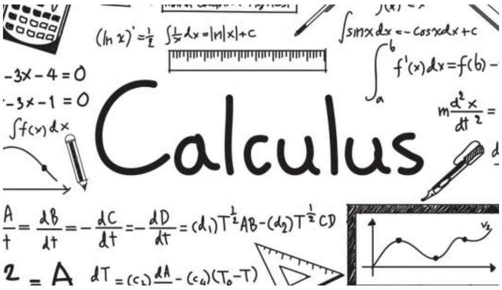

An introduction to the big O and small o notation

the fashion of approach in calculus

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May 9, 2021

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Chapter 1

Introduction

We have seen notations like

$$o(x)$$

in Taylor series many times. For example,

$$arcsin(x) = x + \frac{1}{3!}x^3 + \frac{9}{5!}x^5 + o(x^6)$$

It seems that $o(x^6)$ is an error term for polynomial estimation of $arcsin(x)$. Just writing for error term is not precise, so we need a better notation like the small o. Of course we have a more analytical form of error term derived from

$$\frac{f^{(n+1)}(t)x^{n+1}}{(n+1)!}$$

But that does not show how "large" is the error at the first glance. Is it approaching linearly? Is the error going down very fast, or very slow when we increase the number of terms in Taylor series?

Here are some styles of "approach" in physics and biology:

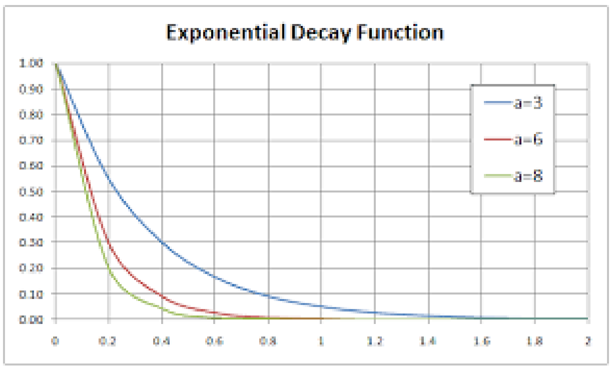

Figure 1.1: exponential decay of radioactive nuclei

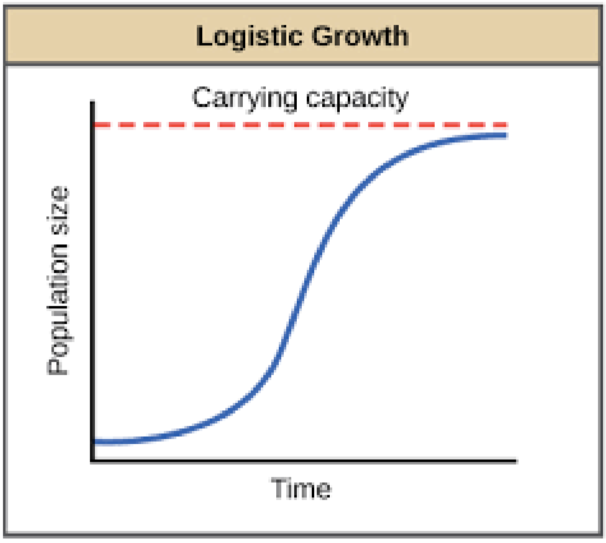

Figure 1.2: logistic growth of population

If you are familiar with signals, you might have heard of Fourier series and Fourier transformation, a way to figure out different components of frequencies from a mixed signal. Error analysis is important for Fourier series as in reality there is no time waiting for hundreds of iterations until 1% accuracy is reached .

Bachmann first used O notation in 1894, this could simplify many steps in calculus. And generally it is very intuitive if you use simple functions like x^n or $log(x)$

Chapter 2

Quantifying approach speed

2.1 Infinite small

Recall that we call a sequence infinite small if no matter how small we want it to be ($< \epsilon$), going far down ($n \geq N$) the sequence would satisfy this target. Similar definition applies to a function, but we are going close to some point a ($x \to a$) instead of going down a sequence.

Let's consider the following functions for $x \to 0$, which one approaches 0 with the fastest speed?

1. x^3
2. x
3. $e^x - 1$

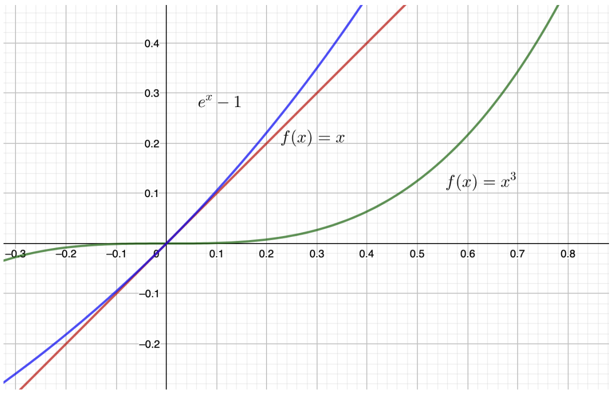

Figure 2.1: three functions drawn on the same pair of axes

It is easy to see that first one approaches in a fast way. One could argue that by intuition the third one is the slowest, as it is always above x. Another may say they seems to have the same gradient when getting when $x \to 0$. A more quantitative way is required to describe the speed of approach.

Recall when comparing which of two numbers is greater, say 2^3 and 3^2, one possible way is to write them as $\frac{2^3}{3^2}$ and observe if the result is:

- exactly 1. Then two numbers are the same

- smaller than 1. Then 2^3 is smaller
- greater than 1. Then 2^3 is larger

We can put functions into fractions into a similar manner! And when $\frac{f(x)}{g(x)} = 1$, we say two functions are the same. But we are looking at regions around $x = a$, so a limit should be added here: $\lim_{x \to a}(\frac{f(x)}{g(x)}) = 1$. And that means two functions approach a value in the same manner.

Definition 1. Equivalent infinite small

$f(x)$, $g(x)$ are two functions defined around $x \to a$, and both have limit 0. We say $f(x)$, $g(x)$ are infinite-smalls at $x = a$. If $\lim_{x \to a}(\frac{f(x)}{g(x)}) = 1$, we call them equivalent, denoted by $f \sim g(x \to a)$The bracket cannot be omitted, as f and g may not be equivalent around another point.

We can also define equivalent infinite large for $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$.

Question

Try to prove that $f \sim g$ is an equivalence relationship for the set of all functions defined around $x = a$.

For example, $e^x - 1 \sim x$. (This can be proved by L'Hopital's rule)

Here are some important equivalent functions.

- $sinx \sim x(x \to 0)$
- $ln(1 + x) \sim x(x \to 0)$
- $1 - cosx \sim \frac{1}{2}x^2(x \to 0)$
- $arctan(x) \sim x(x \to 0)$
- $(1 + x)^\alpha \sim 1 + \alpha x(x \to 0)$

Question 1. Try to prove the above 5 pairs of equivalent infinite-smalls.

We can see that $f(x) = x$ is equivalent to many functions around $x = 0$.

One may soon say that we can claim if $\lim_{x \to a}(\frac{f(x)}{g(x)}) < 1$, $f(x)$ approaches 0 faster than $g(x)$. This is right but cannot apply to general case. Think of $f(x) = x$, $g(x) = 3x$, indeed when x approaches 0, $f(x)$ is always closer to 0 than $g(x)$, but actually $g(x)$ descends faster.

It is not meaningful to say there is any difference between two linear decreases $f(x) = x$ and $g(x) = 3x$. Especially in light of the fact that something descends much slower exists (like $\sqrt{x}$). So we say f, g are of the same order.

Definition 2. infinite-small of the same order

If f, g are infinite-small at $x = a$, $\lim_{x \to a}(\frac{f(x)}{g(x)}) = A$, where $A \neq 0$, we say f, g are infinite-smalls of the same order.

What does "smaller" or "approach faster" really mean in this case? Compare x and $\sqrt{x}$ as a quotient, we get $\lim_{x \to 0}(\frac{x}{\sqrt{x}}) = \lim_{x \to 0} \sqrt{x} = 0$. And we know x descends way more faster than $\sqrt{x}$ when x approaches 0.

Definition 3. small o

If $\lim_{x \to a}(\frac{f(x)}{g(x)}) = 0$, denote $f(x) = o(g(x))(x \to a)$ or $f \ll g$. We say $f(x)$ is an infinite-small of higher order compared to $g(x)$ if f, g are infinite-small at $x = a$,

Remark. Actually, the real meaning of $f(x) = o(g(x))$ is $f(x) \in o(g(x))$ where we treat $o(g(x))$ as the set of all infinite-smalls of higher order compared to $g(x)$. So you cannot read it as $o(g(x)) = f(x)$.

Remark. Sometimes we see $o(1)$, it just means $\lim_{x \to a} f(x) = 0$. i.e. $f(x)$ is infinite small. The set of functions $\{g(x) = c | c \in \mathbb{R}\}$ is of the lowest order as it does not approach 0 at all.

Examples to help you get used to this symbol:

- $ln(x) = o(1)(x \to 1)$
- $ln(x) = o(x)(x \to \infty)$
- $ln(x) = o(\frac{1}{x})(x \to 0^+)$

There is even a more precise way to quantify the idea of order.

$f(x) = k(x - a)$ $(k \in \mathbb{R}, k \neq 0)$ goes to 0 when $x \to a$. And if we compare all such $f(x)$ with $g(x) = x - a$, the result $\lim_{x \to a}(\frac{f(x)}{g(x)})$ is always a non-zero real number. (i.e. they have the same order) We can use the functions $(x - a)^n, n \in \mathbb{R}^+$ as a representative for an order!

Definition 4. order of infinite-small

If $\lim_{x \to a}(\frac{f(x)}{(x-a)^\alpha}) = l$ $(l \neq 0)$, f is infinite-small. We say $f(x)$ is an α'th order infinite-small when $x = a$. Here α is any positive real number.

The idea of small O and order of infinite-small both apply to infinite-large.

Remark (no order). Some infinite-small is not comparable to any of $(x - a)^\alpha$. Say the function $f(x) = xsin(\frac{1}{x})$ when $x \to 0$, if we observe $\lim_{x \to 0}(\frac{xsin(\frac{1}{x})}{x^\alpha})$. Take $\alpha \geq 1$, the limit does not exist; take $\alpha < 1$, the limit is always 0. $xsin(\frac{1}{x}) = o(x^\alpha) \forall \alpha < 1 (x \to 0)$

We can think of this as the infinite-small $xsin(\frac{1}{x})$ at $x = 0$ has higher order than any of $(x-a)^\alpha$. Or it has infinite order.

There are two famous equivalence relationships:

1. Stirling formulae
$$n! \sim (\frac{n}{e})^n \sqrt(2)$$
2. Theorem of prime numbers
$$\pi(x) \sim \frac{x}{lnx}(x \to \infty)$$

Some important relationships between infinite-large are worth to know:

$$\ln x \ll x^\epsilon \ll a^x \ll x^x (a > 1, \epsilon > 0)$$

2.2 Big O

Big O studies the size of each function instead of how infinite-smalls, infinite-larges are interacting. For example, on the whole real line, $f(x) = sin(x)$ is always bounded by $g(x) = 1$.

Definition 5. Big O

We say $f(x) = O(g(x))(x \to a)$ if there is a constant $M > 0$ such that $\left\|\frac{f(x)}{g(x)}\right\| \leq M$ for a small neighbourhood around(not including) $x = a$.

Remark. The form is equivalent to $|f(x)| \leq M|g(x)|$ Here we allow M to be greater than 1. i.e. $f(x)$ cannot be bounded by $g(x)$, but it can be bounded by stretching $g(x)$ along y-axis. There are some functions where we can never achieve this. `e.g.` $f(x) = x$, $g(x) = 1$ for $x \to \infty$.

Remark. If $f(x) = O(1)(x \to a)$, by definition, it is locally bounded.

Question 2. Prove $cos(x) = 1 + O(x^2)(x \to 0)$

Proof. It is easy to prove that $\lim_{x\to 0}(\frac{1-cosx}{x^2}) = \frac{1}{2}$. So $\exists \delta > 0, s.t. \frac{1-cosx}{x^2}$ is bounded. That means $cosx - 1 = O(x^2)(x \to 0)$. Move -1 to the other side proves the question. ∎

2.3 Substitutions with equivalent infinite-smalls

Question 3. $\alpha \neq 0$, find $\lim_{x\to 0}(\frac{(1+x)^\alpha - 1}{x})$

Theorem of prime numbers is an important idea in number theory. It was first given by Legendre and Gauss by experiment in 1896.

Answer 3.1. Let $y = (1+x)^\alpha - 1$. When $x \to 0$, $y \to 0$. And $1 + y = (1+x)^\alpha$, so $ln(1+y) = \alpha ln(1+x)$.

$$\lim_{x\to 0}(\frac{(1+x)^\alpha - 1}{x}) = \lim_{x\to 0}(\frac{(1+x)^\alpha - 1}{ln(1+x)}) = \lim_{y\to 0}(\frac{\alpha y}{ln(1+y)}) = \lim_{y\to 0}(\frac{\alpha y}{y}) = \alpha$$

.

Remark. Without substitution, we can only use binomial expansion to deal with rational α. We can try to change complex functions to x to create a polynomial, where the limit is easier to find.

When is this substitution valid?

Theorem 1 (Multiplication substitution rule)**.** *If $u \sim u_1$, $\lim_{x\to a} uv = \lim_{x\to a} u_1 v \frac{u}{u_1} = \lim_{x\to a} u_1 v$ by algebra of limit.*

Theorem 2 (Addition substitution rule)**.** *If $u \sim u_1$, $v \sim v_1$, $\lim_{x\to a} u + v = \lim_{x\to a} u_1 + v_1$ when $\lim_{x\to a} u + v \neq 0$*

Remark. $\lim_{x\to a} u + v \neq 0$ prevents $u + v$ from being an infinite-small of a higher order than u, v. (think of why that is a problem?)

2.4 Exercise

1. Are these infinite small? 10^{-10000}, $e^{-10^{10}}$, $x(x \to 0)$. Are these infinite large? 10^{10000}, $e^{10^{10}}$, $x(x \to \infty)$
2. Find the following limits if they exist(you may use a graph calulator to help you):

   1. $\lim_{x\to 0}(\frac{\sin x}{x})$
   2. $\lim_{x\to\infty}(\frac{\sin x}{x})$
   3. $\lim_{x\to 0} x\sin(\frac{1}{x})$
   4. $\lim_{x\to\infty}(x\sin(\frac{1}{x}))$
   5. $\lim_{x\to 0}(\frac{1}{x}\sin(\frac{1}{x}))$
   6. $\lim_{x\to\infty}(\frac{1}{x}\sin(\frac{1}{x}))$

3. Which of the following are true when $x \to 0$?

   1. If $f = o(1)$, then $f = O(1)$
   2. If $f = o(x^2)$, then $f = O(x)$
   3. If $f = O(1)$, then $f = o(1)$
   4. If $f = O(x^2)$, then $f = o(x)$
   5. If $f = o(x^2)$, $xf(x) = o(x^3)$
   6. If $f = O(x^2)$, $\frac{f(x)}{x} = o(x)$

4. f is monotone increasing on $(0, \infty)$, and $\lim_{x\to\infty}(\frac{f(2x)}{f(x)}) = 1$. Prove that for all $a > 0$, $\lim_{x\to\infty}(\frac{f(ax)}{f(x)}) = 1$

2.4.1 Challenging set

1. Dirichlet function is defined as

$$D(x) = \begin{cases} 1 & \text{for } xisrational \\ 0 & \text{for } xisirrational \end{cases}$$

Prove that $D(x) = \lim_{m\to\infty}\{\lim_{n\to\infty}[cos(\pi m!x)]^{2n}\}$

2. If $\lim_{x\to 0} f(x) = 0, f(x) - f(\frac{x}{2}) = o(x)(x \to 0)$. Prove $f(x) = o(x)(x \to 0)$