Math 3398 Review # 1

- **Be able to use each of the following rules to solve counting problems.**

  1. the sum rule and the product rule.
  2. the principle of inclusion-exclusion:
     The following is the principle of inclusion-exclusion with two sets:
     $$|A \cup B| = |A| + |B| - |A \cap B|$$
  3. the pigeonhole principle

- **Be familiar with permutation and combination problems and tell the difference between these two types of problems.**

- **Memorize the following formulas and be able to use them.**

  1. permutation without repetitions:
     $$P(n, r) = \frac{n!}{(n-r)!}$$
  2. combinations without repetitions:
     $$C(n, r) = \frac{n!}{r!(n-r)!}$$
  3. permutation with repetitions:
     $$P_r(n, r) = n^r$$
  4. combinations with repetitions:
     $$C_r(n, r) = C(n + r - 1, r)$$
     (The donut-problem)
  5. permutations of a set with identical elements:
     $$\frac{n!}{n_1! n_2! \cdots n_k!}$$
     (The MISSISSIPPI-problem)

- **Memorize the Binomial Theorem**

$$(a + b)^n = \sum_{k=0}^{n} C(n, k) a^{n-k} b^k$$

- **Be able to use the Binomial Theorem to do the following things:**

  1. Expand a binomial to certain power. For example, expanding $(x - 3y)^5$.
  2. Find the coefficient of $x^{n-k}y^k$ in the expansion of $(cx + dy)^n$.
  3. Prove identities. For example, proving $\sum_{k=0}^{n} (-1)^k C(n, k) = 0$.

- **Sample problems**

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1. Write T, if it is true. Write F, otherwise.

   (1) There are 45 ways to select two students from a group of 10 students.

   (2) The number of r-permutations from a set of n elements is r^n, if the repetition is allowed.

   (3) $2C(n, 2) = P(n, 2)$.

   (4) There are $P(52, 5)$ ways to line up 5 cards chosen from a deck of 52 cards.

   (5) There are 64 bit strings of length eight that begin with 0.

   (6) There are 126 ways to select 6 donuts from 4 types of donuts.

   (7) $(x - y)^n = \sum_{k=0}^{n} C(n, k)x^{n-k}y^k$.

   (8) There are 8! ways to arrange all letters in HONOLULU.

   (9) There are $C(4, 1)C(51, 4)$ ways to select five cards from a deck of 52 cards so that each selection contains at least one King.

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2. Expand $(1 - 4x)^4$ using the binomial theorem.

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3. Find the coefficient of x^8y^3 in $(2x - 3y)^{11}$.

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4. A man, a woman, a boy, a girl, a dog, and a cat are walking down a long and winding road one after the other.

   (1) In how many ways can this happen if the dog comes first?

   (2) In how many ways can this happen if the dog immediately follows the boy?

   (3) In how many ways can this happen if the dog (and the only dog) is between the man and the boy?

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5. How many bit strings of length ten are there

   (1) that contain at least two 1s?

   (2) that begin with 0 or end with 1?

   (3) that contain four 1s and 6 0s so that each 1 is followed by one 0 immediately?

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6. How many ways are there to arrange all letters from the word COMMEMORATE in a row,

   (1) so that all M's are together?

   (2) so that two O's must be separated?

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7. There are five boys and eight girls in a class.

   (1) How many ways are there to form a team with three boys and two girls?

   (2) How many ways are there to select an executive board consisting of president, vice president, secretary, treasurer, and one other officer?

   (3) In how many ways to arrange all boys and girls in a row so that no two boys are next to each other?

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8. A piggy bank contains 50 pennies, 40 nickels, 30 dimes, and 20 quarters.

   (1) How many ways are there to choose 10 coins with at least one of each type?

   (2) How many ways are there to choose 10 coins with at least 3 nickels but no more than 2 quarters?

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9. Prove that for any nonnegative integer n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

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10. Prove that if n and k are integers with $1 \le k \le n$, then

$$C(n + 1, k) = C(n, k) + C(n, k - 1)$$

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11. How many ways are there to draw a five-card poker hand that contains five cards of the same suit?

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12. How many ways are there to draw a five-card poker hand that contains at least one ace?

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13. How many ways are there to arrange five cards in a row so that each arrangement

(1) contains at least one ace?

(2) begins and ends with a king?

(3) begins or ends with a club?

(4) contains exactly two kings?

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14. A box contains 5 red balls, 3 blue balls, and 2 yellow balls. Assume that all balls are different. In how many ways can you select 4 balls so that each selection contains

(1) exactly two red balls?

(2) exactly two red balls and 2 blue balls?

(3) at least two red balls?

(4) no yellow balls?

(5) at most two red balls?

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15. A club has 6 women and 7 men.

(1) How many ways are there to award three different people with three different prizes so that exactly one woman is awarded?

(2) How many ways are there to award three different people with three different prizes so that no woman is awarded?

(3) How many ways are there to choose 5 people with exactly four men ?

(4) How many ways are there to choose 5 people with at least two women?

(5) How many ways are there to arrange all people in two rows so that each row contains at least one people?

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16. How many nonnegative integer solutions does the following equation has

$$x + y + z = 15$$

(1) if $x \geq 4$?

(2) if $1 \leq x \leq 6$?

(3) if $x \geq 2$, and $y \leq 3$?

(4) if $x \geq 3$, $y \geq 2$, and $1 \leq z \leq 3$?

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17. A jar contains five quarters, 2 nickels, and 4 pennies. In how many ways can you arrange all coins in a row so that each arrangement

(1) begins with a quarter?

(2) all quarters together?

(3) two nickels are not next to each other?

(4) no two pennies are next to each other?

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18. ALL QUIZ PROBLEMS

ANSWERS

1. T F T T F F F F F
2. $1 - 16x + 96x^2 - 256x^3 + 256x^4$
3. -1140480
4. (1) 120 (2) 120 (3) 48
5. (1) 1013 (2) 768 (3) 15
6. (1) 90720 (2) 1360800
7. (1) 280 (2) 154440 (3) $8!C(9,5)5!$
8. (1) 84 (2) 85
9. Use the binomial theorem by choosing $x = 1$ and $y = 1$.
10. Simplify the right hand side by using the formula and combining two terms to get the left hand side.
11. 5148
12. 886656
13. (1) 106398720 (2) 1411200 (3) 137592000 (4) 12453120
14. (1) 100 (2) 30 (3) 155 (4) 70 (5) 155
15. (1) 756 (2) 210 (3) 210 (4) 1056 (5) (13!)(12)
16. (1) 78 (2) 107 (3) 50 (4) 27
17. (1) 3150 (2) 105 (3) 5670 (4) 1470