Differential Equation Cheat Sheet

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Based on notes by Dr Martin Rasmussen

We study solutions of the d-dimensional first-order differential equation (possibly with initial value $x(t_0) = x_0$, in this case we call the problem IVP, initial value problem):

$$\dot{x} = f(t, x)$$

where x can be in $\mathbb{R}^d$

Integral condition (This requires f to be continuous) A function $\lambda(t) : I \to \mathbb{R}^d$ is a solution of IVP:

$$\lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))\, ds \quad \forall t \in I$$

Picard iteration is based on this condition, by picking initial function $\lambda_0(t) \equiv x_0$. And for higher dimension, the integration is defined element-wise. One can show using induction that $\lambda_n(t)$ are differentiable, so integral term in $\lambda_{n+1}(t)$ is valid for all $t \in D$. (D is domain of t in $f(t, x)$), therefore $\lambda_n(t)$ are defined for all $t \in D$.

Theorem (Convergence of Picard iteration)**.** *Given continuous $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, consider ODE with IV*

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

If for compact interval J containing t_0, the Picard iterates $\{\lambda_n : J \to \mathbb{R}^d\}_{n \in \mathbb{N} \cup \{0\}}$ converges uniformly to λ_∞. Then λ_∞ is a solution.

Autonomous equation $\dot{x} = f(x)$ has constant solution iff $f(a) = 0$ for some $a \in \mathbb{R}^d$. And if $\lambda(t)$ is a solution, any $\mu(t) = \lambda(t + \tau), \tau \in \mathbb{R}$ is also a solution (time-translation invariance). And $\mu'(t) = \lambda(t) + k$ is also a solution with a different initial value. (space-translation invariance)

Time-reversal of solution Given solution $\lambda : I \to \mathbb{R}^d$ of $\dot{x} = f(x)$, $\mu(t) := \lambda(-t)$ defined in $-I$ is solution to $\dot{x} = -f(x)$
Separation of variables Given ODE

$$\dot{x} = g(t)h(x), \quad x(t_0) = x_0$$

where $g : I \to \mathbb{R}, h : J \to \mathbb{R}$ for some open intervals $I, J \subset \mathbb{R}$ and $h(x_0) \neq 0$. The solution can be obtained by solving

$$\int_{x_0}^{x} \frac{1}{h(y)}\ dy = \int_{t_0}^{t} g(s)\ ds$$

Autonomous DE to polar coordinates If $\dot{x} = f(x)$ where $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$. It is equivalent to solving

$$\dot{r} = f_1(r\cos\phi, r\sin\phi)\cos\phi + f_2(r\cos\phi, r\sin\phi)\sin\phi$$

$$\dot{\phi} = f_2(r\cos\phi, r\sin\phi)\cos\phi - f_1(r\cos\phi, r\sin\phi)\sin\phi$$

Solution portrait/extended phase space the (t, x) space of dimension $d+1$. We can visualise solution by drawing vector field $(t, x) \to (1, f(t, x))$
Phase Portrait(Projected) the x space of dimension d. Solutions of autonomous equations are visualised by vector field $x \to f(x)$

Theorem (Comparison)**.** *Given interval $I \subset \mathbb{R}$ and solution $\lambda : I \to \mathbb{R}$ to ODE $\dot{x} = f(t, x)$ where f is continuous. If differentiable $\alpha : I \to \mathbb{R}$ satisfies $\alpha(t) > f(t, \alpha(t))\ \forall t \in I$ with $\alpha(t_0) \geq \lambda(t_0)$ for some $t_0 \in I$, then $\alpha(t) > \lambda(t)$ for all $t \in I$ with $t > t_0$*

1 Existence and uniqueness of solutions

Lipschitz Continuity map $f : V \to W$, it is Lipschitz continuous if there exists constant $K > 0$ s.t. $d_W(f(x), f(y)) \leq K d_V(x, y)$. $K < 1$ means that the map is **contracting**. Mean value theorem/Mean value inequality is the best way to tell Lipschitz continuity. For 1-dimensional case, the mean value equality means that unbounded derivative implies not Lipschitz continuous, but in higher dimension, we cannot disprove in this way. However, one can attempt to find sequences x_n, y_n s.t. $\frac{\|f(x_n)-f(y_n)\|}{\|x_n-y_n\|} \to \infty$ as $n \to \infty$ to disprove Lipschitz continuity.

For the function $f(t, x) : D \to \mathbb{R}^d$ mentioned in the differential equation,

1. It is globally Lipschitz continuous if exists $K > 0$ s.t.

$$\|f(t, x) - f(t, y)\| \leq K\|x - y\| \quad \forall (t, x), (t, y) \in D$$

2. It is locally Lipschitz continuous if for any $(t_0, x_0) \in D$, exists $K > 0$ and neighbourhood $U \subset D$ s.t.

$$\|f(t, x) - f(t, y)\| \leq K\|x - y\| \quad \forall (t, x), (t, y) \in U$$

Def. Operator Norm Given matrix $A \in \mathbb{R}^{m\times n}$,

$$\|A\| = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|}{\|x\|} = \sup_{x\in\mathbb{R}^n, \|x\|=1} \|Ax\|$$

any linear map A is Lipschitz continuous with Lipschitz constant $\|A\|$.
In case of small matrix, operator norm $\|A\|$ can be conveniently calculated by finding smallest Eigenvalue of A^TA, and take SQUARE ROOT.
Operator norm satisfies $\|BC\| \leq \|B\|\|C\|$.
Max norm $\|A\|_{\max} = \max |a_{ij}|$. And we have inequalities

$$\|A\|_{\max} \leq \|A\| \leq m\sqrt{n}\|A\|_{\max}$$

this can be used to bound $\|A\|$.

Mean value inequality. $D \subset \mathbb{R}^n, f : D \to \mathbb{R}^m$ is continuous differentiable. For all $x, y \in D$ with the line segment $[x, y]$ joining x, y contained in D, exists $\xi \in [x, y]$ s.t.

$$\|f(x) - f(y)\| \leq \|f'(\xi)\|\|x - y\|$$

Triangle-like inequality. If f is continuous on interval $I \subset \mathbb{R}$,

$$\left\| \int_a^b f(s)\ ds \right\| \leq \left| \int_a^b \|f(s)\|\ ds \right| \quad \forall a, b \in I$$

we can remove the absolute value on RHS if $b > a$.
So restriction of continuously differentiable function f on any compact and convex set is Lipschitz continuous.

1.1 Picard-Lindelöf

(global) Picard-Lindelöf Theorem If $f(t, x)$ is continuous, and satisfies global Lipschitz continuity, then $\dot{x} = f(t, x), x(t_0) = x_0$ has unique solution $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$ where $h \leq \frac{1}{2K}$. Hence, we may push this to a global solution by applying the theorem again at endpoints.
(local) Picard-Lindelöf Theorem If $f(t, x)$ is continuous, and satisfies locally Lipschitz continuity, then $\dot{x} = f(t, x), x(t_0) = x_0$ has locally unique solution $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$ where h may depend on x_0, t_0. (In contrast to the constant h is global version)
Continuously differentiable functions (if f is multi-variable, requires Df exists and is continuous) defined on open set are locally Lipschitz continuous. So every initial value problem $\dot{x} = f(t, x)$ with continuously differentiable f has locally unique solution.
A weaker condition for f to be locally Lipschitz is $(t, x) \mapsto \frac{\partial f}{\partial x}(t, x)$ is continuous. As Lipschitz continuity in this course does not concern variation in t.

Solutions of $\dot{x} = f(t, x)$ where f is continuous and locally Lipschitz continuous cannot cross.

1.2 Maximal Solution

Given f defined on open $D \subset \mathbb{R} \times \mathbb{R}^d$ which is continuous and locally Lipschitz continuous. Consider $\dot{x} = f(t, x), \quad x(t_0) = x_0$

$$I_+(t_0, x_0) := \sup t \geq t_0 \text{ s.t. solution exists on} [t_0, t]$$

$$I_-(t_0, x_0) := \inf t \leq t_0 \text{ s.t. solution exists on} [t, t_0]$$

$I_{\max}(t_0, x_0) := (I_-(t_0, x_0), I_+(t_0, x_0))$ is the maximal existence interval. (Maximal interval is only guaranteed to exist when f is local Lipschitz continuous)
Maximal solution $\lambda_{\max} : I_{\max}(t_0, x_0) \to \mathbb{R}^d$ exists. And if $I_+(t_0, x_0))$ is finite, either solution is unbounded for $t \geq t_0$ or we are touching the boundary of domain of f(which will be non-empty in this case) in the extended phase space i.e.

$$\lim_{t \uparrow I_+(t_0, x_0)} \operatorname{dist}((t, \lambda_{\max}(t)), \partial D) = 0$$

where for $A \neq \emptyset$, $\operatorname{dist}(y, A) := \inf \|y - a\| : a \in A$. Same applies when $I_-(t_0, x_0))$ is finite.

General Solution
In this part, assumes the setting of local Picard-Lindelof theorem.
General solution combines all maximal solutions subjected to initial condition.
$\lambda(t, t_0, x_0) = \lambda_{\max}(t, t_0, x_0)$ for $t \in I_{\max}(t_0, x_0), (t_0, x_0) \in D$.
Properties of General solution:

- general solution satisfies initial value: $\lambda(t_0, t_0, x_0) = x_0$.
- taking another general solution at another initial value given by $(s, \lambda(s, t_0, x_0))$ gives the same solution. $\lambda(t, s, \lambda(s, t_0, x_0)) = \lambda(t, t_0, x_0)$ $\forall t \in I_{\max}(t_0, x_0)$.
- And the new maximal interval of the solution taken above is the same as original maximal interval. $I_{\max}(s, \lambda(s, t_0, x_0)) = I_{\max}(t_0, x_0)$.

By translational invariance of solution of autonomous ODE, we have $\lambda(t - t_0)$ is also a solution. But the initial condition is $(0, x_0)$ instead, so

$$\lambda(t, t_0, x_0) = \lambda(t - t_0, 0, x_0) \quad \forall t \in I_{\max}(t_0, x_0)$$

$$I_{\max}(t_0, x_0) = I_{\max}(0, x_0) + t_0$$

i.e. solution only depends on elapse of time, not on time/initial time separately.
Def. flow flow of autonomous ODE is

$$(t, x_0) \mapsto \varphi(t, x_0)$$

where $\varphi(t, x_0) := \lambda(t, 0, x_0)\ \forall t \in J_{\max}(x_0) := I_{\max}(0, x_0)$. $\varphi(t, x_0)$ is usually denoted $\varphi(t, x)$ for convenience.
Properties of flow

- It satisfies initial value: $\varphi(0, x) = x$.
- (Group Property) Taking another flow using existing solution as initial x, gives the same solution with shifted time. $\varphi(t, \varphi(s, x)) = \varphi(t+s, x)\ \forall t, s$ with $s, t+s \in J_{\max}(x)$.
- And the maximal interval is also shifted in time: $J_{\max}(\varphi(s, x)) = J_{\max}(x) - s$.
- $\varphi(-t, \varphi(t, x)) = x\ \forall t \in J_{\max}(x)$

Orbits φ is flow of an autonomous ODE, $\forall x \in D$

$$O(x) := \{\varphi(t, x) \in D : t \in J_{\max}(x)\}$$

is called orbit through x, which is actually the projection of solution in extended phase space to phase space(x space). O^+ is the orbit with positive t and O^- is orbit with negative t.
Types of orbits

- singleton. $f(x) = 0$, $J_{\max}(x) = \mathbb{R}$. x is called equilibrium.
- Closed curve. i.e $\exists t > 0\ s.t.\ \varphi(t, x) = x$, but $f(x) \neq 0$. $J_{\max}(x) = \mathbb{R}$ and x is called periodic.
- not closed curve. i.e. $t \mapsto \varphi(t, x)$ is injective on $J_{\max}(x)$.

If $d = 1$, solution to autonomous DE (no requirement on f) must be monotone. As a consequence, closed curve is not possible when $d = 1$.
If $f : D \to \mathbb{R}^d$ is locally Lipschitz, and there exists a solution $\lambda : I \to \mathbb{R}^d$ where I is unbounded above. And $\lim_{t\to\infty} \lambda(t) = c \in D$. Then $\mu(t) = c$ is also a solution.

If $f(t, x)$ or $f(x)$ satisfies settings of local Picard-Lindelof, then general solution and flow are continuous. If further f is continuously differentiable, general solution and flow are continuously differentiable.

Theorem (Variational Equation). *If λ is general solution to $\dot{x} = f(t, x)$, for fixed (t_0, x_0), define $\mu : I_{\max}(t_0, x_0) \to \mathbb{R}$ by $\mu(t) := \frac{\partial\lambda}{\partial x_0}(t, t_0, x_0)$. μ is the maximal solution to variational equation:*

$$\dot{y} = \frac{\partial f}{\partial x}(t, \lambda(t, t_0, x_0)) \cdot y, \quad y(t_0) = 1$$

2 Linear system

Consider solutions to

$$\dot{x} = Ax \quad x(0) = x_0$$

where A is constant matrix, $x \in \mathbb{R}^d$. One can use Picard iteration to obtain a solution $\lambda(t) = e^{At}x_0$ where

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

using group property of flow, it can be shown that $\phi(t, x_0) := e^{At}x_0$ indeed solves the ODE globally. If somehow you can obtain e^{At}, recover A by $\frac{d}{dt}\Big|_{t=0} (e^{At})$

A trick for double summation
When dealing with double summation of the form $\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}$, we can define $n := k + l$. Now the summation becomes

$$\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{k,n-k} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{k,n-k}$$

the equality is obtained by changing summing order, and it yields a finite sum that ease our work.

Properties of Matrix exponential

- $e^{B(t+s)} = e^{Bt}e^{Bs}$ for all $t, s \in \mathbb{R}$.
- If $C = T^{-1}BT$, then $e^C = T^{-1}e^B T$.
- $e^{-B} = (e^B)^{-1}$
- If $BC = CB$, $e^{B+C} = e^B e^C$
- If $B = \text{diag}(B_1, ..., B_k)$, then $e^B = \text{diag}(e^{B_1}, ..., e^{B_k})$

With the help of Jordan canonical form, $A = T^J T^{-1}$,

$$e^{At} = Te^{Jt}T^{-1}$$

For example, there are four possible forms of J if A is 2 by 2

- $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where a, b are two different real Eigenvalues.
- $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ where a is double Eigenvalue with $g(a) = 2$.
- $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ where a is double Eigenvalue with $g(a) = 1$.

- $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ s.t. $a \pm ib$ is a pair of Eigenvalues.

The following graph shows the corresponding phase portraits:

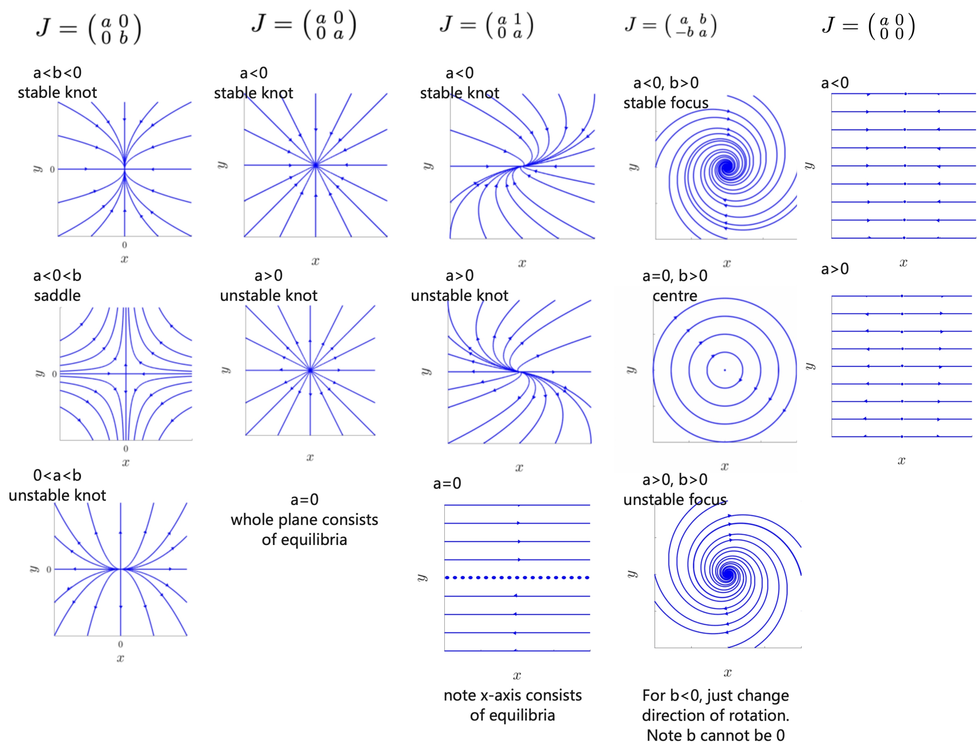

Figure 1: JCF and phase portrait

Then phase portrait of $\dot{x} = Ax$ can be obtained by applying transformation T to phase portrait of $\dot{x} = Jx$.

The matrix exponential for the last two cases are:

$$e^{\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} t} = \begin{pmatrix} e^{at} & te^{at} \\ 0 & e^{at} \end{pmatrix}, \quad e^{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} t} = e^{at} \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$$

Lyapunov exponent If initial condition is not $\mathbf{0}$,

$$\sigma_{\text{Lyap}}(\lambda) := \lim_{t \to \infty} \frac{\ln \|\lambda(t)\|}{t}$$

is the Lyapunov exponent if limit exist. This indicates rate of exponential growth. Positive Lyapunov exponent means exponential increase, vice versa.

Theorem (Real JCF)**.** *There is a transformation T s.t.*

$$J := T^{-1}AT = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_p \end{pmatrix}$$

where the Jordan blocks J_i are either as usual (if the Eigenvalue is real) or if corresponding Eigenvalue is $a+ib$, the Jordan block is

$$\begin{pmatrix} C & I_2 & & 0 & 0 \\ 0 & C & I_2 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & C & I_2 \\ 0 & 0 & & 0 & C \end{pmatrix}$$

where $C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

They way to find T which transforms A into JCF, is to first find a complex T consisting of Eigenvectors s.t. A is transformed to complex JCF. For columns w_1, w_2 in T which corresponds to a pair of conjugate complex values $a \pm bi$, replace them by $\mathrm{Re}(w_1), \mathrm{Im}(w_1)$. Do the same for every pair of columns. Then we obtain a real matrix T giving real JCF.

We have

$$e^{At} = Te^{Jt}T^{-1} = T \begin{pmatrix} e^{J_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_p t} \end{pmatrix} T^{-1}$$

And for each Jordan block, assume the shape is (d,d)

$$e^{\begin{pmatrix} \rho & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \rho \end{pmatrix}} = e^{\rho t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{d-1}}{(d-1)!} \\ 0 & 1 & t & \ddots & \dots \\ & & \ddots & \ddots & \frac{t^2}{2} \\ 0 & & & 1 & t \\ 0 & 0 & & 0 & 1 \end{pmatrix}$$

for the other form of Jordan block, assume shape is $(2d, 2d)$, and C is defined as above in the theorem.

$$e^{\begin{pmatrix} C & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & C \end{pmatrix}} = e^{at} \begin{pmatrix} G(t) & tG(t) & \frac{t^2}{2}G(t) & \dots & \frac{t^{d-1}}{(d-1)!}G(t) \\ 0 & G(t) & tG(t) & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2}G(t) \\ 0 & & & G(t) & tG(t) \\ 0 & 0 & & 0 & G(t) \end{pmatrix}$$

$$G(t) := \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$$

note,

$$e^{Ct} = e^{at}G(t)$$

so each entry of e^{Jt} is of the form $g(t)t^n e^{\rho t}$ where $n \in \{0, ..., n-1\}$ and $g(t)$ is constant or $\cos(bt)$ or $\sin(bt)$.

Spectrum of matrix $\Sigma(A) := \{\text{Re } \rho : \rho \text{ is Eigenvalue of } A\} = \{s_1, ..., s_q\}$
Assume $|\Sigma(A)| = q$, consider $\dot{x} = Ax$, there is a decomposition of $\mathbb{R}^d$:

$$E_1 \oplus \cdots \oplus E_q$$

s.t. E_j are all invariant
And if $x \in E_j \setminus \{0\}$, then $\sigma_{\text{Lyap}}(\phi(\cdot, x)) = s_j$.

Semi-simple An Eigenvalue λ is semi-simple if $a(\lambda) = g(\lambda)$. i.e. corresponding Jordan blocks are of 1-dimension if $\lambda \in \mathbb{R}$ and 2-dimensional if $\lambda \in \mathbb{C} \setminus \mathbb{R}$.
Spectrum can be used to give exponential bound for the operator norm of matrix exponential:
Pick $\gamma > \max \Sigma(A)$. If all Eigenvalues ρ of A with $\rho = \max \Sigma(A)$ are semi-simple, then take $\gamma = \max \Sigma(A)$. Then exists $K > 0$ s.t.

$$\|e^{At}\| \leq Ke^{\gamma t} \quad \forall t \geq 0$$

Variation of constant formulae
The general solution to

$$\dot{x} = Ax + g(t)$$

where $g : I \to \mathbb{R}^d$ is continuous on $I \subset \mathbb{R}$ is given by

$$\lambda(t, t_0, x_0) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}g(s)\, ds \quad \forall t, t_0 \in I,\ x_0 \in \mathbb{R}^d$$

in general, if A depends on t, i.e. $t = A(t)$, $\dot{x} = A(t)x$ is not explicitly solvable. Unless $d = 1$, $\dot{x} = a(t)x + g(t)$,

$$\lambda(t, t_0, x_0) = e^{\int_{t_0}^{t} a(s)\, ds}x_0 + \int_{t_0}^{t} e^{\int_{s}^{t} a(\tau)\, d\tau}g(s)\, ds$$

3 Non-linear Systems

Given autonomous DE $\dot{x} = f(x)$(assume f is locally Lipschitz for Section 3), equilibrium is defined as x^* s.t. $f(x^*) = 0$. Definitions for equilibrium:

- x^* is stable if $\forall \epsilon > 0,\ \exists \delta > 0$ s.t.

$$\|\varphi(t, x) - x^*\| < \epsilon \quad \forall x \in B_\delta(x^*), t \geq 0$$

- x^* is attractive $\exists \delta > 0$ s.t.

$$\lim_{t\to\infty} \varphi(t, x) = x^* \quad \forall x \in B_\delta(x^*)$$

- asymptotically stable is stable + attractive
- x^* is exponentially stable if $\exists \delta > 0, K \geq 1$ and $\gamma < 0$ s.t.

$$\|\varphi(t, x) - x^*\| \leq K e^{\gamma t} \|x - x^*\| \quad \forall x \in B_\delta(x^*), t \geq 0$$

- x^* is repulsive $\exists \delta > 0$ s.t.

$$\lim_{t\to-\infty} \varphi(t, x) = x^* \quad \forall x \in B_\delta(x^*)$$

Remark. Stability does not imply attractivity, and attractivity does not imply stability (unless $d = 1$ and f is locally Lipschitz) Exponentially stable implies attractive.

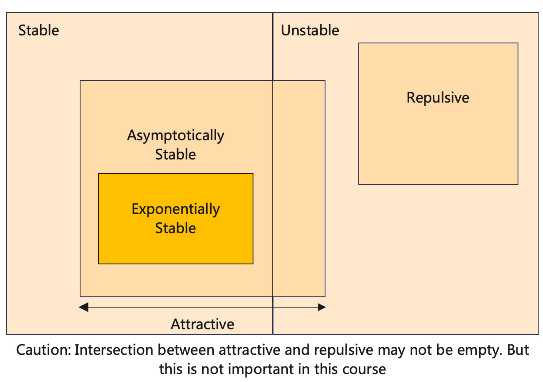

Figure 2: Relationship between types of equilibria

Definition (Homoclinic and heteroclinic orbits)**.** Consider ODE $\dot{(x)} = f(x)$ where $f : D \to \mathbb{R}$ $D \subset \mathbb{R}^d$.
Orbit $O(x)$ $(x \in D)$ is homoclinic if exists equilibrium $x^* \in D \setminus \{x\}$ s.t.

$$\lim_{t\to\infty} \varphi(t,x) = x^*, \quad \lim_{t\to-\infty} \varphi(t,x) = x^*$$

Orbit is heteroclinic if there exists different equilibria x_1^*, x_2^* s.t.

$$\lim_{t\to\infty} \varphi(t,x) = x_1^*, \quad \lim_{t\to-\infty} \varphi(t,x) = x_2^*$$

For linear system $\dot{x} = Ax$, following are relationships between Eigenvalues and stability of equilibrium

Theorem. *Trivial equilibrium $x^* = 0$ is stable iff following holds*

- *real part of all Eigenvalues of A are non-positive*
- *all Eigenvalues with $Re(\rho) = 0$ are semi-simple ($a(\rho) = g(\rho)$)*

and x^* is exponentially stable iff $Re(\rho) < 0$ for all Eigenvalues ρ.

Moreover, attractivity of autonomous linear system is equivalent to exponential stability.

Theorem (Trace-determinant rule)**.** *For matrix $A \in \mathbb{R}^{2\times 2}$ there is an easy method to tell its spectrum. Define $p := tr(A), q := \det A$, then all Eigenvalues*

- *Have negative real part $\Leftrightarrow p < 0, q > 0$ (stable)*
- *have positive real part $\Leftrightarrow p, q > 0$ (unstable)*
- *are non-real if $p^2 - 4q < 0$. (focus/centre)*
- *are real and have opposite signs $\Leftrightarrow q < 0$ (saddle)*

Definition (Hyperbolic)**.** Matrix A is hyperbolic if all Eigenvalues of A have non-zero real part. Equilibrium x^* of $\dot{x} = f(x)$ (f is continuously differentiable) is hyperbolic if matrix $Df(x^*) \in \mathbb{R}^{d\times d}$ is hyperbolic

Exponential stability and stability of hyperbolic equilibrium are locally preserved if small non-linear perturbation is added. i.e. $\dot{x} = Ax + p(x)$ where p is perturbation.

Lemma (Gronwall lemma)**.** *$u : [a,b] \to \mathbb{R}$ is continuous, $c, d \geq 0$. If*

$$0 \leq u(t) \leq c + d \int_a^t u(s)\, ds \quad \forall t \in [a,b]$$

then

$$u(t) \leq ce^{d(t-a)}$$

Theorem (Linearised stability)**.** *Given continuously differentiable $f : D \to \mathbb{R}^d$ and consider $\dot{x} = f(x)$ with equilibrium x^* s.t. all Eigenvalues $\lambda \in \mathbb{C}$ of linearisation $Df(x^*)$ satisfies Re$(\lambda) < 0$, then x^* is exponentially stable.*

Note, linearisation of a non-linear differential equation is $\dot{x} = A(x-x^*)+r(x)$, where $r = o(\|x - x^*\|)$, $A = Df(x^*)$. Remainder function r is continuously differentiable and $r(x^*) = r'(x^*) = 0$. Using time-reversal, theorem above implies hyperbolic equilibria with Eigenvalues with only positive real parts are exponentially repulsive.

Theorem (Stability of linear transform)**.** *Given ODE $\dot{x} = f(x)$ with stable equilibrium x^* and invertible matrix T. Then $y^* := T^{-1}x^*$ is a stable equilibrium to ODE $\dot{y} = T^{-1}f(Ty)$.*

Theorem (Stability of one-dimensional ODE)**.** *Given n-times differentiable function f and ODE $\dot{x} = f(x)|$. Consider equilibrium x^* s.t. $f^{(k)}(x^*) = 0$ for $k = 1, 2, ..., n-1$ and $f^{(n)}(x^*) \neq 0$.*

- *If n is odd and $f^{(n)}(x^*) < 0$, then x^* is attractive and stable (attractivity implies stability when $d = 1$)*
- *otherwise, x^* must be unstable.*

Note sign of $f^{(k)}(x^*)$ for even n indicates whether x^* is local minimum or maximum and for odd n it indicates local increase/decrease.

Definition (Stable and unstable sets)**.** For $\dot{x} = f(x)$ with flow φ and equilibrium x^*,

$$W^s(x^*) := \left\{x \in D : \lim_{t\to\infty} \varphi(t, x) = x^*\right\}$$

unstable set is

$$W^u(x^*) := \left\{x \in D : \lim_{t\to-\infty} \varphi(t, x) = x^*\right\}$$

if x^* is attractive, $W^s(x^*)$ is called domain of attraction, and it is an open neighbourhood of x^*.

For linear system $\dot{x} = Ax$ (A is hyperbolic), consider the decomposition using spectrum:

$$\mathbb{R}^d = E_1 \oplus \cdots \oplus E_q$$

where $x \in E_j \setminus \{0\}$ gives solution Lyapunov exponent s_j. By hyperbolicity, there is $1 \leq k \leq q + 1$ s.t. $s_l < 0$ for all $l < k$ and $s_l > 0$ if $l \geq k$. So

$$W^s(0) = \bigoplus_{i=1}^{k-1} E_i \quad W^u(0) = \bigoplus_{i=k}^{q} E_i$$

so $\mathbb{R}^d = W^s(0) \oplus W^u(0)$.

Definition. For non-linear ODE $\dot{x} = f(x)$, set $M \subset D$ is positive invariant if for all $x \in M$, $O^+(x) \subset M$.
M is negative invariant if $\forall x \in M$, $O^-(x) \subset M$.
M is invariant if $\forall x \in M, O(x) \subset M$.

The following sets are all invariant

- Sets of equilibria
- periodic orbits
- stable set, unstable sets.
- union of any orbits
- Omega and alpha sets
- Unions of half-orbits $O^+(x)$ are positive invariant and unions of half-orbits $O^-(x)$ are negative invariant.

Note union of any invariant sets is invariant. And if $\dot{x} = f(x)$ with locally Lipschitz continuous f, M is invariant, then boundary ∂M is also invariant.

Given $\dot{x} = f(x)$, $f : D \to \mathbb{R}^d$ and its flow φ

Definition (Omega and alpha)**.** Given $x \in D$, $x_\omega \in D$ is omega limit point of x if there is a sequence $(t_n)_{n\geq 1}$ s.t. $\lim_{n\to t_n} = \infty$ and

$$x_\omega = \lim_{n\to\infty} \varphi(t_n, x)$$

Let $\omega(x)$ be the set of omega limit points of x.

$x_\alpha \in D$ is alpha limit point of x if there is a sequence $(t_n)_{n\geq 1}$ s.t. $\lim_{n\to t_n} = -\infty$ and

$$x_\alpha = \lim_{n\to\infty} \varphi(t_n, x)$$

Let $\alpha(x)$ be the set of alpha limit points.

Properties of ω, α sets:

- maximal interval should be unbounded for alpha/omega limit point to exist.
- $$\omega(x) = \bigcap_{t\geq 0} \overline{O^+(\varphi(t,x))} \qquad \alpha(x) = \bigcap_{t\leq 0} \overline{O^-(\varphi(t,x))}$$
- And $\omega(x), \alpha(x)$ are invariant.
- If $O^+(x)$ is bounded with $\overline{O^+(x)} \subset D$ then $\omega(x)$ is non-empty, compact and connected. Same applies to $O^-(x)$ and $\alpha(x)$
- By time-reversal, $\omega(x)$ of $\dot{x} = f(x)$ is the $\alpha(x)$ for $\dot{x} = -f(x)$.

3.1 Lyapunov functions

Definition (Orbital Derivative)**.** Given ODE $\dot{x} = f(x)$ and $V : D \to \mathbb{R}$ is a continuously differentiable function. Orbital derivative $\dot{V}$ is

$$\dot{V} := V'(x) \cdot f(x) = \sum_{i=1}^{d} \frac{\partial V}{\partial x_i}(x) f_i(x)$$

where $V'(x)$ is gradient of V. This is the derivative along solution i.e. $\frac{d}{dt}V(\mu(t)) = \dot{V}(\mu(t))$.

Usually we pick positive V. In physics-related problem we use energy function and for $\mathbb{R}^2$ plane we usually use $ax^2+bxy+cy^2$. (Where a, b, c are constant chosen s.t. $\dot{V} > 0$ or $\dot{V} < 0$ for all points) Sometimes $V(x, y) = x^4 + y^4$ may have to be used.

Definition (Lyapunov function)**.** Consider ODE $\dot{x} = f(x)$ and continuously differentiable $V : D \to \mathbb{R}$, V is Lyapunov function if $\dot{V}(x) \leq 0$ for all $x \in D$. i.e. Lyapunov function decreases along the solution.

Since Lyapunov function decreases along solution i.e. $V(\varphi(t, x)) \leq V(x)$ for $t \in [0, \sup J_{\max}(x))$ (proved by fundamental theorem of calculus and $\frac{d}{dt}V(\mu(t)) = \dot{V}(\mu(t))$. This trick is very useful), so $S_c := \{x \in D : V(x) \leq c\}$ is positively invariant.

Here is a useful application of Lyapunov function:

Theorem. *Consider equilibrium $x^* \in D$ of $\dot{x} = f(x)$ and Lyapunov function V s.t. $V(x^*) = 0$ and $V(x) > 0$ for all $x \in D \setminus \{x^*\}$, x^* must be a stable equilibrium.*

Existence of Lyapunov function informs location of omega set

Theorem (La Salle's invariance principle)**.** *If there is Lyapunov function $V : D \to \mathbb{R}$, then*

$$\omega(x) \subset \{y \in D : \dot{V}(y) = 0\} \quad \forall x \in D$$

Extended version: Further, $\omega(x)$ is contained in the largest invariant subset of $\{y \in D : \dot{V}(y) = 0\}$ i.e. union of all invariant subsets.

The following is important consequence of La Salle's principle regarding asymptotic stability

Theorem. *If V is Lyapunov function and equilibrium $x^* \in D$ satisfies:*

$$V(x^*) = 0, V(x) > 0 \quad \forall x \in D \setminus \{x^*\}$$

$$\dot{V}(x^*) = 0, \dot{V}(x) < 0 \quad \forall x \in D \setminus \{x^*\}$$

then x^* is asymptotically stable.
Further if sublevel $S_c := \{x \in D : V(x) \leq c\}$ where $c > 0$ is compact, then $S_c \subset W^s(x^*)$.

Remark. If it is not possible to ensure $\dot{V}(x) < 0$ for all points except x^*, that is when set of points s.t. $\dot{V}(x) = 0$ is not singleton, one can show using extended version of La Salle's principle that $\omega(x) = \{x^*\}$ for x in a small neighbourhood around x^*, then the result of theorem still hold.

Theorem (Poincaré-Bendixson theorem)**.** *Consider $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^2$ is continuously differentiable on open set $D \subset \mathbb{R}^2$. And for some $x \in D$, $O^+(x) \subset K$ where K is compact subset of D which contains finitely many equilibria (or no equilibrium). Then one of following holds for $\omega(x)$:*

1. *$\omega(x)$ is singleton with an equilibrium*
2. *$\omega(x)$ is periodic*
3. *$\omega(x)$ consists of equilibria and non-closed orbits. non-closed orbits converge forward and backward in time to equilibria in $\omega(x)$ i.e. either homoclinic or heteroclinic.*

Similarly if the conditions hold for $O^-(x)$ we have the same conclusion for $\alpha(x)$.

Corollary of Poincaré-Bendixson, criteria for existence of periodic orbit:

Corollary. *Consider $\dot{x} = f(x)$ where $f : D \to \mathbb{R}^2$ is continuously differentiable on open set $D \subset \mathbb{R}^2$. And for some $x \in D$, $O^+(x) \subset K$ where K is compact subset of D which does not contain equilibrium, them $\omega(x)$ is periodic.*

Note ODE in $\mathbb{R}^3$ do not have such regular behaviours.

4 Examples in notes and sheets

With examples of ODEs in hand, especially the higher dimensional ones, you may gain more intuitions. The numbering of examples matches with lecture notes.

Example (No solution, Example 1.6)**.**

$$\dot{x} = f(x) := \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases} \quad x(0) = 0$$

Example (Multiple solutions, Example 1.7)**.**

$$\dot{x} = f(x) := \sqrt{|x|} \quad x(0) = 0$$

for any $b \geq 0$,

$$\lambda_b(t) := \begin{cases} 0 & \text{if } t \leq b \\ \frac{1}{4}(t-b)^2 & \text{if } t > b \end{cases}$$

is a solution to the ODE

Example (Only local solution exists, Example 1.8)**.**

$$\dot{x} = tx^2, \quad x(t_0) = x_0$$

where $x_0 \neq 0$. The solution is given by

$$\lambda(t) = \frac{2x_0}{2 + x_0(t_0^2 - t^2)}$$

which cannot exist at $t = \pm\sqrt{t_0^2 + 2/x_0}$

Example (Exponent, Example 2.18)**.** For fixed $\alpha > 0$, consider $\dot{x} = x^\alpha, x(0) = 1$ where $f(t,x) := x^\alpha$ is defined on $D := \mathbb{R} \times \mathbb{R}^+$
If $\alpha = 1$, the maximal solution is $\lambda_{1,\max}(t) = e^t$ which exists for all $t \in \mathbb{R}$. If $\alpha \neq 1$,

$$\lambda_{\alpha,\max} = (1 + (1-\alpha)t)^{\frac{1}{1-\alpha}}$$

this is not defined at $t = \frac{1}{\alpha - 1}$.

Example (Solution with periodic orbit, Example 2.26)**.** Consider 2-dimensional ODE

$$\dot{x} = y + x(1 - x^2 - y^2)$$
$$\dot{y} = -x + y(1 - x^2 - y^2)$$

with polar form $\dot{r} = r(1 - r^2), \dot{\phi} = -1$. The flow is given by

$$\varphi(t,x,y) = \frac{1}{\sqrt{x^2 + y^2 + (1 - x^2 - y^2)e^{-2t}}} \begin{pmatrix} x\cos t + y \sin t \\ y \cos t - x \sin t \end{pmatrix}$$

From the graph below we can see $O((0,1)^T)$ is a periodic orbit $\mathbb{S}^1$. And for any point x other than $(0,0)$, $\omega(x) = \mathbb{S}^1$.

(a) The phase portrait of Example 4 (b) The phase portrait of Example 4

Figure 3: Phase portraits

Example (Homoclinic and heteroclinic orbit, Example 4.3)**.**

$$\dot{x} = x + xy - (x+y)\sqrt{x^2+y^2}$$
$$\dot{y} = y - x^2 + (x-y)\sqrt{x^2+y^2}$$

is a two-dimensional ODE with polar form

$$\dot{r} = r(1-r), \qquad \dot{\phi} = r(1-\cos\phi)$$

Equilibrium $(0,0)$ is unstable knot with many tangents and attractive equilibrium $(1,0)$ is on a homoclinic orbit (attractive forward and backward), which is the unit circle. But $(1,0)$ is not stable. All orbits starting inside unit circle are heteroclinic. The phase portrait is shown in Figure 3b

Example (Nonlinear perturbation to hyperbolic linear system, Example 4.6)**.** Let matrix $A := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ consider linear system and its perturbed version below:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{5}y^2 \\ \frac{3}{10}x^2 + \frac{1}{5}y^2 \end{pmatrix}$$

The following are phase portraits of two systems We can see the behaviours at

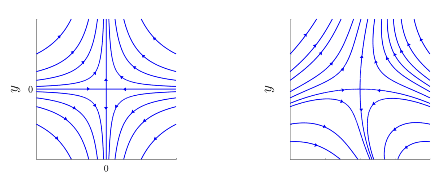

Figure 4: Phase portraits of linear and non-linear system

trivial equilibrium $x^* := (0,0)$ is similar. And stable, unstable sets at x^* are both two curves tangent at x^*.

Example (Nonlinear perturbation to non-hyperbolic linear system, Example 4.6)**.** Let matrix $A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ consider linear system and its perturbed version below:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5x(x^2 + y^2) \\ -5y(x^2 + y^2) \end{pmatrix}$$

Matrix A has Eigenvalues with 0 real part, so it is not hyperbolic. Stability of perturbed system is changed as shown below

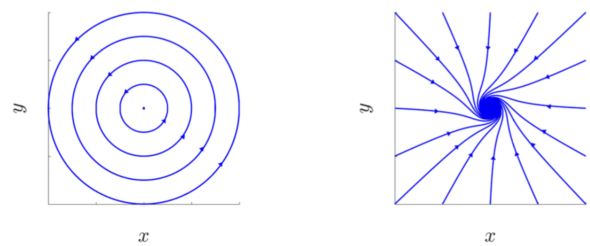

Figure 5: Phase Portrait Comparison

Example (Pendulum, Example 4.11, 4.16, 4.23, 4.29)**.** This example is visited 4 times in lecture notes.
If $r > 0$ is radius of the string, $k > 0$ is coefficient of friction and $m > 0$ is mass of object, let x be angle from the vertical line,

$$\ddot{x} = -\frac{g}{r} \sin x - \frac{k}{m} \dot{x}$$

It can be transferred into first order system:

$$\dot{x} = y, \quad \dot{y} = -\frac{g}{r} \sin x - \frac{k}{m} y$$

Equilibria: $(n\pi, 0), n \in \mathbb{Z}$. For even n, equilibrium is exponentially stable(as the linearisation is hyperbolic with Eigenvalues with negative real part). As for odd n, linearisation is hyperbolic but Eigenvalues $\lambda_1 < 0 < \lambda_2$. Just like previous examples, the stable and unstable set are curves tangent at equilibrium.

One can assign Lyapunov function $V(x, y) := \frac{1}{2} m(ry)^2 + mgr(1 - \cos x)$ which is sum of kinetic energy and potential energy, $\dot{V}(x, y) = -kr^2 y^2$. At equilibria $(2k\pi, 0), k \in \mathbb{Z}$, $V(2k\pi, 0) = 0$, it is positive for other points. It can be proved using La Salle's principle that omega set of any point is singleton consisting of an equilibrium.

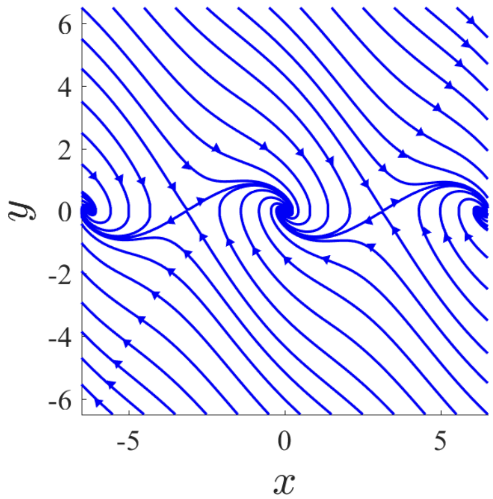

Figure 6: Phase portrait of pendulum system

Example (Another system with periodic orbit, Example 4.36)**.**

$$\dot{x} = y, \quad \dot{y} = -x + y(1 - x^2 - 2y^2)$$

using Lyapunov function $V(x, y) := x^2 + y^2$ that $M := \{(x, y) : \frac{1}{3} \leq x^2 + y^2 \leq 2\}$ is positively invariant. M is compact without any equilibrium so by corollary of Poincaré-Bendixson, there is a periodic orbit within M. (We do not have to draw phase portrait to find it) The following is phase portrait of this system:

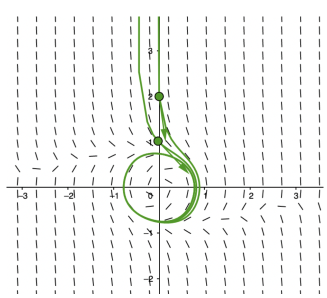

Figure 7: Phase portrait

Example (Limit value of a solution is not constant solution, PS4 Ex16ii)**.** Consider

$$\dot{x} = -x + \frac{1}{t} - \frac{1}{t^2}$$

with domain $D := \mathbb{R}^+ \times \mathbb{R}$ and general solution $\lambda(t, t_0, x_0) = \frac{1}{t} - e^{-(t-t_0)}(\frac{1}{t_0} - x_0)$ exists for all $t, t_0 > 0, x_0 \in \mathbb{R}$. $\lim_{t\to\infty} \lambda(t, t_0, x_0) = 0$ but 0 is not a solution.