Math 3398 Homework – Recurrence relation (Part 1)

1. Find the first five terms of the sequence defined by each of the following recurrence relations and initial conditions

(1) $a_n = 6a_{n-1}$, for $n \geq 1$, $a_0 = 2$

(2) $a_n = 2na_{n-1}$, for $n \geq 1$, $a_0 = -3$

(3) $a_n = a_{n-1}^2$, for $n \geq 2$, $a_1 = 2$

(4) $a_n = a_{n-1} + 3a_{n-2}$, for $n \geq 3$, $a_0 = 1, a_1 = 2$

(5) $a_n = na_{n-1} + n^2 a_{n-2}$, for $n \geq 2$, $a_0 = 1, a_1 = 1$

(6) $a_n = a_{n-1} + a_{n-3}$, for $n \geq 3$, $a_0 = 1, a_1 = 2, a_2 = 0$

---

2. Is the sequence $\{a_n\}$ a solution of the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

if

(1) $a_n = 0$?

(2) $a_n = 1$?

(3) $a_n = 2^n$?

(4) $a_n = 4^n$?

(5) $a_n = n4^n$?

(6) $a_n = (-4)^n$?

(7) $a_n = n^2 \cdot 4^n$?

---

3. Solve each of the following recurrence relation with the given initial condition(s).

(1) $a_n = a_{n-1} + 2$, for $n \geq 1$, $a_0 = 3$

(2) $a_n = a_{n-1} - 5$, for $n \geq 2$, $a_1 = 2$

(3) $a_n = a_{n-1} + n$, for $n \geq 1$, $a_0 = 1$

(4) $a_n = a_{n-1} + n - 7$, for $n \geq 1$, $a_0 = 2$

(5) $a_n = a_{n-1} + 2n + 3$, for $n \geq 1$, $a_0 = 4$

(6) $a_n = a_{n-1} - 2n + 1$, for $n \geq 2$, $a_1 = 1$

(7) $a_n = 3a_{n-1}$, for $n \geq 1$, $a_0 = 2$

(8) $a_n = 2a_{n-1} - 1$, for $n \geq 1$, $a_0 = 3$

(9) $a_n = 3a_{n-1} + 1$, for $n \geq 1$, $a_0 = 1$

(10) $a_n = 5a_{n-1} + 2$, for $n \geq 2$, $a_1 = 2$

---

4. Solve each of the following recurrence relation with the given initial condition(s).

(1) $a_n = 5a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 8$

(2) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

(3) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 9$, $a_1 = 6$

(4) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$

(5) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$

(6) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$

(7) $a_n = \frac{a_{n-2}}{4}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$

Answers

1. Find the first five terms of the sequence defined by each of the following recurrence relations and initial conditions

(1) $a_n = 6a_{n-1}, \quad a_0 = 2$

Answer: $\boxed{a_0 = 2,\ a_1 = 12,\ a_2 = 72,\ a_3 = 432,\ a_4 = 2592}$

(2) $a_n = 2na_{n-1}, \quad a_0 = -3$

Answer: $\boxed{a_0 = -3,\ a_1 = -6,\ a_2 = -24,\ a_3 = -144,\ a_4 = -1152}$

(3) $a_n = a_{n-1}^2, \quad a_1 = 2$

Answer: $\boxed{a_0 = 2,\ a_1 = 4,\ a_2 = 16,\ a_3 = 256,\ a_4 = 65536}$

(4) $a_n = a_{n-1} + 3a_{n-2}, \quad a_0 = 1, a_1 = 2$

Answer: $\boxed{a_0 = 1,\ a_1 = 2,\ a_2 = 5,\ a_3 = 11,\ a_4 = 26}$

(5) $a_n = na_{n-1} + n^2 a_{n-2}, \quad a_0 = 1, a_1 = 1$

Answer: $\boxed{a_0 = 1,\ a_1 = 1,\ a_2 = 6,\ a_3 = 27,\ a_4 = 204}$

(6) $a_n = a_{n-1} + a_{n-3}, \quad a_0 = 1, a_1 = 2, a_2 = 0$
Answer: $\boxed{a_0 = 1,\ a_1 = 2,\ a_2 = 0,\ a_3 = 1,\ a_4 = 3}$

---

2. Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

(1) $a_n = 0$?

Answer: $\boxed{\text{yes}}$
Note that $LHS = 0$ and $RHS = 8(0) - 16(0) = 0$. Since the both sides are the same, $a_n = 0$ is a solution.

(2) $a_n = 1$?

Answer: $\boxed{\text{no}}$
Note that $LHS = 1$ and $RHS = 8(1) - 16(1) = -8$. Since the both sides are NOT the same, $a_n = 0$ is not a solution.

(3) $a_n = 2^n$?

Answer: $\boxed{\text{no}}$
Note that $LHS = 2^n$ and $RHS = 8(2^{n-1}) - 16(2^{n-2}) = 8 \cdot 2^{n-1} - 8 \cdot 2 \cdot 2^{n-2} = 8 \cdot 2^{n-1} - 8 \cdot 2^{n-1} = 0$. Since the both sides are NOT the same, $a_n = 2^n$ is not a solution.

(4) $a_n = 4^n$?

Answer: $\boxed{\text{yes}}$

Note that $LHS = 4^n$ and $RHS = 8(4^{n-1}) - 16(4^{n-2}) = 2 \cdot 4 \cdot 4^{n-1} - 4^2 \cdot 4^{n-2} = 2 \cdot 4^n - 4^n = 4^n$. Since the both sides are the same, $a_n = 4^n$ is a solution.

(5) $a_n = n4^n$?

Answer: $\boxed{\text{yes}}$

Note that $LHS = n4^n$ and $RHS = 8((n-1)4^{n-1}) - 16((n-2)4^{n-2}) = 2 \cdot 4(n-1) \cdot 4^{n-1} - (n-2)4^2 \cdot 4^{n-2} = 2(n-1) \cdot 4^n - (n-2)4^n = 4^n[2(n-1) - (n-2)] = n4^n$. Since the both sides are the same, $a_n = n4^n$ is a solution.

(6) $a_n = (-4)^n$?

Answer: $\boxed{\text{no}}$

Note that $a_1 = -4$, $a_2 = 16$, and $a_3 = -64$. But $a_3 \neq 8a_2 - 16a_1$. So $a_n = (-4)^n$ is not a solution.

(7) $a_n = n^2 \cdot 4^n$?

Answer: $\boxed{\text{no}}$

Note that $a_1 = 4$, $a_2 = 64$, and $a_3 = 574$. But $a_3 \neq 8a_2 - 16a_1$. So $a_n = n^2 \cdot 4^n$ is not a solution.

---

3. Solve each of the following recurrence relation with the given initial condition(s).

(1) $a_n = a_{n-1} + 2$, for $n \geq 1$, $\quad a_0 = 3$

Answer: $\boxed{a_n = 2n + 3, \quad n \geq 0}$

(2) $a_n = a_{n-1} - 5$, for $n \geq 2$, $\quad a_1 = 2$

Answer: $\boxed{a_n = -5n + 7, \quad n \geq 1}$

(3) $a_n = a_{n-1} + n$, for $n \geq 1$, $\quad a_0 = 1$

Answer: $\boxed{a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1, \quad n \geq 0}$

(4) $a_n = a_{n-1} + n - 7$, for $n \geq 1$, $\quad a_0 = 2$

Answer: $\boxed{a_n = \frac{1}{2}n^2 - \frac{13}{2}n + 2, \quad n \geq 0}$

(5) $a_n = a_{n-1} + 2n + 3$, for $n \geq 1$, $\quad a_0 = 4$

Answer: $\boxed{a_n = n^2 + 4n + 4, \quad n \geq 0}$

(6) $a_n = a_{n-1} - 2n + 1$, for $n \geq 2$, $\quad a_1 = 1$

Answer: $\boxed{a_n = -n^2 + 2, \quad n \geq 1}$

(7) $a_n = 3a_{n-1}$, for $n \geq 1$, $\quad a_0 = 2$

Answer: $\boxed{a_n = 2 \cdot 3^n, \quad n \geq 0}$

(8) $a_n = 2a_{n-1} - 1$, for $n \geq 1$, $\quad a_0 = 3$

Answer: $\boxed{a_n = 2^{n+1} + 1, \quad n \geq 0}$

(9) $a_n = 3a_{n-1} + 1$, for $n \geq 1$, $\quad a_0 = 1$

Answer: $\boxed{a_n = \frac{1}{2} \cdot 3^{n+1} - \frac{1}{2}, \quad n \geq 0}$

(10) $a_n = 5a_{n-1} + 2$, for $n \geq 2$, $\quad a_1 = 2$

Answer: $\boxed{a_n = \frac{1}{2} \cdot 5^n - \frac{1}{2}, \quad n \geq 1}$

---

4. Solve each of the following recurrence relation with the given initial condition(s).

(1) $a_n = 5a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $\quad a_0 = 1$, $\quad a_1 = 8$

Answer: $\boxed{a_n = \frac{9}{7} \cdot 6^n - \frac{2}{7}(-1)^n, \quad n \geq 0}$

(2) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $\quad a_0 = 6$, $\quad a_1 = 8$

Answer: $\boxed{a_n = (6 - 2n) \cdot 2^n, \quad n \geq 0 \quad \text{or} \quad a_n = (3 - n) \cdot 2^{n+1}, \quad n \geq 0}$

(3) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $\quad a_0 = 9$, $\quad a_1 = 6$

Answer: $\boxed{a_n = (9 - 11n) \cdot (-3)^n, \quad n \geq 0}$

(4) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $\quad a_0 = 2$, $\quad a_1 = 1$

Answer: $\boxed{a_n = 3 \cdot 2^n - 5^n, \quad n \geq 0}$

(5) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $\quad a_0 = 4$, $\quad a_1 = 1$

Answer: $\boxed{a_n = 4 - 3n, \quad n \geq 0}$

(6) $a_n = 4a_{n-2}$ for $n \geq 2$, $\quad a_0 = 0$, $\quad a_1 = 4$

Answer: $\boxed{a_n = 2^n - (-2)^n, \quad n \geq 0}$

(7) $a_n = \dfrac{a_{n-2}}{4}$ for $n \geq 2$,
$a_0 = 4$, $\quad a_1 = 1$

Answer: $\boxed{a_n = 3 \cdot \left(\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^n, \quad n \geq 0}$