An Introduction to Differential equations
(Chapter 8)

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Chapter 1

What is a Differential equation

Differential equations arise largely from physics, and probably some areas in biology and sociology. For example, we have seen $\frac{dP}{dt} = rP$ (r is a rate of growth, P denotes population) that represents an exponential growth. From its appearance, it seems no different from a normal equation, just involving a derivative. Yes! Differential equations are equations, they are used when situation is easier to be described with rate of change instead of actual amount. In this case, the equation says ”rate of change is proportional to the population”. There is not a strict definition for differential equations as they can take any form. But in general, differential equations are equations involving derivatives.

But in reality, no population really growth exponentially forever. With limited resources, the growth is more reasonably estimated by logistic model:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

. Here K is called the carrying capacity. We can see that the population is growing when $P < K$, and shrinking when $P > K$.

The final solution is $P(t) = \frac{K}{1+ce^{-rt}}$. (c is to be determined) Below is a graph when $K = 1$ from Wikipedia.

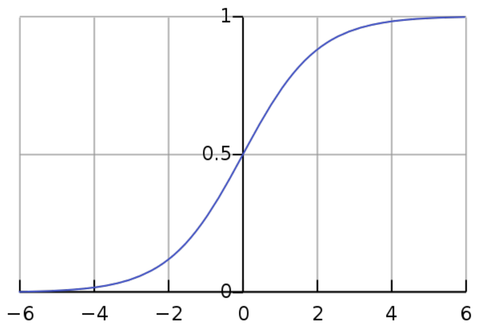

Figure 1.1: Logistic growth

We can see that the growth rate is decreasing, and when it is approaching capacity

K, the growth almost stops.

The famous Newton's second law, can also be treated as a differential equation:

$$F = ma = m\frac{d^2s}{dt^2}$$

Hence, if we know $F(t)$, we can find an expression for position/displacement with respect to time $s(t)$ by solving this differential equation. For example, definition of simple harmonic motion is that $F = -kx$. (force is proportional to negative displacement) Pendulums/old ticking clocks with swings are good examples of simple harmonic motion. With out any difficult we can write down:

$$\frac{d^2s}{dt^2} = -\frac{k}{m}x$$

. The solution is some form of sine wave, which should not be surprising.

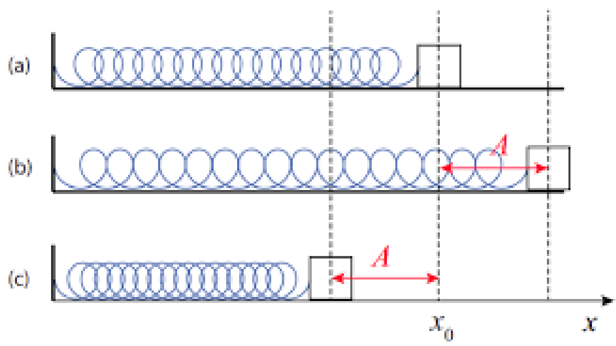

Figure 1.2: An object connected to a spring, doing simple harmonic motion

There are much more fascinating differential equations that describes waves, conservation, diffusion, heat and quantum. Search for them if you are interested, especially the solutions.

By the way, the image in the cover page is the solution of Lorenz equation, which describes butterfly effect.(Small change in initial condition can result in drastic difference in the result)

The equations are actually simple:

- $\frac{dx}{dt} = \sigma(y - x)$
- $\frac{dy}{dt} = x(\rho - z) - y$
- $\frac{dz}{dt} = xy - \beta z$

But the result is really chaotic.

Chapter 2

Separable equations

There are few cases where we can solve differential equations by inspection. $\frac{dy}{dx} = 0$ immediately yields $y = c$ (c is a constant), $\frac{dy}{dx} = x$ means $y = \frac{1}{2}x^2$. But for $\frac{dy}{dx} = y$, inspection no longer works. The only equations that can be solved by, directly integrating both sides, are $\frac{dy}{dx} = f(x)$.

It is very difficult to solve equations like $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = xe^x$. It turns out that differential equations are much more complex than equations with numbers. So in this course, A-level math, the only type of equation we will solve are the **separable equations**.

Separable equations are the equations that can rearranged to the form $f(y)\ dy = g(x)\ dx$. Just like implicit differentiation, we can imagine an underlying wheel t driving x, y. Then we write $x = x(t), y = y(t)$ to indicate they depend on t. So

$$f(y(t))\ \frac{dy}{dt} = g(x(t))\ \frac{dx}{dt}$$

Now this is just an equation with t as the only independent variable! It is perfectly okay to integrate both sides with respect to t:

$$\int f(y(t))\ \frac{dy}{dt}\ dt = \int g(x(t))\ \frac{dx}{dt}\ dt$$

Then using change of variable for integration, we change t on two sides to x, y. This operation eliminates $\frac{dy}{dt}$ and $\frac{dx}{dt}$. We are left with:

$$\int f(y(t))\ dy = \int g(x(t))\ dx$$

This make sense as t is what we built, there must be a way to eliminate t ultimately. When you are actually working on a question, do not include all the steps with t. After getting $f(y)\ dy = g(x)\ dx$, simply integrate to $\int f(y(t))\ dy = \int g(x(t))\ dx$! But you have to understand that it is not just adding a $\int$ sign at both sides of equations. As we know, this $\int$ has no meaning when it stands on its own, we have to use a pair $\int ...\ dt$ to specify which variable are we integrating with.

Example. Solve $\frac{dP}{dt} = 2P$.
First change to the form $\frac{1}{2P}\,dP = dt$, then integrate both sides:

$$\int \frac{1}{2P}\,dP = \int dt$$

$$\frac{1}{2}\ln 2P = t + c$$

In theory, we should add constant c_1, c_2 on both sides. But this constant is just something to be determined, so we can move them to one side and claim a new constant $c = c_1 + c_2$. This will not change the final solution. Or we say c_1 absorbs c_2

$$\ln 2P = 2t + C$$

here we use $C = 2c$ to replace the old constant.

$$2P = Ae^{2t} \text{ where } A = e^C$$

so the final solution is $P = \frac{1}{2}Ae^{2t}$. We can see that the population grows exponentially all the way up with no end.

Warning. do not ignore the constant at first and then solve the equation without it, but finally add that back. You can see that it yields different result. $+c$ may become $\cdot c$ in the end, so we should not ignore c. But you can replace it with other symbols when convenient.

Finally, what exactly is A? We need an **initial condition** to find it. That is, what the population was at the beginning: $t = 0$. Let's say initial population is 10, we can plug this into the equation to get $A = 10$.
Question. What is A if instead we are given a condition $P(2) = 10$?

2.1 Small Remark

Sometimes you will be asked to find differential equation from the context. Do not be terrified. Just try to use letters to define some quantities in the question(even though you may not use all of them), and extract every equality relation you can find from the question. Translate them into differential equations.

Chapter 3

Exercises

Q1. A spherical ice melts gradually in water. At first, it has radius 5 mm, but after 2 minutes, the radius decreased to 3 mm. For simplicity, assume that rate of decrease of radius is inversely proportional to radius square.
(a) Find an equation linking radius and time.
(b) How long does it take for the ice to melt completely? Give your answer to the nearest second.

Q2. A balloon is spherical, and when it is blown up, it is still spherical. At $t = 3$, the radius is 9. And rate of increase at this instance is $1.08cms^{-1}$. It is known that rate of radius increase is inversely proportional to square root of radius.
(a) Write a differential equation to model this situation. Solve it.
(b) What is the volume of air in the balloon initially? (at $t = 0$)

Q3. Solve $\frac{dy}{dx} = \frac{5-x}{12}$. It is given that when $x = 0$, $y = 16$. Find the x-coordinate when $y = 10$.

Q4. Solve $\frac{dy}{dx} = e^{x+y}$.

Q5. Solve $\frac{dy}{dx} = \frac{3}{y \sin^2 x}$.

Q6.
(a) Express $\frac{1}{P(1-\frac{P}{K})}$ in partial fraction(K is a constant).
(b) Solve the logistic growth equation $\frac{dP}{dt} = cP(1 - \frac{P}{K})$, check if it is the same as the solution mentioned above. (c) If we write $c = e^{rt_0}$ for some constant t_0 instead, interpret the meaning of t_0 by drawing the graph of $P(t)$. (You may use a graphing calculator or software to help you)