An Introduction of Euclidean Distance and Algebra of Polynomials (Chapter 1 of Pure 3)

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Preface

If reading official textbooks(A-level pure math, or any other high school math textbooks) gives you the feeling: ”Hey? Why this make sense?” or ”Why the writer did that? Why not the other way?”, then this is the right place to consult. I wrote these notes to enhance your understanding of these important topics, provide a more natural, more intuitive way of defining math objects, and provide some extensional theorems or materials to tell you the applications of these topics.

One of the features of this series of notes is that I would guide you through the way of thinking behind each definition/theorem, just like the mathematicians developed them did. (We cannot go through the full history as that is really long for some topics) This enables you to really understand why are some definitions abstracted from phenomenons, and the meaning of some theorems. Advantage of such intuitional thinking makes definitions, formulas easier to memorise and allows you to solve more flexible problems instead of just blindly apply methods on the textbook. So this book is for students aiming A^*. Of course, if you are just aiming A, you are welcome to read it. But you may achieve A without reading this series of notes. The grades of math can be interpreted as follows: U - *Not understanding the materials*, E - *Know some basic definitions, know some basic formulas*, C - *can apply formulas and theorems in simple context*, A - *Very strong base, can solve some non-standard problems*, A^* - *Have a deep understanding of the materials, can solve many flexible, extensional problems*. So I would definitely say if you are just aiming grades below A or wishing to get a quick review of formulas, definitions before the exam, this is not the right place. Z-notes or other summaries perhaps serve your purpose.

The way to use this book is very easy, reading IN ORDER, complete EXERCISES both in the content and at the end of each note. This is to force you actively think about the topics instead of blindly reading what's given. Also, please do try the slightly harder questions marked with * if you have time. They are very good extensional materials, especially for students pursuing STEM majors. You can read these notes before learning them in the school, or during the semester chapter by chapter. Note that the chapter about Numerical Analysis is not included, because that requires not much profound understanding to learn.

If you are prepared, just begin sailing over the sea of mathematics!

Chapter 1

Euclidean distances

Distance is a concept we are already familiar with since elementary school. It describes a property between two points in space(either 1-D, 2-D or 3-D). We can actually treat it as a function $d(P, Q)$ which takes two points P, Q and spits out a positive number indicating the distance between two points. The following are three fundamental properties of distance:

1. $d(P, Q) \geq 0$, any distance is non-negative

2. $d(P, Q) = d(Q, P)$, this should be obvious.

3. $d(P, Q) + d(Q, R) \geq d(P, R)$. This is just the triangular inequality.

But now we need a way to calculate this! Let's first consider 1-D case. (i.e. the real number line) If I ask for a point sitting at value x, what is its distance to point 0. One may spot immediately that $|x|$ gives the desired result. Geometrically, it folds the whole real line around the point 0.

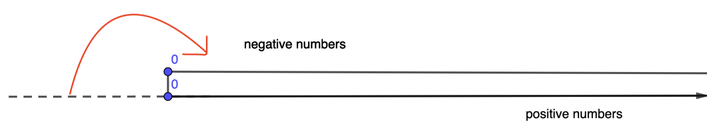

Figure 1.1: Folding of real line

Algebraically, we can interpret $|x|$ as: find $x - 0$ if x is greater, find $0 - x$ if 0 is greater. Can you guess a formulae for distance from point x to a given point a on real line?

Answer: $|x - a|$. (Think of why, interpret it algebraically and geometrically)

Absolute values have very close relationships with squares, as the both spits out non-negative number!

Proposition 1. $(x-a)^2 = b^2 \Leftrightarrow |x-a| = b$ *where* a, b *are constants.*

Try to prove this by yourself. (Hint: consider possible relations of x, a, $\Leftrightarrow$ basically means you can go from left to right and from right to left) Can you find similar results for $(x-a)^2 < b^2$ and $(x-a)^2 > b^2$?

Interpretation of $|x-a| = b$ is the distance from x to a is b, for 1-D case, there should be only two points satisfying this condition: $a \pm b$(two points merge to one if $b = 0$) For inequalities, say $|x| < 3$ where we are only looking at regions to the left of 3 on the folded real line. The 2 red lines in the figure are what we want.

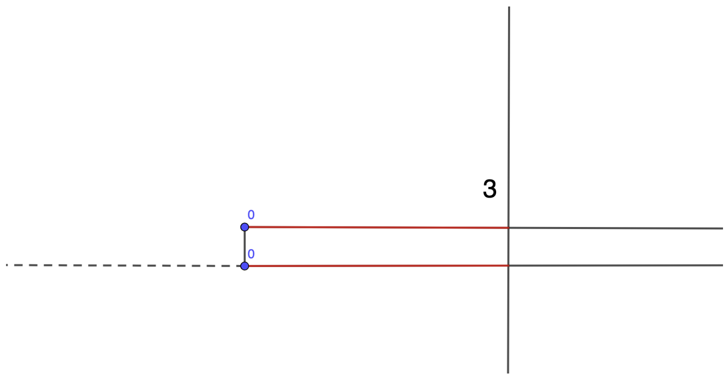

Figure 1.2: $|x| < 3$

So the solutions are $-3 < x \leq 0$ and $0 \leq x < 3$, and combining them gives $-3 < x < 3$. This should remind you of quadratic inequalities learnt in Pure 1. You can find interpretations for $|x-a| > b$, $|x-a| < b$ similarly.
`Exercise.` find all possible x satisfying $|4x-5| = |2x+8|, 7x-2 < |6x+2|$.

1.1 2-D distance and higher dimensions

To describe a point in 2-D plane, we actually need 2 coordinates. So the distance function $d(P, Q)$ takes 4 numbers as input here! Using Pythagoras's theorem, we can easily deduce a formulae for distance between (x_1, y_1) and (x_2, y_2):

$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

go to Figure 1.3 if you do not understand.
With a little bit more work on geometry, we can deduce the distance formulae for 3-D case:

$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$

.

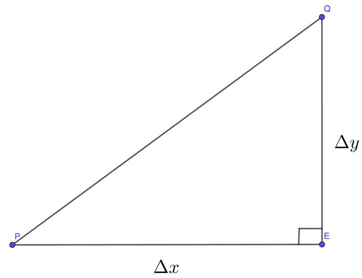

Figure 1.3: 2-D distance

So it is not surprising that mathematicians generalise the concept of distance to higher dimensions between $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$:

$$\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

Try to check that 3 fundamental properties of distance are satisfied by the generalised distance.
There is another representation for distance: $|\boldsymbol{x} - \boldsymbol{y}|$ which looks just like the 1-D case, but $\boldsymbol{x}, \boldsymbol{y}$ are vectors. The formulae is exactly the same. If you are not familiar with vectors, just treat that as coordinates. For example, if you see (x, y), you can change that to a vector by writing them in a column: $\begin{pmatrix} x \\ y \end{pmatrix}$. But this is not really the essence of vectors, see *An Introduction to Vectors* for more.

1.2 Circles, spheres and more?

If we are looking at 2-D plane, what would the equation $|\boldsymbol{x} - \boldsymbol{a}| = R$ represent? A circle with radius R centred at $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ (you should treat $\boldsymbol{a}$ as a point) as absolute value represents distance here. So actually in 1-D case, $|x - a| = R$ is like a circle! In 3-D space, $|\boldsymbol{x} - \boldsymbol{a}| = R$ forms a sphere. (Careful! Here the vectors have 3 entries x, y, z) Of course this generalises to higher dimensions, though you and me cannot imagine a hyper-sphere.

Exercise. Try to write the equation $|\boldsymbol{x} - \boldsymbol{a}| = R$ where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ out using the formulae we found for 2-D distance in Section 1.1. Is this equation familiar?

Chapter 2

Algebra of Polynomials

2.1 Integers and Polynomials

Integers	Polynomials
Integer plus Integer is Integer	Poly. plus Poly. is Poly.
Integer times Integer is Integer	Poly. times Poly. is Poly.
$a + b = b + a$	$f + g = g + f$
$(a + b) + c = a + (b + c)$	$(f + g) + h = f + (g + h)$
$ab = ba$	$fg = gf$
$0 + a = a + 0 = a$	$0 + f = f + 0 = f$
$1a = a1 = a$	$1f = f1 = f$

Table 2.1: Basic Similarities

Remark 2.1.1. *It is just a convention to write f, g, h for polynomials/functions and a, b, c for numbers. Note that 0 for polynomials is not a number, but really a "zero" polynomial $0 + 0x + 0x^2 + ...$ Similar for 1.*

Definition 2.1.1 (formal definition of a polynomial)**.** *A polynomial in x is $\sum_{i=0}^{n} a_i x^i$ where a_i are constants called coefficients. The degree of a polynomial$f$, $deg(f)$, is defined as n, the highest power of x with non-zero coefficient. For example, $x^3 + 2x + 1$ has degree 3.*

Remark 2.1.2. *We can treat polynomials as sliders, each slider controls one coefficient. Say slider 1 controls the term x, slider 2 controls the term x^2, slider 0 controls constant term.*

Multiplication and addition always work for 2 integers, and they also works for polynomials. Unfortunately, division does not for neither integers nor polynomials. This is because multiplicative inverse does not always exist for integers/polynomials.

- We say a number $-a$ is additive inverse of a if $a + (-a) = 0$. For integers, $-a$ is just minus a. (do not confuse the two symbols)

- As an exercise, please try to write down the additive inverse $-f$ for the polynomial $f(x) = \sum_{i=1}^{n} a_i x^i$.
- Similarly, multiplicative inverse of a is defined as the number a^{-1} such that $aa^{-1} = 1$. (Note: here a^{-1} is just a notation, it does not necessarily mean reciprocal. For example for a function, f^{-1} means the inverse function)
- Try to find a counterexample from integers and polynomials to prove that not all numbers/polynomials have multiplicative inverse.

Exercise. Find all integers that have a multiplicative inverse, and find all polynomials that have a multiplicative inverse.

Back to division, $6 \div 2 = 3$, it works! But not for $7 \div 2$, at least when we restrict ourselves to integers. We pick all the numbers that divide a number n factors of n. But how to deal with divides like $7 \div 2$? If 7 represents 7 cakes, to be allocated to two children. Well, you may say cut one cake into two halves, so the result is $3\frac{1}{2}$. What if we are separating 7 people to 2 groups? Do we cut one person into two halves? The idea is, we pick two people each time, and place them into to two groups. Keep picking until we cannot pick two people any more. The rest (in this case, only one person possible) of the people are called remainders. This process can be done for 3 times, so we write $7 \div 2 = 3...1$.
1 is remainder and 3 is the quotient. This is called division with remainder.

Theorem 1 (Division with remainder)**.** *For any two integers a and b, exists unique integers q, r such that $aq + r = b$. We call q the quotient of b divided by a and r the remainder.*

There is a similar definition for polynomials:

Theorem 2 (Division with remainder)**.** *For any two polynomials $f(x)$ and $g(x)$, exists unique integers $h(x), r(x)$ such that $f(x)h(x) + r(x) = g(x)$. We call $h(x)$ the quotient of $g(x)$ divided by $f(x)$ and $r(x)$ the remainder.*

It should not be surprising that the method to do division for integers can be used for polynomials:

Here is an integer division
$$\begin{array}{r}
12 \\
12\overline{)152} \\
\underline{12} \\
32 \\
\underline{24} \\
8
\end{array}$$

Here is how to do it for polynomials:
$$\begin{array}{rr}
 & x+3 \\
x+2\,\big) & \overline{x^2+5x+2} \\
 & \underline{-x^2-2x} \\
 & 3x+2 \\
 & \underline{-3x-6} \\
 & -4
\end{array}$$

If you write the integers out, they are actually $1 \times 10^2 + 5 \times 10^1 + 2 \times 10^0$ and $1 \times 10^1 + 2 \times 10^0$. If you replace 10 with x, you just get the polynomials! Of course we are not saying that you can just simply write polynomials as integers in this way and then find the result (Think of why?). But indeed the principles are the same. No carry(-propagation) taking place makes calculation easier for polynomial divisions.

Definition 2.1.2 (Factor). *An integer a is a factor of b if there exists an integer c such that $ac = b$. Abbreviated as $a|b$.*

Definition 2.1.3 (Companion). *Two integers a, b are companion if $a|b$ and $b|a$.*

It is only very few cases satisfy this property in integers. Actually just $a = \pm b$. The proof is left as an exercise. But for polynomials, any f, g satisfying $f(x) = cg(x)$ for some constant c are companion! Such polynomials have the same roots, their derivatives are also companion. Companion polynomials always "travel" together. So it is enough for you to just consider one of them. Usually the representative chosen is the one with coefficient on the higher power of x being 1.

Example 2.1.1. *The representative of $4x+8$, $2x+4$, $9x+18$ and many other companion ones is $x + 2$.*

2.1.1 Some Links between degree of polynomial and integers

If I ask if 6 is a factor of 7, you would say no without a hesitation. What if I ask if $x^6 + 3$ is a factor of $x^7 + 3$. Do you have to do long division to find out the answer?

Theorem 3 (divisibility and degree). *If polynomial f has a lower degree than g. f is not a factor of g.*

Example 2.1.2. *Try to prove these conclusions about degree of a polynomial: $deg(f + g) = max\{deg(f), deg(g)\}$, $deg(fg) = deg(f) + deg(g)$.*

2.2 Factor and remainder theorem

We can investigate root of polynomials, this is one of the major differences between polynomials and integers. Let's say after you used the long division for polynomials in the last section, you found that $x - a$ is actually a factor of $f(x)$, what can you say about roots of $f(x)$?

$$f(x) = (x - a)h(x)$$

for some polynomial $h(x)$. So if we calculate $f(a)$, it must be 0! What about the other way around? If $x = a$ is a solution to $f(x) = 0$, is $x - a$ a factor of $f(x)$?

- Let's begin with polynomials of degree 1 (Degree 0 cannot have root $x = a$ unless the polynomial is just 0): $f(x) = c_1x + c_2$, then $c_1a + c_2 = 0$. So $c_2 = -c_1a$. That means $f(x) = c_1(x - a)$. Right for degree 1.

- If this conclusion is true for polynomials of degree lower than k (k is a natural number), then either it has no factor(except constant polynomials), or it has two factors of lower degree. The polynomial in the first case cannot have any root(take it as granted), the second case, by our assumption, must yield a factor $x - a$.

- By the above arguments, we can deduce that the statement is true for 2(because it is true for degree 1), so 3 because it is true for degree 1 and 2, and then $4, 5, \ldots$. This is true for all polynomials!

Remark 2.2.1. *The above method of proof is called proof by strong induction.*

Theorem 4 (Factor Theorem)**.** *$x - a$ is a factor of $f(x)$ if and only if(it means you can go both directions) $f(x) = 0$ has a root $x = a$.*

Remainder theorem is just a small corollary form factor theorem:

Theorem 5 (Remainder theorem)**.** *$f(x) = (x - a)h(x) + c$ for some polynomial h and constant c if and only if $f(a) = c$.*

Exercise. Prove remainder theorem from factor theorem. (Hint: try the substitution $g(x) = f(x) - c$)

2.3 Ring

If you pursue your study in mathematics, you will meet something called a ring. Not the one for wedding.

It is basically an algebraic structure extracted from integers, because as we an see, polynomials and integers are so alike! So we call any system with such structure ring. Namely, a system with (commutative and associative)addition and (commutative and associative)multiplication defined, both with an identity($0, 1$ for integers) and addition has inverse. Also, the distribution law holds. (Try to prove $f(g+h) = fg+fh$ for polynomials) And you will learn more about how ideas of prime numbers, factors, divisors are generalised there. And polynomials with integer/rational/real/complex coefficients have really distinct but interesting behaviours. Also you would learn how different things can be inserted into a polynomial while equations still hold, like complex number i or matrix A(e.g. Cayley-Hamilton theorem). Rings can even contain smaller rings(subring).

Remark 2.3.1. *You may see that partial fractions and a special binomial expansion are also covered in chapter one. But I would leave the first one for notes on calculus and second one for notes on exponential and logarithm. You will see why when you see them.*

Chapter 3

Exercises

Q1. Plot $|f(x)|$ for $f(x) = 3x^2 + 6x$ and $f(x) = cos(x - \frac{\pi}{2})$

Q2.
(a) Prove that for two integers a, b, $a|b$ and $b|a$(a is a factor of b and b is a factor of a) if $a = \pm b$. And $a = \pm b$ if $a|b$, $b|a$.
(b) Prove that if $a|b$ and $b|c$, then $a|c$. (Divisibility is transitive)
(c) Prove that for two polynomials $f(x), g(x)$, if $f(x) = cg(x)$(c is constant), then $f(x)|g(x)$ and $g(x)|f(x)$(equivalently, they are companion). Try to prove if f, g are companion, then the only possibility is that $f(x) = cg(x)$ for some constant c using the definition of $f|g$. (Recall if there is a function $h(x)$ such that $f(x)h(x) = g(x)$, then we say $f|g$ i.e. $f(x)$ is a factor of $g(x)$)

Q3. Calculate $x^4 + x^3 - 2x + 3$ divided by $3x^2 - x + 2$ using division with remainder.

Q4. Show that $x + 2$ is a factor of $f(x) = x^3 + 4x^2 + x - 6$, hence factorise $f(x)$ completely. Write down all the solutions for $f(x) = \sin^3\theta + 4\sin^2\theta + \sin\theta - 6 = 0$ if $0 \leq \theta \leq 4\pi$.

Q5.
(a) Prove that $x + 3$ is a factor of $f(x) = x^3 + 3^3$. Hence find the quotient $\frac{f(x)}{x+3}$.
(b) Let's generalise part (a), prove that for any positive integer m and constant a, $x + a$ is a factor of $f(x) = x^{2m+1} + a^{2m+1}$. Hence find the quotient $\frac{f(x)}{x+a}$.
(c) Why can't we let the power be even? Try part (b) with $f(x) = x^{2m} + a^{2m}$.