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Revision Note

MATH40007 Applied Math

2021

1 Basic skills and knowledge

1.1 Linear Algebra

Rank-nulity

Theorem 1.1 (Rank-nulity theorem)**.** *Right null space* $N(A) = \{A\vec{x} = \vec{0}\}$*, left null space* $= N(A^T) = \{A^T\vec{y} = \vec{0} = \vec{y}^T A\}$.
1. Dimension relationship: $dim(N(A^T))+rank(A) =$ *number of rows,* $dim(N(A))+rank(A) =$ *number of columns*
2. Intersection: $CSp(A) \bigcap N(A^T) = \{\vec{0}\}$*,* $RSp(A) \bigcap N(A) = \{\vec{0}\}$

Positive definite matrix

Definition 1.2 (Positive definite matrix)**.** A symmetric matrix A is called positive definite matrix if $\mathbf{x}^T A\mathbf{x} > 0\ \forall \mathbf{x} \neq \mathbf{0}$.

This definition comes from quadratic form $\mathbf{x}^T A\mathbf{x}$ which is used to represent 3-D quadratic surfaces(or even higher dimensions) in the form:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + g = 0$$

As an exercise, you may try to write this equation in the matrix form, by finding entries of A in terms of $a, b, ..., g$ and assuming $\mathbf{x} = (x, y, z)^T$.

Recall that a quadratic function $f(x) = x^2$ has only one solution as $x^2 > 0$ $\forall x \in \mathbb{R}, x \neq 0$. The definition of positive definition is motivated from here.

Here are some properties of Positive definite matrix:

Proposition 1.3 (Eigenvalues of positive definite matrix)**.** *If A is positive definite, all the Eigenvalues of A are real positive numbers. (recall that if $A_{n\times n}$ is symmetric, it has n real Eigenvalues)*

Actually, A is symmetric and all Eigenvalues are positive $\Leftrightarrow$ A is positive definite.

Proposition 1.4 (Diagonalisation)**.** *If A is a symmetric matrix, A can be diagonalised to the form $C^T AC = D$. Where C is orthogonal, and D is diagonal.*
If A is positive definite, by 1.3, $D = D'^2$ for some diagonal matrix D'. So $C^T AC = D'^2$.

Remark. The reverse process of diagonalisation can be used if you know all the Eigenvalues and Eigenvectors of a matrix.

Proposition 1.5. *If A, B are positive definite matrices:*

- *$A + B$ is positive definite.*
- *A^{-1} is positive definite.*
- *$\exists$ positive definite matrix C s.t. $A = C^2$*

- $tr(A) > 0$
- $AB = BA \Rightarrow AB$ *is positive definite matrix*

Here are some ways to create a symmetric matrix:

1. AA^T, A^TA must be symmetric for any matrix A.
2. If A, B are symmetric, then $ABA^T = ABA$ is symmetric.

Circulant Matrix

Recall that a circulant matrix is a matrix whose columns are just rotations of the first column. So it is enough to just have the first column in the definition. Here is a 3 by 3 example:

$$\begin{pmatrix} a_1 & a_3 & a_2 \\ a_2 & a_1 & a_3 \\ a_3 & a_2 & a_1 \end{pmatrix}$$

Proposition 1.6 (Commutativity of circulant matrices)**.** *If A, B are circulant matrices, then $AB = BA$*

Commutative matrices have a good property using this result:

Corollary 1.7 (Linear system with circulant)**.** *$A\boldsymbol{x} = \boldsymbol{0}$ is a system of linear equations($\boldsymbol{c}$ is a constant vector, A is a circulant matrix). If B is another circulant matrix, $\boldsymbol{x_0}$ is a solution to the system, $B\boldsymbol{x_0}$ is also a solution.*

Remark. For non-homogeneous case $A\boldsymbol{x} = \boldsymbol{c}$, only specific choices of matrix and vector will work. One example is when

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \boldsymbol{c} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$$

then we can see $A\boldsymbol{c} = \boldsymbol{c}$. The matrix A above corresponds to a permutation in the symmetric group S_n, we can define other matrices similarly to solve problems. But usually to give it a physical meaning, we should choose a permutation from D_{2n}. Moreover, if you choose any rotation, then the corresponding matrix is guaranteed to be circulant! This technique is used to prove proposition 2.11 and 1.9.

Proposition 1.8 (Commutative and Eigenvector)**.** *If $AB = BA$, an Eigenspace of A has dimension 1, then this Eigenspace is also an Eigenspace of matrix B. (But Eigenvalues may not be the same!)*

Proposition 1.9 (Eigenvectors of circulant matrix)**.** *If C is a n by n circulant matrix,*

$$\boldsymbol{x}_k = \begin{pmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \dots \\ \omega^{(n-1)k} \end{pmatrix}, \lambda_k = c_1 + c_n\omega^k + c_{n-1}\omega^{2k} + \dots + c_2\omega^{(n-1)k}$$

where $\omega = e^{2\pi i/n}$, $(c_1, c_2, ..., c_n)^T$ *is the first column.*

Remark. $\boldsymbol{x}_k = \overline{\boldsymbol{x}_{n-k}}$, as $\omega^n = 1$.

Remark. If we know that a circulant matrix is also symmetric (that is $c_n = c_1, c_{n-1} = c_2, ...$), we know all the Eigenvalues should be real. As $\omega^{n-j} + \omega^j = 2\cos\frac{2\pi}{n}j$ and $\omega^{\frac{n}{2}} = e^{\pi i} = -1$ if n is even. By symmetric of cos over $[0, 2\pi]$, $\lambda_k = \lambda_{n-k}$, so $\boldsymbol{x}_k, \boldsymbol{x}_{n-k}$ sit in the same Eigenspace. That means real and imagery parts of $\boldsymbol{x}_k$ (as linear combinations of $\boldsymbol{x}_k, \boldsymbol{x}_{n-k} = \overline{\boldsymbol{x}_k}$) are also (independent) Eigenvectors, and they are real! So all the Eigenvectors are actually real in this case.

Real Eigenvectors of symmetric circulant matrix:

$$\mathbf{x}_k = \begin{pmatrix} 1 \\ \cos\frac{2\pi}{n}k \\ \cos\frac{2\pi}{n}2k \\ ... \\ \cos\frac{2\pi}{n}(n-1)k \end{pmatrix}, \boldsymbol{y}_k = \begin{pmatrix} 0 \\ \sin\frac{2\pi}{n}k \\ \sin\frac{2\pi}{n}2k \\ ... \\ \sin\frac{2\pi}{n}(n-1)k \end{pmatrix}$$

Remark. Surprisingly, they are the Eigenvectors for any symmetric circulant matrix. (Though the Eigenvalues may be different) Since the example on the book for circulant matrix is frequently used, you should write that down on a piece of paper if you find the general form difficult to remember.

Others

Here is a way to extract an entry for a (column)vector:

$$\varepsilon_i^T \mathbf{v} = v_i$$

where ε_i is the i'th canonical basis.

Proposition 1.10. *If you write a matrix-vector equation in submatrix form, then you can do matrix multiplication as usual by treating submatrices as elements.*

1.2 Vector calculus

Differentiation rules including addition, product, quotient etc. can all be used for vectors. If there is a matrix in the equation independent of t, they can be treated as you have to be careful not to change the order of multiplication. So following change cannot occur:

$$\frac{d}{dt}(AB\mathbf{x}) = BA\frac{d\mathbf{x}}{dt}$$

Eigenvalues and Eigenvectors are frequently used to solve linear differential equations, especially when the ansatz is of the form:

$$\mathbf{x} = \phi e^{i\omega t}$$

as derivative of e^x is itself.

1.3 Multivariate calculus

Partial differential equations may have more constants than they look like. Sometimes there are no enough initial conditions, you may leave some of the constants there.

Using ansatz of only one variable can bring PDE to ODE.

Here is a special type of integral defined on a rectangle: (iterated integrals)

Theorem 1.11 (Fubini's theorem)**.** *If we have a region $R = [a, b] \times [c, d]$ (a rectangle),* $\int_a^b \int_c^d f(x, y)dy\,\mathrm{dx} = \int_c^d \int_a^b f(x, y)dx\,\mathrm{dy}$.
This theorem should be very intuitive. And if we know further that $f(x, y) = g(x)h(y)$ defined on rectangle R(i.e. f is separable), it is easy to prove that

$$\iint_R f(x, y)dA = \left\{\int_a^b g(x)dx\right\}\left\{\int_c^d h(y)dy\right\}$$

.

But we have to be careful when switching integrals with general area D! This is not true generally.

For a general area D, the trick is to imagine you are a CT machine scanning through the area. First you are sliding along y-axis, scanning parallel to x-axis, so for each position on y-axis, we treat y as constant and compute the range of x, $I_{x|y}$. But you can also scan along another direction, and for each fixed x, find $I_{y|x}$.

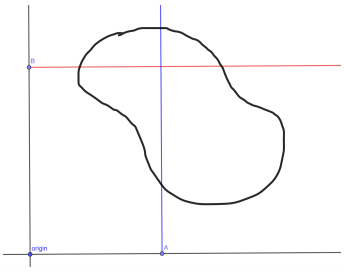


Figure 1: Two-scanner model

Then we have

$$\int_a^b \left\{\int_{I_{y|x}} f(x, y)dy\right\} dx = \int_c^d \left\{\int_{I_{x|y}} f(x, y)dx\right\} dy$$

.

Proposition 1.12. *You may assume that addition and scalar multiplication rule still applies for double or even triple integrals.*

But note that you need Jacobin if you want to use substitution. The following are two results derived using Jacobin:

Cartesian to polar conversion:
recall that we had learnt $dA = r\ dr\ d\theta$ if we want to integrate with polar

coordinates. This is particularly convenient when you are studying circles or discs. Note: usually in polar coordinates, we integrate r before θ.

Polar arc length:

$$ds = r\ d\theta$$

, this transformation allow us to convert integrals on arc length to polar coordinates.

You might have seen integrals like this:

$$\oint f(x,y)\vec{ds}$$

This just means that you are integrating on a circle. This can differ from normal integrals so we give it a special name. This is used for physical properties on a ring like heat energy, electricity or radiation. Fourier started his research of Fourier series on thermal energy of a ring.

Actually, this integral is just a member of a family of integrals: line integrals, and closely related to directional derivatives. (Integrating along closed lines) It is time consuming to learn about all of this, so I will just state a quick result here:

$$\oint f(x,y)\vec{ds} = \int_0^{2\pi} f(r\cos\theta, r\sin\theta)(r\ d\theta)$$

not surprising that polar coordinates simplify the problem.

1.4 complex analysis

Definition 1.13 (Complex logarithm)**.** $log'(z) = log|z| + arg(z)i, z \in \mathbb{Z}$. The function log' is called complex logarithm. (From now on, I will remove the slash)

Remark. You can check by yourself that all identities for real logarithm apply here.

Proposition 1.14. *Complex logarithm is analytic unless $z \in \mathbb{R}$ and $z \leq 0$, and its derivative is $\frac{d}{dz}log(z) = \frac{1}{z}$*

Proposition 1.15. *Operators $Re(*), Im(*)$ are linear.*

Conjugacy

Definition 1.16. Conjugate of a complex number z is defined as $Re(z) - i\ Im(z)$

Proposition 1.17 (relationship between components and conjugates)**.** *$Re(z) = \frac{z+\overline{z}}{2}, Im(z) = \frac{z-\overline{z}}{2}$, and for the form $z = |z|e^{i\theta}$: $|z| = \sqrt{z\overline{z}}, arg(z) = \frac{\ln z - \ln \overline{z}}{2i}$*

Proposition 1.18 (Properties of conjugate)**.** *Conjugate is distributive over addition, multiplication and division. And $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$. And it is commutative with $\exp *, \log *$ operators.*

Proposition 1.19. *$\overline{z} = \frac{1}{z} \Leftrightarrow |z| = 1$. This is useful for rewriting equations in many situations.*

Definition 1.20 (Inner product on $\mathbb{C}$)**.** If $\boldsymbol{a} = (a_1, a_2)^T, \boldsymbol{b} = (b_1, b_2)^T$, then $\boldsymbol{a} \cdot \boldsymbol{b} = Re(\overline{z_1} z_2) = Re(z_1 \overline{z_2})$ where $z_1 = a_1 + ia_2, z_2 = b_1 + ib_2$. This is possible as $\mathbb{R}^2 \cong \mathbb{C}$. Thus $\overline{z_1} z_2$ is called the inner product on $\mathbb{C}$.

Differentiation on $\mathbb{C}$

Example 1.21. Show that the derivative of $f(z) = \overline{z}, z \in \mathbb{C}$ at $z = 0$ cannot be defined as usual in real analysis.

$$f'(0) = \lim_{\Delta z \to 0} = \frac{f(\Delta z) - f(0)}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

. But even if we define $\Delta z \to 0$ to be $\Delta x \to 0, \Delta y \to 0$, there are many possibilities.

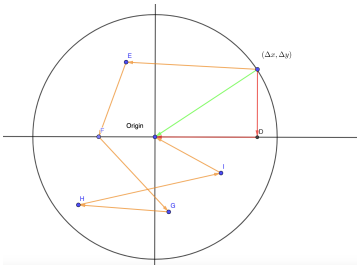


Figure 2: Ways to approach origin

Let's take the red path in figure 1 as an example, after reaching x-axis($\Delta y = 0$), we have $f'(0) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$. But if we reach y-axis first($\Delta x = 0$), we will have $f'(0) = \lim_{\Delta y \to 0} \frac{-\Delta y}{\Delta y} = -1$.

Oops! It seems that we cannot do differentiation anymore in $\mathbb{C}$. Luckily, if instead we define the function limit in a different manner, we can differentiate!

Definition 1.22 (Function limit for multi-variate function)**.** $\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = \boldsymbol{A}$ if $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $|\boldsymbol{x} - \boldsymbol{a}| < \delta, |f(\boldsymbol{x}) - \boldsymbol{A}| < \epsilon$. Where $z \sim \boldsymbol{x} = (x, y)^T, f(\boldsymbol{x}) \sim f(x + iy) = f(z)$. ($\sim$ is used here to emphasise that they are not equal! It is just an map based on isomorphism between $\mathbb{R}^2$ and $\mathbb{C}$)

Remark. What? I am just changing every symbol into bold? This definition has a much meaningful interpretation. The region $|\boldsymbol{x} - \boldsymbol{a}| < \delta$ is called an open disc around $\boldsymbol{x} = \boldsymbol{a}$.

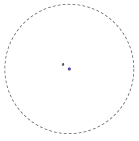


Figure 3: Open disc

Similarly there is an target open disc for $f(\boldsymbol{x})$. The meaning for every point $\boldsymbol{x}$

in the open disc satisfies our target is that for any path you take(like all the three paths in figure 2), your function $f(\boldsymbol{x})$ approaches to the same point $\boldsymbol{A}$. There is also an "open-disc" for infinity(or a neighbourhood of infinity), but that actually looks like a complement of open disc. search more if you are interested. And there is a closed disc defined, but really not relevant to us.

Definition 1.23 (Analytic function(1))**.** A function is analytic if $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists. Where the limit is defined as in Definition 1.22.

Remark. Obviously under this definition, $f(z) = \overline{z}$ is not analytic. But what functions are analytic, do we have to compute this limit every time?

Proposition 1.24. *You may assume that derivatives of common functions(exponential, polynomials) and differentiation rules apply for derivatives of analytical function. (Addition, multiplication, quotient, chain rule)*

Recall from A-level or your other experiences, that usually when writing things in z and $\overline{z}$ is easier than using x and y. (i.e. $(1,1)^T, (1,-1)^T$ is a better orthogonal basis than $(1,0)^T, (0,1)^T$ on complex planes) Conjugates appears as a pair in solutions to polynomial equations, their specific linear combinations can be linearly to "pull out" real and imagery components. (1.17) And you can see that in the matrix representation of complex numbers

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

The first row represents conjugate. This is because this "conjugate basis" is symmetric about the real line! Everything becomes convenient on the real line. And $z\overline{z} \in \mathbb{R}$.

So we can build a relationship between complex differentiation(Wirtinger derivatives) and real differentiation upon conjugacy, this time we are getting rid of vectors.

We begin by total differentiation of a function $p(x, y)$: $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$. Using proposition 1.17:

$$\frac{\partial}{\partial z} = \frac{1}{2}[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}], \frac{\partial}{\partial \overline{z}} = \frac{1}{2}[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}]$$

Then we can write that as:

$$\Delta p = \frac{\partial p}{\partial z}\Delta z + \frac{\partial p}{\partial \overline{z}}\Delta \overline{z}$$

There are two formulae derived from this equation:

$$\frac{\Delta p}{\Delta z} = \frac{\partial p}{\partial z} + \frac{\partial p}{\partial \overline{z}}\frac{\Delta \overline{z}}{\Delta z}$$

$$\frac{\Delta p}{\Delta \overline{z}} = \frac{\partial p}{\partial z}\frac{\Delta z}{\Delta \overline{z}} + \frac{\partial p}{\partial \overline{z}}$$

.

It is easy to see that the fraction $\frac{\Delta z}{\Delta \overline{z}}$ is not constant as z changes. So we have two conditions correspondingly for two derivatives to be well defined:

$$\frac{\partial p}{\partial \overline{z}} = 0(\text{analytic/holomorphic}), \frac{\partial p}{\partial z} = 0(\text{anti-analytic})$$

Definition 1.25 (Analytic function(2))**.** A function $p(z)$ is analytic if $\frac{\partial p}{\partial \overline{z}} = 0$. And we define its derivative by $\frac{dp}{dz} = \frac{\partial p}{\partial z}$. Similarly $q(\overline{z})$ (the overline is just to align with equations) is anti-analytic if $\frac{\partial p}{\partial z} = 0$, $\frac{dp}{d\overline{z}} = \frac{\partial p}{\partial \overline{z}}$

You may try to prove that two definitions are equivalent.

Definition 1.26 (Schwartz conjugate)**.** For a function $p(z)$, its conjugate $q(u) = q(\overline{z})$ is found by taking conjugate of EVERY term in $p(z)$, and then substitute $\overline{z}$ with u.

Remark. Be careful with the argument of $q(u)$, q itself is actually not equal to $\overline{p(z)}$. They are only equal under the transformation $u \to \overline{z}$

Proposition 1.27. *If $p(z)$ is an analytic function, then Schwartz conjugate of $p(z)$ is anti-analytic.*

Theorem 1.28 (Cauchy-Riemann equations)**.** *If we write $h(z) = \Phi(x, y) + i\Psi(x, y)$, using the definition of $\frac{dh}{dz}$, we can get two equations.*

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}$$

.

Finally we have a better way to tell if a function is analytic or not.

2 Electric circuit

2.1 Basics

Symbol	***Meaning***	***Formulae***	***Node/Edge***
n	number of nodes		Node
m	number of edges		Edge
$A_{m\times n}$	Connections of the nodes(incidence matrix)		Node & edge
$C_{m\times m}$	conductance		Edge
$\mathbf{x}$	voltages		Node
$\mathbf{e}$	potential differences	$A\mathbf{x}$	Edge
$\mathbf{w}$	fluxes	$-CA\mathbf{x}$	Edge
$\mathbf{f}$	net flux out	$-A^T\mathbf{w}$	Node

Table 1: Some important quantities

Remark. The last row means whether this quantity is defined for a node or an edge.

Remark. Combine formulae for $\vec{\omega}$ and $\mathbf{f}$ gives $\mathbf{f} = A^TCA\mathbf{x} = K\mathbf{x}$. Here K is called weighted Laplacian of graph.

Remark. Formulae for $\mathbf{w}$ is equivalent to Ohm's law. Formulae for $\mathbf{e}$ and $\mathbf{f}$ are just manipulations of other quantities on the graph. So real physical quantities are only fluxes and voltages. (which are connected by conductance)

Physical meanings of left and right null spaces of A:

- $N(A)$ consists of $\mathbf{x}$ that will make no potential difference across any edge in the circuit. i.e. All connected nodes have equal potential. So $dim(N(A))$ = number of disconnected components of a graph
- $N(A^T)$ consists of the choice of fluxes on edges $\vec{\omega}$ that gives 0 net flux for each node. (`e.g.` for a node j, if one edge has flux 1 flowing in, this flux must flow out from some other edge) Obviously, a circular graph satisfies this requirement if all arrows are pointing in the same direction(c.c. or anti c.c.). If not, using -1 for that entry in $\vec{\omega}$ aligns it with the correct direction. So $dim(N(A^T))$ = number of independent loops on a graph.

To generalise observations found for circuits, I will define a flow system here:

Definition 2.1 (flow system)**.** A flow system is a graph consists of nodes and edges connecting nodes(one on each side). There are two quantities: flow w_j assigned to each edge, and potential x_i assigned to each node. There may or may not be a connection between potential and flow. We can define potential difference w_j on edges and net flux out f_i on nodes for any flow system. If potential difference and flow has a linear relationship $e_j = -c_jw_j$ (c_j is a real positive constant) for any edge, we call c_j conductance on edge j. (i.e. the flow system satisfies Ohm's law)

Definition 2.2 (Degree of node)**.** degree(node i) is defined to be number of nodes connected to node i by edges.

Definition 2.3 (Interior and boundary nodes)**.** Any node of flow system with net flux flowing $f_i = 0$ is called an interior node(i.e. the node satisfies KCL). Other nodes are called boundary nodes. Though this definition is not meaningful if all nodes are boundary node.

Proposition 2.4 (Kirchhoff's current law(KCL))**.** *When KCL property ($\boldsymbol{f} = \boldsymbol{0}$) holds, $\boldsymbol{x} = \boldsymbol{0}$.*

Proof. Assume such ω exists. By definition of $\mathbf{f}$, $\vec{\omega} \in N(A^T)$, by definition of $\vec{\omega}$, $\vec{\omega} \in CSp(A)$. But by rank-nulity, intersection of these two spaces is only $\{\mathbf{0}\}$. ■

Luckily, if we allow two of the points to not follow KCL(flux disappears/appears to/from nowhere), we can get a graph with not all-zero voltages! This is called a **two-point source current**

Proposition 2.5 (Harmonic potential)**.** *If all interior nodes satisfy KCL and the flow system has unit conductance, $x_i = \frac{1}{degree(nodei)} \sum_{adjacent\ node\ j} x_j$. i.e. potential of a node is average of potentials of nodes connected to it.*

We call any flow system with potential satisfying this property "having harmonic potentials". (or harmonic potential system)

Proposition 2.6 (General law of harmonic potential)**.** *This for the case where conductance may not be unit(KCL still needs to hold!): For any interior node i:*

$$\left(\sum_{adjacentj} c_{ij} \right) x_i = \sum_{adjacentj} (c_{ij} x_j)$$

Principles of harmonic potential:

Proposition 2.7 (Boundary principles)**.** *Maximum and minimum potentials of harmonic potential system appears at the boundaries.*

Proposition 2.8 (Combined Flows)**.** *If two flow systems have the same shape(i.e. same incidence matrix A) and both have harmonic potentials (x_i, y_i), then if we define $z_i = x_i \pm y_i$ to be a new set of potentials, $\boldsymbol{z}$ is also harmonic potential.*

Proposition 2.9 (Uniqueness)**.** *If two flow systems have same shapes and same boundary conditions(potentials for boundary nodes), all potentials on corresponding interior nodes are the same among two systems.*

2.2 Two types of circuit(two-point source) problems

1. Neumann problem: given fluxes at boundaries, work out voltages. i.e.

$$\mathbf{f} = K\mathbf{x}, \mathbf{f} = (+1, -1, 0, ..., 0)^T$$

- Solution: use reduced Laplacian $\hat{K}$ by deleting the row and column corresponding to the grounded(or negative) node. $\hat{K}$ must be invertible as it has rank $n-1$.
- By linear algebra, the final general solution should be in the form $\mathbf{x}+c\mathbf{x}_0$ ($\mathbf{x}$ is the particular solution found by above method, c is a real number, $\mathbf{x}_0$ is any vector from $N(K)=N(A)$)
- Brief explanation of why we can delete a row and a column: the column corresponds to entry of $\mathbf{x}$ for grounded node, which is 0. So this column does not affect the result. The row will be automatically satisfied as sum of all f_i in $\mathbf{f}=0$ must be satisfied by the deleted equation(row).

2. Dirichlet's problem: given potentials at boundaries, find other potentials.
   - Solution1: Write the equation $\mathbf{f}=K\mathbf{x}$ in sub-block form. Sub-blocks should separate the sources and interior points. This can also be done on a reduced Laplacian $\hat{K}$.
   - Solution2: Method of relaxation (As long as KCL is satisfied for all interior nodes in the circuit)
   - Solution3: Use law of harmonic potentials 2.6 (As long as KCL is satisfied for all interior nodes in the circuit)

Remark (The idea of unit). If the graph of a flow system is connected, and the flow system has harmonic potential and conductance(i.e. flows and potentials are related). Suppose the solution to Dirichlet's principle is $\vec{x_D}$, then if instead we applied a voltage of V at positive node instead, the voltage vector will become $V\vec{x_D}$. Because the potential is harmonic! Positive node passes this voltage(or coefficient) throughout the graph. Similarly if we have a solution to Neumann problem $\vec{x_N}$, then current I entering at positive node would give new potential vector $I\vec{x_N}$. Doesn't this mean if we solve any of the two problems, we have solved another one?

- If we have $\vec{x_D}$(solution to Dirichlet's problem) and this gives current I at positive node. Then we just need to divide I to get back a "unit current" system. So $\vec{x_N}=\frac{1}{I}\vec{x_D}$ is the solution to Neumann problem.
- If we have $\vec{x_N}$ and voltage at positive node is V. Then we divide V to get back a "unit potential" system. So $\vec{x_D}=\frac{1}{V}\vec{x_N}$.

Remark. Solutions to Neumann problem are not unique, but it is just because they have different "base potential". (i.e. voltage of "grounded node", which is constant c in the equation $\mathbf{x}+c\mathbf{x}_0$) But for convenience, we usually investigate the case where "grounded node" is really grounded, so it has potential 0. However, Dirichlet's problem has only one unique solution. (potential drives current, so potential determines current but current does not determine potential)

2.3 Finding effective conductance

Definition 2.10 (Effective Conductance)**.** When we treat the whole circuit as a box, specifying an entry and an exist. Then the conductance of the whole box as

a conductor is called effective conductance. By Ohm's law, $f = -c_{\text{eff}}(x_- - x_+)$ where x_- is the voltage at negative node, x_+ is the voltage at positive node. For two-point source circuit with unit voltage, $f = c_{\text{eff}}$ (i.e. flux flowing into the circuit)

Using above formulae, we can derive two important results by writing $\mathbf{f} = A^T CA\mathbf{x} = K\mathbf{x}$ to find $\mathbf{f}$.
Effective conductance for:

- conductors in series: $\frac{c_1 c_2}{c_1 + c_2}$
- conductors in parallel: $c_1 + c_2$

For two-point source circuit:

- Reduce the circuit using the above two rules
- Use Schur complement for Laplacian/reduced Laplacian

If the circuit has multiple grounded node, you should first use the following rule:

Proposition 2.11. *Nodes at the same voltage can be merged together, but all the edges should be kept. So the new graph may have several edges connected between two nodes.*

For multiple grounded node, the flux out at grounded notes sums up to c_{eff}.

Of course if you already have all the potentials solved by methods in last section, you do not need to use any of the method above, just use the definition!

2.4 Energy Dissipation

From physics, we know that the power of a circuit can be determined by $P = VI = I^2R = \frac{V^2}{R}$.(Derived by Ohm's law) (V is voltage, I is current/flux) As $c = \frac{1}{R}$ (c is conductance), $P = cV^2$. For a circuit, energy dissipated on edge i is $e_i^2 c_i$. Total energy dissipation for the circuit is

$$\sum_{\text{edge i}} c_i e_i^2 = \mathbf{e}^T C\mathbf{e} = (A\mathbf{x})^T C(A\mathbf{x}) = \mathbf{x}^T K\mathbf{x}$$

Intermediate term is a quadratic form.

Proposition 2.12 (Dirichlet's principle)**.** *If energy dissipation depends on voltages on the nodes, it is minimised when all interior nodes satisfy KCL.*

Proof. By combining Schur complement method for the equation $K\mathbf{x} = \mathbf{f}$ to get an equation (*) for $\mathbf{x}$ that is only satisfied when KCl holds. Then do sub-block decomposition for energy dissipation formulae $\xi(\mathbf{x}) = \mathbf{x}^T K\mathbf{x}$ WITHOUT using KCL, we can prove that equation (*) of $\mathbf{x}$ minimises dissipation. ■

But we can also see that there is another way to find total energy dissipation using $P = VI$:

$$\xi(\mathbf{x}) = \sum_{\text{edge i}} -e_i w_i = -\mathbf{e} \cdot \mathbf{w}$$

where the minus sign is due to conventional orientation of the graph. Energy dissipation must be positive.

Using the formulae $P = I^2R = \frac{I^2}{C}$, we have a formulae for energy dissipation completely dependent on currents:

$$\tilde{\xi}(\mathbf{w}) = \sum_{\text{edge f}} \frac{w_i^2}{c_i}$$

Thomson's principle states the same idea as Dirichlet's principle for energy dissipation in terms of edge fluxes. (Dirichlet's principle is in terms of edge potential differences)

Proposition 2.13 (Tellegen's principle)**.** *It is a powerful statement without support of Ohm's law, just requires divergence* $\boldsymbol{f} = \boldsymbol{0}$ *and the graph is connected.*

$$\boldsymbol{e} \cdot \boldsymbol{w}' = 0$$

. In words: sum of edge current and potential differences equals 0 for any flow system(current, random walk, etc.) satisfying KCL.

Proposition 2.14 (General Tellegen's principle)**.** *If instead* $\boldsymbol{f} = \boldsymbol{f}'$ *for some vector* $\boldsymbol{f}' \in \mathbb{R}^n$*.*

$$\boldsymbol{e} \cdot \boldsymbol{w}' = -\boldsymbol{x} \cdot \boldsymbol{f}'$$

.

Corollary 2.15. *Minimum energy dissipation of a two-point source current with unit voltage is* $\xi(\boldsymbol{x}_*) = c_{\mathit{eff}}$

Table 2 is a table stating underlying assumptions of each theorem/proposition/formulae used for electric circuits(or flow systems)

Theorem/prop	***Assumption***
harmonic potential principles	KCL
$\boldsymbol{w} = -CA\boldsymbol{x}$	Ohm's law
$\boldsymbol{f} = K\boldsymbol{x}$	Ohm's law
Everything about Effective conductances	Ohm's law
Dirichlet principle (& Thomsons's)	Ohm's law, KCL
Tellegen's principle	connected graph, KCL
General Tellegen	connected graph
$\boldsymbol{e} = A\boldsymbol{x}$	no assumption
$\boldsymbol{f} = -A^T\boldsymbol{w}$	no assumption

Table 2: Underlying assumptions

3 Connection between random walks and electric circuit

Definition 3.1 (Hitting probability)**.** p_i is the probability starting at node i, to reach node 1(or node +) before node 2(or grounded node). (You can change numbering of nodes WOLG)

Since hitting probability defined on each node has harmonic potential property, it can be a potential on this flow system. And $p_1 = 1$, $p_2 = 0$. So they can be viewed as sources! Therefore by uniqueness principle, two system have same potentials for all interior nodes, that is $x_i = p_i$

Definition 3.2 (Hopping probability)**.** $p_{i,j}$ is the probability of jumping from node i to node j in one random walk.

For a harmonic potential system with uniform conductance, by intuition the hopping probability should be $\frac{1}{\text{degree(node } i)}$. For non-uniform conductance, we change it to the weighted form:

$$\frac{c_{ij}}{\sum_{\text{adjacent node } j} c_{ij}}$$

where c_{ij} is the conductance for the edge connecting node i and j. $c_{ij} = 0$ if i and j are not connected.(It is impossible to jump to a not connected node!)

Definition 3.3 (Escape probability)**.** p_{esc} is probability for leaving node 1 and jump to node 2 (or grounded node) before returning to node 1 (or node +). (You can change the numbering of nodes WOLG)

Proposition 3.4. $p_{esc} = \frac{c_{eff}}{\sum_{adjacent\ node\ j} c_{+j}}$. *i.e. effective conductance of the flow system divided by sum of conductance of all edges connected to node +.*

Definition 3.5 (Altered Hitting probability)**.** p'_i is the probability starting at node i, to reach node 1(or node +) before reaching any node $g \in J$ (J is the set of grounded nodes). (You can change numbering of nodes WOLG)

Definition 3.6 (Altered Escape probability)**.** p'_{esc} is probability for leaving node 1 and jump to any node $g \in J$ (J is the set of grounded nodes) before returning to node 1 (or node +). (You can change the numbering of nodes WOLG)

You need altered source circuit (multiple grounded nodes) to solve problems like this. Or you may choose to use probability models(law of total probability) for altered hitting probability, but that does not work for altered escape probability.
We call initial 3 probabilities a simple random walk model. Of course you can alter further by creating more positive nodes, but then you need to do analysis on a circuit with multiple positive nodes.(Or use law of harmonic potential)

Remark. As we know, idea of complement is useful in probability. So if you have a version of hitting probability: probability of hitting node $h \in J$(J is a set of nodes) before hitting a specific node g. This may sounds like a circuit

with multiple + nodes. But taking complement solves the problem! Of course, only flow system with unit voltage is possible with this operation, you have to bring voltage down to "unit" for general case.

Analogy between two-point source circuit and simple random walk:

walk	***Electric***
node 1	positive node(+)
node 2	grounded node(-)
hitting probability p_i	voltages x_i
law of total probability	KCL (harmonic potential)
hopping probability p_{ij}	normalised conductance
escape probability p_{esc}	normalised effective conductance

Table 3: Analogy by treating node 1 as +, node 2 as grounded node

Remark. Normalised conductance means $\frac{c_{ij}}{\sum_{\text{adjacent node } j} c_{ij}}$

Remark. Potential difference and flow for this system does not give any particular meaning, but they can still be defined anyway.

4 Spring-mass system

$\boldsymbol{Spring-mass}$	$\boldsymbol{Electric}$	$\boldsymbol{Mass/Spring}$
mass	node	
spring	edge	
Displacement ϕ_i	Voltage x_i	Mass i
Relative displacements $\boldsymbol{\Phi}$	potential differences $\boldsymbol{e}$	Spring
Tension T_a	flux w_a	Spring a
Internal force $\boldsymbol{f_I}$	net flux in $-\boldsymbol{f}$	Mass
Hooke's law	Ohm's law	Spring
Spring constants	Conductance	Spring

Table 4: Analogy between spring-mass system and electric circuit

Remark. The formulae for these quantities are exactly the same as that for electric circuits. Similarly spring constant matrix C and Laplacian K are defined. Where at mechanical equilibrium, $\boldsymbol{f} = -\boldsymbol{f_I} = K\boldsymbol{\Phi}$ is the vector of external forces on each mass.

Remark. The ceiling/walls should also be treated as nodes. But there displacements must be 0.

But spring-mass system is a dynamic system, unless the system is at equilibrium, displacements may change over time. That is not the case for electric circuit. So we have to first investigate how it changes, and the relationship with external forces $\boldsymbol{f}$. Then probably we can apply some results from electric circuit after that.

Proposition 4.1 (Equation for acceleration)**.** $\boldsymbol{F} = \boldsymbol{f} + \boldsymbol{f_I} = \boldsymbol{f} - K\boldsymbol{\Phi} = M\frac{d^2\boldsymbol{\Phi}}{dt^2}$

Remark. $M_{n\times n}$ is the positive definite diagonal matrix of masses, the derivative here is vector derivative, it corresponds to the acceleration of masses. Here we have ignored all entries of ceiling or walls.

There are two famous dynamics: vertically hung springs and horizontal springs. They may or may not be oscillations.

Equilibrium state is easy to analyse. For vertical springs, the only two external forces are gravity and reaction force by the ceiling(can be found in terms of two masses). As a revision exercise, you can find all the displacement and reaction force by yourself.

4.1 Masses between walls

There are n masses hanging horizontally between two walls, connected by n+1 springs

The displacements at two sides: ϕ_0 and $\phi_(n+1)$ are 0. So we can derive a reduced case formulae just as we did before for electric circuit by deleting all corresponding entries of ϕ_0 and ϕ_n.

$$\hat{\boldsymbol{f}} - \hat{K}\hat{\boldsymbol{\Phi}} = M\frac{d^2\hat{\boldsymbol{\Phi}}}{dt^2}$$

Remark. Do not be shocked about the fact that M is missing a little hat here. Because we excluded the edge nodes when defining matrix M.

General form of reduced Laplacian is:

$$\hat{K} = \begin{pmatrix} 2 & -1 & 0 & 0 & . & . & 0 & 0 \\ -1 & 2 & -1 & 0 & . & . & 0 & 0 \\ 0 & -1 & 2 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 2 & -1 \\ 0 & 0 & 0 & 0 & . & . & -1 & 2 \end{pmatrix}$$

Definition 4.2 (Free Oscillation)**.** When $\hat{\boldsymbol{f}} = 0$, that is, there is no external force acting on any of the masses, we say the system is under free oscillation.

If a system is under free oscillations and all the masses are unit, we can WOLG derive the following formulae for oscillations:

$$-\hat{K}\hat{\mathbf{\Phi}} = \frac{d^2\hat{\mathbf{\Phi}}}{dt^2}$$

This can be solved using the substitution $\hat{\mathbf{\Phi}} = \mathbf{\Phi}_0 e^{i\omega t}$ where ω is a real number, $\mathbf{\Phi}_0 \in \mathbb{R}^n$.(complex exponential used here to eliminate the minus sign on LHS) The final solution has $2n$ constants, determined by n initial positions and n initial velocities. We say that ω is the natural frequency of this system. But be careful, ω^2 is the Eigenvalue, not ω!

But as n gets larger, the calculation becomes difficult.

The technique is to use a double circulant matrix and rearrange it to create sub-blocks of the form $\hat{K}$. Since we are already very familiar with Eigenvalues and Eigenvectors for circulant matrix, we can use sub-block decomposition to find those for $\hat{K}$. Hence we can solve the differential equation in general cases.

Remark. It is a convention to put the rows and columns for boundaries at the upper left corner of Laplacian.

I will just put the final result here. (Check the notes if you forget):

$$\mathbf{\Phi_m} = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\frac{m\pi}{n+1} \\ \sin\frac{2m\pi}{n+1} \\ ... \\ \sin\frac{nm\pi}{n+1} \end{pmatrix}, \lambda_m = 2 - 2\cos\frac{m\pi}{n+1}, m = 1, 2, ..., n$$

The case is more complex if the system is not under free oscillation, you may assume that Eigenvectors of K spans whole vectors space(reduced), so any solution for this can be written as $\sum a_j \vec{e}_j$ where a_j are constants to be found.

4.2 One wall, one force

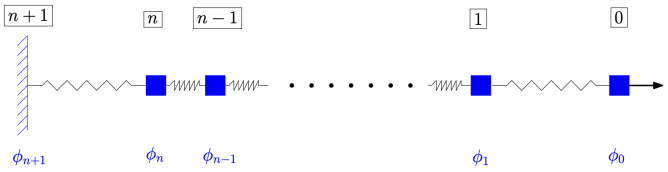


Figure 4: One wall with one external force at the other side

Assumptions:

- Displacement caused by external force f_0 is 1. $\phi_0 = 1$
- All interior masses are under free oscillation. $f_i = 0, i = 1, 2, ..., n$

This problem can be solved using sub-block(contains $\hat{K}_n$) decomposition and another technique from linear algebra: finding solution to a system $\hat{K}_n \boldsymbol{x} = \boldsymbol{c}$ by writing $\boldsymbol{x}$ in terms of an orthonormal basis consists of Eigenvectors $\boldsymbol{\Phi_m}$ derived above. Again I will just put the final result below:

$$\phi_m = \frac{1}{n+1} \sum_{j=1}^{n} \frac{\sin \frac{j\pi}{n+1}}{1 - \cos \frac{j\pi}{n+1}} \sin \frac{mj\pi}{n+1}, m = 1, 2, 3, ..., n$$

Another equivalent solution comes from harmonic potential system,

$$\phi_m = p_m = 1 - \frac{m}{n+1}$$

By considering the case $n \to \infty$, using Taylor series of $\cos x, \sin x$ and a suitable substitution, Fourier series of $1 - \frac{x}{\pi}$ can be derived here.

5 Continuum

Motivation: In reality, there is not circuit looking like nodes and edges. Conductors are 3-D solids, and there are infinite number of nodes there, each having a potential. Current flows in trajectories in the solid instead of flowing along segments(or edges) So we need to extend to continuous case.

Definition 5.1 (conductivity)**.** If a cubic conductor made of a material has uniform property, and cross sectional area of the pair of faces connected to electric sources is A, length of conductor in this orientation is L. $\tilde{c} = \frac{cL}{A}$ is called conductivity, where c is the conductance as defined previously by Ohm's law.

Remark. $c \propto A$ (the wider the "road" is, the easier it is to pass), $c \propto \frac{1}{L}$. (the shorter the "road" is, the easier it is to pass)

Remark. Conductivity is an intrinsic property of material, but conductance refers to a specific shape of conductor and a specific direction of flow.

Definition 5.2 (conductivity on $\mathbb{R}$)**.** If $c = \frac{\hat{c}}{L'}$, $\hat{c}$ is called conductivity

Definition 5.3 (conductivity of a wire $\mathbb{R}^2$)**.** If $c = \frac{\hat{C}H}{L}$, $\hat{c}$ is called conductivity. (H is thickness of the wire)

Remark. Note the use of $\hat{c}$, it means it is not the first definition of conductivity, it is just a convenient representation here to absorb meaningless constants.

5.1 Continuum Conversion

The following is a summary table of how to change discrete quantities into Δ form and then by partitions(this is a basic idea in calculus, here node partition points, edge partitioned intervals), to get a continuous function instead of a vector. Here we assume range of x is $[0, 1]$, 0 is positive node.

Discrete	**1*D***	**2*D***
x_i	$\Phi(x)$	$\Phi(x, y)$
$\hat{f}_i$	$F(x)$	$F(x, y)$
$\hat{c}_i$	$C(x)$	$(j_{i,j}^{(x)}, j_{i,j}^{(y)})$
w_i	$W(x) = W(x_i - \frac{\Delta x}{2})$	$U(x, y) = u_{i,j} = j_{i,j}^{(x)}\Delta y$, $V(x, y) = v_{i,j} = j_{i,j}^{(y)}\Delta x$
c_i	$\frac{C(x_i - \frac{\Delta x}{2})}{\Delta x}$	$c^{(x)} = \frac{\hat{c}\Delta y}{\Delta x}$, $c^{(y)} = \frac{\hat{c}\Delta x}{\Delta y}$
f_i	$F(x_i)\Delta x = \hat{f}_i\Delta x$	$F(x, y)\Delta x\Delta y = \hat{f}_{i,j}\Delta x\Delta y$

Table 5: Continuum of quantities in 1-D and 2-D case

Remark. Since x_i represents a point on $[0, 1]$ now, we replace every x_i with x. The functions with argument $x_i - \frac{\Delta x}{2}$ are edge functions.

Remark (Explanation of symbols).

- $\hat{f}_i$ is current source density for node i, $f_i = \hat{f}_i\Delta x$. This quantity is also defined for 2D similarly, but with area instead. This definition brings current on edges to a current "density" on a point. (Just like conductivity, but conductivity is not a "density") We will be using $\hat{f}_i, \hat{c}_i$ instead of f_i, c_i in continuum case, as you can see the last two rows are not defined as functions.
- $U(x, y), V(x, y)$ are currents along x direction and along y direction at position (x, y). Current densities $j_{i,j}^{(x)}, j_{i,j}^{(y)}$ are as defined in terms of U, V as in the table. In continuum case, they are determined by $j^{(x)} = -\hat{c}\frac{\partial\Phi}{\partial x}, j^{(y)} = -\hat{c}\frac{\partial\Phi}{\partial y}$ when conductivity is uniform.

The following is a table of correspondence between formulae, and correspondence between matrix and operator

Discrete	**1*D***	**2*D***
A	$\frac{d}{dx}$ on(Φ)	∇ Gradient operator
$-A^T$	$\frac{d}{dx}$ on(W)	$\nabla.$ Divergence Operator
K	$-\hat{c}\frac{d^2}{dx^2}$	$-\hat{c}\nabla^{(2)}$ Laplacian Operator
$\boldsymbol{w} = -CA\boldsymbol{x}$	$W(x) = -C(x)\frac{d\Phi}{dx}$	$\vec{j} = -\hat{c}\nabla\Phi$
$-A^T\boldsymbol{w} = \boldsymbol{f} \Leftrightarrow$	$\frac{dW}{dx} = F(x)$	$\nabla.\vec{j} = F(x,y)$
$K\boldsymbol{x} = \boldsymbol{f}$	$-\frac{d}{dx}(C(x)\frac{d\Phi}{dx}) = F(x)$	$-\hat{c}\nabla^{(2)}\Phi = F(x,y)$
$f_i = 0$(interior point)	$\frac{d\Phi}{dx} = 0$(uniform conductivity)	$\nabla^{(2)}\Phi = 0$
$w_1 = f_0$(flow in)	$-C(0)\frac{d\Phi(0)}{dx} = f_0$	
$-w_N = f_N$(flow out)	$C(1)\frac{d\Phi(1)}{dx} = f_\infty$	
$w_1 + \sum_{i=1}^{N-1} - w_N = 0$	$\int_0^1 \frac{dW}{dx} = W(1) - W(0)$	
$-K\boldsymbol{x} = \frac{d^2x}{dt^2}$(from spring-mass)	$\frac{\partial^2\phi(x,t)}{\partial x^2} = \frac{\partial^2\phi(x,t)}{\partial t^2}$	

Table 6: Continuum of equations in 1-D and 2-D case

Remark (Explanation of symbols).

- $\nabla^{(n)}$ is the n'th gradient operator $(\frac{\partial^n}{\partial x^n}, \frac{\partial^n}{\partial y^n})^T$.(the second gradient operator is called Laplacian operator)
- $\nabla.\vec{j}$ is divergence operator $\frac{\partial}{\partial x}J^{(x)} + \frac{\partial}{\partial y}J^{(y)}$.
- $\vec{j} = \begin{pmatrix} J^{(x)} \\ J^{(y)} \end{pmatrix}$ is the divergence vector(corresponding to net flux out $\boldsymbol{f}$).
- When we use $\hat{c}$, we mean the conductivity is assumed to be uniform. Otherwise $C(x)$ will be used.
- Any function satisfying $\nabla^{(2)}\Phi = 0$ is called a harmonic function.

Proposition 5.4. *$\nabla^n, \nabla.$ operators are all linear. And they are all commutative with (Schwartz)conjugate. i.e.* $\nabla^n\left(\overline{f(z)}\right) = \overline{\nabla^n f(z)}$

Corollary 5.5. *$\nabla^n, \nabla.$ operators are commutative with $Re(*), Im(*)$ operators.*

5.2 Two-terminal problem

This is analogue to two-point source problem.

- Point sources → terminals(electrodes)
- Interior nodes → points except on boundaries
- Dirichlet's problem → given voltages for electrodes, find distribution of $\Phi(x, y)$ on the region D.
- Finding effective conductance → finding C_{eff} = total current entering the circuit through anode(positive terminal) =

$$\lim_{\Delta x, \Delta y \to 0} \sum_{\text{anode}} \boldsymbol{j}\hat{\boldsymbol{n}}\Delta s = \int_{anode} \boldsymbol{j}\hat{\boldsymbol{n}} ds$$

  - Here, we are finding component of current density parallel to $\hat{\boldsymbol{n}}$(i.e. directly entering the region D) instead of simply x or y. (So we are not using $J^{(x)}, J^{(y)}$). And by definition of current density, we have to multiply the arc length of that small piece on boundary Δs, which, by Pythagoras's theorem, is $\sqrt{\Delta x^2 + \Delta y^2}$.

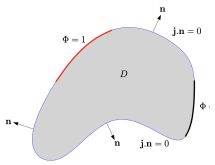


Figure 5: Two-terminal circuit

But as we can see from the figure, there are two more problems to consider (compared to discrete case)

- What is the direction of current flow on region D? (i.e. What is the expression of divergence vector $\vec{j}$ in terms of position vector $(x, y)^T$)
- What happens to the currents(and voltages) on the boundaries except at the terminals?

Proposition 5.6. *On the boundaries (except terminals), the divergence vector is parallel to the tangent of surface. i.e.* $\vec{j} \cdot \boldsymbol{n} = 0$ *(*$\boldsymbol{n}$ *is normal vector of the region boundary)*

Proof. If the $\vec{j}$ points out the conductor, it means we have a current going out not a terminal. This breaks KCL. But you may say what if $\vec{j}$ points inside the conductor. Then by conversation law, there will be a complement vector $\vec{j'}$ in opposite direction with same magnitude. This complement vector is pointing outside! (see figure 6)

But one may argue that the vector $\hat{t}$ seems to be also pointing outside. But remember that our divergence vector $\vec{j}$ is not a true "flow", it is the direction current would travel in a very short time interval. (i.e. vector derivative) So it

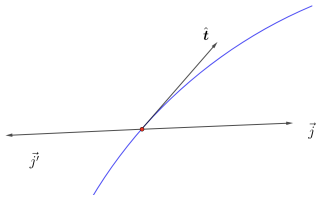


Figure 6: Explanation of orthogonality

is not shocking that if $\vec{j}$ is always parallel to tangent, the "flow" will be kept inside the region. ■

By the above proposition, we can find an interesting relationship:

$$\vec{j} \cdot \boldsymbol{n} = Re((J^{(x)} - iJ^{(y)})(n_1 + n_2 i)) = J^{(x)} n_1 + J^{(y)} n_2 = 0$$

The last equality seems to be just a geometric result, but the second term is interesting. We will investigate the quantity $J^{(x)} - iJ^{(y)}$ now.

From the equation $\vec{j} = -\hat{c}\nabla\Phi$ we know that

$$J^{(x)} - iJ^{(y)} = -\hat{c}(\frac{\partial\Phi}{\partial x} - i\frac{\partial\Phi}{\partial y}) = -\hat{c}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Phi) = -2\hat{c}\frac{\partial\Phi}{\partial z}$$

. Of course similarly we also have $J^{(x)} + iJ^{(y)} = -2\hat{c}\frac{\partial\Phi}{\partial\overline{z}}$. (Not meaningful in our course)

These equations are not friendly. We have a mixture of multivariate function $\Phi(x, y)$ and a derivative from complex plane. So we have to try to change Φ to a complex function, is there any choice?

The best way to jump from real to complex is to use proposition 1.17.

$$\Phi(x) = Re(h(z)) = \frac{h(z) + \overline{h(z)}}{2}$$

where $h(z)$ is some complex-valued function called complex potential.

Proposition 5.7. *$h(z)$ is analytic $\Leftrightarrow$ Φ is a harmonic function.*

Proof. $h(z)$ is analytic $\Leftrightarrow \frac{\partial^2\Phi}{\partial z\partial\overline{z}} = 0 \Leftrightarrow \nabla^{(2)}\Phi = 0 \Leftrightarrow \Phi$ is a harmonic potential. ■

(The details are left as exercises)

Corollary 5.8. *If $\Phi(x, y)$ is a harmonic function, $J^{(x)} - iJ^{(y)} = -2\hat{c}\frac{\partial\Phi}{\partial z} = -\hat{c}\frac{dh}{dz}$. Where $Re(h(z)) = \Phi(x, y)$, $\frac{d}{dz}$ is the Wirtinger derivative.*

So we have reduced the problem of finding $\Phi(x, y)$ to satisfy the boundary condition $\vec{j} \cdot \boldsymbol{n}$ to the problem of:

- finding $h(z)$ so that $J^{(x)}, J^{(y)}$ solved from equation in the above corollary satisfies $\vec{j} \cdot \boldsymbol{n}$.

You may be wondering the physical meaning of $Im(h(z)) = \Psi(z)$ (Many thanks to Mr. Miyoshi for providing the proof behind it on piazza)

Proposition 5.9 (Ψ Stream function)**.** *The Contour lines of $\Psi(z)$ on x-y plane are parallel to derivatives of current density $\vec{j}$. So contour plot of Ψ has exactly the same shape as current lines.*

Proof. Recall that $J^{(x)} = -\frac{\partial \Phi}{\partial x}, J^{(y)} = -\frac{\partial \Phi}{\partial y}$, combine this result with Cauchy-Riemann equations (See theorem 1.28) to the equation $\Psi(z) =$ constant(a contour line),

$$d\Psi = \frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{\partial y}dy = J^{(y)}\, dx - J^{(x)}\, dy = 0$$

$$\Rightarrow \frac{dx}{J^{(x)}} = \frac{dy}{J^{(y)}}$$

so $(dx, dy)^T$ is parallel to $(J^{(x)}, J^{(y)})^T$. ■

partial differentiation of $h(z)$

Sometimes we are required to differentiate a complex-valued function in the direction of real and imagery axes.
As $z = x + iy$, $\frac{\partial z}{\partial x} = 1, \frac{\partial z}{\partial y} = i$.
By chain rule:

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial h}{\partial z} = h'(z)$$

similarly $\frac{\partial h}{\partial y} = i\, h'(z)$.

5.3 Solutions to specific two-terminal circuits

If you have viewed section 1.3 and 1.4 about multivariate calculus and complex analysis, the answers to the questions mentioned above on lecture notes should not be difficult to understand.

Here are some good techniques/important facts summarised from lecture notes:

- You can use symmetry to simplify many calculations
- You have to check consistency of your function $h(z)$ with potentials at terminals after guessing $h(z)$
- Equating $J^{(x)} - iJ^{(y)}$ with $-\hat{c}\frac{dh}{dz}$ solves current density in x and y axes separately.
- If there is a point source z_0 on the conductor, you can be 100% sure that $-\frac{m}{2\pi}\log z - z_0$ is part of $h(z)$. (Derived from limit case of annular conductor) And in rigorous logic, we define a point source in this way. (Logic and ideas are going in the reverse direction)

- *Singularity analysis*: There is another condition for $h(z)$ to be a proper complex potential function for a point-source, that is the rest of the part $\hat{h}(z) = h(z) - (-\frac{m}{2\pi}\log z - z_0)$ should not blow up, at minimum, not blow up for all points $z \in$ interior region(not terminals). But you are still required to find all points of singularity, and then eliminate any singularity outside the region or on terminals.

You can jump to conductors of different shapes using conformal mapping (the idea of mapping a region to another, with the requirement that angle of any vector defined relative to a vector u_0 is not changed after mapping) The method is given as follow:

Proposition 5.10. *If a map ζ is given by $\zeta = f(z)$, where $f(z)$ is analytic and one-to-one function from region D to unit disc on complex plane. Then the function*

$$h(z) = -\frac{m}{2\pi} log(f(z))$$

is a complex potential for a current due to point source of strength m at z_* ($f(z_*) = 0$).

Common conformal mappings:

- $\zeta = z + \beta$ Translation
- $\zeta = e^{i\theta}z$ anti c.c. rotation by angle θ
- $\zeta = e^z$ folding
- $\zeta = \rho z$ scaling
- $\zeta = z^2$ This acts with different effect on different regions. For sectors of a circle centred at origin, it "doubles" the angle subtended at the centre.
- $\zeta = \frac{\alpha z+\beta}{\gamma z+\delta}$ Stereo-graphic Projection(a map from sphere to complex plane, except north pole)
- A special case of stereo-graphic: $\zeta = \frac{1-z}{1+z}$

Of course you can join several maps by finding composites of $\zeta(z)$. A very quick way to find which function could work is to look at how the boundaries change.

Properties of conductors

I will end with a table summarising all complex potential functions of conductors mentioned in the lecture notes.

Conductor	**Point?**	**+ & - Terminals**	$h(z)$	C_{eff}
Rectangle	No	$x = -L$ & $x = 0$	$-\frac{z}{L}$	$\frac{\hat{c}H}{L}$
Annular conductor	No	$r = \rho$ & $r = 1 (\rho < 1)$	$\frac{\log z}{\log \rho}$	$-\frac{\hat{c}}{\rho \log \rho}$
Annular point source	Yes	origin & $r = 1$	$f(z) = z$	$m\hat{c}$
Half-space(x>0)	Yes	$(1, 0)$ & $x = 0$	$f(z) = \frac{1-z}{1+z}$	
Strip(height π)	Yes	origin & $x = \pm\frac{\pi}{2}$	$f(z) = \frac{1-e^z}{1+e^z}$	
Quarter conductor	Yes	$(\frac{1}{\sqrt{(2)}}, \frac{1}{\sqrt{(2)}})$ & positive x and y	$f(z) = \frac{1+i\ z^2}{1-i\ z^2}$	

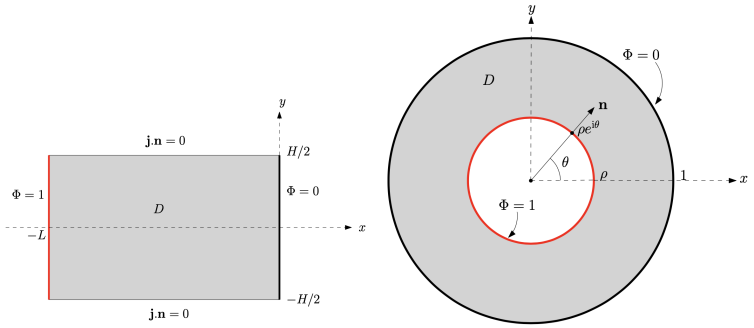
Table 7: Analogy between spring-mass system and electric circuit

Remark. The rows for complex potential of point source conductors uses $f(z)$ instead, actually

$$h(z) = -\frac{m}{2\pi} log(f(z))$$

. Since $c_{\text{eff}} = m\hat{c}$, we say such a conductor with point source has strength m if $h(z)$ has the form stated above.

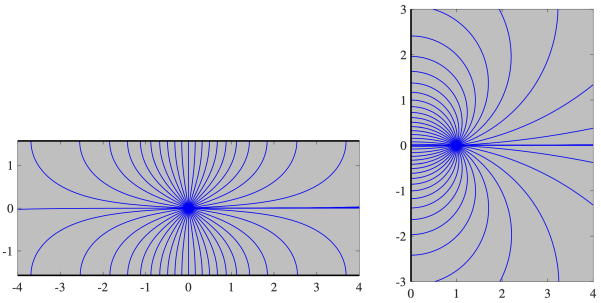
You can see below for graphs of conductors as a reference.
Graphs for first two conductors:



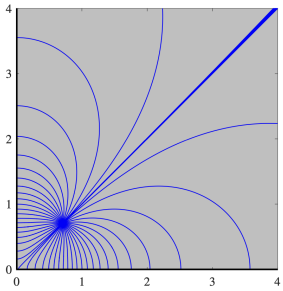
(a) Rectangle conductor (b) Annular conductor

Figure 7: Shapes of Conductors

Current lines for 3 point-source conductors:



(a) strip (b) half plane



(c) Quarter plane

Figure 8: Point-source current lines

6 Conclusion

We began with just a simple model of a discrete electric system, investigated circuit properties, and then by analogy generalised this to spring-mass systems and random walk. Then to make our electric model more realistic, we derived a continuum version and investigated properties of circuit when sources are terminals on a region.

Here are some important tips to remember when solving problems

1. Usually we play with unit conductance but there are cases where conductance is not unit.(similarly for spring constants) Be careful with the following differences:
   - K is in weighted form. A^TCA
   - Harmonic potential formulae should be weighted (general law of harmonic potential)
   - When calculating current on edge, it should be potential difference times conductance! (Ohm's law)
   - When finding effective conductance, you should label conductance on the graph.
   - Escape probability is no longer effective conductance divided by degree of positive node. It is divided by sum of conductances of all edges connected.

   Similarly there are many things to keep in mind when unit voltage is not assumed.
2. Do not just blindly apply the first method you thought of. Look back to previous parts to see if there is any useful information, and quickly try those in mind. Use the best way of approach!
3. Always begin with the formulae/rules in the lecture notes. Even though some adjustments are required after that to suit the problem.
4. After you got an answer, think of the physical meaning! Does it make sense?

Good luck with your exams!

Warning: Please do not share it with anyone else than students of Imperial.

7 Bibliography

Due to the time limit, I have no enough time to create a formal reference list but here are the websites/papers/books I used as reference:

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11. and the lecture notes.