

Tensor Basics *

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1 Tensor Basics

Description 1: A tensor is an object that is invariant under change of the coordinate system basis vectors, and has components that change in a special, predictable way under a change of coordinates, i.e. a change of the basis.

Description 2: A tensor is a collection of vectors and covectors combined together using the tensor product.

The approach presented by “eigenchris” that has been adopted in the first part of the document does not need the introduction of a scalar product.

The approach presented by Klingbeil uses a scalar product and does introduce covectors as vectors with a specific relation to the basis vectors defined by the scalar product, but not as functions.

*Sources: youtube channel “eigengchris”, Klingbeil “Tensorrechnung für Ingenieure”

1.1 Basis Transformation Rules

Old Basis: $\{\vec{e}_1, \vec{e}_2\}$
 New Basis: $\{\tilde{\vec{e}}_1, \tilde{\vec{e}}_2\}$

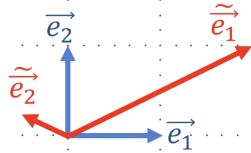


Figure 1: old and new basis vectors

Basis transformation: The basis vectors of the new system expressed as a linear combination of the basis vectors of the old system is considered a forward transformation. The other direction is considered a backward transformation.

Here F stands for Forward transformation. This transformation maps the basis vectors of the old basis to the new basis vectors. The coordinates of the mapped basis vectors in the old system become the columns of the resulting forward transformation matrix. Basis vectors are chosen to be left multiplied to the transformation matrix, which corresponds to summation via the first index of the transformation matrix. Indices are chosen upper and lower to enable the Einstein summation convention later on. Transformation indices are always from “north west” to “south east”:

$$\begin{aligned} [\tilde{\vec{e}}_1 & \quad \tilde{\vec{e}}_2] = [\vec{e}_1 & \quad \vec{e}_2] \begin{bmatrix} F_1^1 & F_2^1 \\ F_1^2 & F_2^2 \end{bmatrix} \\ \tilde{\vec{e}}_1 &= F_1^1 \vec{e}_1 + F_2^1 \vec{e}_2 \quad \text{figure 1} \quad 2\vec{e}_1 + 1\vec{e}_2 \\ \tilde{\vec{e}}_2 &= F_1^2 \vec{e}_1 + F_2^2 \vec{e}_2 \quad \text{figure 1} \quad -\frac{1}{2}\vec{e}_1 + \frac{1}{4}\vec{e}_2 \end{aligned} \quad (1)$$

$$\Rightarrow \tilde{e}_j = F_j^k e_k \quad (\text{summation convention})$$

Here B stands for Backward transformation:

$$\begin{aligned} [\vec{e}_1 & \quad \vec{e}_2] = [\tilde{\vec{e}}_1 & \quad \tilde{\vec{e}}_2] \begin{bmatrix} B_1^1 & B_2^1 \\ B_1^2 & B_2^2 \end{bmatrix} \\ \vec{e}_1 &= B_1^1 \tilde{\vec{e}}_1 + B_1^2 \tilde{\vec{e}}_2 \quad \text{figure 1} \quad \frac{1}{4}\tilde{\vec{e}}_1 + (-1)\tilde{\vec{e}}_2 \\ \vec{e}_2 &= B_2^1 \tilde{\vec{e}}_1 + B_2^2 \tilde{\vec{e}}_2 \quad \text{figure 1} \quad \frac{1}{2}\tilde{\vec{e}}_1 + 2\tilde{\vec{e}}_2 \end{aligned} \quad (2)$$

$$\Rightarrow e_i = B_i^j \tilde{e}_j \quad (\text{summation convention})$$

Between B and F following relation holds:

$$\begin{aligned} B^j_i F^k_j &= F^k_j B^j_i = \delta_i^k = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases} \\ \text{full equation transposed: } &= B^i_j F^j_k = \delta_k^i \quad (3) \\ \Rightarrow B &= F^{-1} \quad (B \text{ is the inverse of } F) \end{aligned}$$

1.2 Vector Definition

- Vectors are **invariant** under a change of the coordinate system.
- Vector components are **not invariant** under a change of the coordinate system.
- Vectors can be defined for spaces of arbitrary dimension n . They form the vector space V with their components $\in \mathbb{R}^n$.
- Not all vectors can be visualized as arrows (only easily done for vectors in Euclidean spaces).

Vectors are members of a vector space. A vector space needs four things:

1. V : a set of vectors
2. S : a set of scalars
3. $+$: an operation to add vectors $\vec{v} + \vec{w} = \overrightarrow{v+w}$
4. \cdot : an operation to scale a vector by a scalar, e.g. $2\vec{v}$

Vectors are basically things that can be added together and scaled by a factor. They are written as columns of numbers of their components in the respective basis.

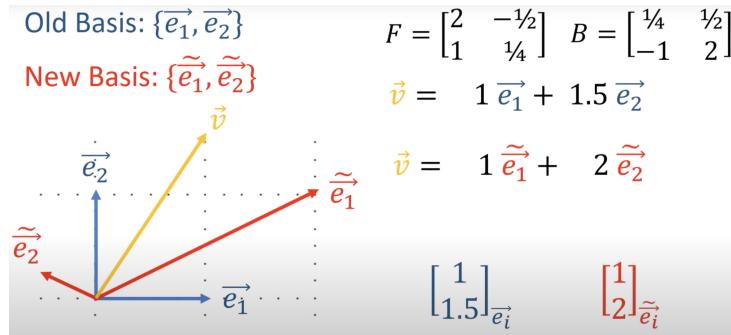


Figure 2: vector example in old and new system

Vectors are formed by linear combinations of their basis vectors with the coordinates as scaling coefficients of the corresponding basis vectors:

$$\vec{v} = \sum_i v^i \vec{e}_i = v^i \vec{e}_i \quad (\text{summation convention}) \quad (4a)$$

$$\vec{v} = \sum_i \tilde{v}^i \tilde{\vec{e}}_i = \tilde{v}^i \tilde{\vec{e}}_i \quad (\text{summation convention}) \quad (4b)$$

Inserting the backward transformation (2) in (4a) and re-arranging the sums by exchanging the indices, we can get the transformation rule of the vector components by comparing with (4b). This yields the transformation rule for the vector components:

$$\begin{aligned}\vec{v} &= v^i \vec{e}_i = v^i B_j^i \vec{e}_j = v^j B_j^i \vec{e}_i \\ \vec{v} &= \tilde{v}^i \tilde{\vec{e}}_i \\ \Rightarrow \tilde{v}^i &= B_j^i v^j\end{aligned}\tag{5}$$

By inserting the forward transformation (1) in (4b) and re-arranging the sums, we can get the transformation rule of the vector components by comparing with (4a). This yields the transformation rule for the vector components in the other direction:

$$\begin{aligned}\vec{v} &= v^i \vec{e}_i \\ \vec{v} &= \tilde{v}^i \tilde{\vec{e}}_i = \tilde{v}^i F_i^j \vec{e}_j = \tilde{v}^j F_j^i \vec{e}_i \\ \Rightarrow v^i &= F_j^i \tilde{v}^j\end{aligned}\tag{6}$$

Summarizing the transformation rules for basis vectors and the respective vector components we get:

$$\begin{array}{lll}\tilde{\vec{e}}_i &= F_i^j \vec{e}_j & \tilde{v}^i = B_j^i v^j \\ \vec{e}_i &= B_i^j \tilde{\vec{e}}_j & v^i = F_j^i \tilde{v}^j\end{array}\tag{7}$$

As can be seen, the basis vectors and their components transform with different transformations. This transformation behaviour is called contravariant. Thus vectors are contravariant tensors.

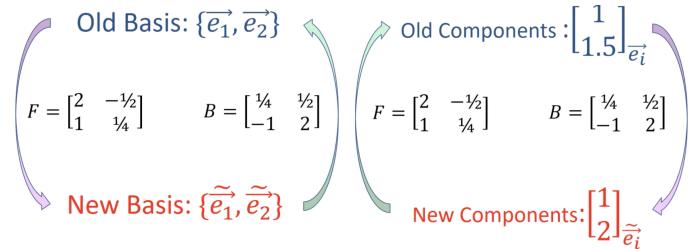


Figure 3: Summary of transformations (example)

For basis transformations the basis vectors are applied to the transformation matrix from the left whereas vector components are multiplied to the

transformation matrix from the right. This can be seen in the index notation as well by the position of the indices. Multiplication from the left corresponds to summation over the first index, while multiplication from the right corresponds to summation over the second index of the transformation matrix.

1.3 Covector Definition:

Covectors and vectors are fundamentally different objects¹. Covectors are functions that take vectors as arguments. They yield a scalar output α on a vector input \vec{v} :

$$\alpha(\vec{v}) = \alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_n v^n = \sum_i \alpha_i v^i \quad (8)$$

Covectors take an input from vector space V and return a real number α , i.e. $\alpha : V \rightarrow \mathbb{R}$. Covectors have the linearity properties, i.e. they are linear functions:

$$\begin{aligned} \alpha(\vec{v} + \vec{w}) &= \alpha(\vec{v}) + \alpha(\vec{w}) \\ \alpha(n\vec{v}) &= n\alpha(\vec{v}) \\ \alpha(n\vec{v} + m\vec{w}) &= n\alpha(\vec{v}) + m\alpha(\vec{w}) \end{aligned} \quad (9)$$

Covectors can also be viewed as elements of their own vector space V^* , i.e. they can be scaled and added:

$$\begin{aligned} (n\alpha)(\vec{v}) &= n\alpha(\vec{v}) \\ (\beta + \gamma)(\vec{v}) &= \beta(\vec{v}) + \gamma(\vec{v}) \end{aligned} \quad (10)$$

The vector space of contravariant vectors is defined by $(V, S, +, \cdot)$. The vector space of covectors is defined by $(V^*, S, +, \cdot)$. The covector operations $+$ and \cdot are different from the corresponding $+$ and \cdot operations of vectors.

Covectors can be visualized as oriented stacks of lines of constant function value. When the coordinates are visualized as vectors they would provide a vector oriented as the normal to the stack height lines:

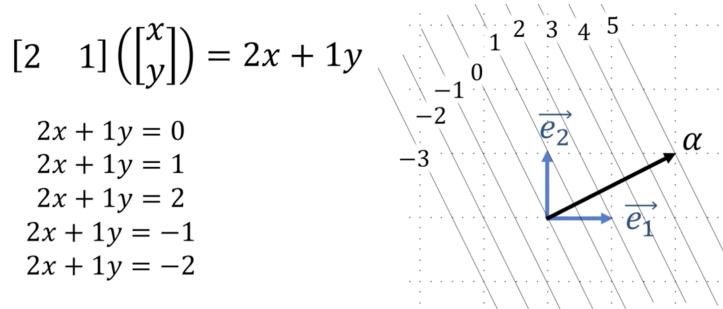


Figure 4: Example: covector visualized as lines of constant function value

¹Covectors can "basically" be regarded as row vectors. However, only in an orthonormal basis there is a trivial link to column vector components. This is not true in the general case of non-orthonormal basis vectors.

The function values returned by the covector can be visualized depending on the vectors provided as arguments. The number of height lines pierced by the vector gives the returned result:

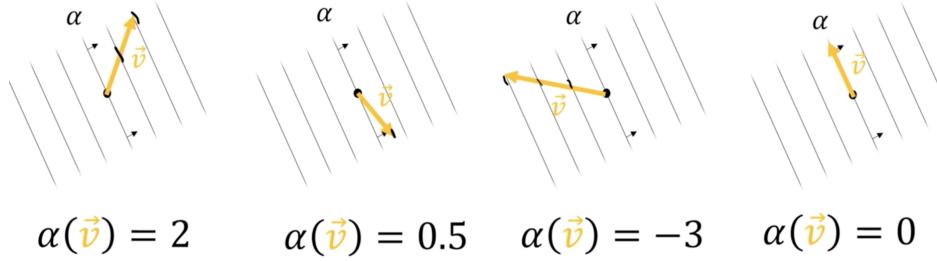


Figure 5: Example: covector return values from given input vectors

- Covectors are **invariant** under a change of the coordinate system.
- Covector components are **not invariant** under a change of the coordinate system.

Covectors are functions $\alpha : V \rightarrow \mathbb{R}$ that take vectors as arguments. Covectors don't live in the vector space V , but in V^* . Thus we cannot use basis vectors like $\{\vec{e}_1, \vec{e}_2\}$ to measure covectors directly. But still the basis $\{\vec{e}_1, \vec{e}_2\}$ can be used to derive a special covector basis $\{\epsilon^1, \epsilon^2\}$ with the specific property:

$$\epsilon^i(\vec{e}_j) = \delta_j^i = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (11)$$

such that for the 2D case:

$\epsilon^1(\vec{e}_1) = 1$	$\epsilon^1(\vec{e}_2) = 0$
$\epsilon^2(\vec{e}_1) = 0$	$\epsilon^2(\vec{e}_2) = 1$

This effectively links the bases ϵ^i and \vec{e}_i and the vector spaces V and V^* , making V^* the dual space of V . It enables us to use these covectors to find the coordinates of arbitrary vectors of V by applying these special covectors to them:

$$\begin{aligned} \epsilon^1(\vec{v}) &= \epsilon^1(v^1\vec{e}_1 + v^2\vec{e}_2) \stackrel{(9)}{=} v^1\epsilon^1(\vec{e}_1) + v^2\epsilon^1(\vec{e}_2) \stackrel{(11)}{=} v^1 \\ \epsilon^2(\vec{v}) &= \epsilon^2(v^1\vec{e}_1 + v^2\vec{e}_2) \stackrel{(9)}{=} v^1\epsilon^2(\vec{e}_1) + v^2\epsilon^2(\vec{e}_2) \stackrel{(11)}{=} v^2 \end{aligned} \quad (12)$$

$$\Rightarrow \epsilon^i(\vec{v}) = v^i$$

This means we select a special covector basis that is aligned with our vector basis such that each covector basis selects it's corresponding basis vector contribution exclusively:



Figure 6: Example: specifically chosen covector basis

If we now want to apply a general covector α to a vector \vec{v} we can transform the equations by defining the covector components in terms of our specially chosen covector basis vectors ϵ^1 and ϵ^2 :

$$\begin{aligned} \alpha(\vec{v}) &= \alpha(v^1 \overrightarrow{e_1} + v^2 \overrightarrow{e_2}) = v^1 \alpha(\overrightarrow{e_1}) + v^2 \alpha(\overrightarrow{e_2}) \\ &\stackrel{(9)}{=} \epsilon^1(\vec{v}) \alpha(\overrightarrow{e_1}) + \epsilon^2(\vec{v}) \alpha(\overrightarrow{e_2}) \end{aligned}$$

$$\text{defining : } \begin{aligned} \alpha(\overrightarrow{e_1}) &= \alpha_1 \\ \alpha(\overrightarrow{e_2}) &= \alpha_2 \end{aligned} \tag{13}$$

$$\begin{aligned} \alpha(\vec{v}) &= \alpha_1 \epsilon^1(\vec{v}) + \alpha_2 \epsilon^2(\vec{v}) \\ &\stackrel{(10)}{=} (\alpha_1 \epsilon^1 + \alpha_2 \epsilon^2)(\vec{v}) \end{aligned}$$

$$\Rightarrow \alpha = \alpha_1 \epsilon^1 + \alpha_2 \epsilon^2 = \alpha_i \epsilon^i$$

Thus every covector can be expressed as a linear combination of our specifically chosen covector basis. We call the basis $\{\epsilon^1, \epsilon^2\}$ the “dual basis” of the vector space V^* . Fig. 7 shows this graphically.

Of course we could also define an arbitrary covector with respect to the vector basis in the new system: The basis $\{\overrightarrow{e_1}, \overrightarrow{e_2}\}$ can be used to derive a special

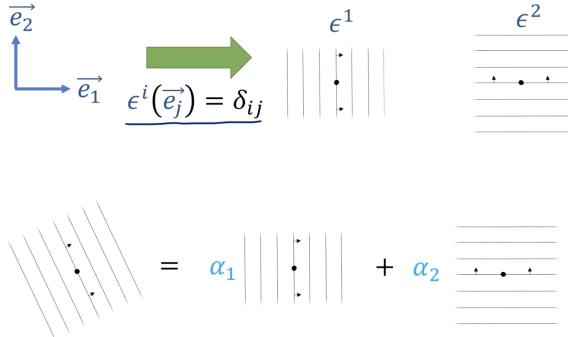


Figure 7: Arbitrary covector (lower row left) expressed as linear combination of the covector basis $\{\epsilon^1, \epsilon^2\}$ after the dual basis has been derived from the vector basis.

covector basis $\{\tilde{\epsilon}^1, \tilde{\epsilon}^2\}$ with the specific property:

$$\tilde{\epsilon}^i(\tilde{e}_j) = \delta_j^i = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (14)$$

such that for the 2D case:

$\tilde{\epsilon}^1(\tilde{e}_1) = 1$	$\tilde{\epsilon}^1(\tilde{e}_2) = 0$
$\tilde{\epsilon}^2(\tilde{e}_1) = 0$	$\tilde{\epsilon}^2(\tilde{e}_2) = 1$

This effectively links the bases $\tilde{\epsilon}^i$ and \tilde{e}_i and the vector spaces V and V^* , making V^* the dual space of V . It enables us to use these covectors to find the coordinates of arbitrary vectors of V by applying these special covectors to them:

$$\begin{aligned} \tilde{\epsilon}^1(\tilde{v}) &= \tilde{\epsilon}^1(\tilde{v}^1 \tilde{e}_1 + \tilde{v}^2 \tilde{e}_2) \stackrel{(9)}{=} \tilde{v}^1 \tilde{\epsilon}^1(\tilde{e}_1) + \tilde{v}^2 \tilde{\epsilon}^1(\tilde{e}_2) \stackrel{(14)}{=} \tilde{v}^1 \\ \tilde{\epsilon}^2(\tilde{v}) &= \tilde{\epsilon}^2(\tilde{v}^1 \tilde{e}_1 + \tilde{v}^2 \tilde{e}_2) \stackrel{(9)}{=} \tilde{v}^1 \tilde{\epsilon}^2(\tilde{e}_1) + \tilde{v}^2 \tilde{\epsilon}^2(\tilde{e}_2) \stackrel{(14)}{=} \tilde{v}^2 \end{aligned} \quad (15)$$

$$\Rightarrow \tilde{\epsilon}^i(\tilde{v}) = v^i$$

This leads to another dual basis specific for the basis vectors in the new system as shown in equation 16.

$$\begin{aligned}\alpha(\tilde{v}) &= \alpha(\tilde{v}^1 \tilde{\vec{e}}_1 + \tilde{v}^2 \tilde{\vec{e}}_2) = \tilde{v}^1 \alpha(\tilde{\vec{e}}_1) + \tilde{v}^2 \alpha(\tilde{\vec{e}}_2) \\ &\stackrel{(9)}{=} \tilde{\epsilon}^1(\tilde{v}) \alpha(\tilde{\vec{e}}_1) + \tilde{\epsilon}^2(\tilde{v}) \alpha(\tilde{\vec{e}}_2)\end{aligned}\quad (15)$$

defining: $\alpha(\tilde{\vec{e}}_1) = \tilde{\alpha}_1$

$$\begin{aligned}\alpha(\tilde{\vec{e}}_2) &= \tilde{\alpha}_2 \\ \alpha(\tilde{v}) &= \tilde{\alpha}_1 \tilde{\epsilon}^1(\tilde{v}) + \tilde{\alpha}_2 \tilde{\epsilon}^2(\tilde{v}) \\ &\stackrel{(10)}{=} (\tilde{\alpha}_1 \tilde{\epsilon}^1 + \tilde{\alpha}_2 \tilde{\epsilon}^2)(\tilde{v}) \\ \Rightarrow \alpha &= \tilde{\alpha}_1 \tilde{\epsilon}^1 + \tilde{\alpha}_2 \tilde{\epsilon}^2 = \tilde{\alpha}_i \tilde{\epsilon}^i\end{aligned}\quad (16)$$

Graphically fig. 8 shows the covectors with respect to the basis in the new system.

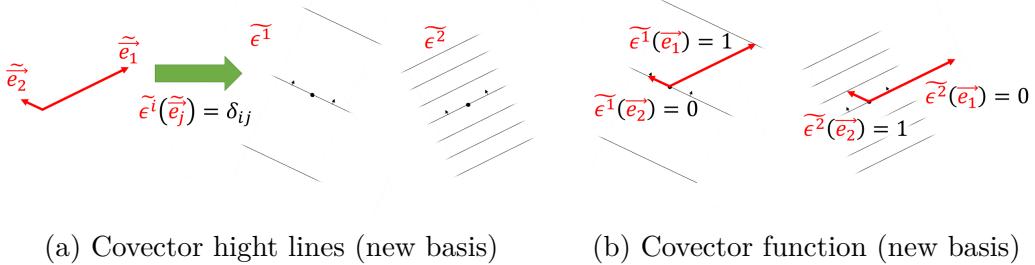


Figure 8: Another dual covector basis expressed with respect to the basis in the new system

In order to derive the transformation rules for the the covector basis we use $\{\tilde{\epsilon}^j\}$ as the old basis for V^* and $\{\tilde{\epsilon}^i\}$ as the new basis for V^* . The new basis can be expressed by a linear combination of the old basis vectors using the coefficients Q^i_j :

$$\tilde{\epsilon}^i = Q^i_j \tilde{\epsilon}^j \quad (17)$$

Using (14) and (17) we can write:

$$\begin{aligned}
\tilde{\epsilon}^i(\tilde{e}_k) &= Q^i_j \tilde{\epsilon}^j(\tilde{e}_k) \\
\delta_k^i &\stackrel{(14),(1)}{=} Q^i_j \tilde{\epsilon}^j(F^l_k \tilde{e}_l) \stackrel{(9)}{=} Q^i_j F^l_k \tilde{\epsilon}^j(\tilde{e}_l) \\
\delta_k^i &\stackrel{(11)}{=} Q^i_j F^l_k \delta_l^j = Q^i_j F^j_k
\end{aligned} \tag{18}$$

comparing with (3): $Q^i_j = B^i_j$

$$\text{inserting in (17)} \Rightarrow \tilde{\epsilon}^i = B^i_j \tilde{\epsilon}^j$$

This shows that covectors transform with the backward transformation from the old to the new basis. Using the same approach it can be shown that:

$$\Rightarrow \epsilon^i = F^i_j \tilde{\epsilon}^j \tag{19}$$

Summarizing the transformation rules for vectors and covectors we get:

$$\begin{aligned}
\tilde{e}_j &= F^i_j e_i & \tilde{\epsilon}^i &= B^i_j \tilde{\epsilon}^j \\
e_j &= B^i_j \tilde{e}_i & \epsilon^i &= F^i_j \tilde{\epsilon}^j
\end{aligned} \tag{20}$$

The covector can be constructed from the basis in the old and new basis:

$$\alpha = \alpha_i \epsilon^i = \tilde{\alpha}_j \tilde{\epsilon}^j \tag{21}$$

By inserting the transformation rule (18) or (19) and comparing with the other part of the equation (21) we can get the transformation rules for the covector components:

$$\begin{aligned}
\tilde{\alpha}_j &= F^i_j \alpha_i \\
\alpha_j &= B^i_j \tilde{\alpha}_i
\end{aligned} \tag{22}$$

This means that covector components transform as the basis vectors do, i.e. with the forward transformation from the old to the new system and with the backward transformation from the new to the old system. This is called covariant transformation behaviour. For both cases the summation goes via the first index of the transformation matrix (equivalent to matrix multiplication from the left).

Summing up everything so far we get:

$\tilde{e}_j = F^i_j e_i$ $e_j = B^i_j \tilde{e}_i$ with: $\tilde{v} = v^i e_i = v^j \tilde{e}_j$ $(\text{contravar.}) \quad \tilde{v}^i = B^i_j v^j$ $v^i = F^i_j \tilde{v}^j$	$\tilde{\epsilon}^i = B^i_j \tilde{\epsilon}^j$ (<u>contravar.</u>) $\epsilon^i = F^i_j \tilde{\epsilon}^j$ $\alpha = \alpha_i \epsilon^i = \tilde{\alpha}_j \tilde{\epsilon}^j$ $\tilde{\alpha}_j = F^i_j \alpha_i$ (<u>covar.</u>) $\alpha_j = B^i_j \tilde{\alpha}_i$
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1.4 Linear Maps

Linear maps take a vector as an input and produce a new vector as an output. Linear maps don't transform the basis vectors. They just transform the input vectors into the output vectors given in the same basis.

Linear maps shown as matrices map copies of the unit basis vectors $[1, 0]^T$ and $[0, 1]^T$ into the vectors their coordinate columns consist of (see matrix multiplication with the corresponding vector from the right side). The first column is the mapped value of the first basis vector and the second column the mapped value of the second basis vector. In general the i^{th} column is the mapped value of the i^{th} basis vector.

From a geometrical point of view linear maps are spacial transforms that keep gridlines parallel, keep gridlines evenly spaced, and keep the origin stationary. That is linear maps can stretch and rotate the input vectors, but they don't shift the origin.

From an abstract point of view linear maps do map vectors to vectors and have the linearity properties, i.e. can be added and scaled. That means:

$$\begin{aligned} L : V \rightarrow V &\quad \text{or} \quad L : V \rightarrow W \\ L(\vec{v} + \vec{w}) &= L(\vec{v}) + L(\vec{w}) \\ L(n\vec{v}) &= nL(\vec{v}) \end{aligned} \tag{24}$$

We can derive the linear map and the resulting matrix multiplication of the vector components directly from the linearity properties (24):

$$\begin{aligned} \vec{w} = L(\vec{v}) &= L(v^1 \vec{e}_1 + v^2 \vec{e}_2) \\ &\stackrel{(24)}{=} v^1 L(\vec{e}_1) + v^2 L(\vec{e}_2) \end{aligned} \tag{25}$$

Assuming $L : V \rightarrow V$, we can express the resulting vector in $\{\vec{e}_1, \vec{e}_2\}$:

$$\begin{aligned} L(\vec{e}_1) &= L^1_1 \vec{e}_1 + L^2_1 \vec{e}_2 \\ L(\vec{e}_2) &= L^1_2 \vec{e}_1 + L^2_2 \vec{e}_2 \\ L(\vec{e}_j) &= L^k_j \vec{e}_k \end{aligned} \tag{26}$$

$$\begin{aligned} \Rightarrow \vec{w} = L(\vec{v}) &= v^1(L^1_1 \vec{e}_1 + L^2_1 \vec{e}_2) + v^2(L^1_2 \vec{e}_1 + L^2_2 \vec{e}_2) \\ &= (L^1_1 v^1 + L^1_2 v^2) \vec{e}_1 + (L^2_1 v^1 + L^2_2 v^2) \vec{e}_2 \\ &= w^1 \vec{e}_1 + w^2 \vec{e}_2 \end{aligned} \tag{27}$$

$$\begin{aligned} w^1 &= L^1_1 v^1 + L^1_2 v^2 \\ w^2 &= L^2_1 v^1 + L^2_2 v^2 \end{aligned}$$

$$\Rightarrow w^i = L^i_j v^j$$

To figure out the transformation rules for linear maps we need to express the linear map as an expansion in the new basis to see how the new basis vector are mapped:

$$\begin{aligned}
 \widetilde{L}_i^l \widetilde{\vec{e}}_l &= L(\widetilde{\vec{e}}_i) \\
 &\stackrel{(1)}{=} L(F_i^j \vec{e}_j) \stackrel{(24)}{=} F_i^j L(\vec{e}_j) \\
 &\stackrel{(26)}{=} F_i^j L_j^k \vec{e}_k \stackrel{(2)}{=} F_i^j L_j^k B_k^l \widetilde{\vec{e}}_l \\
 \widetilde{L}_i^l \widetilde{\vec{e}}_l &= B_k^l L_j^k F_i^j \widetilde{\vec{e}}_l \\
 \Rightarrow \widetilde{L}_i^l &= B_k^l \textcolor{blue}{L}_j^k F_i^j
 \end{aligned} \tag{28}$$

To figure out the transformation to the old basis we can use the fact that the forward and the backward transform are inverses of each other (3). By multiplying (28) from the left with the forward transformation matrix F_l^s and from the right with the backward transformation matrix B_t^i we get:

$$\begin{aligned}
 \widetilde{L}_i^l &= B_k^l L_j^k F_i^j \\
 F_l^s \widetilde{L}_i^l B_t^i &= F_l^s B_k^l L_j^k F_i^j B_t^i \\
 &= \delta_k^s \delta_j^i = L_s^i
 \end{aligned} \tag{29}$$

$$\Rightarrow \textcolor{blue}{L}_s^i = F_l^s \widetilde{L}_i^l B_t^i$$

1.5 Metric Tensor

The length of a vector in an arbitrary coordinate system is defined by its norm which in turn is defined by the scalar product:

$$\begin{aligned}
 \|\vec{v}\|^2 &= \vec{v} \cdot \vec{v} \\
 &= (v^1 \overset{\rightarrow}{e_1} + v^2 \overset{\rightarrow}{e_2}) \cdot (v^1 \overset{\rightarrow}{e_1} + v^2 \overset{\rightarrow}{e_2}) \\
 &= (v^1)^2 (\overset{\rightarrow}{e_1} \cdot \overset{\rightarrow}{e_1}) + v^1 v^2 (\overset{\rightarrow}{e_1} \cdot \overset{\rightarrow}{e_2}) + v^2 v^1 (\overset{\rightarrow}{e_2} \cdot \overset{\rightarrow}{e_1}) + (v^2)^2 (\overset{\rightarrow}{e_2} \cdot \overset{\rightarrow}{e_2}) \\
 &= (v^1)^2 (\overset{\rightarrow}{e_1} \cdot \overset{\rightarrow}{e_1}) + 2v^1 v^2 (\overset{\rightarrow}{e_1} \cdot \overset{\rightarrow}{e_2}) + (v^2)^2 (\overset{\rightarrow}{e_2} \cdot \overset{\rightarrow}{e_2}) \\
 &= (\tilde{v}^1)^2 (\overset{\rightarrow}{\tilde{e}_1} \cdot \overset{\rightarrow}{\tilde{e}_1}) + 2\tilde{v}^1 \tilde{v}^2 (\overset{\rightarrow}{\tilde{e}_1} \cdot \overset{\rightarrow}{\tilde{e}_2}) + (\tilde{v}^2)^2 (\overset{\rightarrow}{\tilde{e}_2} \cdot \overset{\rightarrow}{\tilde{e}_2})
 \end{aligned} \tag{30}$$

In orthogonal coordinate systems the mixed products $(\overset{\rightarrow}{e_1} \cdot \overset{\rightarrow}{e_2})$ and $(\overset{\rightarrow}{\tilde{e}_1} \cdot \overset{\rightarrow}{\tilde{e}_2})$ are equal to 0, but in the more general non-orthogonal case these terms do not disappear.

The scalar product can be written in component form based on the summation convention with an implied summation over i and j as

$$\begin{aligned}
 \|\vec{v}\|^2 = \vec{v} \cdot \vec{v} &= (v^i \overset{\rightarrow}{e_i}) \cdot (v^j \overset{\rightarrow}{e_j}) = v^i v^j (\overset{\rightarrow}{e_i} \cdot \overset{\rightarrow}{e_j}) = v^i v^j g_{ij} \\
 &= (\tilde{v}^i \overset{\rightarrow}{\tilde{e}_i}) \cdot (\tilde{v}^j \overset{\rightarrow}{\tilde{e}_j}) = \tilde{v}^i \tilde{v}^j (\overset{\rightarrow}{\tilde{e}_i} \cdot \overset{\rightarrow}{\tilde{e}_j}) = \tilde{v}^i \tilde{v}^j \tilde{g}_{ij}
 \end{aligned} \tag{31}$$

Here g is the metric tensor given in coordinates of the old and new system respectively. Its components are defined by the scalar products:

$$\begin{aligned}
 g_{ij} &= (\overset{\rightarrow}{e_i} \cdot \overset{\rightarrow}{e_j}) \\
 \tilde{g}_{ij} &= (\overset{\rightarrow}{\tilde{e}_i} \cdot \overset{\rightarrow}{\tilde{e}_j})
 \end{aligned} \tag{32}$$

In component form it can be written as a symmetric matrix, because the sequence of arguments does not influence the result of the dot product.

$$\begin{aligned}
 g_{\overset{\rightarrow}{e_i}} &= \begin{bmatrix} \overset{\rightarrow}{e_1} \cdot \overset{\rightarrow}{e_1} & \overset{\rightarrow}{e_1} \cdot \overset{\rightarrow}{e_2} \\ \overset{\rightarrow}{e_2} \cdot \overset{\rightarrow}{e_1} & \overset{\rightarrow}{e_2} \cdot \overset{\rightarrow}{e_2} \end{bmatrix}_{\overset{\rightarrow}{e_i}} \\
 g_{\overset{\rightarrow}{\tilde{e}_i}} &= \begin{bmatrix} \overset{\rightarrow}{\tilde{e}_1} \cdot \overset{\rightarrow}{\tilde{e}_1} & \overset{\rightarrow}{\tilde{e}_1} \cdot \overset{\rightarrow}{\tilde{e}_2} \\ \overset{\rightarrow}{\tilde{e}_2} \cdot \overset{\rightarrow}{\tilde{e}_1} & \overset{\rightarrow}{\tilde{e}_2} \cdot \overset{\rightarrow}{\tilde{e}_2} \end{bmatrix}_{\overset{\rightarrow}{\tilde{e}_i}}
 \end{aligned} \tag{33}$$

All dot products can be computed in an arbitrary basis system by using the metric tensor:

$$\begin{aligned}
 \vec{v} \cdot \vec{v} &= \|\vec{v}\|^2 &= v^i v^j g_{ij} = \tilde{v}^i \tilde{v}^j \tilde{g}_{ij} \\
 \vec{w} \cdot \vec{w} &= \|\vec{w}\|^2 &= w^i w^j g_{ij} = \tilde{w}^i \tilde{w}^j \tilde{g}_{ij} \\
 \vec{v} \cdot \vec{w} &= \|\vec{v}\| \|\vec{w}\| \cos(\theta) &= v^i w^j g_{ij} = \tilde{v}^i \tilde{w}^j \tilde{g}_{ij}
 \end{aligned} \tag{34}$$

$$\begin{aligned}
 (\vec{v} \cdot \vec{w}) &= (a\tilde{\vec{e}}_1) \cdot (b\tilde{\vec{e}}_2) \\
 &= ab(\tilde{\vec{e}}_1 \cdot \tilde{\vec{e}}_2) \\
 &= ab \cos \theta \\
 &= \|\vec{v}\| \|\vec{w}\| \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\vec{e}}_1 \cdot \tilde{\vec{e}}_1 &= 1 \\
 \tilde{\vec{e}}_1 \cdot \tilde{\vec{e}}_2 &= \cos \theta \\
 \tilde{\vec{e}}_2 \cdot \tilde{\vec{e}}_2 &= 1
 \end{aligned}$$

$$\boxed{\frac{(\vec{v} \cdot \vec{w})}{\|\vec{v}\| \|\vec{w}\|} = \cos \theta}$$

Figure 9: Angle θ between two arbitrary vectors.

To show that the last formula in equation (34) is correct, consider figure 9.

To check how the metric tensor transforms, we use the formulation of the metric tensor in the new system and transform it by replacing the tilde-components by expressing them in the old system via the definitions of the forward transform given by equation (7):

$$\begin{aligned}
 \tilde{g}_{ij} &= \tilde{\vec{e}}_i \cdot \tilde{\vec{e}}_j \\
 &= (F_i^k \tilde{\vec{e}}_k) \cdot (F_j^l \tilde{\vec{e}}_l) \\
 &= F_i^k F_j^l (\tilde{\vec{e}}_k \cdot \tilde{\vec{e}}_l) \\
 \Rightarrow \tilde{g}_{ij} &= F_i^k F_j^l g_{kl} \\
 g_{kl} &= \vec{e}_k \cdot \vec{e}_l \\
 &= (B_k^i \vec{e}_i) \cdot (B_l^j \vec{e}_j) \\
 &= B_k^i B_l^j (\vec{e}_i \cdot \vec{e}_j) \\
 \Rightarrow g_{kl} &= B_k^i B_l^j \tilde{g}_{ij}
 \end{aligned} \tag{35}$$

Metric tensors are **(0,2)-tensors**, because they transform with covariant transformation behaviour using two transformations for change from one system to another. This is also suggested by using lower indices for the metric tensor components.

1.6 Transformation laws of arbitrary tensors

Let's review the definition of a tensor: “*A tensor is an object that is invariant under a change of coordinates, and has components that change in a special, predictable way under a change of coordinates.*”

To sum up everything up to here:

Contravariant (1,0)-tensors transform like this (basis / components) - the components can be written as column vectors and are multiplied to the matrix from the right:

$$\begin{aligned}\tilde{\epsilon}^i &= B^i_j \epsilon^j \\ \epsilon^i &= F^i_j \tilde{\epsilon}^j\end{aligned}\quad \begin{aligned}\tilde{v}^i &= B^i_j v^j \\ v^i &= F^i_j \tilde{v}^j\end{aligned} \quad (36)$$

Covariant (0,1)-tensors transform like this (basis / components) - the components can be written as row vectors and are multiplied to the matrix from the left:

$$\begin{aligned}\tilde{\vec{e}}_j &= \vec{e}_i^i F^i_j \\ \vec{e}_j &= \tilde{\vec{e}}_i^i B^i_j\end{aligned}\quad \begin{aligned}\tilde{\alpha}_j &= \alpha_i F^i_j \\ \alpha_j &= \tilde{\alpha}_i B^i_j\end{aligned} \quad (37)$$

Linear maps are (1,1)-tensors, i.e. they have a vector and a covector part and transform like this - the matrix of the coefficients of the linear map is multiplied from the left and the right with the respective forward or backward transformation matrices:

$$\begin{aligned}\tilde{L}^i_j &= B^i_k L^k_l F^l_j \\ L^i_j &= F^i_k L^k_l B^l_j\end{aligned} \quad (38)$$

Metric tensors are (0,2)-tensors, i.e. they have two covector parts and transform like this:

$$\begin{aligned}\tilde{g}_{ij} &= F_i^k F_j^l g_{kl} \\ g_{kl} &= B_k^i B_l^j \tilde{g}_{ij}\end{aligned} \quad (39)$$

The general transformation laws for tensor components for an arbitrary tensor of type (m, n) with m upstairs indices and n downstairs indices are:

$$\begin{aligned}\tilde{T}_{xyz\dots}^{abc\dots} &= (B^a_i B^b_j B^c_k \dots) \tilde{T}_{rst\dots}^{ijk\dots} (F^r_x F^s_y F^t_z \dots) \\ T_{rst\dots}^{ijk\dots} &= (F^i_a F^j_b F^k_c \dots) \tilde{T}_{xyz\dots}^{abc\dots} (B^x_r B^y_s B^z_t \dots)\end{aligned} \quad (40)$$

The upstairs indices represent the contravariant components of the tensor and the downstairs indices represent the covariant components of the tensor. The transformation itself is a series of backward and forward transforms. To go from old to new system the covariant components use forward transformations, while the contravariant components use backward transformations.

Anything that follows these transformation rules during change of coordinates is a tensor.