NCERT Solutions for Class 12

Maths

Chapter 5 – Continuity and Differentiability

Exercise 5.1

1. Prove that $f(x)=5x-3$ is a continuous function at $x=0$, $x=-3$ and $x=5$.

Ans: The given function is $f(x)=5x-3$.

At $x=0$, $f(0)=5\times 0-3=3$.

Taking limit as $x\to 0$ both sides of the function give

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}(5x-3)=5\times 0-3=3$$

$$\therefore \lim_{x\to 0} f(x)=f(0).$$

Thus, f satisfies continuity at $x=0$.

Again, at $x=-3, f(-3)=5\times(-3)-3=-18$.

Now, taking limit as $x\to 3$ both sides of the function give

$$\lim_{x\to 3} f(x)=\lim_{x\to 3} f(5x-3)=5\times(-3)-3=-18$$

$$\therefore \lim_{x\to 3} f(x)=f(-3).$$

Therefore, f satisfies continuity at $x=-3$.

Also, at $x=5, f(x)=f(5)=5\times 5-3=25-3=22$.

Taking limit as $x\to 5$ both sides of the function give

$$\lim_{x\to 5} f(x)=\lim_{x\to 5}(5x-3)=5\times 5-3=22$$

$$\therefore \lim_{x\to 5} f(x)=f(5).$$

Hence, f satisfies continuity at $x=5$.

2. Verify whether the function $f(x)=2x^2-1$ is continuous at $x=3$.

Ans: The given function is $f(x)=2x^2-1$.

Now, at $x=3, f(3)=2\times 3^2-1=17$.

Taking limit as $x\to 3$ both sides of the function give

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\left(2x^2-1\right)=2\times 3^2-1=17$$

$$\therefore \lim_{x\to 3} f(x)=f=(3).$$

Hence, f satifies continuity at $x=3$.

3. Verify whether the following functions are continuous.

(a) $f(x)=x-5$

Ans: The given function is $f(x)=x-5$.

It is assured that for every real number k, f is defined and its value at k is $k-5$. Also, it can be noted that

$$\lim_{x\to k} f(x)=\lim_{x\to k} f(x-5)=k=k-5=f(k).$$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, f satisfies continuity at every real number and so, it is a continuous function.

(b) $f(x)=\frac{1}{x-5}, x\neq 5$

Ans: The given function is

$$f(x)=\frac{1}{x-5}.$$

Let $k \neq 5$ is any real number, then taking limit as $x \to k$ both sides of the function give

$$\lim_{x\to k} f(x)=\lim_{x\to k}\frac{1}{x-5}=\frac{1}{k-5}$$

Also, $f(k)=\frac{1}{k-5}$, since $k \neq 5$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Therefore, f satisfies continuity at every point in the domain of f and so, it is a continuous function.

(c) $\mathbf{f(x)=\frac{x^2-25}{x+5}, x \neq 5}$

Ans: The given function is

$$f(x)=\frac{x^2-25}{x+5},\ x \neq 5$$

Now let $c \neq -5$ be any real number, then taking limit as $x \to c$ on both sides of the function give

$$\lim_{x\to c} f(x)=\lim_{x\to c}\frac{x^2-25}{x+5}=\lim_{x\to c}\frac{(x+5)(x-5)}{x+5}=\lim_{x\to c}(x-5)=(c-5)$$

Again, $f(c)=\frac{(c+5)(c-5)}{c+5}=c(c-5)$, since $c \neq 5$.

Hence, f satisfies continuity at every point in the domain of f and so it is a continuous function.

(d) $\mathbf{x \neq 5=|x-5|}$

Ans: The given function is $f(x)=|x-5|=\begin{cases} 5-x, \text{ if } x<5 \\ x-5, \text{ if } x>5 \end{cases}$.

Note that, f is defined at all points in the real line. So, let assume c be a point on a real line.

Then, we have $c<5$ or $c=5$ or $c>5$.

Now, let discuss these three cases one by one.

Case (i): $c<5$

Then, the function becomes $f(c)=5-c$.

Now, $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(5-x) = 5-c$.

$\therefore \lim\limits_{x \to c} f(x) = f(c)$.

Therefore, f is continuous at all real numbers which are less than $.5$..

Case (ii): $c=5$

Then, $f(c)=f(5)=(5-5)=0$.

Now,

$\lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5}(5-x) = (5-5) = 0$ and

$\lim\limits_{x \to 5^+} f(x) = \lim\limits_{x \to 5}(x-5) = 0$.

Therefore, we have

$\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x) = f(c)$.

Thus, f satisfies continuity at $x=5$, and so f is continuous at $x=5$.

Case (iii): $c>5$

Then we have, $f(c)=f(5)=c-5$.

Now,

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c}(x-5) = c-5$.

Therefore,

$\lim_{x \to c} f(x) = f(c)$.

So, f is continuous at all real numbers that are greater than 5.

Thus, f satisfies continuity at every real number and hence, it is a continuous function.

4. Prove that $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Ans: The given function is $f(x) = x^n$.

We noticed that the function f is defined at all positive integers n and also its value at $x = n$ is n^n.

Therefore, $\lim_{x \to n} f(n) = \lim_{x \to n} f(x^n) = n^n$.

So, $\lim_{x \to n} f(x) = f(n)$.

Thus, the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

5. Verify whether the following function f is continuous at $x = 0$, $x = 1$ and at $x = 2$.

$$\mathbf{f(x) = \begin{cases} x, \text{ if } x \le 1 \\ 5, \text{ if } x > 1 \end{cases}}$$

Ans: The given function is $f(x) = \begin{cases} x, \text{ if } x \le 1 \\ 5, \text{ if } x > 1 \end{cases}$.

It is obvious that the function f is defined at $x = 0$ and its value at $x = 0$ is 0.

Now, $\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0$.

So, $\lim_{x \to 0} f(x) = f(0)$.

Hence, the function f satisfies continuity at $x = 0$.

It can be observed that f is defined at $x=1$ and its value at this point is 1.

Now, the left-hand limit of the function f at $x=1$ is

$$\lim_{x\to1^-} f(x) = \lim_{x\to1-} x = 1.$$

Also, the right-hand limit of the function f at $x=1$ is

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} f(5)$$

Therefore, $\lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$.

Thus, f is not continuous at $x=1$

It can be found that f is defined at $x=2$ and its value at this point is 5.

That is, $\lim_{x\to2} f(x) = \lim_{x\to2} f(5) = 5$.

Therefore, $\lim_{x\to2} f(x) = f(2)$

Hence, f satisfies continuity at $x=2$.

6. Locate all the discontnuity points for the function f, where f is given by

$$\mathbf{f(x)=\begin{Bmatrix} 2x+3, \text{ if } x\le 2 \\ 2x\text{-}3, \text{ if } x>2 \end{Bmatrix}.}$$

Ans: The given function is $f(x)=\begin{Bmatrix} 2x+3, \text{ if } x\le 2 \\ 2x\text{-}3, \text{ if } x>2 \end{Bmatrix}$.

It can be observed that the function f is defined at all the points in the real line.

Let consider c be a point on the real line. Then, three cases may arise.

I. $c<2$

II. $c>2$

III. $c=2$

Case (i): When $c<2$

Then, we have $\lim_{x\to c} f(x)=\lim_{x\to\infty}(2x+3)=2c+3$.

Therefore,

$\lim_{x\to c} f(x)=f(c)$.

Hence, f attains continuity at all points x, where $x<2$.

Case (ii): When $c>2$

Then, we have $f(c)=2c-3$.

So,

$\lim_{x\to c} f(x)=\lim_{x\to\infty}(2x-3)=2c-3$.

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Hence, f satisfies continuity at all points x , where $x>2$.

Case(iii): When $c=2$

Then, the left-hand limit of the function f at $x=2$ is

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x+3)=2\times 2+3=7$ and

the right-hand limit of the function f at $x=2$ is,

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2+}(2x+3)=2\times 2-3=1$.

Thus, at $x=2$, $\lim_{x\to 2^-} f(x)\neq\lim_{x\to 2^+} f(x)$.

So, the function f does not satisfy continuity at $x=2$.

Hence, $x=2$ is the only point of discontinuity of the function $f(x)$.

7. Locate all the discontinuity points for the function f, where f is given by

$$\mathbf{f(x)=\begin{Bmatrix} |x|+3, \text{ if } x\le -3 \\ -2x, \text{ if } -3<x<3 \\ 6x+2, \text{ if } x\ge 3 \end{Bmatrix}.}$$

Ans: The given function is $f(x)=\begin{Bmatrix} |x|+3, \text{ if } x\le -3 \\ -2x, \text{ if } -3<x<3 \\ 6x+2, \text{ if } x\ge 3 \end{Bmatrix}$.

Observe that, f is defined at all the points in the real line.

Now, let assume c as a point on the real line.

Then five cases may arise. Either $c<-3$, or $c=-3$ or $-3<c<3$, or $c=3$, or $c>3$.

Let's discuss the five cases one by one.

Case I: When $c<-3$

Then, $f(c)=-c+3$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-x+3)=-c+3$.

Therefore, $\lim\limits_{x\to c} f(x)=f(c)$.

Hence, f satisfies continuity at all points x, where $x<-3$.

Case II: When $c=-3$

Then, $f(-3)=-(-3)+3=6$.

Also, the left-hand limit

$\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}(-x+3)=-(-3)+3=6$.

and the right-hand limit

$\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+} f(-2x)=2x(-3)=6$.

Therefore, $\lim\limits_{x\to 3} f(x)=f(-3)$.

Hence, f satisfies continuity at $x=-3$.

Case III: When $-3<c<3$

Then, $f(c)=-2c$ and also $\lim_{x\to c} f(x)=\lim_{x\to 3c}(-2x)=-2c$.

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Hence, f satisfies continuity at x, where $-3<x<3$.

Case IV: When $c=3$

Then, the left-hand limit of the function f at $x=3$ is

$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-} f(-2x)=-2\times 3=6$ and

the right-hand limit of the function f at $x=3$ is

$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+} f(6x+2)=6\times 3+2=20$.

Thus, at $x=3$, $\lim_{x\to 3^-} f(x)\neq \lim_{x\to 3^+} f(x)$.

Hence, f does not satisfy continuity at $x=3$.

Case V: When $c>3$.

Then $f(c)=6c+2$ and also

$\lim_{x\to c} f(x)=\lim_{x\to c}(6x+2)=6c+2$.

Therefore, $\lim_{x\to c} f(x)=f(c)$.

So, f satisfies continuity at all points x, when $x>3$.

Thus, $x=3$ is the only point of discontinuity of the function f.

8. Locate all the discontnuity points for the function f, where f is given by

$$f(x)=\left\{\begin{array}{l}\frac{|x|}{x}, \text{ if } x\neq 0 \\ 0, \text{ if } x=0\end{array}\right\}.$$

Ans: The given function is $f\left(x\right)=\begin{cases}\dfrac{\left|x\right|}{x}, \text{ if } x\neq 0\\ 0, \text{ if } x=0\end{cases}$.

Now, $f\left(x\right)$ can be rewritten as

$$f\left(x\right)=\begin{cases}\dfrac{\left|x\right|}{x}=\dfrac{-x}{x}=-1 \text{ if } x\neq 0\\ 0, \text{ if } x=0\\ \dfrac{\left|x\right|}{x}=\dfrac{x}{x}=1 \text{ if } x>0\end{cases}$$

It can be noted that the function f is defined at all points of the real line.

Now, let assume c as a point on the real line.

Then three cases may arise, either $c<0$, or $c=0$, or $c>0$.

Let discuss three cases one by one.

Case I: When $c<0$.

Then, $f\left(c\right)=-1$ and

$\lim\limits_{x\to c} f\left(x\right)=\lim\limits_{x\to c}\left(-1\right)=-1$.

Therefore, $\lim\limits_{x\to c} f\left(x\right)=f\left(c\right)$.

Hence, f satisfies continuity at all the points x where $x<0$.

Case II: When $c=0$.

Then, the left-hand limit of the function f at $x=0$ is

$\lim\limits_{x\to 0^-} f\left(x\right)=\lim\limits_{x\to 0^-}\left(-1\right)=-1$ and

the right-hand limit of the function f at $x=0$ is

$\lim\limits_{x\to 0+} f\left(x\right)=\lim\limits_{x\to 0^+}\left(1\right)=1$.

At $x=0$, $\lim\limits_{x\to 0^-} f\left(x\right)\neq\lim\limits_{x\to 0^+} f\left(x\right)$.

Hence, the function f does not satisfy continuity at $x=0$.

Case III: When $c>0$.

Then $f(c)=1$ and also

$\lim_{x\to c} f(x)=\lim_{x\to c}(1)=1.$

Therefore, $\lim_{x\to c} f(x)=f(c)$.

So, the function f is continuous at all the points x, for $x>0$.

Thus, $x=0$ is the only point of discontinuity for the function f.

9. Locate all the discontnuity points for the function f, where f is given by $f(x)=\begin{Bmatrix} \frac{x}{|x|}, \text{ if } x<0 \\ -1, \text{ if } x\geq 0 \end{Bmatrix}$.

Ans: The given function is $f(x)=\begin{Bmatrix} \frac{x}{|x|}, \text{ if } x<0 \\ -1, \text{ if } x\geq 0 \end{Bmatrix}$.

Now, we know that, if $x<0$, then $|x|=-x$.

Therefore, the $f(x)$ can be written as

$$f(x)=\begin{Bmatrix} \frac{x}{|x|}, \text{ if } x<0 \\ -1, \text{ if } x\geq 0 \end{Bmatrix}.$$

$\Rightarrow f(x)=-1$ for all positive real numbers.

Now, let assume c as any real number.

Then, we have $\lim_{x\to c} f(x)=\lim_{x\to c}(-1)=-1$ and

$f\left(c\right)=-1=\lim_{x \to e} f\left(x\right).$

Therefore, the function $f\left(x\right)$ is a continuous function.

Thus, there does not exist any point of discontinuity.

10. Locate all the discontnuity points for the function f, where f is given by $f(x)=\begin{cases} x+1, \text{ if } x \geq 1 \\ x^2+1, \text{ if } x<1 \end{cases}$.

Ans: The given function is

$$f(x) = \begin{cases} x+1, \text{ if } x^{3}1 \\ x^2+1, \text{ if } x<1 \end{cases}.$$

Note that, $f\left(x\right)$ is defined at all the points of the real line.

Now, let assume c as a point on the real line.

Then three cases may arise, either $c<1$, or $c=1$, or $c>1$.

Let discuss the three cases one by one.

Case I: When $c<1$.

Then, $f\left(c\right)=c^2+1$ and also

$$\lim_{x \to c} f\left(x\right) = \lim_{x \to c} f\left(x^2+1\right)=c^2+1.$$

Therefore, $\lim_{x \to c} f\left(x\right)=f\left(c\right).$

Hence, f satisfies continuity at all the points x, where $x<1$.

Case II: When $c=1$.

Then, we have $f\left(c\right)=f\left(1\right)=1+1=2$.

Now, the left-hand limit of f at $x=1$ is

$\lim\limits_{x\to1^-}f(x)=\lim\limits_{x\to1^-}(x^2+1)=1^2+1=2$ and the right-hand limit of f at $x=1$ is,

$\lim\limits_{x\to1^+}f(x)=\lim\limits_{x\to1^+}(x^2+1)=1^2+1=2$.

Therefore, $\lim\limits_{x\to1}f(x)=f(c)$.

Hence, f satisfies continuity at $x=1$.

Case III: When $c>1$.

Then, we have $f(c)=c+1$ and

$\lim\limits_{x\to c}f(x)=\lim\limits_{x\to c}(x+1)=c+1$.

Therefore,

$\lim\limits_{x\to c}f(x)=f(c)$.

So, f satisfies continuity at all the points x, where $x>1$.

Hence, there does not exist any discontnuity points.

11. Locate all the discontnuity points for the function f, where f is given by $f(x)=\begin{Bmatrix} x^3-3, \text{ if } x\le 2 \\ x^2+1, \text{ if } x>2 \end{Bmatrix}$.

Ans: The given function is $f(x)=\begin{Bmatrix} x^3-3, \text{ if } x\le 2 \\ x^2+1, \text{ if } x>2 \end{Bmatrix}$.

Observe that, the function f is defined at all points in the real line.

Now, let assume c as a point on the real line.

Case I: When $c<2$.

Then, we have $f(c)=c^3-3$ and also $\lim\limits_{x\to c}f(x)=\lim\limits_{x\to c}(x^3-3)=c^3-3$.

Therefore, the function f attains continuity at all the points x, where $x<2$.

Case II: When $c=2$.

Then, we have $f(c)=f(2)=2^3-3=5$.

Now the left-hand limit of the function is

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}(x^3-3)=2^3-3=5$ and the right-hand limit is

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}(x^2+1)=2^2+1=5$.

Therefore, $\lim\limits_{x\to 2} f(x)=f(2)$.

Hence, the function f is continuous at $x=2$.

Case III: When $c>2$.

Then, $f(c)=c^2+1$ and

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x^2+1)=c^2+1$.

Therefore, $\lim\limits_{x\to c} f(x)=f(c)$.

So, f attains continuity at all the points x, where $x>2$.

Thus, the function f is continuous at all the points on the real line.

Hence, f does not have any point of discontinuity.

12. Locate all the discontnuity points for the function f, where f is given by $f(x)\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$.

Ans: The given function is $f(x)\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$.

Observe that, the function f is defined at every point of the real line.

Now, let assume c as a point on the real number line.

Case I: When $c<1$.

Then $f(c)=c^{10}-1$.

Also, $\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^{10}-1\right)=c^{10}-1$

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Hence, the function f attains continuity at every point x, for $x<1$.

Case II: When $c=1$.

Then the left-hand limit of the function $f(x)$ at $x=1$ is

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^{10}-1\right)=10^{10}-1=1-1=0$ and

the right-hand limit of the function f at $x=1$ is

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(x^{2}\right)=1^{2}=1$.

So, we can notice that, $\lim_{x\to 1^-} f(x)\neq \lim_{x\to 1^+} f(x)$.

Hence, the function f does not satisfy continuity at $x=1$.

Case III: When $c>1$.

Then, $f(c)=c^{2}$.

Also, $\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^{2}\right)=c^{2}$.

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Thus, the function f attains continuity at every point x, for $x>1$.

Hence, we can conclude that $x>1$ is the only point of discontinuity for the function f.

13. Verify whether the function $f(x)=\begin{cases} x+5, \text{ if } x\le 1 \\ x-5, \text{ if } x>1 \end{cases}$ is continuous.

Ans: The given function is $f(x)=\begin{cases} x+5, \text{ if } x\le 1 \\ x-5, \text{ if } x>1 \end{cases}$.

It can be noted that the function f is defined at every point on the real line.

Now, let assume c as a point on the real line.

Case I: When $c<1$.

Then, $f(c)=c^{10}-1$.

Also, $\lim_{x\to c} f(x)=\lim_{x\to c}(x^{10}-1)=c^{10}-1$.

Hence, f satisfies continuity at every point x, for $x<1$.

Case II: When $c=1$.

Then, $f(1)=1+5=6$.

Now, the left-hand limit of the function f at $x=1$ is

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+5)=1+5=6$ and

the right-hand limit of the function at $x=1$ is $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x-5)=1-5=4$.

Thus, it is seen that, $\lim_{x\to 1^-} f(x)\ne \lim_{x\to 1^+} f(x)$.

Hnece, f does not attain continuity at $x=1$.

Case III: When $c>1$.

Then $f(c)=c-5$.

Also, $\lim_{x\to c} f(x)=\lim_{x\to c}(x-5)=c-5$.

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Thus, the function f is continuous at every point x, for $x>1$.

Hence, we can conclude that $x=1$ is the only point of discontinuity for the function f.

14. Verify whether the following function f is continuous.

$$f(x)=\begin{cases} 3, \text{ if } 0\le x\le 1 \\ 4, \text{ if } 1<x<3 \\ 5, \text{ if } 3\le x\le 10 \end{cases}.$$

Ans: The given function is $f(x)=\begin{cases} 3, \text{ if } 0\le x\le 1 \\ 4, \text{ if } 1<x<3 \\ 5, \text{ if } 3\le x\le 10 \end{cases}$.

Therefore, f is defined in the interval $[0,10]$.

Now let assume c as a point in the interval $[0,10]$.

Then there may arise five cases.

Case I: When $0\le c<1$.

Then $f(c)=3$.

Also, $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(3)=3$.

Therefore, $\lim\limits_{x\to c} f(x)=f(c)$.

Hence, the function f attains continuity at the interval $[0,1]$.

Case II: When $c=1$.

Then $f(3)=3$.

Also, the left-hand-limit of the function at $x=1$ is

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(3)=3$ and the right-hand-limit of the function at $x=1$ is

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(4)=4$.

Thus, it is noticed that $\lim\limits_{x\to1^-} f(x) \neq \lim\limits_{x\to1^+} f(x)$.

Hence, the function f does not satisfy continuity at $x=1$.

Case III: When $1<c<3$.

Then $f(c)=4$.

Also, $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(4)=4$.

Thus, $\lim\limits_{x\to c} f(x)=f(c)$.

Hence, the function f attains continuity at every point in the interval $[1,3]$.

Case IV: When $c=3$.

Then $f(c)=5$.

Now, the left-hand-limit of the function f at $x=3$ is

$\lim\limits_{x\to3^-} f(x)=\lim\limits_{x\to3^-}(4)=4$ and the right-hand-limit of the function f at $x=3$ is

$\lim\limits_{x\to3^+} f(x)=\lim\limits_{x\to3^+}(5)=5$.

Therefore, it is noted that $\lim\limits_{x\to3^-} f(x) \neq \lim\limits_{x\to3^+} f(x)$.

Hence, the function f is not continuous at $x=3$.

Case V: When $3<c\leq10$.

Then $f(c)=5$.

Also, $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(5)=5$.

Therefore, $\lim\limits_{x\to c} f(x)=f(c)$.

So, the function f attains continuity at every point in the interval $[3,10]$.

Hence, the function f is not continuous at $x=1$ and $x=3$.

15. Verify whether the following function f is continuous. f such that $f(x)=\begin{cases} 2x, \text{ if } x<0 \\ 0, \text{ if } 0\le x\ne 1 \\ 4x, \text{ if } x>1 \end{cases}$.

Ans: The given function is $f(x)=\begin{cases} 2x, \text{ if } x<0 \\ 0, \text{ if } 0\le x\ne 1 \\ 4x, \text{ if } x>1 \end{cases}$.

Now, let consider c be a point on the real number line.

Then, five cases may arrive.

Case I: When $c<0$.

Then, $f(c)=2c$.

Also, $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(2x)=2c$.

Therefore, $\lim\limits_{x\to c} f(x)=f(c)$.

Hence, the function f attains continuity at every point x whenever $x<0$.

Case II: When $c = 0$.

Then, $f(c)=f(0)=0$.

Now, the left-hand-limit of the function f at $x = 0$ is

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}(2x)=0$ and the right-hand limit of the function f at $x = 0$ is,

$\lim\limits_{x\to 0^+}(x)=\lim\limits_{x\to 0^+}(0)=0$.

Therefore, $\lim\limits_{x\to 0} f(x)=f(0)$.

Thus, the function f attains continuity at $x = 0$.

Case II: When $0<c<1$

Then, $f(x)=0$.

Also, $\lim_{x\to c} f(x)=\lim_{x\to c}(0)=0$.

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Hence, f attains continuity at every point in the interval $(0,1)$.

Case IV: When $c=1$.

Then, $f(c)=f(1)=0$.

Now, the left-hand-limit at $x=1$ is

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(0)=0$ and the right-hand-limit at $x=1$ is

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(4x)=4\times 1=4$.

Thus, it is noticed that, $\lim_{x\to 1^-} f(x)\neq \lim_{x\to 1^+} f(x)$.

Hence, the function f is not continuous at $x=1$.

Case V: When $c<1$.

Then, $f(c)=f(1)=0$.

Also, $\lim_{x\to c} f(x)=\lim_{x\to c}(4x)=4c$

Therefore, $\lim_{x\to c} f(x)=f(c)$.

So, the function f attains continuity at every point x, for $x>1$.

Hence, the function f is discontinuous only at $x=1$.

16. Verify whether the function f is continuous. Provided that f is defined by $f(x)=\begin{cases} -2, \text{ if } x\leq -1 \\ 2x, \text{ if } -1<x\leq 1 \\ 2, \text{ if } x>1 \end{cases}$.

Ans: The given function is $f(x)=\begin{cases} -2, \text{ if } x\le -1 \\ 2x, \text{ if } -1<x\le 1 \\ 2, \text{ if } x>1 \end{cases}$.

Note that, f is defined at every point in the interval $[-1,\infty)$.

Now, let assume c is a point on the real number line.

Case I: When $c<-1$.

Then, $f(c)=-2$.

Also, $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-2)=-2$.

Therefore, $\lim\limits_{x\to c} f(x)=f(c)$.

Hence, the function f attains continuity at every point x, for $x<-1$.

Case II: When $c=-1$.

Then, $f(c)=f(-1)=-2$.

Now, the left-hand-limit of the function at $x=-1$ is

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(-2)=-2$ and the right-hand-limit at $x=-1$ is

$\lim\limits_{x\to 1^+}(x)=\lim\limits_{x\to 1^+}=2\times(-1)=-2$.

Therefore, $\lim\limits_{x\to -1} f(x)=f(-1)$.

Hence, the function f satisfies continuity at $x=-1$.

Case III: When $-1<c<1$.

Then, $f(c)=2c$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(2x)=2c$.

Therefore, $\lim\limits_{x\to c} f(x)=f(c)$.

Hence, the function f attains continuity at every point in the interval $(-1,1)$.

Case IV: When $c=1$.

Then, $f(c)=f(1)=2\times1=2$

Now, the left-hand-limit of the function at $x=1$ is

$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}(2x)=2\times1=2$ and the right-hand-limit at $x=1$ is

$\lim_{x\to1^+} f(x)=\lim_{x\to1^+} 2=2$.

Therefore, $\lim_{x\to1} f(x)=f(c)$.

Thus, the function f attains continuity at $x=2$.

Case V: When $c>1$.

Then $f(c)=2$.

Also, $\lim_{x\to2} f(x)=\lim_{x\to2}(2)=2$.

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Hence, the function f is continuous at every point x, for $x>1$.

17. Formulate a relationship between a and b so that the function f defined by $f(x)=\begin{Bmatrix} ax+1, \text{ if } x\le3 \\ bx+3, \text{ if } x>3 \end{Bmatrix}$ is continuous at $x=3$.

Ans: The given function is $f(x)=\begin{Bmatrix} ax+1, \text{ if } x\le3 \\ bx+3, \text{ if } x>3 \end{Bmatrix}$.

The function f will be continuous at $x=3$ if

$\lim_{x\to3^-} f(x)=\lim_{x\to3^-} f(x)=f(3)$, ……. (1)

$\lim_{x\to3^-} f(x)=\lim_{x\to3^-} f(ax+1)=3a+1$,

$\lim_{x\to3^+} f(x)=\lim_{x\to3^+} f(bx+1)=3b+3$, …… (2)

and

$f(3)=3a+1.$ …… (3)

Therefore, from the equation (1), (2), and (3) gives

$3a+1=3b+3=3a+1$

$\Rightarrow 3a+1=3b+3$

$\Rightarrow 3a=3b+2$

$\Rightarrow a=b+\dfrac{2}{3}$

Hence, the required relationship between a and b is given by $a=b+\dfrac{2}{3}$.

18. Determine the value of λ for which the function defined by $f(x)=\begin{cases}\lambda(x^2-2x), \text{if } x\le 0\\ 4x+1, \quad \text{if } x>0\end{cases}$ is continuous at $x=0$. Also discuss the continuity of f at $x=1$?

Ans: The given function is $f(x)=\begin{cases}\lambda(x^2-2x), \text{ if } x\le 0\\ 4x+1, \quad \text{if } x>0\end{cases}$.

Now the function will be continuous at $x=0$ if

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} f(x)=f(0)$

$\Rightarrow \lim\limits_{x\to 0^-} \lambda(x^2-2x)=\lim\limits_{x\to 0^-}(4x+1)=\lambda(0^2-2\times 0)$

$\Rightarrow \lambda(0^2-2\times 0)=4\times 0+1=0$

$\Rightarrow 0=1=0$, which is impossible.

Thus, there does not exist any value of λ for which f is continuous at $x=0$.

Now, at $x=1$,

$f(1)=4x+1=4\times 1+1=5$ and

$\lim_{x\to 1}(4x+1)=4\times 1+1=5$.

Therefore, $\lim_{x\to 1} f(x)=f(1)$.

Hence, the function f is continuous at $x=1$, for all values of λ.

19. Prove that the function $g(x)=x-[x]$ is not continuous at any integral point, where $[x]$ denotes the greatest integer value of x that are less than or equal to x.

Ans: The given function is $g(x)=x-[x]$.

Note that, the function is defined at every integral point.

Now, let assume that n be an integer.

Then, $g(n)=n-[n]=n-n=0$.

Now taking left-hand-limit as $x\to n$ to the function g gives

$\lim_{x\to n^-} g(x)=\lim_{x\to n^-}[x-[x]]=\lim_{x\to n^-}(x)-\lim_{x\to n-}[x]=n-(n-1)=1$.

Again, the right-hand-limit on the function at $x=n$ is

$\lim_{x\to n^+} g(x)=\lim_{x\to n^+}[x-[x]]=\lim_{x\to n^+}(x)-\lim_{x\to n^+}[x]=n-n=0$.

Note that, $\lim_{x\to n^-} g(x)\neq \lim_{x\to n^+} g(x)$.

Thus, the function f is cannot be continuous at $x=n$,

Hence, the function g is not continuous at any integral point.

20. Verify whether the function $f(x)=x^2-\sin x+5$ is continuous at $x=p$.

Ans: The given function is $f(x)=x^2-\sin x+5$.

Now, at $x=\pi$,

$f(x)=f(\pi)=\pi^2\text{-}\sin\pi+5=\pi^2\text{-}0+5=\pi^2+5$.

Taking limit as $x\to\pi$ on the function $f\left(x\right)$ gives

$\lim\limits_{x\to\pi}f(x)=\lim\limits_{x\to\pi}(x^2\text{-}\sin x+5)$.

Now substitute $x=\pi+h$ into the function $f\left(x\right)$.

When $x\to\pi$, then $h\to0$.

Therefore,

$\lim\limits_{x\to\pi}f(x)=\lim\limits_{x\to\pi}(x^2\text{-}\sin x)+5$.

$=\lim\limits_{h\to0}\left[(\pi+h^2)\text{-}\sin(\pi+h)+5\right]$

$=\lim\limits_{h\to0}(\pi+h)^2\text{-}\lim\limits_{h\to0}\sin(\pi+h)+\lim\limits_{h\to0}5$

$=(\pi+0)^2\text{-}\lim\limits_{h\to0}[\sin\pi\cos h+\cos\pi\sin h]+5$

$=\pi^2\text{-}\lim\limits_{h\to0}\sin\pi\cos h\text{-}\lim\limits_{h\to0}\cos\pi\sin h+5$

$=\pi^2\text{-}\sin\pi\cos0\text{-}\cos\pi\sin0+5$

$=\pi^2\text{-}0\times1\text{-}(\text{-}1)\times0+5=\pi^2+5$.

So, $\lim\limits_{x\to x}f(x)=f(\pi)$.

Hence, it is concluded that the function f is continuous at $x=n$.

21. Determine whether the follwing functions are continuous.

(a) $f(x)=\sin x+\cos x$ (b) $f(x)=\sin x-\cos x$ (c) $f(x)=\sin x\times\cos x$.

Ans: It is known that if two functions g and h are continuous, then $g+h$, $g\text{-}h$ and g,h are also continuous.

So, let us assume that, $g(x)=\sin x$ and $h(x)=\cos x$ are two continuous functions.

Now, as $g(x)=\sin x$ is defined for every real number, so let c be a real number. Substitute $x=c+h$ into the function g.

When $x \to c$, then $h \to 0$.

So, $g(c)=\sin c$.

Also,

$$\begin{aligned}\lim_{x\to c} g(x) &= \lim_{x\to c} \sin x \\ &= \lim_{h\to 0} \sin(c+h) \\ &= \lim_{h\to 0} [\sin c\cos h+\cos c\sin h] \\ &= \lim_{h\to 0} (\sin c\cos h)+\lim_{h\to 0} (\cos c\sin h) \\ &= \sin c\cos 0+\cos c\sin 0 \\ &= \sin c+0 \\ &= \sin c\end{aligned}$$

Therefore, $\lim_{x\to c} g(x)=g(c)$.

Hence, the function g is a continuous.

Again, let us assume that $h(x)=\cos x$.

Note that, the function $h(x)=\cos x$ is defined for every real number.

Now, let c be a real number.

Substitute $x=c+h$ into the function.

When $x \to c$, then $h \to 0$.

So, $h(c)=\cos c$ and

$$\lim_{x\to c} h(x)=\lim_{x\to c} \cos x$$
$$=\lim_{h\to 0} \cos(c+h)$$
$$=\lim_{h\to 0}[\cos c\cos h-\sin c\sin h]$$
$$=\lim_{h\to 0} \cos c\cos h-\lim_{h\to 0} \sin c\sin h$$
$$=\cos c\cos 0-\sin c\sin 0$$
$$=\cos c\times 1-\sin c\times 0$$
$$=\cos c$$

Therefore, $\lim_{h\to 0} h(x)=h(c)$.

Thus, the function h is continuous.

Hence, we conclude that all the following functions are continuous.

(a) $f(x)=g(x)+h(x)=\sin x+\cos x$.

(b) $f(x)=g(x)-h(x)=\sin x-\cos x$.

(c) $f(x)=g(x)\times h(x)=\sin x\times \cos x$.

22. Verify whether the following trigonometric functions are continuous. sine, cosine, cosecant, secant and cotangent.

Ans: We know that if two functions say g and h are continuous, then

i. $\frac{h(x)}{g(x)}, g(x)\neq 0$ is continuous.

ii. $\frac{1}{g(x)}, g(x)\neq 0$ is continuous.

iii. $\frac{1}{h(x)}, h(x)\neq 0$ is continuous.

It can be observed that the function $g(x)=\sin x$ is defined for all real numbers.

Now, let consider c be a real number and substitute $x=c+h$ into the function g.

When, $x \to c$, then $h \to 0$.

So, $g(c)=\sin c$ and

$$\lim_{x\to c} g(x)=\lim_{x\to c}\sin x$$
$$=\lim_{h\to 0}\sin(c+h)$$
$$=\lim_{h\to 0}[\sin c\cos h+\cos c\sin h]$$
$$=\lim_{h\to 0}(\sin c\cos h)+\lim_{h\to 0}(\cos c\sin h)$$
$$=\sin c\cos 0+\cos c\sin 0$$
$$=\sin c+0$$
$$=\sin c$$

Therefore, $\lim_{x\to c} g(x)=g(c)$.

Thus, the function $g\left(x\right)=\sin x$ is continuous.

Again, let $h(x)=\cos x$.

It can be noted that $h(x)=\cos x$ is defined for all real numbers.

Now, let consider c be a real number and substitute $x=c+h$ into the function h.

Then, $h(c)=\cos c$ and

$$\lim_{x\to c} h(x)=\lim_{x\to c}\cos x$$
$$=\lim_{h\to 0}\cos(c+h)$$
$$=\lim_{h\to 0}[\cos c\cos h-\sin c\sin h]$$
$$=\lim_{h\to 0}\cos c\cos h-\lim_{h\to 0}\sin c\sin h$$
$$=\cos c\cos 0-\sin c\sin 0$$
$$=\cos c\times 1-\sin c\times 0$$
$$=\cos c$$

Therefore, $\lim_{h\to 0} h(x)=h(c)$.

Thus, the function $h(x)=\cos x$ is continuous.

Now note that,

$\text{cosec } x=\frac{1}{\sin x}$, and $\sin x \neq 0$ is a continuous function.

$\Rightarrow \text{cosec } x, x \neq n\pi (n \in Z)$ is also a continuous function.

Also, secant function is continuous except at $x=(2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

Therefore, $\sec x=\frac{1}{\cos x}$, $\cos x \neq 0$ is continuous.

$\Rightarrow \sec x$, $x \neq (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$ is a continuous function.

Thus, secant function is also continuous except at $x=(2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

And the cotangent function is

$\cot x=\frac{\cos x}{\sin x}$, and where $\sin x \neq 0$ is a continuous function.

$\Rightarrow \cot x$, $x \neq n\pi$, $n \in \mathbb{Z}$ is a continuous function.

Hence, the cotangent function is continuous except at $x=n\pi$, $n \in \mathbb{Z}$.

23. Determine all the discontnuity points for the following function f defined by $f(x)=\begin{cases} \frac{\sin x}{x}, \text{ if } x<0 \\ x+1, \text{ if } x \geq 0 \end{cases}$.

Ans: The given function is $f(x)=\begin{cases} \frac{\sin x}{x}, \text{ if } x<0 \\ x+1, \text{ if } x \geq 0 \end{cases}$.

Note that, the function f is defined at every point on the real number line.

Now, let consider c be a real number.

Then there may arise three cases, either $c<0$, or $c>0$, or $c=0$.

Let us discuss one after another.

Case I: When $c<0$.

Then, $f(c)=\frac{\sin c}{c}$.

Also, $\lim_{x\to c} f(x)\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$.

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Hence, the function f is continuous at every point x, for $x<0$.

Case II: When $c>0$.

Then $f(c)=c+1$.

Also, $\lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$.

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Hence, the function f is continuous at every point, where $x>0$.

Case III: When $c=0$.

Then $f(c)=f(0)=0+1=1$.

Now, the left-hand-limit of the function f at $x=0$ is

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{\sin x}{x}=1$ and the right-hand-limit is

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(x+1)=1$

Therefore, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} f(x)=f(0)$.

So, the function f is continuous at $x=0$.
Thus, the function f is continuous at every real point.

Hence, the function f does not have any point of discontinuity.

24. Discuss the continuity of the function f defined by

$$\mathbf{f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}}.$$

Ans: The given function is $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$.

We can observe that the function f is defined at every point on the real number line.

Now, let consider c be a real number.

Then, there may arise two cases, either $c \neq 0$ or $c=0$.

Let us discuss the cases one after another.

Case I: When $c \neq 0$.

Then $f(c)=c^2\sin\frac{1}{c}$.

Also,

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\sin\frac{1}{x}\right)=\left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c}\sin\frac{1}{x}\right)=c^2\sin\frac{1}{c}.$$

Therefore, $\lim_{x\to c} f(x)=f(c)$.

Hence, the function f is continuous at every point $x \neq 0$.

Case II: When $c = 0$.

Then $f(0)=0$ and also

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left(x^2\sin\frac{1}{x}\right)=\lim_{x\to 0}\left(x^2\sin\frac{1}{2}\right).$$

Now, we know that,

$-1 \le \sin\frac{1}{x} \le 1,\ x \ne 0.$

$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$

$\Rightarrow \lim_{x\to 0}(-x^2) \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$

$\Rightarrow 0 \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$

$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

Therefore, $\lim_{x\to 0^-} f(x)=0$.

Similarly, we have,

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right)=\lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)=0$

Therefore, $\lim_{x\to 0^-} f(x)=f(0)=\lim_{x\to 0^+} f(x)$.

Thus, the function f is continuous at the point $x = 0$.

So, the function f is continuous at all real points.

Hence, the function f is continuous.

25. Determine whether the following function f is continuous.

f such that $f(x)=\begin{cases} \sin x-\cos x, & \text{if } x \ne 0 \\ 1 & \text{if } x=0 \end{cases}$.

Ans: The given function is $f(x)=\begin{cases} \sin x-\cos x, & \text{if } x \ne 0 \\ 1 & \text{if } x=0 \end{cases}$.

It can be observed that the function f is defined at every point on the real number line.

Now, let consider c be a real number.

Then, there may arise two cases, either $c \neq 0$ or $c=0$.

Let us discuss the cases one after another.

Case I: When $c \neq 0$.

Then, $f(c)=\sin c-\cos c$.

Also, $\lim_{x \to c} f(x)=\lim_{x \to c}(\sin x-\cos x)=\sin c-\cos c$.

Therefore, $\lim_{x \to c} f(x)=f(c)$.

Hence, the function f is continuous at every point x for $x \neq 0$.

Case II: When $c=0$.

Then, $f(0)=-1$.

Now the left-hand-limit of the function f at $x=0$ is

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(\sin x-\cos x)=\sin 0-\cos 0=0-1=-1$ and the right-hand-limit is

$\lim_{x \to 0^+} f(x)= \lim_{x \to 0^+}(\sin x-\cos x)=\sin 0-\cos 0=0-1=-1$.

Therefore, $\lim_{x \to 0^-} f(x)= \lim_{x \to 0^+} f(x)=f(0)$.

So, the function f is continuous at $x=0$.

Thus, the function f is continuous at all real points.

Hence, the function f is continuous.

26. Calculate the values of k for which the function f attains continuity at the given points.

$$f(x)\begin{cases} \dfrac{k\cos x}{\pi-2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x=\dfrac{\pi}{2} \end{cases}$$

Ans: The given function is $f(x)\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$.

Observe that, f is defined and continuous at $x=\dfrac{\pi}{2}$, since the value of the f at $x=\dfrac{\pi}{2}$ is equal with the limiting value of f at $x=\dfrac{\pi}{2}$.

Since, f is defined at $x=\dfrac{\pi}{2}$ and $f\left(\dfrac{\pi}{2}\right)=3$, so

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}.$$

Substitute $x=\dfrac{\pi}{2}+h$ into the function $f(x)$.

So, we have, $x\to\dfrac{\pi}{2}\Rightarrow h\to 0$.

Then,

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}.$$

$$\Rightarrow k\lim_{h\to 0}\frac{-\sinh}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sinh}{h}=\frac{k}{2}.1=\frac{k}{2}$$

Therefore, $\lim_{x\to\frac{\pi}{2}} f(x)=f\left(\dfrac{\pi}{2}\right)$

$\Rightarrow \dfrac{k}{2}=3$

$\Rightarrow k=6$

Hence, the value of k is 6 for which the function f is continuous.

27. Determine the values of k for which the following function f satisfies continuity at the given points.

$$\mathbf{f(x)=\begin{cases} kx^2, \text{ if } x \le 2 \\ 3, \quad \text{if } x>2 \end{cases} \text{ at } x=2.}$$

Ans: The given function is $f(x)=\begin{cases} kx^2, \text{ if } x \le 2 \\ 3, \quad \text{if } x>2 \end{cases}$.

Note that, f is continuous at $x=2$ only if f is defined at $x=2$ and if the value of f at $x=2$ is equal with the limiting value of f at $x=2$.

Since, it is provided that f is defined at $x=2$ and $f(2)=k(2)^2=4k$, so

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$
$$\Rightarrow \lim_{x\to 2^-}(kx^2)=\lim_{x\to 2^+}(3)=4k$$
$$\Rightarrow k\times 2^2=3=4k$$
$$\Rightarrow 4k=3=4k$$
$$\Rightarrow 4k=3$$
$$\Rightarrow k=\frac{3}{4}$$

Hence, the value of k is $\frac{3}{4}$ for which the function f is continuous.

28. Determine the values of k for which the following function f attains continuity at the given point.

$$\mathbf{f(x)=\begin{cases} kx^2, \text{ if } x \le 2 \\ 3, \quad \text{if } x>2 \end{cases} \text{ at } x=\pi.}$$

Ans: The given function is $f(x)=\begin{cases} kx^2, \text{ if } x \le 2 \\ 3, \quad \text{if } x>2 \end{cases}$.

Note that, f is continuous at $x=\pi$ only if the value of f at $x=\pi$ is equal with the limiting value of f at $x=\pi$.

Now, since it is provided that the function f is defined at $x=\pi$ and $f(\pi)=k\pi+1$, so

$$\lim_{x\to\pi^-} f(x)=\lim_{x\to\pi^+} f(x)=f(\pi)$$

$$\Rightarrow \lim_{x\to\pi^-} (k\pi+1)=\lim_{x\to\pi^+} \cos x=k\pi+1$$

$$\Rightarrow k\pi+1=\cos\pi=k\pi+1$$

$$\Rightarrow k\pi+1=-1=k\pi+1$$

$$\Rightarrow k\pi+1=-1=k\pi+1$$

$$\Rightarrow k=-\frac{2}{\pi}$$

Hence, the value of k is $-\frac{2}{\pi}$ for which the function f is continuous at $x=\pi$.

29. Determine the values of k for which the following function f attains continuity at the provided point.

$$\mathbf{f(x)=\begin{cases} kx+1, \text{ if } x\le 5 \\ 3x-5, \text{ if } x>5 \end{cases} \text{ at } x=5}$$

Ans: The given function is $f(x)=\begin{cases} kx+1, \text{ if } x\le 5 \\ 3x-5, \text{ if } x>5 \end{cases}$.

Now, note that, the function f is continuous at $x=5$ only if the value of f at $x=5$ is equal to the limiting value of f at $x=5$.

Since it is given that, the function f is defined at $x=5$ and $f(5)=kx+1=5k+1$, so

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} (3x-5)=5k+1$$

$$\Rightarrow 5k+1=15-5=5k+1$$

$$\Rightarrow 5k+1=10$$

$\Rightarrow 5k=9$

$\Rightarrow k=\dfrac{9}{5}$

Hence, the value of k is $\dfrac{9}{5}$ for which the function f is continuous at $x=5$.

30. Determine the values of constants a and b for which the following function f is continuous.

f such that $f(x)=\begin{Bmatrix} 5, & \text{if} & x\le 2 \\ ax+b, & \text{if} & 2<x<10 \\ 21, & \text{if} & x\ge 10 \end{Bmatrix}$.

Ans: The given function is $f(x)=\begin{Bmatrix} 5, & \text{if} & x\le 2 \\ ax+b, & \text{if} & 2<x<10 \\ 21, & \text{if} & x\ge 10 \end{Bmatrix}$.

Note that, f is defined at every point on the real number line.

Now, realise that if the function f is continuous then f is continuous at every real number.

So, let f satisfies continuity at $x=2$ and $x=10$.

Then, since f is continuous at $x=2$, so

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^+} f(x)=f(2)$

$\Rightarrow \lim\limits_{x\to 2^-}(5)=\lim\limits_{x\to 2^+}(ax+b)=5$

$\Rightarrow 5=2a+b=5$

$\Rightarrow 2a+b=5$ ...... (1)

Again, since f attains continuity at $x=10$, so

$\lim\limits_{x\to 10^-} f(x)=\lim\limits_{x\to 10^+} f(x)=f(10)$

$\Rightarrow \lim\limits_{x\to 10^-}(ax+b)=\lim\limits_{x\to 10^+}(21)=21$

$\Rightarrow 10a+b-21=21$

$\Rightarrow 10a+b=21$ …… (2)

Subtracting the equation (1) from the equation (2), gives

$8a=16 \Rightarrow a=2$

Substituting $a=2$ in the equation (1), gives

$2\times 2+b=5$
$\Rightarrow 4+b=5 \Rightarrow b=1$

Hence, the values of a and b are 2 and 1 respectively for which f is a continuous function.

31. Prove that the following function is continuous.

$\mathbf{f(x)=\cos(x^2)}$

Ans: The given function is $f(x)=\cos(x^2)$.

Note that, f is defined for all real numbers and so f can be expressed as the composition of two functions as, $f=g\circ h$, where $g(x)=\cos x$ and $h(x)=x^2$.

$[\therefore (goh)(x)=g(h(x))=g(x^2)=\cos(x^2)=f(x)]$

Now, it is to be Proven that, the functions $g(x)=\cos x$ and $h(x)=x^2$ are continuous.

Since the function g is defined for all the real numbers, so let consider c be a real number.

Then, $g(c)=\cos c$.

Substitute $x=c+h$ into the function g.

When, $x\to c$, then $h\to 0$.

Then we have,

$$\lim_{x\to c} g(x)=\lim_{x\to c}\cos x$$
$$=\lim_{h\to 0}\cos(c+h)$$
$$=\lim_{h\to 0}[\cos c\cos h-\sin c\sin h]$$
$$=\lim_{h\to 0}\cos c\cos h-\lim_{h\to 0}\sin c\sin h$$
$$=\cos c\cos 0-\sin c\sin 0$$
$$=\cos c\times 1-\sin c\times 0$$
$$=\cos c$$

Therefore, $\lim_{x\to c} g(x)=g(c)$.

Hence, the function $g(x)=\cos x$ is continuous.

Again, $h(x)=x^2$ is defined for every real point.

So, let consider k be a real number, then $h(k)=k^2$ and

$$\lim_{x\to k} h(x)=\lim_{x\to k} x^2=k^2.$$

Therefore, $\lim_{x\to k} h(x)=h(k)$.

Hence, the function h is continuous.

Now, remember that for real valued functions g and h, such that $(g\circ h)$ is defined at c, if g is continuous at c and f is continuous at $g(c)$, then $(f\circ h)$ is continuous at c.

Hence, the function $f(x)=(g\circ h)(x)=\cos(x^3)$ is continuous.

32. Prove that the following function is continuous.

$\mathbf{f(x)=|\cos x|}$

Ans: The given function is $f(x)=|\cos x|$.

Note that, the function f is defined for all real numbers. So, the function f can be expressed as the composition of two functions as, $f=g \circ h$, where $g(x)=|x|$ and $h(x)=\cos x$.

$[\because (goh)(x)=g(h(x))=g(\cos x)=|\cos x|=f(x)]$

Now, it is to be proved that the functions $g(x)=|x|$ and $h(x)=\cos x$ are continuous.

Remember that, $g(x)=|x|$, can be written as

$$g(x)=\left\{\begin{matrix} -x, \text{ if } x<0 \\ x, \text{ if } x \geq 0 \end{matrix}\right\}.$$

Now, since the function g is defined for every real number, so let consider c be a real number.

Then there may arise three cases, either $c<0$, or $c>0$, or $c=0$.

Let discuss the cases one after another.

Case I: When $c<0$.

Then, $g(c)=-c$.

Also, $\lim_{x \to c} g(x)=\lim_{x \to c}(-x)=-c$.

Therefore, $\lim_{x \to c} g(x)=g(c)$.

Hence, the function g is continuous at every point x, for $x<0$.

Case II: When $c>0$.

Then, $g(c)=c$.

Also, $\lim_{x \to c} g(x)=\lim_{x \to c} x=c$.

Therefore, $\lim_{x \to c} g(x)=g(c)$.

Hence, the function g is continuous at every point x for $x>0$.

Case III: When $c=0$.

Then, $g(c)=g(0)=0$.

Now, the left-hand-limit of the function g at $x=0$ is

$\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^-}(-x)=0$ and the right-hand-limit is

$\lim\limits_{x\to 0^+} g(x)=\lim\limits_{x\to 0^+}(x)=0$.

Therefore, $\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^+} g(x)=g(0)$.

Hence, the function g is continuous at $x=0$.

By observing the above three discussions, we can conclude that the function g is continuous at every real points.

Now, since the function $h(x)=\cos x$ is defined for all real numbers, so let consider c be a real number. Then, substitute $x=c+h$ into the function h.

So, when $x\to c$, then $h\to 0$.

Then, we have

$h(c)=\cos c$ and

$$\lim_{x\to c} h(x)=\lim_{x\to c}\cos x$$

$$=\lim_{h\to 0}\cos(c+h)$$

$$=\lim_{h\to 0}[\cos c\cos h-\sin c\sin h]$$

$$=\lim_{h\to 0}\cos c\cos h-\lim_{h\to 0}\sin c\sin h$$

$$=\cos c\cos 0-\sin c\sin 0$$
$$=\cos c\times 1-\sin c\times 0$$
$$=\cos c$$

Therefore, $\lim\limits_{x\to c} h(x)=h(c)$.

Hence, the function $h(x)=\cos x$ is continuous.

Now remember that, for real valued functions g and h, such that $(g \circ h)$ is defined at $x=c$ only if g is continuous at c and f is continuous at $g(c)$, then the composition functions $(f \circ g)$ is continuous at $x=c$.

Thus, the function $f(x)=(goh)(x)=g(h(x))=g(\cos x)=|\cos x|$ is continuous.

33. Verify whether the trigonometric function $\sin|x|$ is continuous.

Ans: First suppose that, $f(x)=\sin|x|$.

Now, note that the function f is defined for all real numbers and so f can be expressed as the composition of functions as, $f=g \circ h$, where $g(x)=|x|$ and $h(x)=\sin x$.

$$\left[(g \circ h)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$$

So, it is to be proved that the functions $g(x)=|x|$ and $h(x)=\sin x$ are continuous.

Now, remember that, the function $g(x)=|x|$ can be written as

$$g(x)\begin{Bmatrix} -x, \text{ if } x<0 \\ x, \quad \text{if } x \geq 0 \end{Bmatrix}.$$

Note that, the function g is defined for every real number, and so let consider c be a real number.

Then, there may arise three cases, either $c<0$, or $c>0$, or $c=0$.

Let us discuss the cases one after another.

Case I: When $c<0$.

Then $g(c)=-c$.

Also, $\lim\limits_{x \uparrow c}(-x)=\lim\limits_{x \uparrow c} x=-c$.

Therefore, $\lim\limits_{x \uparrow c} g(x)=g(c)$.

Hence, the function g is continuous at every point x for $x<0$.

Case II: When $c>0$.

Then, $g(c)=c$

Also, $\lim\limits_{x \uparrow c}(-x)=\lim\limits_{x \uparrow c} x=c$.

Therefore, $\lim\limits_{x \uparrow c} g(x)=g(c)$.

Thus, the function g is continuous at every point x for $x>0$.

Case III: When $c = 0$.

Then, $g(c)=g(0)=0$.

Also, the left-hand-limit of the function g at $x=0$ is

$\lim\limits_{x \uparrow 0^-} g(x)=\lim\limits_{x \uparrow 0^-}(-x)=0$ and the right-hand -limit is

$\lim\limits_{x \uparrow 0^+} g(x)=\lim\limits_{x \uparrow 0^+}(x)=0$.

Therefore, $\lim\limits_{x \uparrow 0^-} g(x)=\lim\limits_{x \uparrow 0^+}(x)=g(0)$.

Thus, the function g is continuous at $x = 0$.

By observing the above three discussions, we can conclude that that the function g is continuous at every points.

Again, since the function $h(x)=\sin x$ is defined for all real numbers, so let consider c be a real number and substitute $x=c+k$ into the function.

Now, when $x \uparrow c$ then $k \uparrow 0$.

Then, we have

$h(c)=\sin c$.

Also,

$$\lim_{x \to c} h(x) = \lim_{x \to c} \sin x$$
$$= \lim_{k \to 0} \sin(c+k)$$
$$= \lim_{k \to 0} [\sin c \cos k + \cos c \sin k]$$
$$= \lim_{k \to 0} (\sin c \cos k) + \lim_{h \to 0} (\cos c \sin k)$$
$$= \sin c \cos 0 + \cos c \sin 0$$
$$= \sin c + 0$$
$$= \sin c$$

Therefore, $\lim_{x \to c} h(x) = g(c)$.

Hence, the function h is continuous.

Now, remember that, for any two real valued functions g and h, such that the cmposition of functions $g \circ h$ is defined at c, if g is continuous at c and f is continuous at $g(c)$, then the composition function $f \circ h$ is continuous at c.

Thus, the function $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous.

34. Determine all the discontinuity points of the following function f defined by $f(x) = |x| - |\sin x|$.

Ans: The given function is $f(x) = |x| - |\sin x|$.

Let consider two functions

$g(x) = |x|$ and $h(x) = |x+1|$.

Then we get, $f = g - h$.

Now, the function $g(x) = |x|$ can be written as

$$g(x) = \begin{Bmatrix} -x, \text{ if } x < 0 \\ x, \text{ if } x \geq 0 \end{Bmatrix}.$$

Note that, the function g is defined for every real number and so let consider c be a real number.

Then there may arise three cases, either $c<0$, or $c>0$, or $c=0$.

Let us discuss the cases one after another.

Case I: When $c<0$.

Then, $g(c)=g(0)=-c$.

Also, $\lim_{x \to c} g(x)=\lim_{x \to c}(-x)=-c$.

Therefore, $\lim_{x \to c} g(x)=g(c)$.

Hence, the function g is continuous at every point x for $x<0$.

Case II: When $c>0$.

Then $g(c)=c$.

Also, $\lim_{x \to c} g(x)=\lim_{x \to c} x=c$.

Therefore, $\lim_{x \to c} g(x)=g(c)$.

Hence, the function g is continuous at every point x, where $x>0$.

Case III: When $c = 0$.

Then $g(c)=g(0)=0$.

Also, the left-hand-limit of the function g at $x=0$ is

$\lim_{x \to 0^-} g(x)=\lim_{x \to 0^-}(-x)=0$ and the right-hand-limit is

$\lim_{x \to 0^+} g(x)=\lim_{x \to 0^+}(x)=0$.

Therefore, $\lim_{x \to 0^-} g(x)=\lim_{x \to 0^+}(x)=g(0)$.

Hence, the function g is continuous at $x = 0$.

Thus, we can conclude by observing the above three discussions that g is continuous at every real point.

Now, remember that, the function $h(x)=|x+1|$ can be written as

$$h(x)=\begin{cases} -x(x+1), \text{ if, } x<-1 \\ x+1, \text{ if, } x \geq -1 \end{cases}.$$

Note that, the function h is defined for all real numbers, and so let consider c be a real number.

Case I: When $c<-1$.

Then $h(c)=-(c+1)$.

Also, $\lim\limits_{x \to c}[-(x+1)]=-(c+1)$.

Therefore, $\lim\limits_{x \to c} h(x)=h(c)$.

Hence, the function h attains continuity at every real point x, where $x<-1$.

Case II: When $c>-1$.

Then, $h(c)=c+1$.

Also, $\lim\limits_{x \to c} h(x)=\lim\limits_{x \to c}(x+1)=(c+1)$.

Therefore, $\lim\limits_{x \to c} h(x)=h(c)$.

Hence, the function h satisfies continuity at every real point x for $x>-1$.

Case III: When $c=-1$.

Then, $h(c)=h(-1)=-1+1=0$.

Also, the left-hand-limit of the function h at $x=1$ is

$\lim\limits_{x \to -1^-} h(x)=\lim\limits_{x \to -1^-}[-(x+1)]=-(-1+1)=0$ and the right-hand-limit is

$\lim\limits_{x \to -1^+} h(x)=\lim\limits_{x \to -1^+}(x+1)=(-1+1)=0$.

Therefore, $\lim_{x\to 1^-} h=\lim_{x\to 1^+} h(x)=h(-1)$.

Thus, the function h satisfies continuity at $x=-1$.

Hence, by observing the above three discussions, we can conclude that the function h is continuous for every real point.

Now, since the functions g and h are both continuous, so the function $f=g-h$ is also continuous.

Hence, the function f does not have any discontinuity points.

Exercise 5.2

1. Compute the derivative of the function $f(x)=\sin(x^2+5)$ with respect to x.

Ans: Let $f(x)=\sin(x^2+5)$, $u(x)=x^2+5$, and $v(t)=\sin t$

Then, $(v\circ u)(x)=v(u(x))=v(x^2+5)=\tan(x^2+5)=f(x)$

Therefore, f is a composition of two functions u and v.

Substitute $t=u(x)=x^2+5$.

Then, it gives

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(x^2+5)$$

$$\frac{dt}{dx}=\frac{d}{dx}(x^2+5)=\frac{d}{dx}(x^2)+\frac{d}{dx}(5)=2x+0=2x$$

Applying the chain rule of derivatives gives

$$\frac{df}{dx}=\frac{dv}{dt}.\frac{dt}{dx}=\cos(x^2+5)\times 2x=2x\cos(x^2+5)$$

An alternate method:

$$\frac{df}{dx}=\frac{dv}{dt}.\frac{dt}{dx}=\cos(x^2+5).\frac{d}{dx}(x^2+5)$$

$$=\cos(x^2+5).\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(5)\right]$$

$$=\cos(x^2+5).[2x+0]$$

$$=2x\cos(x^2+5)$$

Hence, the derivative of the function $f(x)=\sin(x^2+5)$ is $2x\cos(x^2+5)$.

2. Compute the derivative of the function $f(x)=\cos(\sin x)$ with respect to x.

Ans: Let suppose that, $f(x)=\cos(\sin x)$, $u(x)=\sin x$, and $v(t)=\cos t$

Then, $(v\circ u)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$

Therefore, it is observed that f is the composition of two functions u and v.

Now, substitute $t=u(x)=\sin x$.

Then,

$$\frac{dv}{dt}=\frac{d}{dt}(\cos t)=-\sin t=-\sin(\sin x) \text{ and}$$

$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x.$$

Applying the chain rule of derivatives gives

$$\frac{df}{dx}=\frac{dv}{dt}.\frac{dt}{dx}=\sin(\sin x).\cos x=-\cos x\sin(\sin x)$$

An alternate method:

$$\frac{d}{dx}[\cos(\sin x)]=-\sin(\sin x).\frac{d}{dx}(\sin x)=-\sin(\sin x)-\cos x=-\cos x\sin(\sin x).$$

Hence, the derivative of the function $f(x)=\cos(\sin x)$ is $-\cos x\sin(\sin x)$.

3. Compute the derivative of the function $f(x) = \sin(ax+b)$ with respect to x

Ans: Let suppose that, $f(x)=\sin(ax+b)$, $u(x)=ax+b,$, and $v(t)=\sin t$

Then we get, $(v\circ u)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=f(x)$.

It is observed that the function f is the composition of two functions u and v .

Now, substitute $t=u(x)=ax+b$.

Therefore,

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b) \text{ and}$$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{dt}{dx}(ax)+\frac{d}{dx}(b)=a+0=a.$$

Applying the chain rule deriavtives, gives

$$\frac{df}{dx}=\frac{dv}{dt}.\frac{dt}{dx}=\cos(ax+b).a=a\cos(ax+b).$$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin(ax+b)\right]&=\cos(ax+b).\frac{d}{dx}(ax+b)\\&=\cos(ax+b)\times\left[\frac{d}{dx}(ax)+\frac{d}{dt}(b)\right]\\&=\cos(ax+b)\times(a+0)\\&=a\cos(ax+b)\end{aligned}$$

Hence, the derivative of the function $f(x)=\sin(ax+b)$ is $a\cos(ax+b)$.

4. Compute the derivative of the function $f(x)=\sec(\tan(\sqrt{x}))$ with resepcet to x.

Ans: Let suppose that, $f(x)=\sec(\tan(\sqrt{x}))$, $u(x)=\sqrt{x}, v(t)=\tan t$, and $w(s)=\sec s$

Then, we get, $(w \circ v \circ u)(x)=w[v(u(x))]=w[v(\sqrt{x})]=w(\tan\sqrt{x})=f(x)$.

It is observed that the function g is the composition of three functions u, v and w.

Now, substitute $s=v(t)$ and $t=u(x)=\sqrt{x}$.

Then, we get

$$\frac{dw}{ds}=\frac{d}{ds}(\sec s)=\sec s=\sec(\tan t)\times\tan(\tan t) \qquad [s=\tan t]$$

$$=\sec(\tan\sqrt{x})\times\tan(\tan\sqrt{x}) \qquad [t=\sqrt{x}]$$

Thus, applying the chain rule of derivatives gives

$$\frac{dt}{dx}=\frac{dw}{ds}.\frac{ds}{dt}\times\frac{dt}{dx}$$

$$=\sec(\tan(\sqrt{x})\times(\tan(\sqrt{x})x\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}(\tan\sqrt{x})\tan(\tan\sqrt{x})$$

$$=\frac{\sec^2\sqrt{x}\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})}{2\sqrt{2}}$$

An alternate method:

$$\frac{d}{dx}\left[\sec(\tan(\sqrt{x}))\right]=\sec(\tan(\sqrt{x}).(\tan(\sqrt{x}).\frac{d}{dx}(\tan(\sqrt{x})$$

$$=\sec(\tan\sqrt{x})\times\tan(\tan\sqrt{x})\times\sec^2(\sqrt{x})\times\frac{d}{dx}(\sqrt{x})$$

$$=\sec(\tan\sqrt{x})\times\tan(\tan\sqrt{x})\times\sec^2(\sqrt{x})\times\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec(\tan\sqrt{x})\times\tan(\tan\sqrt{x})\times\sec^2(\sqrt{x})}{2\sqrt{x}}$$

Hence, the derivative of the function $f(x)=\sec\left(\tan\left(\sqrt{x}\right)\right)$ is

$$\frac{\sec^2\sqrt{x}\sec(\tan\sqrt{x})\tan(\tan\sqrt{x})}{2\sqrt{2}}.$$

5. Compute the derivative of the function $f\left(x\right)=\frac{\sin(ax+b)}{\cos(cx+d)}$ with respect to x.

Ans: The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}$.

Now, let $g(x)=\sin(ax+b)$ and $h(x)=\cos(cx+d)$.

Here we will use the divide formula of derivatives $f'=\frac{g'h-gh'}{h^2}$. …… (1)

First, consider the function $g(x)=\sin(ax+b)$.

Let assume $u(x)=ax+b$, and $v(t)=\sin t$.

Then, we get $(v\circ u)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$.

Therefore, we observe that the function g is the composition of two functions, u and v.

So, substitute $t=u(x)=ax+b$.

Then,

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b) \text{ and}$$

$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{dt}{dx}(ax)+\frac{d}{dx}(b)=a+0=a.$$

Therefore, applying the chain rule of derivatives gives

$$g'=\frac{dg}{dx}=\frac{dv}{dt}.\frac{dt}{dx}=\cos(ax+b).a=a\cos(ax+b).$$

Now, consider the function $h(x)=\cos(cx+d)$.

Let suppose $p(x)=cx+d$, and $q(t)=\cos y$.

Then, we have $(q\circ p)(x)=q(p(x))=q(cx+d)=\cos(cx+d)=h(x)$.

Therefore, the function h is the composition of two functions p and q.

Now, substitute $y=p(x)=cx+d$.

Then we have,

$$\frac{dq}{dy}=\frac{d}{dy}(\cos y)=-\sin y=-\sin(cx+d) \text{ and}$$

$$\frac{dy}{dx}=\frac{d}{dx}(cx+d)=\frac{d}{dx}(cx)+\frac{d}{dx}(d)=c.$$

Therefore, applying the chain rule of derivatives gives

$$h'=\frac{dh}{dx}=\frac{dq}{dy}.\frac{dy}{dx}=-\sin(cx+d)\times c=-c\sin(cx+d).$$

Now, substituting all the obtained derivatives into the formula (1) gives

$$f'=\frac{a\cos(ax+b)\times\cos(cx+d)-\sin(ax+b)(-c\sin cx+d)}{[\cos(cx+d]^2}$$

$$=\frac{a\cos(ax+b)}{\cos(cx+d)}+c\sin(ax+b)\times\frac{\sin(cx+d)}{\cos(cx+d)}\times\frac{1}{\cos(cx+d)}$$

$$=a\cos(ax+b)\sec(cx+d)+c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Hence, the derivative of the function $f(x)=\dfrac{\sin(ax+b)}{\cos(cx+d)}$ is

$a\cos(ax+b)\sec(cx+d)+c\sin(ax+b)\tan(cx+d)\sec(cx+d)$.

6. Compute the derivative of the function $f(x)=\cos(x^3)\times\sin^2(x^5)$ with resepct to x.

Ans: The given function is $f(x)=\cos(x^3)\times\sin^2(x^5)$.

Then,

$$\frac{d}{dx}[\cos x^3\times\sin^2(x^5)]=\sin^2(x^5)\times\frac{d}{dx}(\cos x^3)+\cos x^3 x\frac{d}{dx}[\sin^2(x^5)]$$

$$=\sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3)+\cos x^3+2\sin(x^5)\times\frac{d}{dx}[\sin x^5]$$

$$=\sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3)+\cos x^3+2\sin(x^5)\times\frac{d}{dx}[\sin x^5]$$

$$=\sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3)+\cos x^3+2\sin(x^5)\times\frac{d}{dx}[\sin x^5]$$

$$=\sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3)+\cos x^3+2\sin(x^5)\times\frac{d}{dx}[\sin x^5]$$

$$=\sin x^3\sin^2(x^5)\times 3x^2+2\sin x^5\cos x^3\times\cos x^5\times\frac{d}{dx}(x^5)$$

$$=3x^2\sin x^3\times\sin^3(x^5)+2\sin x^5\cos x^5\cos x^3\times 5x^4$$

$$=10x^4\sin x^5\cos x^5\cos x^3-3x^2\sin x^3\sin^2(x^5)$$

Hence, the derivative of the function $f(x)=\cos\left(x^3\right)\times\sin^2\left(x^5\right)$ is $10x^4\sin x^5\cos x^5\cos x^3-3x^2\sin x^3\sin^2(x^5)$.

7. Compute the derivative of the function $f(x)=\sqrt[2]{\cot(x^2)}$ with respect to x

Ans: The given function is $f(x)=\sqrt[2]{\cot(x^2)}$.

Then,

$$\frac{d}{dx}\left[\sqrt[2]{\cot(x^2)}\right]$$

$$=2\times\frac{1}{\sqrt[2]{\cot(x^2)}}\times\frac{d}{dx}\left[\cot(x^2)\right]$$

$$=\sqrt{\frac{\sin(x^2)}{\cot(x^2)}}\times\operatorname{cosec}^2(x^2)\times\frac{d}{dx}(x^2)$$

$$=\sqrt{\frac{\sin(x^2)}{\cot(x^2)}}\times\frac{1}{\sin^2(x^2)}\times(2x)$$

$$=\frac{-2\sqrt{2x}}{\sqrt{\cos x^2\sqrt{\sin ix^2 \sin x^2}}}$$

$$=\frac{-2\sqrt{2x}}{\sqrt{2\sin x^2\cos x^2\sin x^2}}$$

$$=\frac{-2\sqrt{2x}}{\sin x^2\sqrt{\sin 2x^2}}$$

Hence, the derivative of the function $f(x)=\sqrt[2]{\cot(x^2)}$ is $\frac{-2\sqrt{2x}}{\sin x^2\sqrt{\sin 2x^2}}$.

8. Compute the derivative of the function $f(x)=\cos(\sqrt{x})$ with respect to x.

Ans: The given function is $f(x)=\cos(\sqrt{x})$

Now, let $u(x)=\sqrt{x}$ and $v(t)=\cos t$.

Then, we have, $(v\circ u)(x)=v(u(x))=v(\sqrt{x})=\cos\sqrt{x}=f(x)$.

It is observed that the function f is the composition of two functions u and v. So, let $t=u(x)=\sqrt{x}$.

Then,

$$\frac{dt}{dx}=\frac{d}{dx}(\sqrt{x})=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{\frac{1}{2}}=\frac{1}{2\sqrt{x}}.$$

Also,

$$\frac{dv}{dt}=\frac{d}{dt}(\cos t)=-\sin t=\sin(\sqrt{x}).$$

Now, by applying the chain rule of derivatives, gives

$$\frac{dt}{dx}=\frac{dv}{dt}\times\frac{dt}{dx}$$

$$=-\sin(\sqrt{x})\times\frac{1}{2\sqrt{x}}$$

$$=-\frac{1}{2\sqrt{x}}\sin(\sqrt{x})$$

$$=-\frac{\sin(\sqrt{x})}{2\sqrt{x}}$$

An alternate method:

$$\frac{d}{dx}\left[\cos(\sqrt{x})\right]$$

$$=-\sin(\sqrt{x}).\frac{d}{dx}(\sqrt{x})$$

$$=-\sin(\sqrt{x})\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$=-\sin\sqrt{x}\times\frac{1}{2}x^{\frac{1}{2}}$$

$$=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Hence, the derivative of the function $f(x)=\cos(\sqrt{x})$ is $-\frac{\sin(\sqrt{x})}{2\sqrt{x}}$.

9. Prove that the function $f(x)=|x-1|,\ x\in R$ is not differentiable at $x=1$.

Ans: The given function is $f(x)=|x-1|, x\in R$.

We know that a function f is called differentiable at a point $x=c$ in its domain if both the $\lim\limits_{k\to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim\limits_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

Now verify the differentiability for the function f at the point $x=1$.

First, the left-hand-derivative is

$$\lim_{h \to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h \to 0^-}\frac{f|1+h-1||1-1|}{h}$$

$$\lim_{h \to 0^-}\frac{f|h|-0}{h}=\lim_{h \to 0^+}\frac{-h}{h}=1, \text{ since } h<0 \Rightarrow |h|=-h.$$

Now the right-hand-derivative is

$$\lim_{h \to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h \to 0^+}\frac{f|1+h-1||1-1|}{h}$$

$$\lim_{h \to 0^+}\frac{f|h|-0}{h}=\lim_{h \to 0^+}\frac{-h}{h}=-1, \text{ since } h>0 \Rightarrow |h|=h.$$

From the above, it is noted that $\lim_{h \to 0^-}\frac{f(1+h)-f(1)}{h} \neq \lim_{h \to 0^+}\frac{f(1+h)-f(1)}{h}$.

Hence, the function $f(x)=|x-1|, x \in R$ is not differentiable at the point $x=1$.

10. Prove that $f(x)=[x], 0<x<3$, the greatest integer function is not differentiable at the points $x=1$ and $x=2$.

Ans: The given function is $f(x)=[x],\ 0<x<3$.

Remember that a function f is called differentiable at a point $x=c$ in its domain if both the limits, $\lim_{h \to 0^-}\frac{f(c+h)-f(c)}{h}$ and $\lim_{h \to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

First, take the left-hand-derivative of the function f at $x=1$ such that

$$\lim_{h \to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h \to 0^-}\frac{[1+h]-[1]}{h}=\lim_{h \to 0^-}\frac{(0-1)}{h}=\lim_{h \to 0^+}\frac{-h}{h}=\infty.$$

Now, take the right-hand-derivative of the function f at $x=1$ such that

$$\lim_{h \to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h \to 0^+}\frac{[1+h][1]}{h}=\lim_{h \to 0^+}\frac{1-1}{h}=\lim_{h \to 0^+}0=0.$$

Therefore, it is being noticed that, $\lim\limits_{h\to 0^-}\frac{f(1+h)-f(1)}{h}\neq\lim\limits_{h\to 0^+}\frac{f(1+h)-f(1)}{h}$.

Thus, the function f is not differentiable at $x=1$.

Now, justify the differentiability of the function f at $x=2$.

First, take the left-hand-derivative of the function f at $x=2$, such that

$$\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^-}\frac{[2+h]-[2]}{h}=\lim_{h\to 0^-}\frac{(1-2)}{h}=\lim_{h\to 0^+}\frac{-1}{h}=\infty$$

Now, take the right-hand-derivative of the function f at $x=2$, such that

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0^+}\frac{[2+h][2]}{h}=\lim_{h\to 0^+}\frac{1-2}{h}=\lim_{h\to 0^+}0=0$$

It is observed from the above discussion that, $\lim\limits_{h\to 0^-}\frac{f(2+h)-f(2)}{h}\neq\lim\limits_{h\to 0^+}\frac{f(2+h)-f(2)}{h}$.

Thus, the function f is not differentiable at the point $x=2$.

Exercise 5.3

1. Determine $\frac{dy}{dx}$ from equation $2x+3y=\sin x$.

Ans: The given equation is $2x+3y=\sin x$.

Differentiating both sides of the equation with respect to x, gives

$$\frac{d}{dy}(2x+3y)=\frac{d}{dx}(\sin x)$$

$\Rightarrow\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$, applying the addition rule of derivatives.

$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$

$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$

Therefore, $\frac{dy}{dx}=\frac{\cos x-2}{3}$.

2. Determine $\frac{dy}{dx}$ from the equation $2x+3y=\sin y$.

Ans: The given equation is $2x+3y=\sin y$.

Differentiating both sides of the equation with respect to x, gives

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx}$, applying the chain rule of derivatives.

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

Therefore, $\frac{dy}{dx}=\frac{2}{\cos y-3}$.

3. Determine $\frac{dy}{dx}$ from the equation $ax+by^2=\cos y$.

Ans: The given function is $ax+by^2=\cos y$.

Differentiating both sides of the equation with respect to x, gives

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$\Rightarrow a\cdot 1+b\frac{d}{dy}(y^2)\frac{dy}{dx}=\frac{d}{dy}(\cos y)\frac{dy}{dx}$, applying the chain rule of derivatives.

$$\Rightarrow a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$\Rightarrow (2by+siny)\frac{dy}{dx}=a$

Therefore, $\frac{dy}{dx}=\frac{-a}{2by+siny}$.

4. Determine $\frac{dy}{dx}$ from the equation $xy+y^2=tanx+y$.

Ans: The given equation is $xy+y^2=tanx+y$.

Differentiating both sides of the equation with respect to x, gives

$\frac{d}{dx}(xy+y^2)=\frac{d}{dx}(tanx+y)$

$\Rightarrow \frac{d}{dx}(xy)+\frac{dy}{dx}(y^2)=\frac{d}{dx}(tany)+\frac{d}{dx}$

$\Rightarrow \left[y\times\frac{d}{dx}(x)+x\times\frac{dy}{dx}\right]+2y\frac{d}{dx}=sec^2x+\frac{dy}{dx}$, applying chain rule of derivatives.

$\Rightarrow y\times1+x\frac{dy}{dx}+2y\frac{dy}{dx}=sec^2x+\frac{dy}{dx}$

Therefore, $\frac{dy}{dx}=\frac{sec^2x-y}{(x+2y-1)}$.

5. Determine $\frac{dy}{dx}$ from the equation $x^2+xy+y^2=100$.

Ans: The given equation is $x^2+xy+y^2=100$.

Differentiating both sides of the equation with respect to x, gives

$\frac{dy}{dx}(x^2+xy+y^2)=\frac{d}{dx}100$

$\Rightarrow \frac{dy}{dx}(x^2)+\frac{dy}{dx}(xy)+\frac{dy}{dx}(y^2)=0$

$$\Rightarrow 2x+\left[y\times\frac{d}{dx}(x)+x\times\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0$$, applying the chain rule of derivatives.

$$\Rightarrow 2x+y\times1+x\times\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+y+(x+2y)\frac{dy}{dx}=0$$

Therefore, $\frac{dy}{dx}=-\frac{2x+y}{x+2y}$.

6. Determine $\frac{dy}{dx}$ from the equation $x^2+x^2y+xy^2+y^3=81$.

Ans: The given equation is $x^2+x^2y+xy^2+y^3=81$.

Differentiating both sides of the equation with respect to x, gives

$$\frac{dy}{dx}(x^2+x^2y+xy^2+y^3)=\frac{d}{dx}(81)$$

$$\Rightarrow \frac{dy}{dx}(x^2)+\frac{dy}{dx}(x^2y)+\frac{dy}{dx}(xy^2)+\frac{dy}{dx}(y^3)=0$$

$$\Rightarrow 3x^2+\left[y\frac{d}{dx}(x^2)+x^2\frac{dy}{dx}\right]+\left[y^2\frac{d}{dx}(x)+x\frac{d}{dx}(y^2)\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow 3x^2+\left[y\times2+x^2\frac{dy}{dx}\right]+\left[y^2\times1+x\times2y\times\frac{dy}{dx}\right]+3y^2\frac{dy}{dx}=0$$, applying chain rule.

$$\Rightarrow (x^2+2xy+3y^2)\frac{dy}{dx}+(3x^2+2xy+y^2)=0$$

Therefore, $\frac{dy}{dx}=\frac{-(3x^2+2xy+y^2)}{(x^2+2xy+3y^2)}$.

7. Determine $\frac{dx}{dy}$ from the equation $\sin^2 y+\cos xy=\pi$.

Ans: The given equation is $\sin^2 y+\cos xy=\pi$.

Differentiating both sides of the equation with respect to x, gives

$$\frac{d}{dx}(\sin^2 y+\cos xy)=\frac{d}{dx}\pi$$

$$\Rightarrow \frac{d}{dx}(\sin^2 y)+\frac{d}{dx}(\cos xy)=0 \qquad \text{…. (1)}$$

Applying the chain rule of derivatives gives

$$\frac{d}{dx}(\sin^2 y)=2\sin y\frac{d}{dx}(\sin y)=2\sin y\cos y\frac{dy}{dx} \qquad \text{….. (2)}$$

$$\Rightarrow \frac{d}{dx}(\cos xy)=-\sin xy\frac{d}{dx}(xy)=-\sin xy\left[y\frac{d}{dx}(x)+x\frac{dy}{dx}\right]=-y\sin xy-x\sin xy\frac{dy}{dx} \qquad \text{…. (3)}$$

From (1), (2) and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx}-y\sin xy-x\sin xy\frac{dy}{dx}=0$$

$$\Rightarrow (2\sin y\cos y-x\sin xy)\frac{dy}{dx}=y\sin xy$$

$$\Rightarrow (\sin 2y-x\sin xy)\frac{dx}{dy}=y\sin xy$$

Therefore, $\frac{dx}{dy}=\frac{y\sin xy}{\sin 2y-x\sin xy}$.

8. Determine $\frac{dy}{dx}$ from the equation $\sin 2x+\cos 2y=1$.

Ans: The given equation is $\sin 2x+\cos 2y=1$.

Differentiating both sides of the equation with respect to x, gives

$$\frac{dy}{dx}(\sin^2x+\cos^2y)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}(\sin^2x)+\frac{d}{dx}(\cos^2y)=0$$

$$\Rightarrow 2\sin x\times\frac{d}{dx}(\sin x)+2\cos y\times\frac{d}{dx}(\cos y)=0$$

$$\Rightarrow 2\sin x\cos x+2\cos y(-\sin y)\times\frac{dy}{dx}=0$$

$$\Rightarrow \sin2x-\sin2y\frac{dy}{dx}=0$$

$$\therefore \frac{dx}{dy}=\frac{\sin2x}{\sin2y}$$

9. Determine $\frac{dy}{dx}$ from the equation $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Ans: The given equation is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Now, $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$\Rightarrow \sin y=\frac{2x}{1+x^2}.$$

Differentiating both sides of the equation with respect to x, gives

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad \ldots\ldots (1)$$

Now, the function $\frac{2x}{1+x^2}$ is of the form of $\frac{u}{v}$.

Applying the quotient rule, gives

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{(1+x^2)\frac{d}{dx}(2x)-2x\times\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$=\frac{(1+x^2)\times2-2x\times[0+2x]}{(1+x^2)^2}=\frac{2+2x^2-4x^2}{(1+x^2)^2}$$

Therefore, $\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{2(1-x^2)}{(1+x^2)^2}$ ……. (2)

It is given that,

$$\sin y=\frac{2x}{1+x^2}$$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{(1+x^2)^2}}$$

$$\Rightarrow \cos y=\sqrt{\frac{\left(1-x^2\right)^2}{\left(1-x^2\right)^2}}=\frac{1-x^2}{1+x^2}$$ ….. (3)

From the equation (1), (2) and (3), gives

$$\frac{1-x^2}{1+x^2}\frac{dy}{dx}=\frac{2(1-x^2)}{(1+x^2)^2}$$

Therefore, $\frac{dy}{dx}=\frac{2}{1+x^2}$.

10. Determine $\frac{dx}{dy}$ from the equation $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$.

Ans: The given function is $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$.

Now, $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$\Rightarrow \tan y=\frac{3x-x^3}{1-3x^2} \qquad \text{...... (1)}$$

According to the trigonometric formulas,

$$\tan y=\frac{3\tan\frac{y}{3}-\tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}} \qquad \text{...... (2)}$$

By comparing the equations (1) and (2), gives

$$x=\tan\frac{y}{3}. \qquad \text{...... (3)}$$

Differentiating both sides of the equation (3) with respect to x, gives

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\times\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\times\frac{1}{3}\times\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{\sec^2\frac{y}{3}}=\frac{3}{1+\tan^2\frac{y}{3}}$$

Therefore, $\frac{dx}{dy}=\frac{3}{1+x^2}$.

11. Determine $\frac{dy}{dx}$ from the equation $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $0<x<1$.

Ans: The given equation is $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\Rightarrow \cos y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}=\frac{1-x^2}{1+x^2}. \qquad \ldots\ldots (1)$$

By comparing both sides of the equation (1) give

$$\tan\frac{y}{2}=x \qquad \ldots\ldots (2)$$

Differentiating both sides of the equation (2) with respect to x, gives

$$\sec^2\frac{y}{2}\times\frac{d}{dx}\left(\frac{y}{2}\right)=\frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{d}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+\tan^2\frac{y}{2}}$$

Therefore, $\frac{dy}{dx}=\frac{1}{1+x^2}$.

12. Determine $\frac{dy}{dx}$ from the equation $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $0<x<1$

Ans: The given equation is $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

Now, $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}. \qquad \ldots\ldots (1)$$

Differentiating both sides of the equation with respect to x, gives

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad \ldots\ldots (2)$$

Using chain rule, we get

$$\frac{d}{dx}(\sin y)=\cos y\times\frac{dy}{dx} \qquad \ldots\ldots (3)$$

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}=\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{1+x^2}}=\sqrt{\frac{4x^2}{(1+x^2)^2}}$$, using the equation (1).

$$\Rightarrow \cos y=\frac{2x}{1+x^2} \qquad \ldots\ldots (4)$$

Therefore, from the equation (3) and (4) gives

$$\frac{d}{dx}(\sin y)=\frac{2x}{1+x^2}\frac{dy}{dx} \qquad \ldots.. (5)$$

Now,

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=\frac{(1+x^2)(1-x^2)'-(1-x^2)(1+x^2)'}{(1+x^2)^2}$$, applying the quotient rule.

$$=\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}$$

$$=\frac{-2x-2x^3-2x+2x^3}{(1-x^2)^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=-\frac{4}{(1+x^2)^2} \qquad \ldots.. (6)$$

Using the equations (2), (5), and (6), gives

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{(1+x^2)^2}$$

Therefore, $\frac{dy}{dx}=\frac{-2}{1+x^2}$.

An alternate method:

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow (1+x^2)\sin y=1-x^2$$

$$\Rightarrow (1+\sin y)x^2=1-\sin y$$

$$\Rightarrow x^2=\frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2=\frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{x}\right)^2}$$

$$\Rightarrow x=\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x=\tan\left(\frac{\pi}{4}-\frac{\pi}{2}\right)$$

Differentiating both sides of the equation with respect to x, gives

$$\frac{d}{dx}(x)=\frac{d}{dx}\left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]$$

$$\Rightarrow 1=\sec^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\times\frac{dy}{dx}\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

$$\Rightarrow 1=\left[1+\tan^2\left(\frac{\pi}{4}-\frac{y}{2}\right)\times\left(-\frac{1}{2}\times\frac{dy}{dx}\right)\right]$$

$$\Rightarrow 1=(1+x^2)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

Therefore, $\frac{dx}{dy}=\frac{-2}{1+x^2}$.

13. Determine $\frac{dy}{dx}$ from the equation $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $-1<x<1$

Ans: The given equation is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$.

Now, $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$\Rightarrow \cos y=\frac{2x}{1+x^2}. \qquad \ldots\ldots (1)$$

Differentiating both sides of the equation with respect to x, gives

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\times\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y\times\frac{dy}{dx}=\frac{(1-x^2)\times\frac{d}{dx}(2x)-2x\times\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$, applying the quotient rule.

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx}=\frac{(1+x^2)\times 2-2x\times 2x}{(1+x^2)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dx}{dy}=-\left[\frac{2(1-x)^2}{(1+x^2)^2}\right]$$, using the equation (1).

$$\Rightarrow \sqrt{\frac{(1-x^2)^2-4x^2}{(1+x^2)^2}} = \frac{dy}{dx} = \frac{-2(1-x)^2}{(1+x^2)}$$

$$\Rightarrow \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} \frac{dy}{dx} = \frac{-2(1-x)^2}{(1+x^2)}$$

$$\Rightarrow \frac{1-x^2}{1+x^2} \times \frac{dy}{dx} = \frac{-2(1-x)^2}{(1+x^2)}$$

Therefore, $\frac{dy}{dx} = \frac{-2}{(1+x^2)}$.

14. Determine $\frac{dy}{dx}$ from the equation $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$, $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

Ans: The given equation is $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$.

Now, $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$\Rightarrow \sin y = 2x\sqrt{1-x^2}$. …… (1)

Differentiating both sides of the equation with respect to x, gives

$$\cos y \frac{dy}{dx} = 2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow \sqrt{1-\sin^2 y}\frac{dy}{dx} = 2\left[\frac{x}{2} \times \frac{-2}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right]$$

$$\Rightarrow \sqrt{1-(2x\sqrt{1-x^2})^2} = \frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$, using the equation (1).

$$\Rightarrow \sqrt{1-4x^2(1-x^2)^2}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \sqrt{(1-2x)^2}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \left(1-2x^2\right)\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

Therefore, $\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$.

15. Determine $\frac{dy}{dx}$ from the equation $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$, $0<x<\frac{1}{\sqrt{2}}$.

Ans: The given equation is $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$.

Now,

$$y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y=\frac{1}{2x^2-1}$$

$$\Rightarrow \cos y=2x^2-1$$

$$\Rightarrow 2x^2=1+\cos y$$

$$\Rightarrow 2x^2=2\cos^2\frac{y}{2}$$

$$\Rightarrow x=\cos\frac{y}{2} \qquad \ldots\ldots. (1)$$

Differentiating both sides of the equation with respect to x, gives

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\cos\frac{y}{2}\right)$$

$$\Rightarrow 1=\sin\frac{y}{2}\times\frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin\frac{y}{2}} = \frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin\frac{y}{2}} = \frac{-2}{\sqrt{1-\cos^2\frac{y}{2}}} = \frac{-2}{\sqrt{1-x^2}}$$, using the equation (1).

Therefore, $\frac{dy}{dx} = \frac{-2}{\sqrt{1-x}}$.

Exercise 5.4

1. Find the derivative of the function $y=\frac{e^x}{\sin x}$ with respect to x.

Ans: The given function is $y=\frac{e^x}{\sin x}$.

Then, we have

$$\frac{dy}{dx} = \frac{\sin x\frac{d}{dx}(e^x)-e^x\frac{d}{dx}(\sin x)}{\sin^2 x}$$, by applying the quotient rule of derivatives.

$$= \frac{\sin x\times(e^x)-e^x\times(\cos x)}{\sin^2 x}$$

Therefore, the derivative of the function y is

$$\frac{dy}{dx} = \frac{e^x(\sin x-\cos x)}{\sin^2 x},\ \ x \neq n\pi,\ n \in Z.$$

2. Find the derivative of the function $y=e^{\sin^{-1}x}$.

Ans: The given function is $y=e^{\sin^{-1}x}$.

Then, we have

$$\frac{dy}{dx}=\frac{d}{dx}(e^{\sin^{-1}x})$$

$$=e^{\sin^{-1}x}\times\frac{d}{dx}(\sin^{-1}x)$$

$$=e^{\sin^{-1}x}\times\frac{1}{\sqrt{1-x^2}}$$

$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

Therefore, the derivative of the function y is

$$\frac{dy}{dx}=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}},\ x\in(-1,1).$$

3. Prove that the following function f is strictly increasing on R.

f such that $f(x)=e^{2x}$.

Ans: First consider two real numbers x_1 and x_2 such that $x_1<x_2$.

Then,

$$x_1<x_2 \Rightarrow 2x_1<2x_2 \Rightarrow e^{2x_1}<e^{2x_2}$$

Therefore, $f(x_1)<f(x_2)$.

Hence, the function f is strictly increasing on the real number R.

4. Find the derivative of the function $y=e^{x^2}$ with respect to x.

Ans: The given function is $y=e^{x^2}$.

Then by applying the chain rule of derivatives we have,

$$\frac{dy}{dx}=\frac{d}{dx}(e^{x^2})=e^{x^2}\times\frac{d}{dx}(x^3)=e^{x^3}\times3x^2.$$

Therefore, the derivative of the function y is

$$\frac{dy}{dx}=3x^2e^{x^3}.$$

5. Find the derivative of the function is $y=\sin(\tan^{-1}e^{-x})$ with respect to x.

Ans: The given function is $y=\sin(\tan^{-1}e^{-x})$.

Now, applying the chain rule of derivatives, give

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin(\tan^{-1}e^{-x})\right]$$

$$=\cos(\tan^{-1}e^{-x})\times\frac{d}{dx}(\tan^{-1}e^{-x})$$

$$=\cos(\tan^{-1}e^{-x})\times\frac{1}{1+(e^{-x})}\times\frac{d}{dx}(e^{-x})$$

$$=\frac{\cos(\tan^{-1}e^{-x})}{1+(e^{-x})}\times e^{-x}\times\frac{d}{dx}(-x)$$

$$=\frac{e^{-x}\cos(\tan^{-1}e^{-x})}{1+e^{-2x}}\times(-1)$$

Therefore, the derivative of the function y is

$$\frac{dy}{dx}=\frac{-e^{-x}\cos(\tan^{-1}e^{-x})}{1+e^{-2x}}.$$

6. Find the derivative of the function $y=\log\left(\cos\left(e^x\right)\right)$

Ans: Let $y=\log\left(\cos\left(e^x\right)\right)$

Now, by applying the chain rule of derivatives give

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\cos\left(e^x\right)\right)\right]$$

$$=\frac{1}{\cos e^x}\times\frac{d}{dx}\left(\cos\left(e^x\right)\right)$$

$$=\frac{1}{\cos e^x}\times\left(-\sin\left(e^x\right)\right)\times\frac{d}{dx}(e^x)$$

$$=\frac{-\sin e^x}{\cos e^x}\times e^x$$

Therefore, the derivative of the function y is

$$\frac{dy}{dx}=-e^x\tan\left(e^x\right),\ \ x\neq(2n+1)\frac{\pi}{2}, n\in N.$$

7. Find the derivative of the function $y=e^x+e^{x^2}+...+e^{x^5}$ with respect to x.

Ans: The given function is $y=e^x+e^{x^2}+...+e^{x^5}$.

Then, differentiating with respect to x both sides, give

$$\frac{dy}{dx}=\frac{d}{dx}(e^x+e^{x^2}+...+e^{x^5})$$

$$=\frac{d}{dx}(e^x)+\frac{d}{dx}(e^{x^2})+\frac{d}{dx}(e^{x^4})+\frac{d}{dx}(e^{x^5})$$, applying the sum rule of derivatives.

$$=e^x+\left[e^{x^2}\times\frac{d}{dx}\left(x^2\right)\right]+\left[e^{x^3}\times\frac{d}{dx}\left(x^3\right)\right]+\left[e^{x^4}\times\frac{d}{dx}\left(x^4\right)\right]+\left[e^{x^5}\times\frac{d}{dx}\left(x^5\right)\right]$$

$$=e^x+(e^{x^2}\times 2x)+(e^{x^3}\times 3x^2)+(e^{x^4}\times 4x^3)+(e^{x^5}\times 5x^4)$$

Therefore, the derivative of the function y is

$$\frac{dy}{dx}=e^x+2xe^{x^2}+3x^2e^{x^3}+4x^3e^{x^4}+5x^4e^{x^5}.$$

8. Find the derivative of the function $y=\sqrt{e^{\sqrt{x}}}$, $x>0$ with respect to x.

Ans: The given function is $y=\sqrt{e^{\sqrt{x}}}$.

Then squaring both sides both sides of the equation give

$$y^2=e^{\sqrt{x}}$$

Now, differentiating both sides with respect to x gives

$$\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}\left(e^{\sqrt{x}}\right)$$

$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{1}{2}\times\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$, substituting the value of y.

Therefore,

$\dfrac{dy}{dx}=\dfrac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$, $x>0$.

9. Find the derivative of the function $y=\log(\log x), x>1$.

Ans: The given function is $y=\log(\log x)$.

Now, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}[\log(\log x)]$$

$=\dfrac{1}{\log x}\times\dfrac{d}{dx}(\log x)$, by applying the chain rule of derivatives.

$$=\frac{1}{\log x}\times\frac{1}{x}$$

Therefore, $\frac{dy}{dx}=\frac{1}{x\log}$, $x>1$.

10. Find the derivative of the function $y=\frac{\cos x}{\log x}$, $x>0$ with respect to x.

Ans: The given function is $y=\frac{\cos x}{\log x}$.

Differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x)\times\log x-\cos x\times\frac{d}{dx}(\log x)}{(\log x)^2}$$, by applying the quotient rule.

$$=\frac{-\sin x\log x-\cos x\times\frac{1}{x}}{(\log x)^2}$$

Therefore,

$$\frac{dy}{dx}=\frac{-[x\log x\sin x+\cos x]}{x(\log x)^2},\ x>0.$$

11. Find the derivative of the function $y=\cos(\log x+e^x)$, $x>0$ with resepct to x

Ans: The given function is $y=\cos(\log x+e^x)$.

Then differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}\left[\cos\left(\log x+e^x\right)\right].$$

$$\Rightarrow\frac{dy}{dx}=-\sin[\log x+e^x]\times\frac{d}{dx}(\log x+e^x)$$, by applying the chain rule of derivatives.

$$=\sin(\log x+e^x)\times\left[\frac{d}{dx}(\log x)+\frac{d}{dx}(e^x)\right]$$

$$=\sin(\log x+e^x)\times\left(\frac{1}{x}+e^x\right)$$

Therefore, $\frac{dy}{dx}=\left(\frac{1}{x}+e^x\right)\sin(\log x+e^x),\ x>0$.

Exercise 5.5

1. Find the derivative of the function $y=\cos x\times\cos 2x\times\cos 3x$ with respect to x

Ans: The given function is $y=\cos x\times\cos 2x\times\cos 3x$.

First, taking logarithm both sides of the equation give,

$$\log y=\log(\cos x\times\cos 2x\times\cos 3x)$$

$\Rightarrow \log y=\log(\cos x)+\log(\cos 2x)+\log(\cos 3x)$, by the property of logarithm.

Now, differentiating both sides of the equation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\cos x}\times\frac{d}{dx}(\cos x)+\frac{1}{\cos 2x}\times\frac{d}{dx}(\cos 2x)+\frac{1}{\cos 3x}\times\frac{d}{dx}(\cos 3x)$$

$$\Rightarrow \frac{dy}{dx}=y\left[-\frac{\sin x}{\cos x}-\frac{\sin 2x}{\cos 2x}\times\frac{d}{dx}(2x)-\frac{\sin 3x}{\cos 3x}\times\frac{d}{dx}(3x)\right]$$

Therefore,

$$\frac{dy}{dx}=-\cos\times\cos 2x\times\cos 3x\left[\tan x+2\tan 2x+3\tan 3x\right].$$

2. Find the derivative of the function $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ with respect to x.

Ans: The given function is $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$.

First taking logarithm both sides of the equation give

$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y=\frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y=\frac{1}{2}[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}]$$

$$\Rightarrow \log y=\frac{1}{2}[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)]$$

Now, differentiating both sides of the equation with respect to x give

$$\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)].$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-1}\times\frac{d}{dx}(x-1)+\frac{1}{x-2}\times\frac{d}{dx}(x-2)-\frac{1}{x-3}\times\frac{d}{dx}(x-3)-\frac{1}{x-4}\times\frac{d}{dx}(x-4)\right.$$
$$\left.-\frac{1}{x-5}\times\frac{d}{dx}(x-5)\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}+\frac{1}{x-4}+\frac{1}{x-5}\right)$$

Therefore,

$$\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}+\frac{1}{x-4}+\frac{1}{x-5}\right].$$

3. Find the derivative of the function $y=(\log x)^{\cos x}$ with respect to x.

Ans: The given function is $y=(\log x)^{\cos x}$.

First, taking logarithm both sides of the equation give

$$\log y=\cos x.\log(\log x).$$

Now, differentiating both sides of the equation with respect to x give

$$\frac{1}{y}\times\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$\Rightarrow \frac{1}{y}\times\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}\times\frac{d}{dx}(\log x)$, by applying the chain rule.

$$\Rightarrow \frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

Therefore,

$$\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\times\log(\log x)\right].$$

4. Determine the derivative of the function $y=x^x-2^{\sin x}$ with respect to x.

Ans: The given function is $y=x^x-2^{\sin x}$.

Now, let $x^x=u$ …… (1)

and $2^{\sin x}=v$. …… (2)

Therefore, $y=u-v$. …… (3)

Then differentiating the equation (3) with respect to x gives

$$\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx} \quad \text{…… (4)}$$

Now, taking logarithm both sides of the equation (1) give

$$\log(u)=\log\left(x^x\right)$$

$$\Rightarrow \log u=x\log x$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\left[\frac{d}{dx}(x)\times\log x+x\times\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx}=u\left[1\times \log x+x\times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{x}(\log x+1)$$

$$\Rightarrow \frac{du}{dx}=x^{x}(1+\log x) \qquad \ldots\ldots (5)$$

Now, taking logarithm both sides of the equation (2) give

$$\log\left(2^{\sin x}\right)=\log v$$

$$\Rightarrow \log v=\sin x\times \log 2.$$

Differentiating both sides of the equation with respect to x, give

$$\frac{1}{v}\times \frac{dv}{dx}=\log 2\times \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx}=v\log 2\cos x$$

$$\Rightarrow \frac{dv}{dx}=2^{\sin x}\cos x\log 2 \qquad \ldots\ldots (6)$$

Therefore, from the equation (4), (5) and (6) give

$$\frac{dy}{dx}=x^{x}(1+\log x)-2^{\sin x}\cos x\log 2.$$

5. Find the derivative of the function $y=(x+3)^{2}(x+4)^{3}(x+5)^{4}$ with respect to x

Ans: The given function is $y=(x+3)^{2}(x+4)^{3}(x+5)^{4}$.

First, taking logarithm both sides of the equation give

$$\log y=\log\left[(x+3)^{2}(x+4)^{3}(x+5)^{4}\right]$$

$$\Rightarrow \log y=2\log(x+3)+3\log(x+4)+\log 4(x+5)$$

Now, differentiating both sides of the equation with respect to x, give

$$\frac{1}{y}\times\frac{dy}{dy}=2\times\frac{1}{x-3}\times\frac{d}{dz}(x+3)+3\times\frac{1}{x+4}\times\frac{d}{dx}(x+4)+4\times\frac{1}{x+5}\times\frac{d}{dx}(x+5)$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow\frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\times\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$

$$\Rightarrow\frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4\times\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow\frac{dy}{dx}=(x+3)^2(x+4)^2(x+5)^2\text{-}\left[2(x^2+9x+20)+3(x^2+9x+15)+4(x^2+7x+12)\right]$$

Therefore,

$$\frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3(9x^2+70x+133).$$

6. Find the derivative of the function $y=\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$ with respect to x.

Ans: The given function is $y=\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$.

First, let $u=\left(x+\frac{1}{x}\right)^x$ and $v=x^{\left(1+\frac{1}{x}\right)}$

Therefore, $y=u+v$. …… (1)

Differentiating the equation (1) both sides with respect to x give

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}=\frac{dv}{dx} \quad \text{…... (2)}$$

Now, $u=\left(x+\frac{1}{x}\right)^x$

$$\Rightarrow \log u=\log\left(x+\frac{1}{x}\right)^{x}$$

$$\Rightarrow \log u=x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}\times\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx}=u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}\times\left(x+\frac{1}{x^{2}}\right)\right]$$

$$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^{x}\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^{x}\left[\log\left(x+\frac{1}{x}\right)+\frac{x^{2}+1}{x^{2}-1}\right]$$

$$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^{2}\left[\frac{x^{2}+1}{x^{2}-1}+\log\left(x+\frac{1}{x}\right)\right] \qquad \ldots\ldots (3)$$

Also, $v=x^{\left(x+\frac{1}{x}\right)}$

$$\Rightarrow \log v=\log\left[x^{x^{\left(x+\frac{1}{x}\right)}}\right]$$

$$\Rightarrow \log v=\left(x+\frac{1}{x}\right)\log x$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{v}\times\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right)\times\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right] \qquad \ldots\ldots\ (4)$$

Hence, from the equations (2), (3) and (4), give

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(x+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right).$$

7. Determine derivative of the function $y=(\log x)^x+x^{\log x}$ with respect to x.

Ans: The given function is $y=(\log x)^x+x^{\log x}$.

Then, let $u=(\log x)^x$ and $v=x^{\log x}$.

Therefore, $y=u+v$.

Differentiating both sides of the equation with respect to x gives

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots\ldots\ (1)$$

Now, $u=(\log x)^x$

$$\Rightarrow\log u=\log\left[(\log x)^x\right]$$

$$\Rightarrow\log u=x\log(\log x)$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\log x)+x\times\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{du}{dx}=u\left[1\times\log(\log x)+x\times\frac{1}{\log x}\times\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{x}{\log x}\times\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx}=(\log x)^{x}=\left[\frac{\log(\log x)\times\log x+1}{\log x}\right]$$

$$\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x\times\log(\log x)\right] \qquad \ldots\ldots. (2)$$

Again, $v=x^{\log x}$

$$\Rightarrow \log v=\log\left(x^{\log x}\right)$$

$$\Rightarrow \log v=\log x\log x=\left(\log x\right)^{2}$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{v}\times\frac{dx}{dx}=\frac{d}{dx}\left[(\log x)^{2}\right]$$

$$\Rightarrow \frac{1}{v}\times\frac{dx}{dx}=2(\log x)\times\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dv}{dx}=2x^{\log x}\,\frac{\log x}{x}$$

$$\Rightarrow \frac{dv}{dx}=2x^{\log x}\times\log x \qquad \ldots\ldots.. (3)$$

Hence, from the equations (1), (2), and (3), gives

$$\frac{dy}{dx}=(\log x)^{x+1}\left[1+\log x\times\log(\log x)\right]+2x^{\log x-1}\times\log x\,.$$

8. Find the derivative of the function $y=(\sin x)^2+\sin^{-1}\sqrt{x}$ with respect to x

Ans: The given function is $y=(\sin x)^x+\sin^{-1}\sqrt{x}$.

Now, let $u=(\sin x)^x$ and $v=\sin^{-1}\sqrt{x}$.

Therefore, $y=u+v$.

Then, differentiating both sides of the equation with respect to x gives

$$\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx} \qquad \text{........ (1)}$$

Now, $u=(\sin x)^x$

$$\Rightarrow \log u=x\log(\sin x)^x$$
$$\Rightarrow \log u=x\log(\sin x)$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\sin x)+x\times\frac{d}{dx}\left[\log(\sin x)\right]$$
$$\Rightarrow \frac{du}{dx}=u\left[1\times\log(\sin x)+x\times\frac{1}{\sin x}\times\frac{d}{dx}(\sin x)\right]$$
$$\Rightarrow \frac{du}{dx}=(\sin x)^x\left[\log(\sin x)+\frac{x}{\sin x}\times\cos x\right]$$
$$\Rightarrow \frac{du}{dx}=(\sin x)^x(x\cot x+\log\sin x) \qquad \text{....... (2)}$$

Again, $v=\sin^{-1}\sqrt{x}$.

Differentiating both sides of the equation with respect to x gives

$$\frac{dv}{dx}=\frac{1}{\sqrt{1-(\sqrt{x})^2}}\times\frac{d}{dx}(\sqrt{x})$$
$$\Rightarrow \frac{dv}{dx}=\frac{1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{2\sqrt{x-x^2}}$$

Hence, from the equations (1), (2) and (3), gives

$$\frac{dv}{dx}=(\sin x)^2(x\cot x+\log\sin x)+\frac{1}{2\sqrt{x-x^2}}.$$

9. Find the derivative of the function $y=x^{\sin x}+(\sin x)^{\cos x}$ with respect to x.

Ans: The given function is $y=x^{\sin x}+(\sin x)^{\cos x}$.

Then, let $u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$.

Therefore, $y=u+v$.

Differentiating both sides of the equation with respect to x gives

$$\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx} \qquad \ldots\ldots (1)$$

Now, $u=x^{\sin x}$

$$\Rightarrow \log u=x\log(x^{\sin x})$$

$$\Rightarrow \log u=\sin x\log x$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(\sin x)\times\log x+\sin x\times\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u=\left[\cos x\log x+\sin x\times\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{\sin x}\left[\cos x\log x+\frac{\sin x}{x}\right] \qquad \ldots\ldots. (2)$$

Again, $v=(\sin x)^{\cos x}$

$$\Rightarrow \log v=\log(\sin x)^{\cos x}$$
$$\Rightarrow \log v=\cos\log(\sin x)$$

Then, differentiating both sides of the equation with respect to x gives

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}(\cos x)\times\log(\sin x)+\cos x\times\frac{d}{dx}\left[\log(\sin x)\right]$$
$$\Rightarrow \frac{dv}{dx}=v\left[-\sin x\times\log(\sin x)+\cos x\times\frac{1}{\sin x}\times\frac{d}{dx}(\sin x)\right]$$
$$\Rightarrow \frac{du}{dx}=(\sin x)^{\cos x}\left[-\sin x\log\sin x+\cot x\cos x\right]$$
$$\Rightarrow \frac{dv}{dx}=(\sin x)^{\cos x}[\cos x\cot x+\sin x\log\sin x] \qquad \ldots\ldots (3)$$

Hence, from the equations (1), (2) and (3), gives

$$\frac{du}{dx}=x^{\sin x}\left(\cos x\log x+\frac{\sin x}{x}\right)+(\sin x)^{\cos x}[\cos x\cot x+\sin x\log\sin x].$$

10. Find the derivative function $y = x^{x\cos x} + \frac{x^2+1}{x^2-1}$ with respect to x.

Ans: The given function is $y=x^{x\cos x}+\frac{x^2+1}{x^2-1}$.

First, let $u=x^{x\cos x}$ and $v=\frac{x^2+1}{x^2-1}$.

Therefore, $y=u+v$.

Differentiating both sides of the equation with respect to x gives

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx} \qquad \ldots\ldots. (1)$$

Now, $u=x^{x\cos x}$.

Then, differentiating both sides of the equation with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\cos x\log x+x\times\frac{d}{dx}(\cos x)\times\log x+x\cos x\times\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\cos x\times\log x+x\times(-\sin x)\log x+x\cos x\times\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}(\cos x\log x-x\sin x\log x+\cos x) \qquad \ldots\ldots (2)$$

Again, $v=\frac{x^2+1}{x^2-1}$

$\Rightarrow \log v=\log(x^2+1)-\log(x^2-1)$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{v}=\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{2x(x^2-1)-2x(x^2+1)}{(x^2+1)(x^2-1)}\right]$$

$$\Rightarrow\frac{du}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{(x^2+1)(x^2-1)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{-4x}{(x^2-1)^2} \qquad \ldots\ldots.. (3)$$

Hence, from the equations (1), (2) and (3), give

$$\frac{dv}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right]-\frac{4x}{(x^2-1)^2}.$$

11. Find the derivative of the function $y=(x\cos x)^2+(x\sin x)^{\frac{1}{2}}$ with respect to x

Ans: The given function is $y=(x\cos x)^2+(x\sin x)^{\frac{1}{2}}$.

Then, let $u=(x\cos x)^2$ and $v=(x\sin x)^{\frac{1}{2}}$.

Therefore, $y=u+v$.

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \text{....... (1)}$$

Again, $u=(x\cos x)^2$

$$\Rightarrow \log u=\log(x\cos x)^2$$
$$\Rightarrow \log u=u\log(x\cos x)$$
$$\Rightarrow \log u=x[\log x+\log\cos x]$$
$$\Rightarrow \log u=x\log x+x\log\cos x$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x\log\cos x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\left\{\log x\times\frac{d}{dx}(x)+x\times\frac{d}{dx}(\log x)\right\}+\left\{\log\cos x\times\frac{d}{dx}(x)+x\times\frac{d}{dx}(\log\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[\left\{\log x\times 1+x\times\frac{1}{2}\right\}+\left\{\log\cos x\text{-}1+x\times\frac{1}{\cos x}\times\frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[\{\log x+1\}+\left\{\log\cos x\text{-}1+\frac{x}{\cos x}\times(\text{-}\sin x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[(\log x+1)+(\log\cos x\text{-}x\tan x)\right]$$

$$\Rightarrow \frac{du}{dx}=(x\cos x)^x\left[1\text{-}x\tan x+(\log x+\log\cos x)\right]$$

Therefore,

$$\frac{du}{dx}=(x\cos x)^x\left[1\text{-}x\tan x+(\log x(x\cos x)\right] \qquad \text{........ (2)}$$

Again, $v=(x\sin x)^{\frac{1}{x}}$

$$\Rightarrow \log v=\log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v=\frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}(\log x+\log\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}\log x+\frac{1}{x}\log\sin x$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{x}\log x\right)+\frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=$$

$$\left[\frac{1}{x}\log x\times\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\times\frac{d}{dx}(\log x)\right]+\left[\log(\sin x)\times\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\times\frac{d}{dx}\{(\log\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\frac{1}{x}\log x\times\left(-\frac{1}{x^2}\right)+\frac{1}{x}\times\frac{1}{x}\right]+\left[\log(\sin x)\times\left(-\frac{1}{x^2}\right)+\frac{1}{x}\times\frac{1}{\sin x}\times\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\frac{1}{x^2}(1-\log x)+\left[\frac{1-\log x}{x^2}+\frac{1}{x\sin x}\times\cos x\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\frac{1}{x^2}(x\sin x)^{\frac{1}{x}}+\left[\frac{1-\log x}{x^2}+\frac{-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x-\log(\sin x)+x\cot x}{x^2}\right]$$

Therefore,

$$\frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x)+x\cot x}{x^2}\right] \qquad \ldots\ldots. (3)$$

Hence, from the equations (1), (2) and (3), gives

$$\Rightarrow \frac{dy}{dx}=(x\cos x)^2\left[1-x\tan x+\log(x\cos x)\right]+(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x)+x\cot x}{x^2}\right].$$

12. Determine $\frac{dy}{dx}$ from the equation $x^y+y^x=1$.

Ans: The given function is $x^y+y^x=1$.

Then, let $x^y=u$ and $y^x=v$.

Therefore, $u+v=1$.

Differentiating both sides of the equation with respect to x gives

$$\frac{du}{dx}+\frac{dv}{dy}=0$$

Now, $u=x^y$ ……. (1)

$$\Rightarrow \log u=\log(x^y)$$
$$\Rightarrow \log u=y\log x$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\log x\frac{dy}{dx}+y\times\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\log x\frac{dy}{dx}+y\times\frac{1}{x}\right]$$

Therefore, $\frac{du}{dx}=x^y\left[\log x\frac{dy}{dx}+\frac{y}{x}\right]$ ………… (2)

Also, $v=y^x$

Taking logarithm both sides of the equation give

$$\Rightarrow \log v=\log(y^3)$$
$$\Rightarrow \log v=x\log y$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{v}\times\frac{dv}{dx}=\log y\times\frac{d}{dx}(x)+x\times\frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx}=v\left(\log y\times 1+x\times \frac{1}{y}\times \frac{dy}{dx}\right)$$

Therefore, $\frac{dv}{dx}=y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)$ ……... (3)

So, from the equation (1), (2) and (3), gives

$$x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow \left(x^2+\log x+xy^{y-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

Hence, $\frac{dy}{dx}=\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$.

13. Determine $\frac{dy}{dx}$ from the equation $y^x=x^y$.

Ans: The given equation is $y^x=x^y$.

Then, taking logarithm both sides of the equation give

$x\log y=y\log x$.

Differentiating both sides of the equation with respect to x gives

$$\log y\times \frac{d}{dx}(x)+x\times \frac{d}{dx}(\log y)=\log x\times \frac{d}{dx}(y)+y\times \frac{d}{dx}(\log x)$$

$$\Rightarrow \log y\times 1+x\times \frac{1}{y}\times \frac{dy}{dx}=\log x\times \frac{dy}{dx}+y\times \frac{1}{x}$$

$$\Rightarrow \log y+\frac{x}{y}\frac{dy}{dx}=\log x\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow \left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

Therefore, $\frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$.

14. Determine $\frac{dy}{dx}$ from the equation $(\cos x)^y=(\cos y)^x$.

Ans: The given equation is $(\cos x)^y=(\cos y)^x$.

Then, taking logarithm both sides of the equation give

$y\log\cos x=x\log\cos y$.

Now, differentiating both sides of the equation with respect to x gives

$$\log\cos x\times\frac{dy}{dx}+y\times\frac{d}{dx}(\log\cos x)=\log\cos y\times\frac{d}{dx}(x)+x\times\frac{d}{dx}(\log\cos y)$$

$$\Rightarrow\log\cos x\frac{dy}{dx}+\frac{y}{\cos x}\times(-\sin x)=\log\cos y+\frac{x}{\cos y}(-\sin y)\times\frac{dy}{dx}$$

$$\Rightarrow\log\cos x\frac{dy}{dx}-y\tan x=\log\cos y-x\tan y\frac{dy}{dx}$$

$$\Rightarrow(\log\cos x+x\tan y)\frac{dy}{dx}=y\tan x+\log\cos y$$

Therefore, $\frac{dy}{dx}=\frac{y\tan x+\log\cos y}{x\tan y+\log\cos x}$.

15. Determine $\frac{dy}{dx}$ from the equation $xy=e^{(x-y)}$.

Ans: The given equation is $xy=e^{(x-y)}$.

Then, taking logarithm both sides of the equation give

$$\log(xy)=\log(e^{x-y})$$
$$\Rightarrow \log x+\log y=(x-y)\log e$$
$$\Rightarrow \log x+\log y=(x-y)\times 1$$
$$\Rightarrow \log x+\log y=x-y$$

Now, differentiating both sides of the equation with respect to x gives

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$
$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{1}{x}$$
$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$

Therefore, $\frac{dy}{dx}=\frac{y(x-1)}{x(x+1)}$.

16. Determine the derivative of the following function f and hence evaluate $f'(1)$.

$$\mathbf{f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)}.$$

Ans: The given function is $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$.

By taking logarithm both sides of the equation give

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Now, differentiating both sides of the equation with respect to x gives

$$\frac{1}{f(x)}\times\frac{d}{dx}[f(x)]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow \frac{1}{f(x)}\times f'(x)=\frac{1}{1+x}\times\frac{1}{dx}(1+x)+\frac{1}{1+x^2}\times\frac{d}{dx}\log(1+x^2)+\frac{1}{1+x^4}\times\frac{d}{dx}\log(1+x^4)$$

$$+\frac{1}{1+x^8}\times\frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}\times 2x+\frac{1}{1+x^4}\times 4x^3+\frac{1}{1+x^8}\times 8x^7\right]$$

Therefore,

$$f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

So,

$$f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times 1}{1+1^2}+\frac{4\times 1^3}{1+1^4}+\frac{8\times 1^7}{1+1^8}\right]$$

$$=2\times 2\times 2\times 2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\times\left(\frac{1+2+4+8}{2}\right)$$

$$=16\times\frac{15}{2}=120$$

Hence, $f'(1)=120$.

17. Differentiate the function $y=(x^2-5x+8)(x^3+7x+9)$ in three ways as described below. Also, verify whether all the answers are the same.

(a) By using product rules.

Ans: The given function is $y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$.

Now, let consider $u=(x^2-5x+8)$ and $v=(x^3+7x+9)$

Therefore, $y=uv$.

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dv}.v+u.\frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(x^2-5x+8).(x^3+7x+9)+(x^2-5x+8).\frac{d}{dx}(x^3+7x+9)$$

$$\Rightarrow \frac{dy}{dx}=(2x-5)(x^3+7x+9).(x^2-5x+8)(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx}=2x(x^3+7x+9)-5(x^2-5x+8)+x^2(3x^2+7)-5x(3x^2+7)-8(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx}=(2x^4+14x^2+18x)-5x^3-35x-45+(3x^4+7x^2)-15x^3-35x+24x^2+56$$

Hence, $\frac{dy}{dx}=5x^4-20x^3+45x^2+52x+11$.

(b) By expanding the factors as a polynomial.

Ans: The given function is

$y=(x^2-5x+8)(x^3+7x+9)$.

Then, calculating the product, gives

$$y=x^2(x^3+7x+9)-5x^4(x^3+7x+9)+8(x^3+7x+9)$$

$$\Rightarrow y=x^5+7x^3+9x^2-5x^3-26x^2+11x+72$$

Now, differentiating both sides of the equation with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}(x^5+7x^3+9x^2-5x^3-26x^2+11x+72)$$

$$=\frac{d}{dx}(x^5)-5\frac{d}{dx}(x^4)+15\frac{d}{dx}(x^3)-26\frac{d}{dx}(x^3)+11\frac{d}{dx}(x)+\frac{d}{dx}(72)$$

$$=5x^4-5\times 4x^3+15\times 3x^2-26\times 2x+11\times 1+0$$

Hence, $\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$.

(c) By using a logarithmic function.

Ans: The given function is

$y=(x^2-5x+8)(x^3+7x+9)$.

Now, taking logarithm both sides of the function give

$\log y=\log(x^2-5x+8)+\log(x^3+7x+9)$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log(x^2-5x+8)+\frac{d}{dx}\log(x^3+7x+9)$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}.\frac{d}{dx}(x^2-5x+8)+\frac{1}{x^3+7x+9}.\frac{d}{dx}(x^3+7x+9)$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\times(2x-5)+\frac{1}{x^3+7x+9}\times(3x^2+7)\right]$$

$$\Rightarrow\frac{dy}{dx}=(x^2-5x+8)(x^3+7x+9)\left[\frac{2x-5}{x^3-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\Rightarrow\frac{dy}{dx}=(x^2-5x+8)(x^3+7x+9)\left[\frac{(2x-5)(x^3+7x+9)+(3x^2+7)(x^2-5x+8)}{(x^3-5x+8)+(x^3+7x+9)}\right]$$

$$\Rightarrow\frac{dy}{dx}=2x(x^3+7x+9x^2)-5(x^3+7x+9)+3x^2(x^2-5x+8)+7(x^3+7x+9)$$

$$\Rightarrow\frac{dy}{dx}=(2x^4+14x^2+18x)+(5x^3-35x+45)+(3x^4-15x^3+24x^2)+(7x^2+35x+56)$$

Therefore, $\frac{dy}{dx}=5x^2-20x^3+45x^2-52x+11$.

Hence, comparing the above three results, it is concluded that the derivative $\frac{dy}{dx}$ are the same for all methods.

18. Let u, v, and w are functions of x, then prove that $\frac{d}{dx}(u.v.w)=\frac{du}{dx}v.w+u\frac{du}{dx}.w+u.v\frac{dw}{dx}$ in two ways. First by using repeated application of product rule and second by applying logarithmic differentiation.

Ans: Let the function $y=u.v.w=u.(v.w)$.

Then applying the product rule of derivatives, give

$$\frac{dy}{dx}=\frac{du}{dx}.(v.w)+u.\frac{d}{dx}(v.w)$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}v.w+u\left[\frac{dv}{dx}.w+v.\frac{dv}{dx}\right] \qquad \text{(Using the product rule again)}$$

Thus,

$$\frac{dy}{dx}=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v\frac{dw}{dx}.$$

Now, take logarithm both sides of the function $y=u.v.w$.

Then, we have $\log y=\log u+\log v+\log w$.

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{y}.\frac{dy}{dx}=\frac{d}{dx}(\log u)+\frac{d}{dx}(\log v)+\frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx}=y\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=u.v.w\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

Hence, $\frac{dy}{dx}=\frac{du}{dx}v.w+u\frac{du}{dx}.w+u.v\frac{dw}{dx}$.

Exercise 5.6

1. Determine $\frac{dy}{dx}$ from the equations $x=2at^2$, $y=at^4$, without eliminating the parameter t, where a,b are constants.

Ans: The given equations are

$x=2at^2$ …… (1)

and $y=at^4$ …… (2)

Then, differentiating both sides of the equation (1) with respect to t gives

$$\frac{dx}{dt}=\frac{d}{dt}(2at^2)=2a\times\frac{d}{dt}(t^2)=2a\times 2t=4at. \quad \text{…… (3)}$$

Also, differentiating both sides of the equation (2) with respect to t gives

$$\frac{dy}{dt}=\frac{d}{dt}(at^4)=a\times\frac{d}{dt}(t^4)=a\times 4\times t^3=4at^3 \quad \text{…… (4)}$$

Now, dividing the equations (4) by (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{4at^3}{4at}=t^2.$$

Hence, $\frac{dy}{dx}=t^2$.

2. Determine $\frac{dy}{dx}$ from the equations $x=a\cos\theta$, $y=b\cos\theta$, without eliminating the parameter θ, where a,b are constants.

Ans: The given equations are

$x=a\cos\theta$ …… (1)

and $y=b\cos\theta$ …… (2)

Then, differentiating both sides of the equation (1) with respect to θ gives

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(a\cos\theta)=a(-\sin\theta)=-a\sin\theta . \qquad \ldots\ldots (3)$$

Also, differentiating both sides of the equation (1) with respect to θ gives

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(b\cos\theta)=b(-\sin\theta)=-b\sin\theta \qquad \ldots\ldots (4)$$

Therefore, dividing the equation (4) by (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{-b\sin\theta}{-a\sin\theta}=\frac{b}{a}.$$

Hence, $\frac{dy}{dx}=\frac{b}{a}$.

3. Determine $\frac{dy}{dx}$ from the equations $x=\sin t,\ y=\cos 2t$ without eliminating the parameter t.

Ans: The given equations are

$$x=\sin t \qquad \ldots\ldots (1)$$

and $y=\cos 2t \qquad \ldots\ldots (2)$

Then, differentiating both sides of the equation (1) with respect to t gives

$$\frac{dx}{dt}=\frac{d}{dt}(\sin t)=\cos t . \qquad \ldots\ldots (3)$$

Also, differentiating both sides of the equation (2) with respect to t gives

$$\frac{dy}{dt}=\frac{d}{dt}(\cos 2t)=\sin 2t\times\frac{d}{dt}(2t)=-2\sin 2t \qquad \ldots\ldots (4)$$

Therefore, by dividing the equation (4) by (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-2\sin 2t}{\cos t}=\frac{-2\times 2\sin t\cos t}{\cos t}=-4\sin t$$

Hence, $\frac{dy}{dx}=-4\sin t$.

4. Determine $\frac{dy}{dx}$ from the equations $x=4t,\ y=\frac{4}{t}$ without eliminating the parameter t.

Ans: The given equations are

$x=4t$ …… (1)

and $y=\frac{4}{t}$ …… (2)

Now, differentiating both sides of the equation (1) with respect to t gives

$\frac{dx}{dt}=\frac{d}{dt}(4t)=4$. …… (3)

Also, differentiating both sides of the equation (2) with respect to t gives

$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{4}{t}\right)=4\times\frac{d}{dt}\left(\frac{1}{t}\right)=4\times\left(\frac{-1}{t^2}\right)=\frac{-4}{t^2}$ …… (4)

Therefore, dividing the equation (4) by (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-4}{t^2}\right)}{4}=\frac{-1}{t^2}.$$

Hence, $\frac{dy}{dx}=-\frac{1}{t^2}$.

5. Determine $\frac{dy}{dx}$ from the equations $x=\cos\theta-\cos2\theta, y=\sin\theta-\sin2\theta$, without eliminating the parameter θ.

Ans: The given equations are

$x=\cos\theta-\cos2\theta$ ...... (1)

and $y=\sin\theta-\sin2\theta$ ...... (2)

Then, differentiating both sides of the equation (1) with respect to θ gives

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(\cos\theta-\cos2\theta)=\frac{d}{d\theta}(\cos\theta)-\frac{d}{d\theta}(\cos2\theta)=-\sin\theta(-2\sin2\theta)=2\sin2\theta-\sin\theta \quad \text{... (3)}$$

Also, differentiating both sides of the equation (2) with respect to θ gives

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta-\sin2\theta)=\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\sin2\theta)=\cos\theta-2\cos\theta \quad \text{..... (4)}$$

Therefore, dividing the equation (4) by (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{\cos\theta-2\cos\theta}{2\sin2\theta-\sin\theta}.$$

Hence, $\frac{dy}{dx}=\frac{\cos\theta-2\cos\theta}{2\sin2\theta-\sin\theta}$.

6. Determine $\frac{dy}{dx}$ from the equations $x=a(\theta-\sin\theta), y=a(1+\cos\theta)$, without eliminating the parameter θ, where a,b are constants.

Ans: The given equations are

$x=a(\theta-\sin\theta)$ ...... (1)

and $y=a(1+\cos\theta)$ ...... (2)

Then, differentiating both sides of the equation (1) with respect to θ gives

$$\frac{dx}{d\theta}=a\left[\frac{d}{d\theta}(\theta)-\frac{d}{d\theta}(\sin\theta)\right]=a(1-\cos\theta) \qquad \text{...... (3)}$$

Also, differentiating both sides of the equation (2) with respect to θ gives

$$\frac{dy}{d\theta}=a\left[\frac{d}{d\theta}(1)+\frac{d}{d\theta}(\cos\theta)\right]=a\left[0+(-\sin\theta)\right]=-a\sin\theta \qquad \text{...... (4)}$$

Therefore, by dividing the equation (4) by (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=\frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=-\cos\frac{\theta}{2}.$$

Hence, $\frac{dy}{dx}=-\cos\frac{\theta}{2}$.

7. Determine $\frac{dy}{dx}$ from the equations $x=-\frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$, without eliminating the parameter t.

Ans: The given equations are,

$$x=-\frac{\sin^3 t}{\sqrt{\cos 2t}} \qquad \text{...... (1)}$$

and $$y=\frac{\cos^3 t}{\sqrt{\cos 2t}} \qquad \text{...... (2)}$$

Then, differentiating both sides of the equation (1) with respect to t gives

$$\frac{dx}{dt}=\frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}\cdot\frac{d}{dt}(\sin^3 t)-\sin^3 t\times\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos2t}\times3\sin^2t\times\frac{d}{dt}(\sin t)-\sin^3t\times\frac{1}{2\sqrt{\cos2t}}\times\frac{d}{dt}(\cos2t)}{\cos2t}$$

$$=\frac{3\sqrt{\cos2t}\times\sin^2t\cos t-\frac{\sin^3t}{2\sqrt{\cos2t}}\times(-2\sin2t)}{\cos2t\sqrt{\cos2t}}$$

Also, differentiating both sides of the equation (2) with respect to t gives

$$\frac{dx}{dt}=\frac{3\cos2t\sin^2t\cos t+\sin^2t\sin2t}{\cos2t\sqrt{\cos2t}}. \qquad \text{...... (3)}$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[\frac{\cos^3t}{\sqrt{\cos2t}}\right]$$

$$=\frac{\sqrt{\cos2t}\times\frac{d}{dt}(\cos^3t)-\cos^3t\times\frac{d}{dt}(\sqrt{\cos2t})}{\cos2t}$$

$$=\frac{3\sqrt{\cos2t}\cos^2t(-\sin t)-\cos^3t\times\frac{1}{2(\sqrt{\cos2t})}\times\frac{d}{dt}(\cos2t)}{\cos2t}$$

$$\frac{dy}{dt}=\frac{-3\cos2t\times\cos^2t\times\sin t+\cos^3t\sin2t}{\cos2t\times\sqrt{\cos2t}} \qquad \text{...... (4)}$$

Thus, dividing the equation (4) by the equation (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-3\cos2t\times\cos^2t\times\sin t+\cos^3t\sin2t}{3\cos2t\cos t\sin^2t+\sin^3t\sin2t}$$

$$=\frac{\sin t\cos t\left[-3\cos2t\times\cos t+2\cos^3t\right]}{\sin t\cos t\left[3\cos2t\sin t+2\sin^3t\right]}$$

$$=\frac{\left[-3(2\cos^2t-1)\cos t+2\cos^3t\right]}{\left[3(1-2\sin^3t)\sin t+2\sin^3t\right]} \qquad \begin{bmatrix}\cos2t=(2\cos^2t-1)\\ \cos2t=(1-2\sin^2t)\end{bmatrix}$$

$$=\frac{-4\cos^3 t+3\cos t}{3\sin t-4\sin^3 t} \qquad \left[\begin{matrix}\cos 3t=4\cos^3 t-3\cos t\\ \sin 3t=3\sin t-4\sin^2 t\end{matrix}\right]$$

$$=\frac{-\cos 3t}{\sin 3t}$$

Hence, $\frac{dy}{dx}=-\cot 3t$.

8. Determine $\frac{dy}{dx}$ from the parametric equations

$x=a\left(\cos t+\log\tan\frac{t}{2}\right)$, $y=a\sin t$, without eliminating the parameter t.

Ans: The given equations are

$$x=a\left(\cos t+\log\tan\frac{t}{2}\right) \qquad \ldots\ldots (1)$$

and $y=a\sin t$ …… (2)

Then, differentiating both sides of the equation (1) with respect to t gives

$$\frac{dx}{dt}=a\times\left[\frac{d}{d\theta}(\cos t)+\frac{d}{d\theta}(\log\tan\frac{t}{2})\right]$$

$$=a\left[-\sin t+\frac{1}{\tan\frac{t}{2}}\times\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\cot\frac{t}{2}\times\sec^2\frac{t}{2}\times\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$=a\left(-\sin t+\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\times\frac{1}{\cos^2\frac{t}{2}}\times\frac{1}{2}\right)$$

$$=a\left(-\sin t+\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$=a\left(-\sin t+\frac{1}{\sin t}\right)$$

$$=a\left(\frac{-\sin^2 t+1}{\sin t}\right)$$

Therefore, $\frac{dx}{dt}=a\frac{\cos^2 t}{\sin t}$ …… (3)

Also, differentiating both sides of the equation (2) with respect to t gives

$\frac{dy}{dt}=a\frac{d}{dt}(\sin t)=a\cos t$ …… (4)

Thus, dividing the equation (4) by the equation (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)}=\frac{\sin t}{\cos t}=\tan t.$$

Hence, $\frac{dy}{dx}=\tan t$.

9. Determine $\frac{dy}{dx}$ from the parametric equations $x=a\sec\theta,\ y=b\tan\theta$, without eliminating the parameter θ, where a,b are constants.

Ans: The given equations are

$x=a\sec$ …… (1)

and $y=b\tan\theta$ …… (2)

Then, differentiating both sides of the equation (1) with respect to θ gives

$$\frac{dx}{d\theta}=a\times\frac{d}{d\theta}(\sec\theta)=a\sec\theta\tan\theta \qquad \ldots\ldots (3)$$

Also, differentiating both sides of the equation (2) with respect to θ gives

$$\frac{dy}{d\theta}=b\times\frac{d}{d\theta}(\tan\theta)=b\sec^2\theta \qquad \ldots\ldots (4)$$

Thus, dividing the equation (4) by the equation (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}=\frac{b}{a}\sec\theta\tan\theta=-\frac{b\cos\theta}{a\cos\theta\sin\theta}=\frac{b}{a}\times\frac{1}{\sin\theta}=\frac{b}{a}\operatorname{cosec}\theta$$

Hence, $\frac{dy}{dx}=\frac{b}{a}\operatorname{cosec}\theta$.

10. Determine $\frac{dy}{dx}$ from the parametric equations

$x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$, without eliminating the parameter θ, where a,b are constants.

Ans: The given equations are

$$x=a(\cos\theta+\theta\sin\theta) \qquad \ldots\ldots (1)$$

and $y=a(\sin\theta-\theta\cos\theta)$ $\qquad \ldots\ldots (2)$

Then, differentiating both sides of the equation (1) with respect to θ gives

$$\frac{dx}{d\theta}=a\left[\frac{d}{d\theta}\cos\theta+\frac{d}{d\theta}(\theta\sin\theta)\right]=a\left[-\sin\theta+\theta\frac{d}{d\theta}(\sin\theta)+\sin\theta\frac{d}{d\theta}(\theta)\right]$$

$$=a\left[-\sin\theta+\theta\cos\theta+\sin\theta\right].$$

Therefore, $\frac{dx}{d\theta}=a\theta\cos\theta$ $\qquad \ldots\ldots (3)$

Also, differentiating both sides of the equation (2) with respect to θ gives

$$\frac{dy}{d\theta}=a\left[\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\theta\cos\theta)\right]=a\left[\cos\theta-\left\{\theta\frac{d}{d\theta}(\cos\theta)+\cos\theta\times\frac{d}{d\theta}(\theta)\right\}\right]$$

$$\Rightarrow\frac{dy}{d\theta}=a[\cos\theta+\theta\sin\theta-\cos\theta]$$

Therefore, $\frac{dy}{d\theta}=a\theta\sin\theta$ …… (4)

Thus, dividing the equation (4) by the equation (3) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{a\theta\sin\theta}{a\theta\sin\theta}=\tan\theta.$$

Hence, $\frac{dy}{dx}=\tan\theta$.

11. Prove that $\frac{dy}{dx}=-\frac{y}{x}$, where it is provided that $x=\sqrt{a^{\sin^{-1}t}}, y=\sqrt{a^{\cos^{-1}t}}$.

Ans: The given parametric equations are $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$.

Now, $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$$\Rightarrow x=\left(a^{\sin^{-1}t}\right) \text{ and } y=\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}$$

$$\Rightarrow x=a^{\frac{1}{2}\sin^{-1t}} \text{ and } y=a^{\frac{1}{2}\cos^{-1}t}$$

Therefore, first consider $x=a^{\frac{1}{2}\sin^{-1}t}$.

Take logarithms on both sides of the equation.

Then, we have

$$\log x=\frac{1}{2}\sin^{-1}t\log a.$$

Then, differentiating both sides of the equation with respect to t gives

$$\frac{1}{x}\times\frac{dx}{dt}=\frac{1}{2}\log a\times\frac{d}{dt}(\sin^{-1}t)$$

$$\Rightarrow\frac{dx}{dt}=\frac{x}{2}\log a\times\frac{1}{\sqrt{1-t^2}}$$

Therefore, $\frac{dx}{dt}=\frac{x\log a}{2\sqrt{1-t^2}}$. …… (1)

Again, consider the equation $y=a^{\frac{1}{2}\cos^{-1}t}$.

Take logarithm both sides of the equation.

Then, we have

$$\log y=\frac{1}{2}\cos^{-1}t\log a$$

Differentiating both sides of the equation with respect to t gives

$$\frac{1}{y}\times\frac{dx}{dt}=\frac{1}{2}\log a\times\frac{d}{dt}(\cos^{-1}t)$$

$$\Rightarrow\frac{dx}{dt}=\frac{y\log a}{2}\times\left(\frac{1}{\sqrt{1-t^2}}\right)$$

Therefore, $\frac{dx}{dt}=\frac{-y\log a}{2\sqrt{1-t^2}}$. …… (2)

Thus, dividing the equation (2) by the equation (1) gives

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)}=\frac{y}{x}.$$

Hence, $\frac{dy}{dx}=\frac{y}{x}$.

Exercise 5.7

1. Determine the second order derivative for the following function $y=x^2+3x+2$.

Ans: The given function is $y=x^2+3x+2$.

Then, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}(x^2)+\frac{d}{dx}(3x)+\frac{d}{dx}(2)=2x+3+0=2x+3$$

That is,

$$\frac{dy}{dx}=2x+3.$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(2x+3)=\frac{d}{dx}(2x)+\frac{d}{dx}(3)=2+0=2$$

Hence, $\frac{d^2y}{dx^2}=2$.

2. Determine the second order derivative for the following function $y = x^{20}$.

Ans: The given function is $y=x^{20}$.

Then, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}(x^{20})=20x^{19}$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(20x^{19})=20\frac{d}{dx}(x^{19})=20(19)x^{18}=380x^{18}.$$

Hence, $\frac{d^2y}{dx^2}=380x^{18}$.

3. Determine the second order derivative for the following function $y = x \cdot \cos x$.

Ans: The given function is $y=x.\cos x$.

Then, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}(x.\cos x)=\cos x.\frac{d}{dx}(x)+x\frac{d}{dx}(\cos x)=\cos x.1+x(-\sin x)=\cos x-x\sin x$$

That is, $\frac{dy}{dx}=\cos x-x\sin x$.

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(\cos x-x\sin x)=\frac{d}{dx}(\cos x)-\frac{d}{dx}(x\sin x)$$
$$=-\sin x-[\sin x.\frac{d}{dx}(x)+x.\frac{d}{dx}(\sin x)$$
$$=-\sin x-(\sin x+\cos x)$$

Hence, $\frac{d^2y}{dx^2}=-(x\cos x+2\sin x)$.

4. Determine the second order derivative for the following function $y = \log x$

Ans: The given function is $y=\log x$.

Then, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}(\log x)=\frac{1}{x}$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$$

Hence, $\frac{d^2y}{dx^2}=-\frac{1}{x^2}$.

5. Determine the second order derivative for the following function $y=x^3\log x$

Ans: The given function is $y=x^3\log x$.

Then, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}\left[x^3\log x\right]=\log x.\frac{d}{dx}(x^3)+x^3\frac{d}{dx}(\log x)=\log x.3x^2+x^3.\frac{1}{x}=\log x.3x^2+x^2$$

That is, $\frac{dy}{dx}=x^2(1+3\log x)$.

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(x^2(1+3\log x))$$

$$=(1+3\log x).\frac{d}{dx}(x^2)+x^2\frac{d}{dx}(1+3\log x)$$

$$=(1+3\log x).2x+x^3.\frac{3}{x}$$

$$=2x+6\log x+3x$$

$$=5x+6x\log x$$

Hence, $\frac{d^2y}{dx^2}=x(5+6\log x)$.

6. Determine the second order derivative for the following function. $y = e^x\sin 5x$

Ans: The given function is $y=e^x\sin5x$.

Then, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}\left[e^x\sin5x\right]=\sin x\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(\sin5x)$$

$$\Rightarrow\frac{dy}{dx}=\sin5x.e^x+e^x.\cos5x.\frac{d}{dx}(5x)$$

That is, $\frac{dy}{dx}=e^x(\sin5x+5\cos5x)$.

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^x(\sin5x+5\cos5x)\right]$$

$$=(\sin5x+5\cos5x).\frac{d}{dx}(e^x)+e^x.\frac{d}{dx}(\sin5x+5\cos5x)$$

$$=(\sin5x+5\cos5x)(e^x)+e^x\left[\cos5x.\frac{d}{dx}(5x)+5(-\sin5x).\frac{d}{dx}(5x)\right]$$

$$=e^x(\sin5x+5\cos5x)+e^x(5\cos5x-25\sin5x)$$

$$=e^x(10\cos5x-24\sin5x).$$

Hence, $\frac{d^2y}{dx^2}=2e^x(5\text{cox}5x-12\sin5x)$.

7. Determine the second order derivative for the following function. $y=e^{6x}\cos3x$.

Ans: The given function is $y=e^{6x}\cos3x$.

Then, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}(e^{6x}\cos3x)=\cos3x\times\frac{d}{dx}(e^{6x})+e^{6x}\times\frac{d}{dx}(\cos3x)$$

$$\Rightarrow\frac{dy}{dx}=\cos3x\times e^{6x}\times\frac{d}{dx}(6x)+e^{6x}\times(-\sin3x)\times\frac{d}{dx}(3x)$$

Therefore,

$$\frac{dy}{dx}=6e^{6x}\cos 3x-3e^{6x}\sin 3x \qquad \ldots\ldots (1)$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(6e^{6x}\cos 3x-3e^{6x}\sin 3x)=6\times\frac{d}{dx}(e^{6x}\cos 3x)-3\times\frac{d}{dx}(e^{6x}\sin 3x)$$

$$=6\times\left[6e^{6x}\cos 3x-3e^{6x}\sin 3x\right]-3\times\left[\sin 3x\times\frac{d}{dx}(e^{6x})+e^{6x}\times\frac{d}{dx}(\sin 3x)\right] \text{ [using (1)]}$$

$$=36e^{6x}\cos 3x-18e^{6x}\sin 3x-3\left[\sin 3x\times e^{6x}\times 6+e^{6x}\times\cos 3x\text{-}3\right]$$

$$=36e^{6x}\cos 3x-18e^{6x}\sin 3x-18e^{6x}\sin 3x-9e^{6x}\cos 3x$$

Hence, $\frac{d^2y}{dx^2}=9e^{6x}(3\cos 3x-4\sin 3x)$.

8. Determine the second order derivative for the following function. $y = \tan^{-1}x$.

Ans: The given function is $y=\tan^{-1}x$.

Then, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}\tan^{-1}x=\frac{1}{1-x^2}$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{d}{dx}(1+x^2)=(-1)\times(1+x^2)\times\frac{d}{dx}(1+x^2)$$

$$=\frac{1}{(1+x^2)}\times 2x$$

Hence, $\frac{d^2y}{dx^2}=\frac{-2x}{(1+x^2)}$.

9. Determine the second order derivative for the following function.

$y=\log(\log x)$.

Ans: The given function is $y=\log(\log x)$.

Now, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{1}{\log x}\times\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{\log x}=(x\log x)^{-1}$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[(x\log x)^{-1}\right]=(-1)\times(x\log x)^{-2}\frac{d}{dx}(x\log x)$$

$$=\frac{-1}{(x\log x)^2}\times\left[\log x\times\frac{d}{dx}(x)+x\times\frac{d}{dx}(\log x)\right]$$

$$=\frac{-1}{(x\log x)^2}\times\left[\log x\times 1x\times\frac{1}{x}\right]$$

Hence, $\frac{d^2y}{dx^2}=\frac{-(1+\log x)}{(x\log x)^2}$.

10. Determine the second order derivative for the following function. $y=\sin(\log x)$.

Ans: The given function is $y=\sin(\log x)$.

Now, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}[\sin(\log x)]=\cos(\log x)\times\frac{d}{dx}(\log x)=\frac{\cos(\log x)}{x}$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$=\frac{x\left[\cos(\log x)\right]-\cos(\log x)\times\frac{d}{dx}(x)}{x^2}$$

$$=\frac{x\left[-\sin(\log x)\times\frac{d}{dx}(\log x)\right]-\cos(\log x)\times 1}{x^2}$$

$$=\frac{-x\sin(\log x)\times\frac{1}{x}-\cos(\log x)}{x^2}$$

Hence, $\frac{d^2y}{dx^2}=\frac{\left[-\sin(\log x)+(\log x)\right]}{x^2}$.

11. Prove that $\frac{d^2y}{dx^2}+y=0$ when $y=5\cos x-3\sin x$.

Ans: The given equation is $y=5\cos x-3\sin x$.

Then, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}(5\cos x)-\frac{d}{dx}(3\sin x)=5\frac{d}{dx}(\cos x)-3\frac{d}{dx}(\sin x)=5(-\sin x)-3\cos x$$

Therefore, $\frac{dy}{dx}=-(5\sin x+3\cos x)$.

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}[-(5\sin x+3\cos x)]$$

$$=-\left[5\times\frac{d}{dx}(\sin x)+3\times\frac{d}{dx}(\cos x)\right]$$

$$=[5\cos x+3(-\sin x)]$$

$$=-y$$

That is, $\frac{d^2y}{dx^2}=-y$.

Hence, $\frac{d^2y}{dx^2}+y=0$.

12. Determine $\frac{d^2y}{dx^2}$ containing the terms of y only when $y=\cos^{-1}x$.

Ans: The given function is $y=\cos^{-1}x$.

Now, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}=-(1-x^2)^{\frac{-1}{2}}$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-(1-x^2)^{\frac{-1}{2}}\right]$$

$$=\left(\frac{-1}{2}\right)\times(1-x^2)^{\frac{-3}{2}}\times\frac{d}{dx}(1-x^2)$$

$$=\frac{1}{\sqrt[2]{(1-x^2)^3}}\times(-2x)$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{(1-x^2)^3}} \qquad \ldots\ldots (1)$$

Now, $y=\cos^{-1}x\Rightarrow x=\cos y$.

Therefore, substituting $x=\cos y$ into equation (1), gives

$$\frac{d^2x}{dy^2}=\frac{-\cos y}{\sqrt{(1-\cos^2y)^3}}$$

$$=\frac{-\cos y}{\sin^3 y}$$

$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

Hence, $\frac{d^2y}{dx^2}=\cot y\times\operatorname{cosec}^2 y$.

13. Prove that $x^2y_2+xy_1+y=0$ when $y=3\cos(\log x)+4\sin(\log x)$.

Ans: The given equations are $y=3\cos(\log x)+4\sin(\log x)$ …… (1)

and $x^2y_2+xy_1+y=0$ …… (2)

Then, differentiating both sides of the equation (1) with respect to x gives

$$y_1=3\times\frac{d}{dx}[\cos(\log x)]+4\times\frac{d}{dx}[\sin(\log x)]$$

$$=3\times\left[-\sin(\log x)\times\frac{d}{dx}(\log x)\right]+4\times\left[\cos(\log x)\times\frac{d}{dx}(\log x)\right]$$

$$y_1=\frac{-3\sin(\log x)}{x}+\frac{4\cos(\log x)}{x}=\frac{4\cos(\log x)-3\sin(\log x)}{x}$$

Again, differentiating both sides with respect to x gives

$$y_2=\frac{d}{dx}\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)$$

$$=\frac{x\{4\cos(\log x)-3\sin(\log x)\}-\{4\cos(\log x)-3\sin(\log x)\}}{x^2}$$

$$=\frac{x\left[4\{\cos(\log x)\}-\{3\sin(\log x)\}\right]-\{4\cos(\log x)-3\sin(\log x)\}\times 1}{x^2}$$

$$=\frac{x\left[-4\sin(\log x)(\log x)'-3\cos(\log x)(\log x)\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=\frac{x\left[-4\sin(\log x)\frac{1}{x}-3\cos(\log x)\frac{1}{x}\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$

$$=\frac{-4\sin(\log x)-3\cos(\log x)-4\cos(\log x)+3\sin(\log x)}{x^2}$$

Therefore, $y_2=\frac{-\sin(\log x)-7\cos(\log x)}{x^2}$.

Now, substituting the derivatives y_1 , y_2 and y into the LHS of the equation (2) gives

$$x^2y_2+xy_1+y$$
$$=x^2\left(\frac{-\sin(\log x)-7\cos(\log x)}{x^2}\right)+x\left(\frac{4\cos(\log x)-3\sin(\log x)}{x^2}\right)+3\cos(\log x)+4\sin(\log x)$$
$$=-\sin(\log x)-7\cos(\log x)+4\cos(\log x)-3\sin(\log x)+4\sin(\log x)$$
$$=0$$

Hence, it has been proved that $x^2y_2+xy_1+y=0$.

14. Prove that $\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny=0$ when $y=Ae^{mx}+Be^{nx}$.

Ans: The given equations are $y=Ae^{mx}+Be^{nx}$ …… (1)

and $\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny=0$ ……. (2)

Then, differentiating both sides of the equation (1) with respect to x gives

$$\frac{dy}{dx}=A.\frac{d}{dx}(e^{mx})+B.\frac{d}{dx}(e^{mx})=A.e^{mx}.\frac{d}{dx}(mx)+B.e^{nx}.\frac{d}{dx}(nx)=Ame^{mx}+Bne^{nx}$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(Ame^{mx}+Bne^{nx})=Am.\frac{d}{dx}(e^{mx})+Bn.\frac{d}{dx}(e^{nx})$$

$=Am.e^{mx}.\frac{d}{dx}(mx)+Bn.e^{nx}.\frac{d}{dx}(nx)$

Therefore, $\frac{d^2y}{dx^2}=Am^2e^{mx}+Bn^2e^{nx}$.

Thus, substituting the derivatives y_1, y_2 and y into the LHS of the equation (2) gives

$\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny$

$=Am^2ex^{mx}+Bn^2e^{nx}-(m+n).(Ame^{mx}+Bne^{nx})+mn(Ae^{mx}+Be^{nx})$

$=Am^2ex^{mx}+Bn^2e^{nx}-Amex^{mx}+Bmne^{nx}+Amne^{mx}+Bn^2e^{nx}+Amne^{mx}+Bmne^{nx}$

$=0$

Thus, it has been proved that $\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny=0$.

15. Prove that $\frac{d^2y}{dx^2}=49y$ when $y=500e^{7x}+600e^{-7x}$.

Ans: The given equation is $y=500e^{7x}+600e^{-7x}$. …… (1)

Then, differentiating both sides with respect to x gives

$\frac{dy}{dx}=500\times(e^{7x})+600\times\frac{d}{dx}(-7x)$

$=500\times e^{7x}\times\frac{d}{dx}(7x)+600\times e^{-7x}\times\frac{d}{dx}(-7x)$

$=3500e^{7x}-4200e^{-7x}$

Again, differentiating both sides with respect to x gives

$\frac{d^2y}{dx^2}=3500\times\frac{d}{dx}(e^{7x})-4200\times\frac{d}{dx}(e^{-7x})$

$=3500\times e^{7x}\times\frac{d}{dx}(7x)-4200\times e^{-7x}\times\frac{d}{dx}(-7x)$

$=7\times3500\times e^{7x}+7\times4200\times e^{-7x}$

$=49\times500e^{7x}+49\times600e^{-7x}$

$=49(500e^{7x}+600e^{-7x})$

$=49y$, using the equation (1).

Thus, it has been proved that $\frac{d^2y}{dx^2}=49y$.

16. Prove that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$ when $e^y(x+1)=1$.

Ans: The given equation is $e^y(x+1)=1$.

Now, $e^y(x+1)=1\Rightarrow e^y=\frac{1}{x+1}$.

So, taking logarithm bth sides of the equation gives

$$y=\log\frac{1}{(x+1)}$$

Therefore, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)=(x+1)\times\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$

That is,

$$\frac{dy}{dx}=\frac{-1}{x+1} \qquad \text{...... (1)}$$

Again, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}=\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$

$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$, using the equation (1).

Thus, it is proved that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$.

17. Prove that $(x^2+1)^2y_2+2x(x^2+1)y_1=2$ when $y=(\tan^{-1}x)^2$.

Ans: The given equations are $y=(\tan^{-1}x)^2$.

Then, differentiating both sides with respect to x gives

$$y_1=2\tan^{-1}x\frac{d}{dx}(\tan^{-1}x)$$
$$\Rightarrow y_1=2\tan^{-1}x\times\frac{1}{1+x^2}$$
$$\Rightarrow (1+x^2)y_1=2\tan^{-1}x$$

Again, differentiating both sides with respect to x gives

$$(1+x^2)y_2+2xy_1=2\left(\frac{1}{1+x^2}\right)$$
$$\Rightarrow (1+x^2)y_2+2x(1+x^2)y_1=2$$

Thus, it has been proved that $(1+x^2)y_2+2x(1+x^2)y_1=2$.

Exercise 5.8

1. Determine whether the function $f(x)=x^2+2x-8,\ x\in[-4,2]$ satisfies Rolle's Theorem.

Ans: The given function is $f(x)=x^2+2x-8$.

Note that, the $f\left(x\right)$ is a polynomial function and so it is continuous in the closed interval $[-4,2]$ and differentiable in the open interval $(-4,2)$.

Now, $f(-4)=(-4)^2+2x(-4)-8=16-8-8=0$ and

$f(2)=(2)^2+2\times2-8=4+4-8=0$.

Therefore, $f(-4)=f(2)=0$.

Thus, we observed that the value of at $x=-4,2$ are the same.

Now, according to the Rolle’s Theorem, if a function f is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a)=f(b)$, then for any $c\in(a,b)$, $f'(c)=0$

So, differentiating the function $f(x)$ both sides give

$f'(x)=2x+2$.

Substituting $x=c$ into the above equation gives

$$f'(c)=0$$
$$\Rightarrow 2c+2=0$$
$$\Rightarrow c=-1$$
$$\Rightarrow c=-1\in(-4,2)$$

Thus, the function $f(x)=x^2+2x-8$ satisfies the Rolle’s Theorem.

2. Verify whether the following functions satisfy Rolle's Theorem. Also state whether the converse of Rolle's Theorem is applicable for these functions.

(i) $f(x)=[x]$ for $x\in[5,9]$

Ans: According to the Rolle’s Theorem, for $f:[a,b]\to R$, if

a) f is continuous on $[a,b]$

b) f is continuous on (a,b)

c) $f(a)=f(b)$,

then, there exists any $c\in(a,b)$ such that $f'(c)=0$.

Therefore, the functions that do not satisfy any of the three conditions given in the Rolle's Theorem, do not satisfy the Rolle's Theorem.

The given function is $f(x)=[x]$ for $x\in[5,9]$.

Note that, $f(x)$ is not continuous at every integral point.

In fact, $f(x)$ is not continuous at the points $x=5$ and $x=9$.

Thus, $f(x)$ is not continuous in the closed interval $[5,9]$.

Also, $f(5)=[5]=5$ and $f(9)=[9]=9$.

So, $f(5)\neq f(9)$.

For the differentiability of f in $(5,9)$, let n be an integer in the open interval $(5,9)$.

Then, the left-hand-derivative of f at $x=n$ is

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty \text{ and}$$

the right-hand-derivative of f at $x=n$ is

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}\frac{0}{h}=0.$$

Thus, it has been noticed that $\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}\neq\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}$.

So, the function f is not differentiable at $x=n$.

Therefore, the function f is not differentiable in the open interval $(5,9)$.

Hence, from the above discuss it can be concluded that since the function does not satisfy the conditions of Rolle's Theorem, so the function $f(x)=[x]$ for $x\in[5,9]$ does not satisfy Rolle's Theorem.

(ii) $f(x)=[x]$ for $x\in[-2,2]$

Ans: According to the Rolle's Theorem, for $f:[a,b]\to R$, if

a) f is continuous on $[a,b]$
b) f is continuous on (a,b)
c) $f(a)=f(b)$,

then, there exists any $c\in(a,b)$ such that $f'(c)=0$.

Therefore, the functions that do not satisfy any of the three conditions given in the Rolle's Theorem, do not satisfy the Rolle's Theorem.

$f(x)=[x]$ for $x\in[-2,2]$

Note that, $f(x)$ is not continuous at all integral points.

In fact, the function $f(x)$ is not continuous at the end points $x=-2$ and $x=2$.

Therefore, the function $f=(x)$ is not continuous in the closed interval $[-2,2]$.

Also, the function values at the endpoints, that is $f(-2)=[2]=-2$ and $f(2)=[2]=2$.

So, $f\left(-2\right)\neq f\left(2\right)$.

Now, for the differentiability of in $(-2,2)$, let n be an integer in the open interval $(-2,2)$.

So, the left-hand-derivative of the function f at $x=n$ is

$\lim\limits_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim\limits_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim\limits_{h\to 0^-}\frac{n-1-n}{h}=\lim\limits_{h\to 0^-}\frac{-1}{h}=\infty$ and the right-hand-derivative of the function f at $x=n$ is

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0.$$

Thus, it has been noticed that $\lim\limits_{h\to 0^-}\frac{f(n+h)-f(n)}{h}\neq\lim\limits_{h\to 0^+}\frac{f(n+h)-f(n)}{h}$.

So, the function f is not differentiable at the point $x=n$.

Therefore, the function f is not continuous in the open interval $(-2,2)$.

Hence, from the above discuss it can be concluded that since the function does not satisfy the conditions of Rolle’s Theorem, so the function $f(x)=[x]$ for $x\in[-2,2]$ does not satisfy Rolle's Theorem.

(iii) $f(x)=x^2-1$ for $x\in[1,2]$

Ans: According to the Rolle’s Theorem, for $f:[a,b]\to R$, if

a) f is continuous on $[a,b]$
b) f is continuous on (a,b)
c) $f(a)=f(b)$,

then, there exists any $c\in(a,b)$ such that $f'(c)=0$.

Therefore, the functions that do not satisfy any of the three conditions given in the Rolle’s Theorem, do not satisfy the Rolle’s Theorem.

The given function is $f(x)=x^2-1$ for $x\in[1,2]$.

Note that, f is a polynomial function, and so it is continuous in closed interval $[1,2]$ and differentiable on the open interval $(1,2)$.

Also, $f(1)=(1)^2-1=0$ and

$f(2)=(2)^2-1=3$.

Therefore, $f(1)\neq f(2)$.

Hence, from the above discussion it can be concluded that since the function does not satisfy one of the conditions of Rolle’s Theorem, so the function $f(x)=x^2-1$ for $x\in[1,2]$ does not satisfy the Rolle’s Theorem.

3. Let $f:[-5,5]\to R$ is a differentiable function and the function $f(x)\neq 0$ for any x, then prove that. $f(-5)\neq f(5)$.

Ans: The given function $f:[-5,5] \uparrow R$ is differentiable.

Now, remember that every differentiable function is a continuous function.

Therefore, f is continuous on $[-5,5]$ as well as on $(-5,5)$.

So, according to the Mean Value Theorem, there exists any $c \in (-5,5)$ for which

$$f'(c)=\frac{f(5)-f(-5)}{5-(-5)}.$$

Now, it is provided that the function $f(x) \neq 0$ for any x.

Therefore, $f'(c) \neq 0$.

$\Rightarrow 10f'(c) \neq 0$
$\Rightarrow f(5)-f(-5) \neq 0$
$\Rightarrow f(5) \neq f(-5)$

Thus, it has been proved that $f(5) \neq f(-5)$.

4. Determine whether the function $f(x)=x^2-4x-3$ satisfies the Mean Value Theorem in the interval [1,4].

Ans: The given function is $f(x)=x^2-4x-3$.

Note that, the function f is a polynomial function and so, it is continuous in the closed interval $[1,4]$ and differentiable in the open interval $(1,4)$.

Differentiating the function with respect to x gives

$f'(x)=2x-4$.

Also, $f(1)=(1)^2-4\times1-3=6$,

$f(4)=(4)^2-4\times4-3=-3$ and

$$\frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}=\frac{-3-(-6)}{3}=\frac{3}{3}=1.$$

Therefore, according to the Mean Value Theorem there exists a point $c \in (1,4)$ for which $f'(c)=1$.

So, $f'(c)=1$

$\Rightarrow 2c-4=1$

$\Rightarrow c=\frac{5}{2} \in (1,4)$.

Hence, the function $f(x)=x^2-4x-3$ satisfies the Mean Value Theorem in the interval $[1,4]$.

5. Determine whether the function $f(x)=x^2-5x^2-3x$ satisfies the Mean Value theorem in the interval [1,3] . Also, evaluate all $c \in (1,3)$ such that $f'(c)=0$.

Ans: The given function is $f(x)=x^2-5x^2-3x$.

Note that, the function f is a polynomial function, and so it is continuous in the closed interval [1,3] and differentiable in the open interval $(1,3)$.

Now, differentiating the function $f(x)$ with respect to x gives

$f'(x)=3x^2-10x-3$.

Also, the function values at the endpoints are

$f(1)=1^2-5\times1^2-3\times1=-7,$

$f(3)=3^2-5\times3^2-3\times3=27$

Therefore,

$$\frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}=\frac{-27-(-7)}{3-1}=-10.$$

So, according to the Mean Value Theorem there is at least a point $c \in (1,3)$ for which $f'(c)=-10$.

Then,

$f'(c)=-10$

$\Rightarrow 3c^2-10c-3=10$

$\Rightarrow 3c^2-10c+7=0$

$\Rightarrow 3c^2-3c-7c+7=0$

$\Rightarrow 3c(c-1)-7(c-1)=0$

$\Rightarrow (c-1)(3c-7)=0$

$\Rightarrow c=1,\frac{7}{3}$

Therefore, $c=\frac{7}{3}\in(1,3)$.

Hence, the function $f(x)=x^2-5x^2-3x$ satisfies the Mean Value Theorem in the interval $[1,3]$ and $c=\frac{7}{3}\in(1,3)$ is the only point such that $f'(c)=0$.

6. Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Ans:

(i) $f(x)=[x]$ for $x\in[5,9]$

Ans: According to the Rolle's Theorem, for $f:[a,b]\to R$, if

(i) f is continuous on $[a,b]$

(ii) f is continuous on (a,b)

(iii) $f(a)=f(b)$,

then, there exists any $c\in(a,b)$ such that $f'(c)=0$.

Therefore, the functions that do not satisfy any of the three conditions given in the Rolle's Theorem, do not satisfy the Rolle's Theorem.

Now, note that here $f(x)$ is not continuous at all the integral points.

In fact, the function $f(x)$ is not continuous at the endpoints $x=5$ and $x=9$.

Therefore, the function $f(x)$ is not continuous in the closed interval $[5,9]$.

For the differentiability of f in $(5,9)$, let n be an integer in the open interval $(5,9)$.

Then, the left-hand-derivative of f at $x=n$ is

$$\lim_{h \to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h \to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h \to 0^-}\frac{n-1-n}{h}=\lim_{h \to 0^-}\frac{-1}{h}=\infty \text{ and}$$

the right-hand-derivative of f at $x=n$ is

$$\lim_{h \to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h \to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h \to 0^+}\frac{n-n}{h}=\lim_{h \to 0^+}\frac{0}{h}=0.$$

Thus, it has been noticed that $\lim_{h \to 0^-}\frac{f(n+h)-f(n)}{h} \neq \lim_{h \to 0^+}\frac{f(n+h)-f(n)}{h}$.

So, the function f is not differentiable at $x=n$.

From the above discussions we can conclude that f does not satisfy all the conditions of the Mean Value Theorem.

Hence, the function $f(x)=[x]$ does not satisfy the Mean Value Theorem for $x \in [5,9]$.

(ii) $f(x)=[x]$ for $x \in [-2,2]$

Ans: According to the Rolle's Theorem, for $f:[a,b] \to R$, if

(i) f is continuous on $[a,b]$

(ii) f is continuous on (a,b)

(iii) $f(a)=f(b)$,

then, there exists any $c \in (a,b)$ such that $f'(c)=0$.

Therefore, the functions that do not satisfy any of the three conditions given in the Rolle's Theorem, do not satisfy the Rolle's Theorem.

Note that, $f(x)$ is not continuous at all integral points.

In fact, the function $f(x)$ is not continuous at the end points $x=-2$ and $x=2$.

Therefore, the function $f=(x)$ is not continuous in the closed interval $[-2,2]$.

Now, for the differentiability of in $(-2,2)$, let n be an integer in the open interval $(-2,2)$.

So, the left-hand-derivative of the function f at $x=n$ is

$\lim\limits_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim\limits_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim\limits_{h\to 0^-}\frac{n-1-n}{h}=\lim\limits_{h\to 0^-}\frac{-1}{h}=\infty$ and the right-hand-derivative of the function f at $x=n$ is

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0.$$

Thus, it has been noticed that $\lim\limits_{h\to 0^-}\frac{f(n+h)-f(n)}{h}\neq\lim\limits_{h\to 0^+}\frac{f(n+h)-f(n)}{h}$.

So, the function f is not differentiable at the point $x=n$.

From the above discussions we can conclude that f does not satisfy all the conditions of the Mean Value Theorem.

Thus, the function $f(x)=[x]$ does not satisfy the Mean Value Theorem for $x\in[-2,2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,2]$

Ans: According to the Rolle's Theorem, for $f:[a,b]\to R$, if

(i) f is continuous on $[a,b]$

(ii) f is continuous on (a,b)

(iii) $f(a)=f(b)$,

then, there exists any $c \in (a,b)$ such that $f'(c)=0$.

Therefore, the functions that do not satisfy any of the three conditions given in the Rolle's Theorem, do not satisfy the Rolle's Theorem.

The given function is $f(x)=x^2-1$ for $x \in [1,2]$.

Note that, f is a polynomial function, and so it is continuous in closed interval $[1,2]$ and differentiable on the open interval $(1,2)$.

Thus, the function f satisfies all the conditions of the Mean Value Theorem.

Hence, Mean Value Theorem holds for the function $f(x)=x^2-1$ for $x \in [1,2]$.

Now follow the procedure for proving it.

Here, $f(1)=(1)^2-1=0,$ and

$f(2)=(2)^2-1=3$.

Therefore, $\frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{3-0}{1}=3$.

Then, differentiating the function $f\left(x\right)$ with respect to x gives

$f'(x)=2x$

Thus, at $x=c$,

$$f'(c)=3$$
$$\Rightarrow 2c=3$$
$$\Rightarrow c=\frac{3}{2}=1.5 \in [1,2]$$

Hence, it is proved that the Mean Value Theorem holds for the function $f(x)=x^2-1$ for $x \in [1,2]$.

Miscellaneous Exercise

1. Differentiate the function $y=(3x^2-9x+5)^9$ with respect to x.

Ans: The given function is $y=(3x^2-9x+5)^9$.

Differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}=(3x^2-9x+5)^9$$
$$=9(3x^2-9x+5)^8\times\frac{d}{dx}(3x^2-9x+5)$$
$$=9(3x^2-9x+5)^8\times(6x-9x)$$
$$=9(3x^2-9x+5)^8\times3(2x-3)$$
$$=27(3x^2-9x+5)^8(2x-3)$$

2. Differentiate the function $y=\sin^3x+\cos^6x$ with respect to x.

Ans: The given function is $y=\sin^3x+\cos^6x$.

Differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}=(\sin^3x)+\frac{d}{dx}(\cos^6x)$$
$$=3\sin^2x\times\frac{d}{dx}(\sin x)+6\cos^5x\frac{d}{dx}(\cos x)$$
$$=3\sin^2x\times\cos x+6\cos^5x(-\sin x)$$
$$=3\sin x\cos x(\sin x-2\cos^4x)$$

3. Differentiate the function $y=(5x)^{3\cos2x}$ with respect to x.

Ans: The given function is $y=(5x)^{3\cos2x}$.

First, take the logarithm of both sides of the function.

$\log y=3\cos2x\log5x$.

Then, differentiating both sides with respect to x gives

$$\frac{1}{y}\frac{dy}{dx}=3\left[\log 5.\frac{d}{dx}(\cos 2x)+\cos 2x.\frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow \frac{dy}{dx}=3y\left[\log 5x(-\sin 2x).\frac{d}{dx}(2x)+\cos 2x.\frac{1}{5x}.\frac{d}{dx}(5x)\right]$$

$$\Rightarrow \frac{dy}{dx}=3y\left[-2\sin 2x\log 5x+\frac{\cos 2x}{x}\right]$$

$$\Rightarrow \frac{dy}{dx}=3y\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

Hence, $\frac{dy}{dx}=(5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$.

4. Differentiate the function $y=\sin^{-1}\left(x\sqrt{x}\right)$, $0\le x\le 1$ with respect to x.

Ans: The given function is $y=\sin^{-1}\left(x\sqrt{x}\right)$.

Then, differentiating both sides with respect to x by using the chain rule gives

$$\frac{dy}{dx}=\frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^3}}\times\frac{d}{dx}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-x^3}}.\frac{d}{dx}\left(x^{\frac{1}{3}}\right)$$

$$=\frac{1}{\sqrt{1-x^3}}\times\frac{3}{2}.x^{\frac{1}{2}}$$

$$=\frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

Hence, $\frac{dy}{dx}=\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$.

5. **Differentiate the function** $y=\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2+7}}$, **$-2<x<2$ with respect to x.**

Ans: The given function is $y=\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2+7}}$.

Then, differentiating both sides with respect to x using the quotient rule gives

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right)-\left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$=\frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}.\frac{d}{dx}\left(\frac{x}{2}\right)\right]-\left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}.\frac{d}{dx}(2x+7)}{2x+7}$$

$$=\frac{\sqrt{2x+7}\frac{-1}{\sqrt{4-x^2}}-\left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$=\frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)}-\frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

Hence, $\dfrac{dy}{dx}=-\left[\dfrac{1}{\sqrt{4-x^2}\sqrt{2x+7}}+\dfrac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$.

6. **Differentiate the function** $y=\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$, **$0<x<2$ with respect to x.**

Ans: The given function is $y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ .... (1)

Now, $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)-(1-\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin x\frac{x}{2}\cos\frac{x}{2}}$$

Therefore,

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\cot\frac{x}{2}.$$ ...... (2)

So, from the equations (1) and (2) we obtain,

$$y=\cot^{-1}\left(\cot^{\frac{x}{2}}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

Now, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

Hence, $\frac{dy}{dx}=\frac{1}{2}$.

7. Differentiate the function $y=(\log x)^{\log x}$, $x>1$ with respect to x.

Ans: The given function is $y=(\log x)^{\log x}$.

First take logarithm both sides of the function.

$\log y=\log x\times\log(\log x)$.

Now, differentiating both sides with respect to x gives

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[\log x\times\log(\log x)\right]$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=\log(\log x)\times\frac{d}{dx}(\log x)+\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{dy}{dx}=y\left[\log(\log x)\times\frac{1}{x}+\log x\times\frac{1}{\log x}\times\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{1}{x}\log(\log x)+\frac{1}{x}\right]$$

Hence, $\frac{dy}{dx}=(\log x)^{\log x}\left[\frac{1}{x}+\frac{\log(\log x)}{x}\right]$.

8. Differentiate the function $y=\cos(a\cos x+b\sin x)$, where a and b are any constants.

Ans: The given function is $y=\cos(a\cos x+b\sin x)$.

Now, differentiating both sides with respect to x by using the chain rule of derivatives gives

$$\frac{dy}{dx}=\frac{d}{dx}\cos(a\cos x+b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x) \times \frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x) \times [a(-\sin x) + b\cos x]$$

Hence, $\frac{dy}{dx} = (a\sin x + b\cos x) \times \sin(a\cos x + b\sin x)$.

9. Differentiate the function $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$ with respect to x

Ans: The given function is $y = (\sin x - \cos x)^{(\sin x - \cos x)}$.

First take logarithm both sides of the function.

$$\log y = \log\left[(\sin x - \cos x)^{(\sin x - \cos x)}\right]$$

$$\Rightarrow \log y = (\sin x - \cos x) \times \log(\sin x - \cos x)$$

Now, differentiating both sides with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[(\sin x - \cos x) \times \log(\sin x - \cos x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x) \times \frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x) \times \frac{d}{dx}\log(\sin x - \cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x) \times (\cos x + \sin x) + (\sin x - \cos x) \times \frac{1}{(\sin x - \cos x)} \times \frac{d}{dx}(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\left[\left(\cos x + \sin x\right) \times \log(\sin x - \cos x) + (\cos x + \sin x)\right]$$

Hnece, the required derivative is

$$\frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)\left[1 + \log(\sin x - \cos x)\right].$$

10. Differentiate the function $y = x^x + x^a + a^x + a^a$ with respect to x, where for $a > 0$ and $x > 0$ are any fixed constants.

Ans: The given function is $y=x^x+x^a+a^x+a^a$.

Now, assume that $x^x=u$, $x^a=v$, $a^x=w$ and $a^a=s$

Therefore, we have $y=u+v+w+s$.

So, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}+\frac{ds}{dx} \qquad \text{...... (1)}$$

Also, $u=x^x$

$$\Rightarrow \log u=\log x^x$$
$$\Rightarrow \log u=x\log x$$

Then, differentiating both sides with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\log x.\frac{d}{dx}(x)+x.\frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{du}{dx}=u\left[\log x.1+x.\frac{1}{x}\right]$$

Thus, $\frac{du}{dx}=x^x[\log x+1]=x^x(1+\log x)$ .... (2)

Again, $v=x^a$

Then, differentiating both sides with respect to x gives

$$\frac{du}{dx}=\frac{d}{dx}(x^a)$$

$$\Rightarrow \frac{dv}{dx}=ax^{a-1} \qquad \text{...... (3)}$$

Also, $w=a^x$

$$\Rightarrow \log w=\log a^x$$
$$\Rightarrow \log w=x\log a$$

So, differentiating both sides with respect to x gives

$$\frac{1}{w}.\frac{dw}{dx}=\log a.\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx}=w\log a$$

$$\Rightarrow \frac{dw}{dx}=a^x\log a \qquad \text{.........} (4)$$

and

$s=a^a$

Then differentiating both sides with respect to x gives

$$\frac{ds}{dx}=0, \qquad \text{......}(5)$$

as a is constant, and so a^a is also a constant.

Now, from the equations (1), (2), (3), (4), and (5) we have

$$\frac{dy}{dx}=x^2(1+\log x)+ax^{a-1}+a^x\log a+0$$

Hence, $\frac{dy}{dx}=x^2(1+\log x)+ax^{a-1}+a^x\log a$.

11. Differentiate the function $y=x^{x^2-3}+(x-3)^{x^2}$, for $x>3$ with respect to x.

Ans: The given function is $y=x^{x^2-3}+(x-3)^{x^2}$.

Now suppose that $u=x^{x^2-3}$ and $v=(x-3)^{x^2}$

Therefore, $y=u+v$.

Now, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \text{.......} (1)$$

Also, $u=x^{x^2-3}$.

Take logarithm both sides of the equation.

$\Rightarrow \log u=\log(x^{x^2-3})$

$\Rightarrow \log u=(x^2-3)\log x$

Differentiating both sides with respect to x gives

$$\frac{1}{u}\frac{du}{dx}=\log x.\frac{d}{dx}(x^2-3)+(x^2-3).\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\log x.2x+(x^2-3).\frac{1}{x}$$

Hence, $\dfrac{du}{dx}=x^{x^2-3}.\left[\dfrac{x^2-3}{x}+2\times\log x\right]$. …… (2)

Again, $v=(x-3)^{x^2}$.

Take logarithm both sides of the equation.

$\Rightarrow \log v=\log(x-3)^{x^2}$

$\Rightarrow \log v=x^2\log(x-3)$

Now, differentiating both sides with respect to x gives

$$\frac{1}{u}.\frac{dv}{dx}=\log(x-3).\frac{d}{dx}(x^2)+x^2.\frac{d}{dx}[\log(x-3)]$$

$$\Rightarrow \frac{1}{u}.\frac{dv}{dx}=\log(x-3).2x+x^2.\frac{1}{x-3}.\frac{d}{dx}(x-3)$$

$$\Rightarrow \frac{dv}{dx}=v.\left[2x\log(x-3)+\frac{x^2}{x-3}.1\right]$$

Hence, $\dfrac{dv}{dx}=(x-3)^{x^2}\left[\dfrac{x^2}{x-3}+2x\log(x-3)\right]$ …… (3)

Thus, from the equations (1), (2) and (3) we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right].$$

12. Determine $\frac{dy}{dx}$ from the parametric equations

$y=12(1-\cos t), x=10(t-\sin t), \frac{\pi}{2}<t<\frac{\pi}{2}$, without eliminating the parameter t.

Ans: The given equations are $y=12(1-\cos t)$, …… (1)

and $x=10(t-\sin t)$ …… (2)

Then differentiating the equations (1) and (2) with respect to x gives

$$\frac{dx}{dt}=\frac{d}{dt}\left[10(t-\sin t)\right]=10\times\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[12(1-\cos t)\right]=12\times\frac{d}{dt}(1-\cos t)=12\times\left[0-(-\sin t)\right]=12\sin t$$

Therefore, by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$ we have,

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10(1-\cos t)}=\frac{12\times 2\sin\frac{t}{2}\times\cos\frac{t}{2}}{10\times 2\sin^2\frac{t}{2}}$$

Hence, $\frac{dy}{dx}=\frac{6}{5}\cot\frac{t}{2}$.

13. Determine $\frac{dy}{dx}$ from the equation $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$, $-1\le x\le 1$.

Ans: The given equation is $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$.

Differentiating both sides of the equation with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}(\sin^{-1}x)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1\left(\sqrt{1-x^2}\right)}}\times\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2\times\sqrt{1-x^2}}(-2)$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

Hence, $\frac{dy}{dx}=0$.

14. Prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$ when $x\sqrt{1+y}+y\sqrt{1+x}=0$, for $-1<x<1$.

Ans: The given equation is

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$$\Rightarrow x\sqrt{1+y}=y\sqrt{1+x}$$

Now, squaring both sides of the equation, gives

$$x^2(1+y)=y^2(1+x)$$

$$\Rightarrow x^2+x^2y=y^2+xy^2$$

$$\Rightarrow x^2-y^2=xy^2-x^2y$$

$$\Rightarrow x^2-y^2=xy(y-x)$$

$$\Rightarrow (x+y)(x-y)=xy(y-x)$$

$$\therefore x+y=-xy$$

$$\Rightarrow (1+x)y=x$$

$$\Rightarrow y=\frac{-x}{(1+x)}$$

Now, differentiating both sides of the equation with respect to x gives

$$\frac{dy}{dx}=\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}=\frac{(1+x)-x}{(1+x)^2}$$

Hence, $\frac{dy}{dx}=\frac{1}{(1+x)^2}$.

15. prove that $\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent from a and b, when $(x-a)^2+(y-b)^2=c^2$, for some constant $c>0$.

Ans: The given equation is $(x-a)^2+(y-b)^2=c^2$.

Differentiating both sides of the equation with respect to x gives

$$\frac{d}{dx}=[(x-a)^2]+\frac{d}{dx}[(y-b)^2]=\frac{d}{dx}(c^2)$$

$$\Rightarrow 2(x-a).\frac{d}{dx}(x-a)+2(y-b).\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a).1+2(y-b).\frac{dy}{dx}=0$$

Hence, $\frac{dy}{dx}=\frac{-(x-a)}{y-b}$ .... (1)

Again, differentiating both sides of the equation with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$=-\frac{\left[(y-b).\frac{d}{dx}(x-a)-(x-a).\frac{d}{dx}(y-b)\right]}{(y-b)^2}$$

$$=-\left[\frac{(y-b)-(x-a).\frac{dy}{dx}}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a).\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)^2+(x+a)^2}{(y-b)^2}\right]$$

Therefore,

$$\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}}=\frac{\left[\left(1+\frac{(x-a)^2}{(y-b)^2}\right)\right]^{\frac{3}{2}}}{-\left[\frac{(y-a)^2+(x-a)^2}{(y-a)^3}\right]}=\frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-a)^2+(x-a)^2}{(y-a)^3}\right]}=\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$\Rightarrow\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}}=\frac{\frac{c^2}{(y-b)^3}}{\frac{c^2}{(y-b)^3}}=-c$, is a constant, and is independent of a and b.

16. Prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$, $\cos a\neq\pm1$ from the equation $\cos y=x\cos(a+y)$.

Ans: The given equation is $\cos y=x\cos(a+y)$.

Then, differentiating both sides of the equation with respect to x gives

$$\frac{d}{dx}[\cos y]=\frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx}=\cos(a+y).\frac{d}{dx}(x)+x.\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx}=\cos(a+y)+x.[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y)-\sin y]\frac{dy}{dx}=\cos(a+y) \qquad \text{........ (1)}$$

Since $\cos y=x\cos(a+y)\Rightarrow x=\frac{\cos y}{\cos(a+y)}$, so from the equation (1) gives

$$\left[\frac{\cos y}{\cos(a+y)}.\sin(a+y)-\sin y\right]\frac{dy}{dx}=\cos(a+y)$$

$$\Rightarrow [\cos y.\sin(a+y)-\sin y.\cos(a-y)].\frac{dy}{dx}=\cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx}=\cos^2(a+b)$$

Hence, it has been proved that $\frac{dy}{dx}=\frac{\cos^2(a+b)}{\sin a}$.

17. Determine $\frac{d^2y}{dx^2}$ from the parametric equations $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$, without cancelling the parameter t.

Ans: The given equations are

$x=a(\cos t+t\sin t)$ ...... (1)

and $y=a(\sin t-t\cos t)$ ...... (2)

Then, differentiating both sides of the equation (1) with respect to x gives

$$\frac{dx}{dt}=a\left[-\sin t+\sin t.\frac{d}{dx}(t)+t.\frac{d}{dt}(\sin t)\right]$$

$$=a\left[-\sin t+\sin t+\cos t\right]=at\cos t$$

Again, differentiating both sides of the equation (2) with respect to x gives

$$\frac{dy}{dt}=a.\frac{d}{dt}(\sin t-t\cos t)$$

$$a\left[\cos t-\left\{\cos t.\frac{d}{dt}(t)+t.\frac{d}{dt}(\cos t)\right\}\right]$$

$$a[\cos t-\{\cos t-t\sin t\}]=at\sin t$$

Therefore,

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dx}\right)}=\frac{at\sin t}{at\cos t}=\tan t$$

Now, differentiating both sides with respect to x gives

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(\tan t)=\sec^2 t.\frac{dt}{dx}=\sec^2 t.\frac{1}{at\cos t}$$

Hence, $\frac{d^2y}{dx^2}=\frac{\sec^3 t}{at},\ 0<t<\frac{\pi}{2}$ $\quad\left[\because \frac{dx}{dt}=at\cos t\Rightarrow\frac{dt}{dx}=\frac{1}{at\cos t}\right]$

18. Prove that $f''(x)$ exists for all real values of x when $f(x)=|x|^3$ and hence evaluate it.

Ans: Remember that, $|x|=\begin{Bmatrix}x,\text{if } x\geq 0\\ -x,\text{if } x<0\end{Bmatrix}$

Therefore, if $x\geq 0$, then $f(x)=|x|^3=x^3$.

Then, $f'(x)=3x^2$.

Differentiating both sides with respect to x gives

$f''(x)=6x$.

Now, if $x<0$, then $f(x)=|x|^3=(-x^3)=x^3$.

So, $f'(x)=3x^2$.

Therefore, differentiating both sides with respect to x gives

$f''(x)=6x$.

Hence, for $f(x)=|x|^3$, $f''(x)$ exists for all real values of x and is provided as

$$f''(x)=\begin{Bmatrix} 6x, \text{if } x \geq 0 \\ -6x, \text{if } x<0 \end{Bmatrix}.$$

19. Prove that $\frac{d}{dx}(x^n)=nx^{x-1}$ for all positive integers n by applying the principle of Mathematical Induction.

Ans: Let $P(n):\frac{d}{dx}(x^n)=nx^{x-1}$ for all positive integers n.

Now, when $n=1$,

$P(1):\frac{d}{dx}(x)=1=1\times x^{1-1}$

Therefore, $P(n)$ is true when $n=1$.

Assume that the statement $P(k)$ is true for some positive integer k.

So, $P(k):\frac{d}{dx}(x^k)=kx^{k-1}$,

Now, if it can be shown that the statement $p(k+1)$ is also true, then our proof will be complete.

Let $\frac{d}{dx}(x^{k+1})=\frac{d}{dx}(x\times x^{k})$

$=x^{k}\times\frac{d}{dx}(x)+x\times\frac{d}{dx}(x^{k})$

$=x^{k}\times 1+x\times k\times x^{k-1}$

$=x^{k}+kx^{k}$

$=(k+1)\times x^{k}$

$=(k+1)\times x^{(k+1)-1}$

So, $P(k+1)$ is true if $P(k)$ is true.

Hence, by the principal of mathematical induction, it has been proved that the statement $P(n)$ is true for every positive integer n.

That is, $\frac{d}{dx}(x^{n})=nx^{x-1}$ for all positive integers n.

20. Derive the sum formula for cosine from the sum formula of sine $\sin(A+B)=\sin A\cos B+\cos A\sin B$, by using differentiation.

Ans: The given sum formula is $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

Now, differentiating both sides with respect to x gives

$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B)\times\frac{d}{dx}(A+B)=\cos B\times\frac{d}{dx}(\sin A)+\sin A\times\frac{d}{dx}(\cos B)+\sin B\times\frac{d}{dx}(\cos A)$$
$$+\cos A\times\frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B)\times\frac{d}{dx}(A+B)=\cos B\times\cos A\frac{d}{dx}+\sin A(-\sin B)\frac{dB}{dx}+\sin B(-\sin A)\times\frac{dA}{dx}$$
$$+\cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B)\times\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$

Hence the required sum formula for cosines is $\cos(A+B)=\cos A\cos B-\sin A\sin B$.

21. Prove that $\mathbf{\frac{dy}{dx}=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}}$ **when** $\mathbf{y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}}$**.**

Ans: The given function is $y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Evaluate the determinant.

$y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$.

Now, differentiating both sides with respect to x gives

$$\frac{dy}{dx}=\frac{d}{dx}[(mc-nb)f(x)]-\frac{d}{dx}[(lc-na)g(x)]+\frac{d}{dx}[(lb-ma)h(x)]$$
$$=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$
$$=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Hence, $\frac{dy}{dx}=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$.

22. Prove that $\mathbf{(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0}$ **when** $\mathbf{y=e^{a\cos^{-1}x}, -1\le x\le 1}$**.**

Ans: The given equation is $y=e^{a\cos^{-1}x}$.

Then take logarithm both sides of the equation.

$$\log y = a\cos^{-1}x\log e$$
$$\Rightarrow \log y = a\cos^{-1}x$$

Now, differentiating both sides with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = ax\frac{1}{\sqrt{1-x^2}}$$
$$\frac{dy}{dx} = \frac{-ax}{\sqrt{1-x^2}}$$

Therefore, squaring both the sides of the equation, gives

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$
$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$
$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again, differentiating both sides with respect to x gives

$$\left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2)+(1-x^2)\times\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d}{dx}(y^2)$$
$$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x)+(1-x^2)\times 2\frac{dy}{dx}\times\frac{d^2y}{dx^2} = a^2\times 2y\times\frac{dy}{dx}$$
$$\Rightarrow x\frac{dy}{dx}+(1-x^2)\frac{d^2y}{dx^2} = a^2\times y$$

Hence, it is proved that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$.