) the pivot edge may change position as it interacts with the ground.]

Hence, we hypothesize:

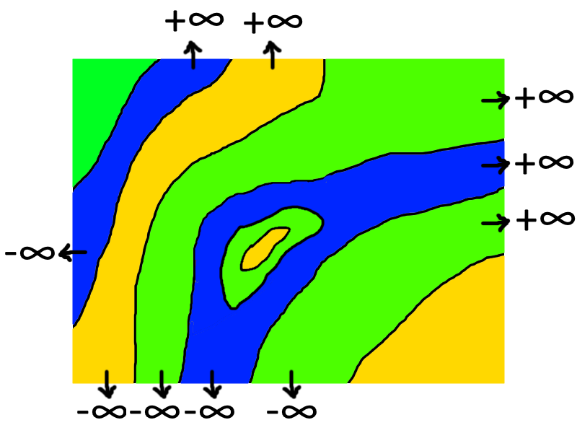
when has definition

is undefined at unstable equilibriums

Sadly, we cannot prove either. Keep in mind that and are unproved assumptions for the rest of this writeup.

Consider the 6-dimentional initial condition space described by and . It involves a topological 6D cell structure. A vertex in 6D must connect to 7 cells. Each cell corresponds to an outcome . Assume that two adjacent cells cannot be of the same . Because there are only 6 faces on a dice, i.e. 6 possible values for , must not have any vertex! That may sound strange, but all it says is that has redundant dimensions.

Here is an intuitive way to understand “redundant dimensions”. Imagine I control the initial linear velocity, and you control the initial angular momentum. As I continuously change initial linear velocity, can you continuously change initial angular momentum, so that stays constant? That feels possible.

 Avoiding vertices in 2D

Here is a more formal statement. If there exists a continuous mapping from to , such that all solutions of leads to the same outcome , and has no vertices, then has no vertices. always has no vertices.

can be easily constructed if we make the gradient of perpendicular to the borders between the cells.

If only we could prove this.