An Introduction to Vectors
(Chapter 7)

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Chapter 1

What is a vector

Many textbooks will define vector as something with a direction and a magnitude. This sounds very clear but does not make many sense. I do not see direction in $3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$ and I do not see magnitude in $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$. Most importantly, this definition is not touching any powerful magic of vectors and linear algebra and the power to solve loads of difficult problems. In this note, we will first introduce vectors visually, and then talk about basic operations, how to use vectors to investigate properties of geometric objects. Finally a little bit about scalar product and matrix.

1.1 Vector, not coordinate

Due to the resemblance between $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ and $(3, 4, 2)$. Many people would think that vectors are coordinates, written in a vertical manner. There is a huge difference between interpretation of numbers here. Imagine we are in a cinema now, with 5 rows of seats, each row has 8 seats. And there are two entrances.

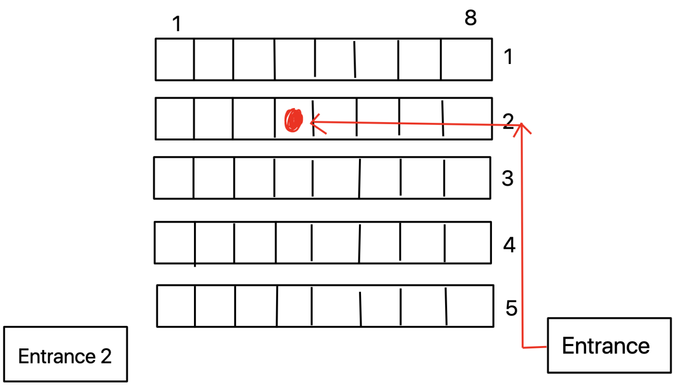


Figure 1.1: Cinema seats

Your seat is at the red dot. Probably the cinema would give you a ticket with ”row 2, seat 4" written on there. This is a coordinate. But there is another way to describe your seat. That is entering from entrance, walk through 4 rows, turn left and walk through 5 seats. Then the place you are standing at is your seat. This is a vector interpretation.

Remark.

1. You must have a starting point to give a vector meaning. If we start at entrance 2 instead, that is a totally different vector.

2. You must strictly define the basis of your direction. Here I used rows and seats, so if walk upwards through one row means $\boldsymbol{i}$, walk through one seat towards left is $\boldsymbol{j}$, the vector I described before is $4\boldsymbol{i} + 5\boldsymbol{j}$ or equivalently, $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$. Here $\boldsymbol{i}, \boldsymbol{j}$ is called a basis. But one may be unsatisfied and instead let $\boldsymbol{i}$ be walking leftwards through one seat, and let $\boldsymbol{j}$ be walking upwards through a row. Then the vector is $5\boldsymbol{i} + 4\boldsymbol{j}$. Without basis, your vector has no meaning.

3. We always write vectors in bold, but for handwriting, use $\vec{i}$ or under tilde. Another representation of vectors use two points, so $\overrightarrow{AB}$ means a vector from point A to point B.

4. Number of vectors required to form a basis(you can describe all points with these vectors) in a plane is 2, so we say a plane is 2-D. Similarly, if there must be n vectors to form a basis, you are looking at a n-dimensional space.

There is even no restrictions to defining $\boldsymbol{i}, \boldsymbol{j}$ to be perpendicular or have unit length. Of course, having unit length and being perpendicular(such basis is called orthonormal) makes life easier in many cases. But for example, following is a map of a place at London below, it is more convenient to allow basis to follow the roads. And of course we wish to describe positions by saying how many blocks you should cross along $\boldsymbol{i}, \boldsymbol{j}$ rather than saying "go to longtitude 51.4918654 and latitude -0.2122188".

Figure 1.2: Map

Chapter 2

Vector Operations

2.1 Scalar multiplication

As the name suggests, it is a multiplication between a number and a vector. (we usually write the number in front) And its geometric effect is scaling. Let's say $3\begin{pmatrix}1\\2\end{pmatrix}$, the result should be

$$\begin{pmatrix}1\times 3\\2\times 3\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix}$$

. It indeed stretches vector $\begin{pmatrix}1\\2\end{pmatrix}$ by factor 3.

Remark. You have to understand that scalar multiplication is not multiplication between numbers. They are different operations as you can see above.

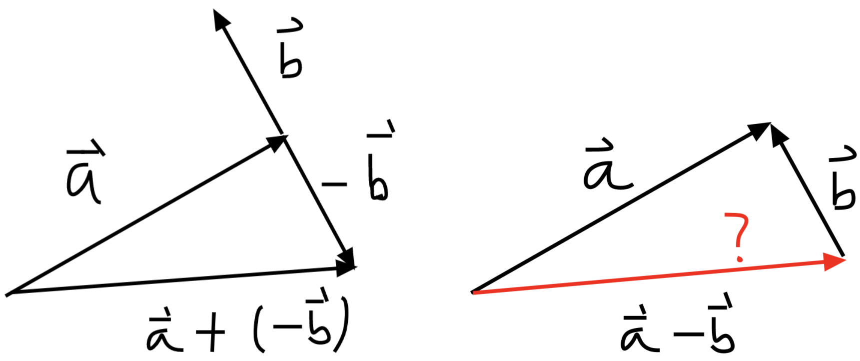
2.2 Addition

Just like scalar multiplication, addition is just a conventional name. It is not the same as number addition. This is used when we want to combine two vectors together. Say $\begin{pmatrix}2\\5\end{pmatrix}$, $\begin{pmatrix}4\\3\end{pmatrix}$. The total steps we walked in $\boldsymbol{i}$ direction is $2+4=6$ units, total steps walk in $\boldsymbol{j}$ is $5+3=8$ units. So the add-up effect of two vectors should be $\begin{pmatrix}6\\8\end{pmatrix}$.

To sum up, if you want to add two vectors, just add there components correspondingly. This works the same for higher dimensions.

It is not easy to prove that vector addition is associative, commutative and has identity 0. (0 vector is the vector with all entries 0) So you can treat it as numerical addition when manipulating equations. And scalar multiplication is distributive over vector addition.
So what does subtraction mean for vectors? There are two interpretations:

1. $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$. This means reverse vector $\boldsymbol{b}$ and then add that to $\boldsymbol{a}$.
2. If $\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{b}$, then $\boldsymbol{d} + \boldsymbol{b} = \boldsymbol{a}$. So we are asking what difference vector $\boldsymbol{d}$ added to $\boldsymbol{b}$ can create vector $\boldsymbol{a}$



(a) 1 (b) 2

Figure 2.1: Interpretations of subtraction

Two ways of thinking should yield the same result.

2.3 Scalar Product

We have learnt in physics that power is defined as $P = Fv$, where F is the force, and v is the distance travelled. But F, v are all scalars here. What if you are standing on an in-extensible car. The car is moving forward, but you keep jumping on the top of car. Yes you have speed relative to the car and you apply your weight to the car. But did you do any work to the car? Did you pass any energy to it? No except a little bit heat energy. So we cannot blindly apply the formulae everywhere.

Now let's imagine that you are using a trolley to carry heavy penguin with force $\boldsymbol{F}$ and trolley travels at velocity $\boldsymbol{v}$ (velocity is vector version of speed) How much effective force are you actually applying to the trolley?

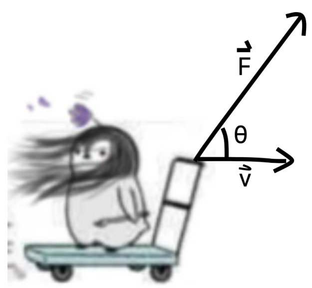


Figure 2.2: Force and velocity of trolley

It should be the projection of $\boldsymbol{F}$ onto the direction of $\boldsymbol{v}$! In this case, effective force is $|\boldsymbol{F}|\cos\theta$. So power $P = |\boldsymbol{F}|\cos\theta|\boldsymbol{v}| = |\boldsymbol{F}||\boldsymbol{v}|\cos\theta$. This kind of calculations also appears in other places, so mathematicians define something called scalar(dot) product of vectors: $\boldsymbol{v} \cdot \boldsymbol{u} = |\boldsymbol{u}||\boldsymbol{v}| \cos\theta$ where θ is the angle between two vectors. Note that inputs are vectors, but output is a scalar. Similarly, work is $\boldsymbol{F} \cdot \boldsymbol{d}$, where $\boldsymbol{d}$ is displacement.

Properties

1. Commutative: $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
2. Distributive over vector addition: $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
3. Also, you can take constants out: $(c\boldsymbol{a}) \cdot \boldsymbol{b} = c(\boldsymbol{a} \cdot \boldsymbol{b})$
4. if two non-zero vectors(it is pointless to talk about perpendicularity of zero vectors) are perpendicular, their scalar product is 0. Reversely, as long as two vectors are not 0, if their scalar product is 0, two vectors are perpendicular. (Think of why)
5. But be careful not to write $(\boldsymbol{a} \cdot \boldsymbol{b}) \cdot \boldsymbol{c}$ (Think of why not)
6. Cancellation is not allowed for scalar product. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{c}$ does not mean $\boldsymbol{b} = \boldsymbol{c}$. Because if we rearrange it using distributive law, $\boldsymbol{a} \cdot (\boldsymbol{b} - \boldsymbol{c}) = 0$. This just means $\boldsymbol{b} - \boldsymbol{c}$ is perpendicular to $\boldsymbol{a}$.

2.3.1 Another way to calculate

Finding the angle between two vectors is quite hard sometimes, there is actually an easier way to find scalar product. If $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2$ as long as the basis you chose is orthonormal. (all perpendicular to each other and have unit length) Our standard basis(the one you used to define coordinates) is one of the orthonormal basis. The rigorous proof is not easy but there is a perfect video explanation in 3Blue1Brown's playlist *Essence of Linear Algebra*. Please do check out if you are interested. But to understand that, you need to check chapter 4 for meaning of a matrix

So now we have found a way to find the angle between two vectors! $\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$. Where you can calculate $\boldsymbol{u} \cdot \boldsymbol{v}$ using the faster method.

2.3.2 An application of scalar product

Given a point $(2, 1)$ and a line $y = 4x + 10$, we learnt how to find the perpendicular foot before using the fact that product of gradients of perpendicular lines is -1. Now we can do this using vectors. First we should find the direction vector of the line: $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Then assume normal direction to the for the line is $\begin{pmatrix} a \\ b \end{pmatrix}$, we know $a + 4b = 0$. So $a = -4b$. Since length of direction vector does not matter, we can simply pick

$b = 1, a = -4$. Then $\begin{pmatrix} 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ is the equation of perpendicular line passing through point $(2, 1)$. You will learn how to find intersection between these two lines in chapter three.

2.3.3 Mysterious Zero Vector

In most of the cases, we can talk about direction of a vector based on a reference line(In standard 2-D basis, the reference line is usually positive x-axis). But zero vector(abbrev. **0**) does not have direction. If we allow zero vector to have a direction, then since dot product of all vectors with zero vector is zero $\mathbf{0} \cdot a = 0$. This means zero vector is perpendicular to any vector. Obviously a mess to define in that way. So we usually avoid talking direction of zero vector, or just say "zero vector has no direction".

2.4 magnitude

We need a way to find magnitude(or length) of a vector. We have investigated this in the first note. Using Pythagoras law, one can prove that for any vector $\boldsymbol{v} = \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$, $|\boldsymbol{v}| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$

2.5 Types of vector

Depending on the use, vectors have two types. One is displacement vector, starting point does not matter. It represents a spatial change. Examples include velocity vector, force vector etc. Another one is position vector, starting from the origin, it represents positions of points.

Chapter 3

Vector and Geometry

3.1 Representing a line

Instead of using an equation of the form $y = ax + b$ to describe a line, there is a more natural way.

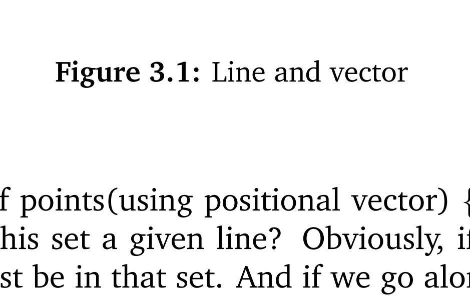


Figure 3.1: Line and vector

Suppose we have a set of points(using positional vector) $\{\boldsymbol{r}\}$, how can we add restrictions on $\boldsymbol{r}$ to make this set a given line? Obviously, if we already know point A($\boldsymbol{a}$) is on the line, $\boldsymbol{a}$ must be in that set. And if we go along direction of the line $\boldsymbol{v}$ from position $\boldsymbol{a}$, obviously we will paint the whole line. So any point on the line(set of points) can be represented by $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$, where t is a parameter that acts like a slider. Remember $\boldsymbol{r}$ here is a moving point, its actual position is determined by parameter t. The "line" is the set of points $\{\boldsymbol{a} + t\boldsymbol{v} | t \text{is a number}\}$.

A great advantage of using vectors is that you can go to higher dimensions with ease. In higher dimensions, you do not even have to alter the definition for a line. Just add more entries to $\boldsymbol{a}$ and $\boldsymbol{v}$. But a line in 4-D space is really difficult to imagine.

3.1.1 Unit Vector

Usually, we choose $\boldsymbol{v}$ to be a unit vector(a vector with length 1). Because in that case, the distance between A and target point is just t! Because $|\overrightarrow{AR}| = |t\boldsymbol{v}| = t|\boldsymbol{v}| = t \cdot 1 = t$.

But given a line $y = 2x + 1$, how can we find the direction and change that to unit vector? First of all, gradient is 2, that means 1 unit increase in x-direction cause 2-unit increase in y-direction. So $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the direction. But this vector has length $\sqrt{1^2 + 2^2} = \sqrt{5}$, so we should divide the vector by $\sqrt{5}$ to make a unit vector in the same direction.

3.1.2 relationships between lines

In 2-D plane, lines either intersect or are parallel. And for numerical equations $y = ax + b$. It is easy to tell by comparing a whether they are parallel. And by finding x, y that satisfies both equations(construct a simultaneous equation), we can find point of intersection between lines.

What about 3-D space? There is an additional relationship: skew. (Not parallel, but not intersecting) Just like one aeroplane travelling south and another travelling north but flying at different altitude. If their routes intersect, that can be a crash. And they are not going towards the same direction so they are not parallel.

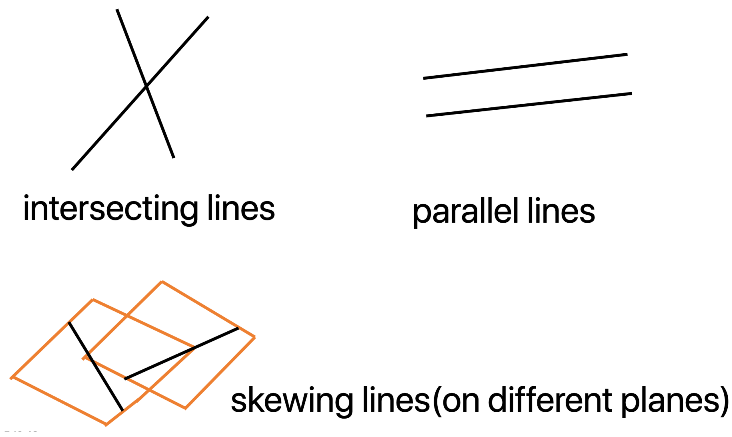


Figure 3.2: Types of relationships

It is not easy to tell this from the equation of 3-D line: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$. What about our vector equation $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$? If $\boldsymbol{v}$ are multiples of each other, the lines are parallel. (e.g. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are parallel) This even works in higher dimensions. But we need to establish an vector equation to tell if the lines are intersecting or skew. That is, assuming two lines are $\boldsymbol{r_1} = \boldsymbol{a_1} + t\boldsymbol{v_1}$ and $\boldsymbol{r_2} = \boldsymbol{a_2} + t\boldsymbol{v_2}$, we equate them to get

$$\boldsymbol{a_1} + t\boldsymbol{v_1} = \boldsymbol{a_2} + \mu\boldsymbol{v_2}$$

. Wait, why is one of t's changed to μ? Remember that parameter t is a slider. Two sliders having the same name t does not mean at a point, they have the same value. So really we should give the parameters different names.

But there are 2 unknowns with one equation! Can we solve it? Well, in fact, a vector equation in n-dimension has n equations! So just equate each entry and solve them. For example,

$$\begin{pmatrix}2\\3\end{pmatrix} + t\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\4\end{pmatrix} + \mu\begin{pmatrix}1\\1\end{pmatrix}$$

yields

$$2 + t = 5 + \mu, 3 + 2t = 4 + \mu$$

If two 2-D vectors are not parallel, you will get two distinct, non-contradicting equations with two variables to solve. That means a unique solution(for t, μ) must exist. This is why two non-parallel lines must intersect(Because one set of solutions represents one intersection).

But with 3-D vectors, you get 3 equations for 2 variables. There is a danger of not getting any solutions even if vectors are not parallel. As you have more restrictions (each equation is a restriction) So if you get no solution but two lines are parallel, this means they are skew.

3.2 Extension: Representing a plane

Because a plane is 2-D, you need two vectors on the plane to represent "direction" of the plane in 3-D space completely. But remember two vectors cannot align! In that case what you can reach is only a line on the plane.

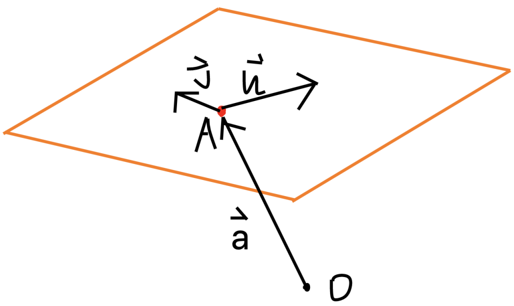


Figure 3.3: Representing a plane

Using a known point on the plane A with positional vector $\boldsymbol{a}$, after picking two vectors $\boldsymbol{u}, \boldsymbol{v}$ on the plane, we can write any point on the plane using $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{u} + \mu\boldsymbol{v}$.

You will learn relationships between planes in further math. Do not worry about them yet. This section is just for you to take a glance at the power of vectors.

Two planes can either intersect in a line or be parallel.

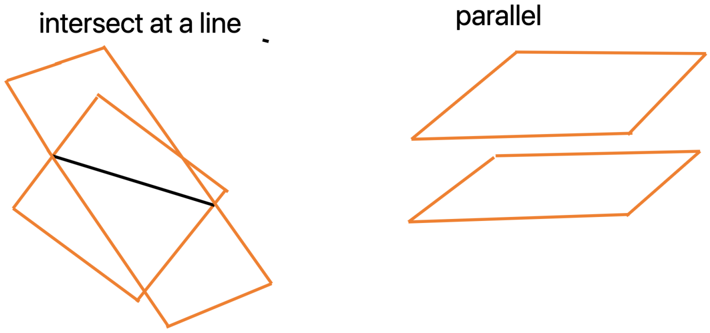


Figure 3.4: Relationships between 2 planes

There are even more exciting relationships between 3 planes, all can be distinguished by vector equations.

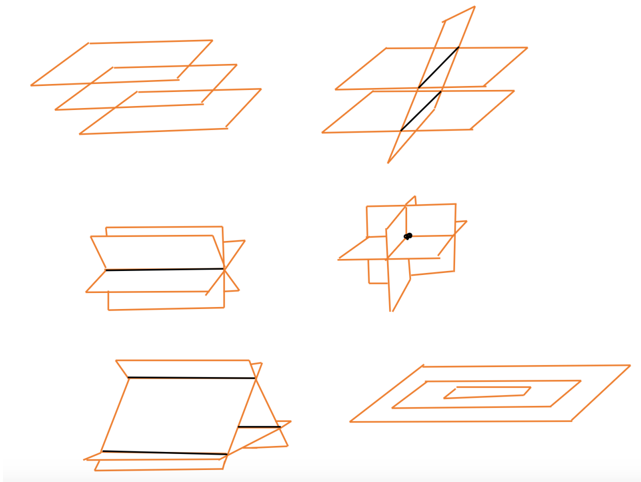


Figure 3.5: Relationships between 3 planes

Chapter 4

Extension: Matrix

In this chapter, we will briefly introduce the concept of matrix.

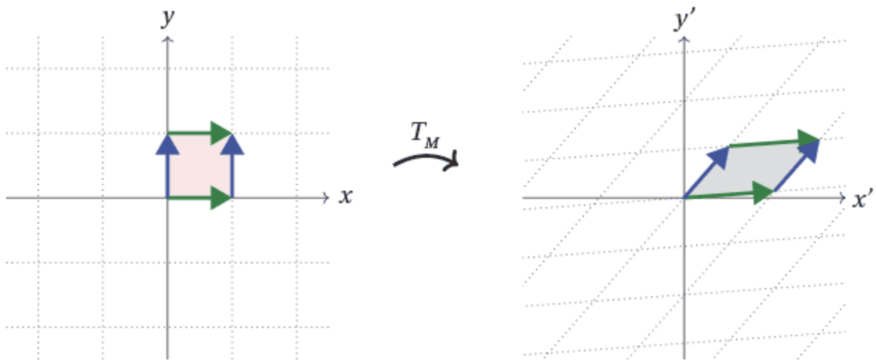


Figure 4.1: Linear transformation

Matrix, in its essence, is a linear transformation. You can imagine stretching the grid into many different shapes, with origin fixed. Rotation is a linear transformation. Other examples include stretching, compressing, rotating and stretching. (Yes! The composition of two linear transformations must be a linear transformations. As origin is never moved and the grid is again a grid.)

Strictly speaking, linear transformation is a function f defined on vectors mapping to another vector satisfying the following properties:

- $f(\boldsymbol{v} + \boldsymbol{w}) = f(\boldsymbol{v}) + f(\boldsymbol{w})$
- $f(c\boldsymbol{v}) = cf(\boldsymbol{v})$

But do not even bother to remember those. From this definition we can see that expected value from statistics is a linear transformation! And differentiation, integration are all linear transformations.

Now suppose I need a quick way to determine the coordinates of $(4, 2)$ after a rotation anti-clockwise by θ. First remember that every expression like $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ has no meaning unless you specify the basis. (What are the units?) For standard coordinates, we just choose the standard basis $B = \{\boldsymbol{i}, \boldsymbol{j}\}$. And from now on, we would express any vector under basis B using a subscript. For example, $\begin{pmatrix} 4 \\ 2 \end{pmatrix}_B$ is positional vector of the point with coordinates $(4, 2)$.

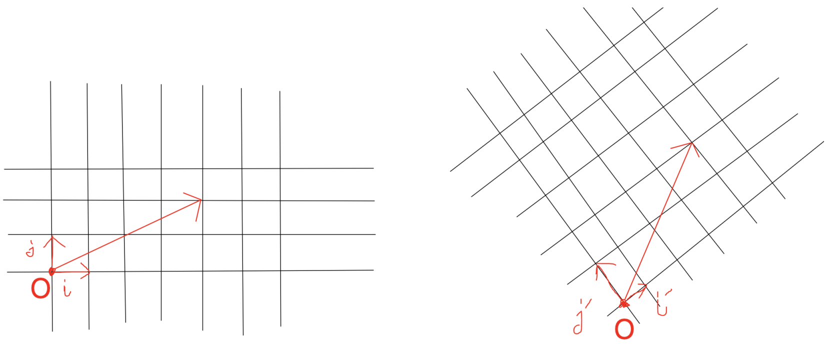


Figure 4.2: Rotation of the grid

Now a trick to do the rotation is, simply change our basis to $B' = \{\boldsymbol{i'}, \boldsymbol{j'}\}$. Where $\boldsymbol{i'}, \boldsymbol{j'}$ are the vectors after original basis $\boldsymbol{i}, \boldsymbol{j}$ rotate by θ. So now, we can directly write down the positional vector for the rotated point: $\begin{pmatrix} 4 \\ 2 \end{pmatrix}_{B'}$. Because the whole grid is rotated, so we can still walk 4 units along new x direction and 2 units along new y direction. You may wonder nothing is done! But we can change this to another form:

$$4\boldsymbol{i'} + 2\boldsymbol{j'}$$

So as long as we have a way to express $\boldsymbol{i'}, \boldsymbol{j'}$ in terms of vectors with basis B, we are done! This is easy, with a little bit of knowledge from trigonometry, we can find:

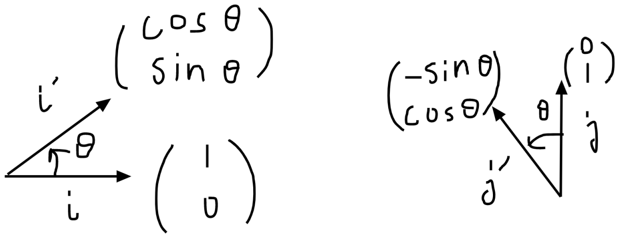


Figure 4.3: Basis change

Therefore:

$$4\boldsymbol{i}' + 2\boldsymbol{j}' = 4\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + 2\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

This process of changing from basis B' to basis B is abstracted into something called a matrix(a block of numbers):

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

The trick to perform matrix calculation is: if you are at row i, column j of the result, you pick row i of the first matrix, and you pick column j of the second matrix/vector. (Yes! Matrix can be multiplied together, this represents composition of linear transformations. Just like composition of function, the matrix on the right goes first) And you multiply the row and column together just like doing dot product. In this case, we call the matrix rotational matrix R_θ

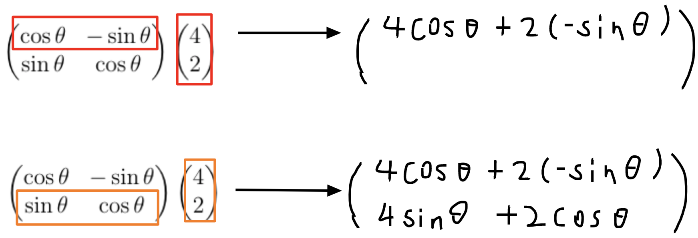


Figure 4.4: Matrix multiplication

Same process works for matrix multiplication. For example, multiplying two rotational matrices R_θ, R_ϕ yields:(try this on your own!)

$$\begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{pmatrix}$$

On the other hand, multiplying these two matrices means do rotation by θ, then do a rotation by ϕ. So this is equivalent to a rotation with angle $\theta + \phi$! Let's write down $R_{\theta+\phi}$, this should be the same matrix as above:

$$\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

So we have proved the two trig equations from here:

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

Since all the matrices we mentioned above has 2 rows, 2 columns, we call them 2-by-2 matrices. Of course you need a 3-by-3 matrices to express transformations in 3-D space. Can you guess what are the transformations represented by 2-by-3 and 3-by-2 matrices?

Chapter 5

Exercises

Q1. Write the following lines in vector form:
(a) $y = 5/7x + 3/4$
(b) $y - 4 = 6(x - 5)$
(c*)challenging $3(x - 1) = 2y = 7(z + 2)$
hint: think of the meaning of each number here, and try to relate them to the direction vector $\boldsymbol{v}$ and the position vector $\boldsymbol{a}$ in the equation $\boldsymbol{r} = t\boldsymbol{v} + \boldsymbol{a}$

Q2. Change the following vector forms to numerical equations(of whatever form you like)

1. $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -4 \end{pmatrix}$
2. $\begin{pmatrix} 4 \\ -9 \end{pmatrix} + \mu \begin{pmatrix} 3/2 \\ -2 \end{pmatrix}$

Q3. In section 2.3.2, we did not find the foot of perpendicular, find that please! And try to find foot of perpendicular from point $(1, 3)$ to the line in Q2-1.

Q4. A Coast Guard ship is located 35 km away from a checkpoint in a direction 42 degree north of west. A distressed sailboat located in still water 20 km from the same checkpoint in a direction 36 degree south of east is about to sink. Draw a diagram indicating the position of both ships. In what direction and how far must the Coast Guard ship travel to reach the sailboat?

Q5. Find the two unit vectors perpendicular to: $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - 5\boldsymbol{j} - 8\boldsymbol{k}$ correspondingly.

Q6. On your own parallel grids(that is, the grid form by two sets of parallel lines, Do not make two sets of lines perpendicular), treating each grid as a unit, pick your base vectors $\boldsymbol{i}, \boldsymbol{j}$. Draw the following vectors:

- $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

- $\boldsymbol{b} = 2\boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$
- $\boldsymbol{a} + \boldsymbol{b}$
- $2\boldsymbol{a} - \boldsymbol{b}$
- $(1.5\boldsymbol{a}) \cdot (0.5\boldsymbol{b})$

Q7. In the methane molecule, CH_4, each hydrogen atom is at the corner of a tetrahedron with the carbon atom at the center. In a coordinate system centered on the carbon atom, if the direction of one of the C-H bonds is described by the vector $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and the direction of an adjacent C-H is described by the vector $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$, what is the angle between these two bonds.

Q8*. Show diagonals of equilateral(all edges have the same length) parallelograms are perpendicular.(Hint: scalar product)

Q9. $\boldsymbol{a}, \boldsymbol{b}$ are unit vectors making angle θ, ϕ with the positive x-axis. Show that $\boldsymbol{a} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \boldsymbol{b} = \cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}$, and prove $\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$ using vectors.

Q10. We know that for the direction vector $\boldsymbol{v}$ in the vector equation of a line: $\boldsymbol{r} = t\boldsymbol{v} + \boldsymbol{a}$, you can have many choices. As long as they point to the same direction(or opposite), as the parameter t will adjust for them. And even $\boldsymbol{a}$ is flexible, as long as it is on the line. Now suppose we have a line $\boldsymbol{r} = t\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}0\\3\end{pmatrix}$. And point A is on the line when $t = 5$. Please write down the value of parameter t in the following equations so that it produces point A. (All equations represent the same line)
(a) $\boldsymbol{r} = t\begin{pmatrix}6\\3\end{pmatrix} + \begin{pmatrix}0\\3\end{pmatrix}$ (b) $\boldsymbol{r} = t\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}8\\7\end{pmatrix}$ (c) $\boldsymbol{r} = t\begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}8\\7\end{pmatrix}$

5.1 Extensional questions

Q1. Try to find a matrix for the following linear transformations:
(a) Stretch along x-axis by factor 3
(b) Shear along positive x-axis by 2 units.

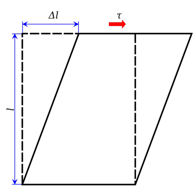


Figure 5.1: Meaning of shear

(c) $f(x, y) = (2x, y)$
(d) Translation along positive y-axis by 4 units. (In fact, if you try this, you will find it is impossible. As translation does not fix the origin so it is not even a linear transformation. But translation is an affine transformation, which is a transformation that keeps the grid as a grid with parallel straight lines but does not necessarily fix the origin. Some translations even twist the grids so that they are not parallel or not even straight lines anymore)

Hint: Try to find the new basis expressed using standard basis. That is the positional vectors of $\boldsymbol{i}, \boldsymbol{j}$ after performing the transformation. And fill them into each column of the matrix.

Q2. $A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$. Write down the following:
(a)AB
(b)BA
(c)AA
(d)BB
Are matrix multiplications commutative? (i.e. is $AB = BA$ true in general?)

Q3. Guess what does the following transformations represent:
(a) $\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$
(c*)

$$\begin{pmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}$$

Do not even bother about part c, that is a general rotation in 3-D space with 3 angles: α: yaw, rotation on x-y plane; β pitch: rotation on x-z plane; γ: row, rotation on y-z plane. 3-D rotations are very important in computer graphics, but to be honest matrices are not the best choice(as you can see from the complexity). Now, people use something called quaternions to represent rotations.