01NAEX - Lecture 04
Latin Squares,
The Greco-Latin Square Design,
Balanced Incomplete Block Design

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Last lesson:
randomized complete block design (**RCBD**) as a tool to reduce the residual error by removing variability due to a known and controllable nuisance variable.

Hint:
If the nuisance source of variability is **known and controllable**, use **blocking** to eliminate its effect on the statistical comparisons among treatments.

Other types of designs that utilize the blocking principle:

- Latin square design;
- Graeco-Latin square design;
- Balanced Incomplete Block Designs.

The Latin Square Design

The $p \times p$ Latin square is a square containing p rows and p columns:

4×4
A B D C
B C A D
C D B A
D A C B

5×5
A D B E C
D A C B E
C B E D A
B E A C D
E C D A B

6×6
A D C E B F
B A E C F D
C E D F A B
D C F B E A
F B A D C E
E F B A D C

- used to systematically take care of two nuisance sources of variability,
- systematically blocks in two directions,
- each cell contains one of the p letters that corresponds to the treatments,
- each letter occurs once and only once in each row and column.
- for larger values, there are many Latin squares to choose between.

The Latin Square Design

The $p \times p$ Latin square is a square containing p rows and p columns:

4×4	5×5	6×6
A B D C	A D B E C	A D C E B F
B C A D	D A C B E	B A E C F D
C D B A	C B E D A	C E D F A B
D A C B	B E A C D	D C F B E A
	E C D A B	F B A D C E
		E F B A D C

- **A Standard Latin Square** obtains the first row and column in alphabetical order,
- A standard Latin square can always be obtained by writing the first row in alphabetical order and then writing each successive row as the row of letters just above shifted one place to the left,
- It is in the design that the Latin square differs from an ordinary block model. The analysis is almost identical.

Notice: see the relation to a **Sudoku** puzzle.

Number of Latin Squares

Standard Latin Squares and Number of Latin Squares of Various Sizes

Size	**3 × 3**	**4 × 4**	**5 × 5**	**6 × 6**	**7 × 7**	**$p \times p$**
Examples of standard squares	$A\ B\ C$ $B\ C\ A$ $C\ A\ B$	$A\ B\ C\ D$ $B\ C\ D\ A$ $C\ D\ A\ B$ $D\ A\ B\ C$	$A\ B\ C\ D\ E$ $B\ A\ E\ C\ D$ $C\ D\ A\ E\ B$ $D\ E\ B\ A\ C$ $E\ C\ D\ B\ A$	$A\ B\ C\ D\ E\ F$ $B\ C\ F\ A\ D\ E$ $C\ F\ B\ E\ A\ D$ $D\ E\ A\ B\ F\ C$ $E\ A\ D\ F\ C\ B$ $F\ D\ E\ C\ B\ A$	$A\ B\ C\ D\ E\ F\ G$ $B\ C\ D\ E\ F\ G\ A$ $C\ D\ E\ F\ G\ A\ B$ $D\ E\ F\ G\ A\ B\ C$ $E\ F\ G\ A\ B\ C\ D$ $F\ G\ A\ B\ C\ D\ E$ $G\ A\ B\ C\ D\ E\ F$	$ABC\ \ldots\ P$ $BCD\ \ldots\ A$ $CDE\ \ldots\ B$ $\vdots$ $PAB\ \ldots\ (P-1)$
Number of standard squares	1	4	56	9408	16,942,080	—
Total number of Latin squares	12	576	161,280	818,851,200	61,479,419,904,000	$p!(p-1)! \times$ (number of standard squares)

The Latin Square Design - Example

Latin Square Design for the Rocket Propellant Problem

Batches of Raw Material	Operators 1	2	3	4	5
1	$A = 24$	$B = 20$	$C = 19$	$D = 24$	$E = 24$
2	$B = 17$	$C = 24$	$D = 30$	$E = 27$	$A = 36$
3	$C = 18$	$D = 38$	$E = 26$	$A = 27$	$B = 21$
4	$D = 26$	$E = 31$	$A = 26$	$B = 23$	$C = 22$
5	$E = 22$	$A = 30$	$B = 20$	$C = 29$	$D = 31$

- An experimenter is studying the effects of five different formulations of a rocket propellant on the observed burn rate.
- Each formulation is mixed from a batch of raw material that is only large enough for five formulations to be tested. And the formulations are prepared by several operators.
- Each formulation once in each batch.
- Each operator uses each formulation once.

The Statistical model for a simple Latin square ($\epsilon \sim \mathcal{N}(0, \sigma^2)$):

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \end{cases}$$

The Latin Square Design - ANOVA

ANOVA consists of partitioning the total sum of squares of the $N = p^2$ observations into components for rows, columns, treatments, and error:

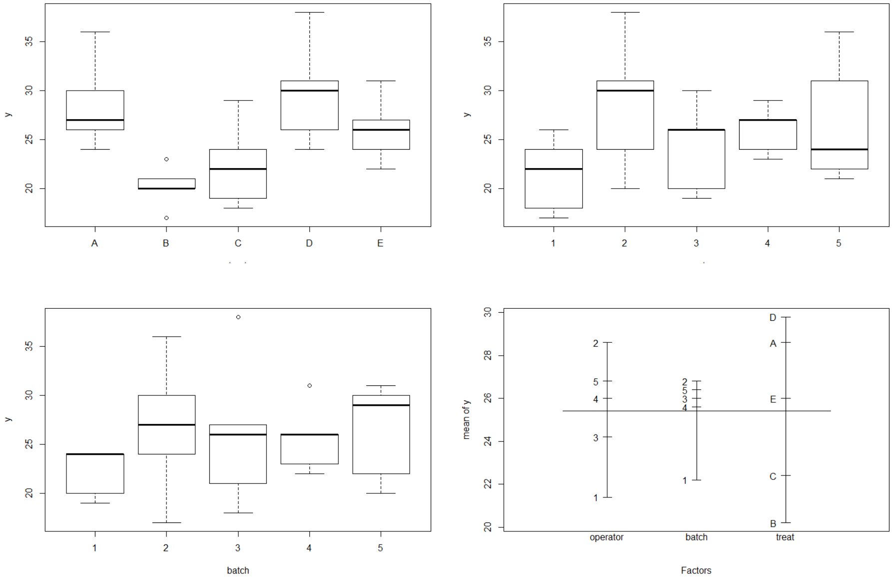
$$SS_{Total} = SS_{Rows} - SS_{Columns} - SS_{Treatments} - SS_{Error}$$

Analysis of Variance for the Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}} = \frac{1}{p}\sum_{j=1}^{p} y_{.j.}^2 - \frac{y_{..}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i..}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{k=1}^{p} y_{..k}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Error	SS_E (by subtraction)	$(p-2)(p-1)$	$\frac{SS_E}{(p-2)(p-1)}$	
Total	$SS_T = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{y_{...}^2}{N}$	p^2-1		

The analysis is a simple extension of the randomized complete block design.

The Latin Square Design - Example



The Latin Square Design - ANOVa of the Example

```
anova(lm(y~operator+batch+treat,rocket))
Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value   Pr(>F)
operator   4    150  37.500  3.5156 0.040373 *
batch      4     68  17.000  1.5937 0.239059
treat      4    330  82.500  7.7344 0.002537 **
Residuals 12    128  10.667
```

There is

- a significant difference in the mean burning rate generated by the different rocket propellant formulations.
- an indication that differences between operators exist.
- no strong evidence of a difference between batches of raw material.

The Latin Square Design - ANOVa of the Example

```
anova(lm(y~operator+batch+treat,rocket))
Analysis of Variance Table
Response: y
Df Sum Sq Mean Sq F value   Pr(>F)
operator   4    150  37.500  3.5156 0.040373 *
batch      4     68  17.000  1.5937 0.239059
treat      4    330  82.500  7.7344 0.002537 **
Residuals 12    128  10.667

anova(lm(y~operator+treat,rocket))
Analysis of Variance Table
Response: y
Df Sum Sq Mean Sq F value   Pr(>F)
operator   4    150   37.50  3.0612 0.047378 *
treat      4    330   82.50  6.7347 0.002237 **
Residuals 16    196   12.25
```

In this particular experiment, blocking on operators factor was a good precaution and we were unnecessarily concerned about the source of variability caused by batches.

However, blocking on batches of raw material is usually a good idea.

The Latin Square Design - Replication

Small Latin squares provide a relatively small number of error DF (3×3 has only 2 and 4×4 only 6) . It's desirable to replicate them.

Replication can be done in several ways (use previous example with batches and operators):

- **Case 1** use same batches and operators in each replicate;
- **Case 2** use same batches but different operators in each replicate (or vice versa);
- **Case 3** use different batches and different operators.

The analysis of variance depends on the method of replication.

Analysis of Variance for a Replicated Latin Square

Case 1: Using same batches and operators in each replicate;

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{np}\sum_{i=1}^{p} y_{i...}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	
Columns	$\frac{1}{np}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)[n(p+1)-3]$	$\frac{SS_E}{(p-1)[n(p+1)-3]}$	
Total	$\sum\sum\sum\sum y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

Analysis of Variance for a Replicated Latin Square

Case 2: Using new batches of raw material but the same operators in each replicate.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{p}\sum_{l=1}^{n}\sum_{i=1}^{p} y_{i..l}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Rows}}}{n(p-1)}$	
Columns	$\frac{1}{np}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)(np-1)$	$\frac{SS_E}{(p-1)(np-1)}$	
Total	$\sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

Note that the source of variation for the rows really measures the variation between rows within the n replicates.

Analysis of Variance for a Replicated Latin Square

Case 3: Using new batches of raw material and new operators in each replicate.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$\frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$\frac{1}{p}\sum_{l=1}^{n}\sum_{i=1}^{p} y_{i..l}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Rows}}}{n(p-1)}$	
Columns	$\frac{1}{p}\sum_{l=1}^{n}\sum_{k=1}^{p} y_{..kl}^2 - \sum_{l=1}^{n}\frac{y_{...l}^2}{p^2}$	$n(p-1)$	$\frac{SS_{\text{Columns}}}{n(p-1)}$	
Replicates	$\frac{1}{p^2}\sum_{l=1}^{n} y_{...l}^2 - \frac{y_{....}^2}{N}$	$n-1$	$\frac{SS_{\text{Replicates}}}{n-1}$	
Error	Subtraction	$(p-1)[n(p-1)-1]$	$\frac{SS_E}{(p-1)[n(p-1)-1]}$	
Total	$\sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	np^2-1		

Note that the variation resulting from both the rows and columns measures the variation resulting from these factors within the replicates.

The Greco-Latin Square Design

Consider one $p \times p$ Latin Square Design superimposed with another $p \times p$ Latin Square Design in which the treatment are denoted by Greek letters:

- If each Greek letter appears once and only once with each Latin letter, the two Latin squares are said to be **orthogonal** and called Greaco-Latin square.
- The Greaco-Latin square design can be used to control systematically three sources of extraneous variability (block in three dimension).

4×4 Graeco-Latin Square Design

	Column			
Row	1	2	3	4
1	$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
2	$B\delta$	$A\gamma$	$D\beta$	$C\alpha$
3	$C\beta$	$D\alpha$	$A\delta$	$B\gamma$
4	$D\gamma$	$C\delta$	$B\alpha$	$A\beta$

The Greco-Latin Square Design

Greek latters appear exactly once in each row and column and exactly once with each Lattin leter.

The analysis is similar to that of a Latin Square.

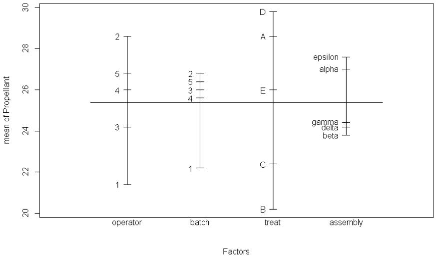
Analysis of Variance for a Graeco-Latin Square Design

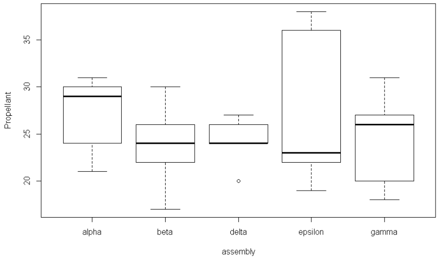
Source of Variation	Sum of Squares	Degrees of Freedom
Latin letter treatments	$SS_L = \frac{1}{p}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Greek letter treatments	$SS_G = \frac{1}{p}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i...}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{l=1}^{p} y_{...l}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Error	SS_E (by subtraction)	$(p - 3)(p - 1)$
Total	$SS_T = \sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	$p^2 - 1$

The null hypotheses of equal row, column, Latin letter, and Greek letter treatments is tested by dividing the corresponding MS by MSE. The rejection region is the upper tail point of the $F_{(p-1)/(p-3)(p-1)}$ distribution.

The Graeco-Latin Square Design - Example

Suppose that in the rocket propellant exp. test assemblies is important.






```
>rocket.lm<-lm(Propellant~operator+batch+treat+assembly)
>anova(rocket.lm)
Analysis of Variance Table:
Response: Propellant
Df Sum Sq Mean Sq F value   Pr(>F)
operator   4    150   37.50  4.5455 0.032930 *
batch      4     68   17.00  2.0606 0.178311
treat      4    330   82.50 10.0000 0.003344 **
assembly   4     62   15.50  1.8788 0.207641
Residuals  8     66    8.25
```

Balanced Incomplete Block Design

If we are not able to run all the treatment combinations in each block, we can apply randomized incomplete block designs.

Balanced Incomplete Block Design (BIBD):

- all treatment comparisons are equally important,
- the treatment combinations in each block are selected in a balanced manner,
- any pair of treatments occur together the same number of times as any other pair.

Suppose that there are a treatments and that each block can hold exactly k ($k < a$) treatments:

- a balanced incomplete block design may be constructed by taking $\binom{a}{k}$ blocks and assigning a different combination of treatments to each block.

Balanced Incomplete Block Design

Let us assume there are

- a number of treatments,
- b number of blocks,
- k number of treatments in each block,
- r number of replications of each treatments in the design
- N number of observations, i.e. $N = ar = bk$,

then the number of times each pair of treatments appears in the same block is

$$\lambda = \frac{r(k-1)}{a-1}.$$

The parameter λ must be an integer and if $a = b$, the design is called **symmetric**.

BIBD - ANOVA

Total variability in the BIBD design can be partitioned into

$$SS_T = SS_{Treatments(adjusted)} + SS_{Block} + SS_E$$

where the sum of squares for treatments is adjusted to separate the treatment and the block effects and it holds:

$$SS_{Treatments(adjusted)} = \frac{k\sum_{i=1}^{a} Q_i^2}{\lambda a} = \frac{k\sum_{i=1}^{a}(y_{i\cdot} - \frac{1}{k}\sum_{j=1}^{b}\delta_{ij}y_{\cdot j})}{\lambda a} \qquad i = 1, \ldots, a$$

with $\delta_{ij} = 1$ if treatment i appears in block j and $\delta_{ij} = 0$ otherwise.

Analysis of Variance for the Balanced Incomplete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments (adjusted)	$\frac{k\sum Q_i^2}{\lambda a}$	$a-1$	$\frac{SS_{\text{Treatments(adjusted)}}}{a-1}$	$F_0 = \frac{MS_{\text{Treatments(adjusted)}}}{MS_E}$
Blocks	$\frac{1}{k}\sum y_{\cdot j}^2 - \frac{y_{\cdot\cdot}^2}{N}$	$b-1$	$\frac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$N-a-b+1$	$\frac{SS_E}{N-a-b+1}$	
Total	$\sum\sum y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{N}$	$N-1$		

BIBD - Example

Suppose that a chemical engineer thinks that the time of reaction for a chemical process is a function of the type of catalyst employed.
The experimental procedure consists of selecting a batch of raw material, applying each catalyst in a separate run, and observing the reaction time.
It has been decided to use batches of raw material as blocks. However, each batch is only large enough to permit three catalysts to be run.

Balanced Incomplete Block Design for Catalyst Experiment

Treatment (Catalyst)	Block (Batch of Raw Material) 1	2	3	4	$y_{i.}$
1	73	74	—	71	218
2	—	75	67	72	214
3	73	75	68	—	216
4	75	—	72	75	222
$y_{.j}$	221	224	207	218	$870 = y_{..}$

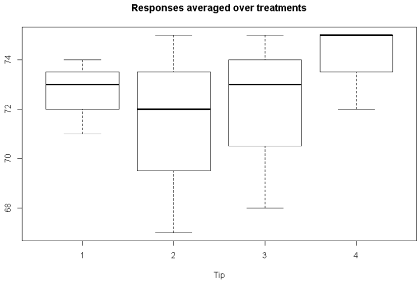
BIBD - Example

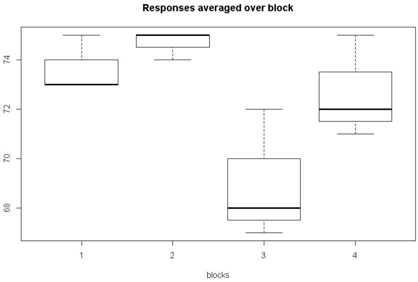
Consider the data for the catalyst experiment, $a = 4$, $b = 4$, $k = 3$, $r = 4$, $\lambda = 2$, and $N = 12$.

$$
\begin{aligned}
SS_T &= \sum_i \sum_j y_{ij}^2 - \frac{y_{..}}{12} = 63.156 - \frac{870^2}{12} = 81.00 \\
SS_{Blocks} &= \frac{1}{3}\sum_{j=1}^{4} y_{.j}^2 - \frac{y_{..}}{12} = \frac{221^2 + 207^2 + 224^2 + 218^2}{3} - \frac{870^2}{12} = 55.00 \\
Q_1 &= 218 - \frac{1}{3}(221 + 224 + 218) = -\frac{9}{3} \\
Q_2 &= 214 - \frac{1}{3}(207 + 224 + 218) = -\frac{7}{3} \\
Q_3 &= 216 - \frac{1}{3}(221 + 207 + 224) = -\frac{4}{3} \\
Q_4 &= 222 - \frac{1}{3}(221 + 207 + 218) = +\frac{20}{3} \\
SS_{Treatments(adj)} &= \frac{k\sum_{i=1}^{a} Q_i^2}{\lambda a} = 22.75 \\
SS_E &= SS_T - SS_{Treatments(adjusted)} - SS_{Block} = 3.25
\end{aligned}
$$

BIBD - Example

Consider the data for the catalyst experiment.






```
> anova(lm(rep~block+treat,Catalists.df))
Analysis of Variance Table
Response: rep
Df Sum Sq Mean Sq F value   Pr(>F)
block      3  55.00 18.3333  28.205 0.001468 **
treat      3  22.75  7.5833  11.667 0.010739 *
Residuals  5   3.25  0.6500
```

Note: Block is unadjusted, to conclude something about blocks we need to compute $SS_{Blocks(adjusted)}$ similar as for treatments.

BIBD - Example

Difference between two ANOVA approaches. Error factor can be expressed separately.

```
> summary(aov(rep~treat+block+Error(block),Catalists.df))
Error: block
Df  Sum Sq Mean Sq
treat   3     55   18.33
Error: Within
Df  Sum Sq Mean Sq F value Pr(>F)
treat       3  22.75   7.583   11.67 0.0107 *
Residuals   5   3.25   0.650
```

```
> summary(aov(rep~block+treat+Error(treat),Catalists.df))
Error: treat
Df  Sum Sq Mean Sq
block   3  11.67   3.889
Error: Within
Df  Sum Sq Mean Sq F value   Pr(>F)
block       3  66.08   22.03   33.89 0.000953 ***
Residuals   5   3.25    0.65
```

BIBD - Example

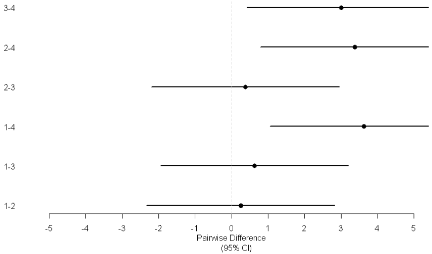
Difference between two ANOVA approaches. Depends which factor is on the first position behind the symbol $\sim$.

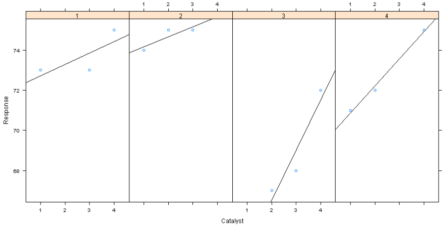
```
> anova(lm(rep~block+treat,Catalists.df))
Analysis of Variance Table
 Response: rep
Df Sum Sq Mean Sq F value   Pr(>F)
block       3  55.00 18.3333  28.205 0.001468 **
treat       3  22.75  7.5833  11.667 0.010739 *
Residuals  5   3.25  0.6500
```

```
> anova(lm(rep~treat+block,Catalists.df))
Analysis of Variance Table
Response: rep
Df Sum Sq Mean Sq F value    Pr(>F)
treat       3 11.667  3.8889  5.9829 0.0414634 *
block       3 66.083 22.0278 33.8889 0.0009528 ***
Residuals  5  3.250  0.6500
```

BIBD - Example

A multiple comparison analysis, using the Tukey method. Confidence intervals on the differences in all pairs of means are displayed.





Notice that the Tukey method would lead us to conclude that catalyst 4 is different from the other three.

We can't use straightly classical `TukeyHSD` R function that is designed for RCBD only.

BIBD - Example by package multcomp

All pairwise comparisons without any multiple testing correction

```
> library(multcomp)
> Catalist_aov <-(aov(rep~block+treat,Catalists.df))
> contr <- glht(Catalist_aov, linfct = mcp(treat = "Tukey"))
> summary(contr, test = adjusted("none"))

Simultaneous Tests for General Linear Hypotheses
Multiple Comparisons of Means: Tukey Contrasts
Fit: aov(formula = rep ~ block + treat, data = Catalists.df)
Linear Hypotheses:
Estimate Std. Error t value Pr(>|t|)
2 - 1 == 0   0.2500     0.6982   0.358   0.73492
3 - 1 == 0   0.6250     0.6982   0.895   0.41173
4 - 1 == 0   3.6250     0.6982   5.192   0.00349 **
3 - 2 == 0   0.3750     0.6982   0.537   0.61424
4 - 2 == 0   3.3750     0.6982   4.834   0.00474 **
4 - 3 == 0   3.0000     0.6982   4.297   0.00774 **
(Adjusted p values reported -- none method)
```

Today Exercise (1)

Problem 4.23 from the chapter 4, D. C. Montgomery DAoE - 8. edition.

An industrial engineer is investigating the effect of four assembly methods (A, B, C, D) on the assembly time for a color television component. Four operators are selected for the study. Furthermore, the engineer knows that each assembly method produces such fatigue that the time required for the last assembly may be greater than the time required for the first, regardless of the method. That is, a trend develops in the required assembly time. To account for this source of variability, the engineer uses the Latin square design shown below. Analyze the data from this experiment (use $\alpha = 0.05$) and draw appropriate conclusions.

	Operator			
Order	1	2	3	4
1	C = 10	D = 14	A = 7	B = 8
2	B = 7	C = 18	D = 11	A = 8
3	A = 5	B = 10	C = 11	D = 9
4	D = 10	A = 10	B = 12	C= 14

Today Exercise (2)

Problem 4.40 from the chapter 4, D. C. Montgomery DAoE - 8. edition.

An engineer is studying the mileage performance characteristics of five types of gasoline additives. In the road test he wishes to use cars as blocks; however, because of a time constraint, he must use an incomplete block design. He runs the balanced design with the five blocks that follow. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Car				
Additive	1	2	3	4	5
1		17	14	13	12
2	14	14		13	10
3	12		13	12	9
4	13	11	11	12	
5	11	12	10		8

Today Exercise (3)

Problem 4.42 from the chapter 4, D. C. Montgomery DAoE - 8. edition.

Seven different hardwood concentrations are being studied to determine their effect on the strength of the paper produced. However, the pilot plant can only produce three runs each day. As days may differ, the analyst uses the balanced incomplete block design that follows. Analyze the data from this experiment (use $\alpha = 0.05$) and draw conclusions.

	Days						
Concentration (%)	1	2	3	4	5	6	7
2	114				120		117
4	126	120				119	
6		137	117				134
8	141		129	149			
10		145		150	143		
12			120		118	123	
14				136		130	127

Try to run, in addition to ANOVA with BIBD, the linear model with concentration as a quantitative response too (on condition there is no day effect).