Sequence and series in complex variables

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This part is omitted by our lecture notes. Though complex sequences are very similar to real sequences, there are slight differences on dealing with sequences and interpretations. This note aims to pull out the theory completely to build a solid understanding, and provide a chance to review some of the contents from real analysis.
This note is based on:

- Complex Analysis, Elias M. Stein
- Complex Analysis, Lars Ahlfors
- Complex variables and applications, James Ward Brown & Ruel V. Churchill

1 Complex Sequences and series

Definition 1.1 (sequence convergence)**.** An infinite sequence $\{z_n\}_{n\geq 1}$ in $\mathbb{C}$ is said to have limit z if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t.

$$|z_n - z| < \epsilon \quad \text{if } n > N$$

Denoted as $\lim_{n\to\infty} z_n = z$. Graphically, for sufficient large n, points z_n lie in a circle around z with arbitrarily small radius.

Theorem 1. *If $z_n = x_n + iy_n$ and $z = x + iy$, then $\lim_{n\to\infty} z_n = z$*

$$\Leftrightarrow \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

Proof. Proof is quite simple. Left as exercise. □

Corollary. *This theorem directly borrows many useful properties from real analysis. For example, the limit is unique, convergent sequences must be bounded ($|z_n| \leq M$ for some $M > 0$ for all n), convergent sequences follow algebraic rules (addition, multiplication, division), every subsequence of convergent sequence must converges to the same limit.*

Example 1.1. The sequence

$$z_n = \frac{1}{n^2} + 3i$$

converges to $0 + 3i$ as $\frac{1}{n^2} \to 0$.

Proposition 1. *If* $\lim_{n\to\infty} z_n = z$, *then* $\lim_{n\to\infty} |z_n| = |z|$.

Proof. Direct result using the fact that $||z_n| - |z|| \leq |z_n - z|$. □

Definition 1.2 (series convergence)**.** An infinite series $\sum_{n=1}^{\infty} z_n$ of complex numbers converges to S if sequence

$$S_N = \sum_{n=1}^{N} z_n$$

of partial sums converges to S. Denoted as $\sum_{n=1}^{\infty} z_n = S$. Otherwise we say it diverges.

Theorem 2. *If* $z_n = x_n + iy_n$ *and* $S = X + iY$*, then* $\sum_{n=1}^{\infty} z_n = S$

$$\Leftrightarrow \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y$$

Proof. Direct consequence of Theorem1 □

Corollary. *If a series of complex numbers converges, n-th term z_n as a sequence converges to* 0.

Proof. By the above theorem, series $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n$ converges. So from real analysis we know $x_n, y_n \to 0$. Therefore by Theorem1, $z_n \to 0$. □

Definition 1.3 (Absolute convergence)**.** Series $\sum_{n=1}^{\infty} z_n$ is said to converge absolutely if the real series $\sum_{n=1}^{\infty} |z_n|$ converges.

Corollary. *Absolute convergence implies converges for complex numbers*

Proof. Assume $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, as

$$|x_n| \leq \sqrt{x_n^2 + y_n^2}, \quad |y_n| \leq \sqrt{x_n^2 + y_n^2}$$

we have $\sum_{n=1}^{\infty} |x_n|, \sum_{n=1}^{\infty} |y_n|$ converges. Since in real case, absolute convergence implies convergence, so x_n, y_n converges. Therefore, by Theorem1, z_n converges. □

Absolute convergence uses a real series, so comparison test, ratio test, root test. And due the above theorem, it is often easier to show a complex sequence converges absolutely first.

Also, recall that it is often more convenient to consider remainder $r_N = S - S_N = \sum_{n=N+1}^{\infty} z_n$. As $|S_N - S| = |r_N|$, series converges to S iff $r_N \to 0$.

Example 1.2. We can show

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{if } |z| < 1$$

now. Assume $|z| < 1$, Recall

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

so partial sum $S_N = \frac{1-z^N}{1-z}$. Let $S := \frac{1}{1-z}$, $r_N = \frac{z^N}{1-z}$. Since $|z| < 1$, the sequence

$$|r_N| = \frac{|z|^n}{|1-z|}$$

converges to 0. So indeed $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

Example 1.3. Using the above example, show that if $z = re^{i\theta}$ where $0 \le r < 1$, then

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \qquad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

Example 1.4. Let

$$\theta_n := Arg(2 + i\frac{(-1)^n}{n^2}$$

, prove $\lim_{n\to\infty} \theta_n = 0$.

Proposition 2. *If $\sum_{n=1}^{\infty} z_n = S$, then $\sum_{n=1}^{\infty} \overline{z_n} = \overline{S}, \sum_{n=1}^{\infty} cz_n = cS$ for any constant $c \in \mathbb{C}$.*

Theorem 3 (Abel's second theorem)**.** *If a power series defined by $f(x) := \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1, and the series $\sum_{n=0}^{\infty} a_n$ converges, then $f(x)$ is continuous from the left at 1.*

This is a result of real analysis. The proof is omitted here as it can be found at many places. This theorem has a complex version

Theorem 4 (Abel's second theorem(complex))**.** *If $f(z) := \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1 and $\sum_{n=0}^{\infty} a_n$ converges, then $f(z)$ tends to $f(1)$ as z approaches 1 in a way s.t. $|1-z|/(1-|z|)$ is bounded.*

the bounding conditions means that z stays in an sector formed by two line segments shooting from 1, that are symmetric about real axis. This sector is called Stolz sector.

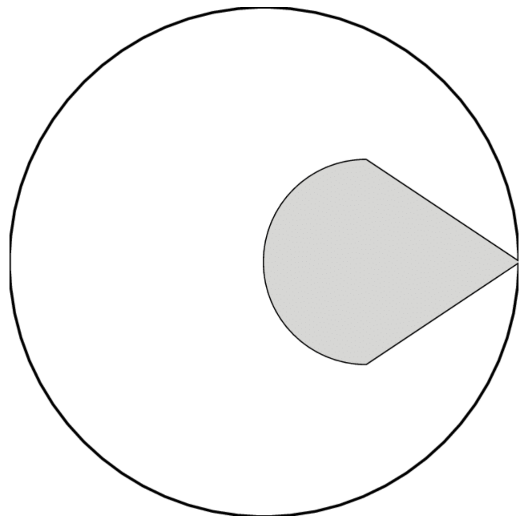

Figure 1: Stolz angle and Stolz sector

Proof. We may assume WOLG $\sum_{n=0}^{\infty} a_n$, this is done by adding a constant to a_0. Let $s_n = a_0 + a_1 + \cdots a_n$.

$$s_N(z) := \sum_{n=0}^{N} a_n z^n = s_0 + \sum_{n=1}^{N} (s_n - s_{n-1}) z^n$$

$$= s_0(1-z) + s_1(z - z^2) + \cdots s_{N-1}(z^{N-1} - z^N) + s_N z^N$$
$$= (1-z)(s_0 + s_1 z + \cdots + s_{N-1} z^{N-1}) + s_N z^N$$

for z with $|z| < 1$, $s_N z^N \to 0$ as $N \to \infty$. So we can conclude

$$f(z) = (1-z) \sum_{0}^{\infty} s_n z^n$$

according to the bound condition, $|1-z| \leq K(1-|z|)$ for some constant K. Also we have $s_N \to 0$. Pick large enough m s.t. $|s_n| < \epsilon$ for any $n \geq m$. So the remainder of series $\sum_0^\infty s_n z^n$ satisfies

$$\sum_{m}^{\infty} |s_n z^n| \leq \epsilon \sum_{m}^{\infty} |z|^n = \epsilon \frac{|z|^m}{1-|z|} < \frac{\epsilon}{1-|z|}$$

so

$$|f(z)| \leq |1-z| \left| \sum_{0}^{m-1} s_k z^k \right| + K\epsilon$$

if z is very close to 1, first term can be arbitrarily small. So $f(z) \to 0$ as z approaches 1 from the left within a Stolz sector. $\square$