Geometric Algebra (GA)

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1 Documentation

Overview (high-level initial intros on youtube):

- *https://www.thestrangeloop.com*
- *https://www.youtube.com/@sudgylacmoe*
- *https://www.youtube.com/@bivector*

Sources:

- `https://projectivegeometricalgebra.org`
- `https://terathon.com/blog/poor-foundations-ga.html`
- `https://bivector.net`
- `https://www.youtube.com/@bivector`
- `https://www.youtube.com/@sudgylacmoe`
- `https://www.youtube.com/watch?v=ItGlUbFBFfc`
- `https://www.youtube.com/watch?v=v-WG02ILMXA&t=1476s`

1.1 Introduction

The ga library is intended to make Geometric Algebra more accessible by providing functionality for numerical calculations required for simulation of physical systems.

It implements functionality for two-dimensional ($2D$) and three-dimensional ($3D$) spaces with Euclidean geometry. Currently it provides data types and operations for non-projective Euclidean geometry ($ega2d, ega3d$) and projective Euclidean geometry ($pga2dp, pga3dp$). The library handles multivectors and their components efficiently. It could be easily extended to provide functionality for handling spacetime algebra (STA) with limited effort.

The main library is a header only library in the ga folder which can be used by including either

```
#include ga/ga_ega.hpp // or
#include ga/ga_pga.hpp
```

and by making the corresponding namespaces accessible for use by stating

```
using namespace hd::ga;      // and either
using namespace hd::ga::ega; // or
using namespace hd::ga::pga;
```

in your application code. For a simple usage example please have a look either at `ga_test/src/ga_ega_test.cpp` or at `ga_test/src/ga_pga_test.cpp`.

All product expressions are generated by the file `ga_prdxpr/ga_prdxpr_main.cpp` and user input provided in the header files of the corresponding algebra. The generated coefficient expressions are used to fill-in the source code for geometric products, wedge and dot products. If certain combinations are not yet implemented in the ga library they can easily be generated within `ga_prdxpr` as needed and then be added to the library.

The complement operation used in this library corresponds to the complement as defined in [Lengyel, 2024]. It is uniquely determined with respect to the outer product (not with respect to the geometric product as commonly used by many autors). The left complement $\underline{u}$ is defined as $\underline{u} \wedge u = I_n$ and the right complement $\bar{u}$ as $u \wedge \bar{u} = I_n$ with I_n as the pseudoscalar of the n-dimensional space modeled by the algebra.

The complement operation is used in turn to define a unique dualization operation that works for cases when the metric is non-degenerate as well as for cases where it is degenerate, like in projective geometric algebra. The right dual $A^\star$ is defined as $A^\star = \overline{GA}$, where A is an arbitrary multivector and G the extended metric. There is also a corresponding left dualization operation defined as $A_\star = \underline{GA}$. $\star$ is used as dualization operator (Hodge star). Relations to the geometric product $A^\star = A^\dagger I_n$ and $A_\star = I_n A^\dagger$ also exist with $\dagger$ as the reversion operation.

TODO: *fill in further introductory notes here*

1.2 Basic formulas

The inner product between multivectors A and B is defined as

$$A \bullet B = A^T G B \tag{1}$$

with G as the extended metric and matrix multiplication on the right hand side. It satisfies the identity

$$A \bullet B = \langle A\tilde{B}\rangle_0 = \langle B\tilde{A}\rangle_0 \tag{2}$$

where juxtaposition within the grade projection operator stands for the geometric product. The norm of a multivector A is defined as

$$\|A\| = \sqrt{A \bullet A} \tag{3}$$

with a positive argument under the square root due to the definition of the inner product.

The contraction is explicitly defined as

$$A \rfloor B = A_\star \vee B \tag{4a}$$
$$A \lfloor B = A \vee B^\star \tag{4b}$$

with $\star$ denoting the Hodge dual with is formed by the respective complement operation (in spaces of even dimension the left or right complement respectively, and in spaces of uneven dimension the complement function regardless of the side of the operand). For blades A and B, they satisfy

$$A \rfloor B = \langle B\tilde{A}\rangle_{gr(B)-gr(A)} \tag{5a}$$
$$A \lfloor B = \langle \tilde{B}A\rangle_{gr(A)-gr(B)} \tag{5b}$$

and fulfill the requirement that $A \rfloor B = A \lfloor B = A \bullet B$ whenever A and B have the same grade. In this case the contractions reduce to the inner product.

The geometric product between and vector v and a blade B is defined in terms of the inner and outer products, i.e. the right contraction $\lfloor$, the left contraction $\rfloor$ and wedge product $\wedge$ as follows:

$$a \mathbin{\wedge\!\!\!\!\wedge} B = B \lfloor a + a \wedge B \tag{6a}$$
$$B \mathbin{\wedge\!\!\!\!\wedge} a = a \rfloor B + B \wedge a \tag{6b}$$

$a = a_\parallel + a_\perp$ can be decomposed into parts $a_\parallel$ parallel to the k-blade B and $a_\perp$ perpendicular to B. For equation (6a): $B \lfloor a$ is a $(k-1)$-blade. If $a_\parallel \neq 0$ then

$B \lfloor a$ represents a subspace of B. $a \wedge B$ is a $(k+1)$-blade. If $a_\perp \neq 0$ then $a \wedge B$ represents $\mathrm{span}(a, B)$.
For arguments of equal grade the contractions reduce to the dot product $\cdot$, so that one can write specifically for two vectors a and b:

$$a \curlywedge b = a \cdot b + a \wedge b \tag{7}$$

Text with a norm $\|x\|$ and an indexed norm as $\|u\|_\bullet$ and $\|u\|_\circ$.

TODO: *fill in basic formulas here*

1.3 Literature

References

[Lengyel, 2024] Lengyel, E. (2024). *Projective Geometric Algebra Illuminated*. Terathon Software LLC.