Math 3398 Homework – Generating Function (RR)

Use generating functions to do each of the following problems

1. Solve each of the following recurrence relation with the given initial condition(s).

   (1) $a_n = a_{n-1} + 2, \quad a_0 = 3$

   (2) $a_n = a_{n-1} - 5, \quad a_1 = 2$

   (3) $a_n = a_{n-1} + n, \quad a_0 = 1$

   (4) $a_n = a_{n-1} + n - 7, \quad a_0 = 2$

   (5) $a_n = a_{n-1} + 2n + 3, \quad a_0 = 4$

   (6) $a_n = a_{n-1} - 2n + 1, \quad a_1 = 1$

   (7) $a_n = 3a_{n-1}, \quad a_0 = 2$

   (8) $a_n = 2a_{n-1} - 1, \quad a_0 = 3$

   (9) $a_n = 3a_{n-1} + 1, \quad a_0 = 1$

   (10) $a_n = 5a_{n-1} + 2, \quad a_1 = 2$

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2. Solve each of the following recurrence relation with the given initial condition(s).

   (1) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$,
   $a_0 = 1, \quad a_1 = 0$

   (2) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$,
   $a_0 = 6, \quad a_1 = 8$

   (3) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$,
   $a_0 = 0, \quad a_1 = 1$

   (4) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$,
   $a_0 = 2, \quad a_1 = 1$

   (5) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$,
   $a_0 = 4, \quad a_1 = 1$

   (6) $a_n = 4a_{n-2}$ for $n \geq 2$,
   $a_0 = 0, \quad a_1 = 4$

   (7) $a_n = \frac{a_{n-2}}{4}$ for $n \geq 2$,
   $a_0 = 1, \quad a_1 = 0$

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3. Solve each of the following recurrence relation with the given initial condition(s).

   (1) $a_n = 2a_{n-1} + 3^n$ for $n \geq 2$,
   $a_1 = 6$

   (2) $a_n = 2a_{n-1} + 2n^2$ for $n \geq 2$,
   $a_1 = 4$

   (3) $a_n = a_{n-1} + 6a_{n-2} - 2n$ for $n \geq 2$,
   $a_0 = 3, \quad a_1 = 6$

   (4) $a_n = a_{n-1} + 2a_{n-2} + 3^n$ for $n \geq 2$,
   $a_0 = \frac{1}{4}, \quad a_1 = \frac{7}{4}$

   (5) $a_n = a_{n-1} + 6a_{n-2} + 5^n$ for $n \geq 2$,
   $a_0 = 3, \quad a_1 = 6$

   (6) $a_n = 6a_{n-1} - 9a_{n-2} + 5^n$ for $n \geq 2$,
   $a_0 = 0, \quad a_1 = 2$

Answers

1. Solve each of the following recurrence relation with the given initial condition(s).

(1) $a_n = a_{n-1} + 2, \quad a_0 = 3$

Answer: $\boxed{a_n = 2n + 3, \quad n \geq 0}$

(2) $a_n = a_{n-1} - 5, \quad a_1 = 2$

Answer: $\boxed{a_n = -5n + 7, \quad n \geq 1}$

(3) $a_n = a_{n-1} + n, \quad a_0 = 1$

Answer: $\boxed{a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1, \quad n \geq 0}$

(4) $a_n = a_{n-1} + n - 7, \quad a_0 = 2$

Answer: $\boxed{a_n = \frac{1}{2}n^2 - \frac{13}{2}n + 2, \quad n \geq 0}$

(5) $a_n = a_{n-1} + 2n + 3, \quad a_0 = 4$

Answer: $\boxed{a_n = n^2 + 4n + 4, \quad n \geq 0}$

(6) $a_n = a_{n-1} - 2n + 1, \quad a_1 = 1$

Answer: $\boxed{a_n = -n^2 + 2, \quad n \geq 1}$

(7) $a_n = 3a_{n-1}, \quad a_0 = 2$

Answer: $\boxed{a_n = 2 \cdot 3^n, \quad n \geq 0}$

(8) $a_n = 2a_{n-1} - 1, \quad a_0 = 3$

Answer: $\boxed{a_n = 2^{n+1} + 1, \quad n \geq 0}$

(9) $a_n = 3a_{n-1} + 1, \quad a_0 = 1$

Answer: $\boxed{a_n = \frac{1}{2} \cdot 3^{n+1} - \frac{1}{2}, \quad n \geq 0}$

(10) $a_n = 5a_{n-1} + 2, \quad a_1 = 2$

Answer: $\boxed{a_n = \frac{1}{2} \cdot 5^n - \frac{1}{2}, \quad n \geq 1}$

2. Solve each of the following recurrence relation with the given initial condition(s).

(1) $a_n = 5a_{n-1} + 6a_{n-2}$ for $n \geq 2$,
$a_0 = 1, \quad a_1 = 8$

Answer: $\boxed{a_n = 2 \cdot 3^n - (-2)^n, \quad n \geq 0}$

(2) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$,
$a_0 = 6, \quad a_1 = 8$

Answer: $\boxed{a_n = (6 - 2n) \cdot 2^n, \quad n \geq 0 \quad \text{or} \quad a_n = (3 - n) \cdot 2^{n+1}, \quad n \geq 0}$

(3) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$,
$a_0 = 9, \quad a_1 = 6$

Answer: $\boxed{a_n = (9 - 11n) \cdot (-3)^n, \quad n \geq 0}$

(4) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$,
$a_0 = 2, \quad a_1 = 1$

Answer: $\boxed{a_n = 3 \cdot 2^n - 5^n, \quad n \geq 0}$

(5) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$,
$a_0 = 4, \quad a_1 = 1$

Answer: $\boxed{a_n = 4 - 3n, \quad n \geq 0}$

(6) $a_n = 4a_{n-2}$ for $n \geq 2$,
$a_0 = 0, \quad a_1 = 4$

Answer: $\boxed{a_n = 2^n - (-2)^n, \quad n \geq 0}$

(7) $a_n = \dfrac{a_{n-2}}{4}$ for $n \geq 2$,
$a_0 = 4, \quad a_1 = 1$

Answer: $\boxed{a_n = 3 \cdot \left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n, \quad n \geq 0}$

3. Solve each of the following recurrence relation with the given initial condition(s).

(1) $a_n = 2a_{n-1} + 3^n$ for $n \geq 2$,
$a_1 = 6$

Answer: $\boxed{a_n = -\frac{3}{2} \cdot 2^n + 3^{n+1}, \quad n \geq 1 Or a_n = -3 \cdot 2^{n-1} + 3^{n+1}, \quad n \geq 1}$

(2) $a_n = 2a_{n-1} + 2n^2$ for $n \geq 2$,
$a_1 = 4$

Answer: $\boxed{a_n = 13 \cdot 2^n - 2n^2 - 8n - 12 \quad n \geq 1}$

(3) $a_n = a_{n-1} + 20a_{n-2} + 16$ for $n \geq 2$,
$a_0 = \frac{1}{5}, \quad a_1 = \frac{6}{5}$

Answer: $\boxed{a_n = \frac{2}{3} \cdot 5^n + \frac{1}{3}(-4)^n - \frac{4}{5}, \quad n \geq 0}$

(4) $a_n = a_{n-1} + 6a_{n-2} - 2n$ for $n \geq 2$,
$a_0 = 3, \quad a_1 = 6$

Answer: $\boxed{a_n = \frac{19}{10} \cdot 3^n + \frac{17}{45}(-2)^n + \frac{13}{n} + \frac{13}{18}, \quad n \geq 0}$

(5) $a_n = a_{n-1} + 2a_{n-2} + 3^n$ for $n \geq 2$,
$a_0 = \frac{1}{4}, \quad a_1 = \frac{7}{4}$

Answer: $\boxed{a_n = -\frac{7}{3} \cdot 2^n + \frac{1}{3}(-1)^n + \frac{9}{4} \cdot 3^n, \quad n \geq 0}$

(6) $a_n = a_{n-1} + 6a_{n-2} + 5^n$ for $n \geq 2$,
$a_0 = 3, \quad a_1 = 6$

Answer: $\boxed{a_n = -\frac{1}{10} \cdot 3^n + \frac{46}{35}(-2)^n + \frac{25}{14} \cdot 5^n, \quad n \geq 0}$

(7) $a_n = 6a_{n-1} - 9a_{n-2} + 5^n$ for $n \geq 2$,
$a_0 = 0, \quad a_1 = 2$

Answer: $\boxed{a_n = \left(-\frac{25}{4} - \frac{7}{2}n\right) \cdot 3^n + \frac{1}{4} \cdot 5^{n+2}, \quad n \geq 0}$