An Introduction to Complex Numbers (Chapter 9)

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Chapter 1

The Marvelous Evolution of Numbers

Back at cave age, people are beginning to realise the magic of counting. Ancient people are very good at counting up to 4 (two hands and two legs). Some smarter tribes may use fingers to count up to 10. But as culture develops, counting becomes important in asset management, trades and resource distributions. Counting up to 10 is not enough. So ”numbers” are recorded on stones, ropes or sticks. But imagine you want to convey that there are 100 people in the tribe, carving 100 notches can be tiring and difficult to read. So decimals appeared. Every 10 would be recorded as a knot on the rope. Of course you can pop up every 5, resulting in ”44” actually represents $4*5+4$. But it is just a convention that decimal systems are used (number system with base 10)

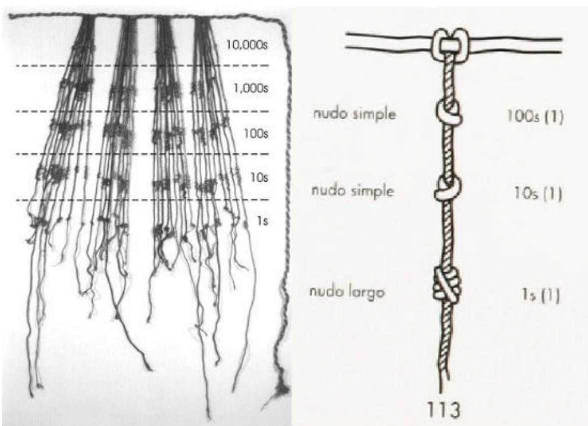

Figure 1.1: Knots on rope

Some cultures admitted the existence of 0, but some did not. To many cultures, 0 is just a placeholder to distinguish 504 and 54. They though 0 does not exist because you can never count 0. Or alternatively you can say 0 corresponds to ”not exists”, ”there are 0 apples” means ”there is no apple”. But let’s now accept 0 as a member of natural numbers(abbreviated as $\mathbb{N}$), because 0 guides us to the world of integers($\mathbb{Z}$). (Strictly speaking, 0 is defined as the starting point of counting)

1.1 Integers

It was difficult for ancient people to understand why we need negative numbers, though they do know subtraction(the reverse process of addition). Let's say another tribe was attacking and you reported to the chief: "There are 10 armed enemies coming! We killed 13 of them. But I do not know how to subtract, how many are still left?". The chief would send you to prison because you can never kill 13 of 10 men. You see? Negative numbers do not exist. But actually, natural numbers do not exist! You can never touch the number 3. You can feel it, you have an idea about what is 3: it is the symbol to abstract the "number" of three apples, three people, three sticks, three trees etc. Though math is closely related to reality sometimes, it is more of symbolic abstractions of reality. So do not doubt that math is never useful, just because many concepts do not directly exist.

Negative numbers are ultimately accepted because traders can use that to distinguish between the money they spent and the money others owed to them. The definition is simple: for each non-zero natural number a, we define its additive inverse to be the number b such that $a+b=0$. We denote additive inverse of a as $-a$, so $a+(-a)=0$. You may have noticed that $+(-a)$ is the same as $-a$, this is another way to view subtractions.

1.2 Rationals and Reals

At first divisions were dealt with residues and quotients, to avoid defining new numbers. For example, $7 \div 3 = 2...1$. We have discussed many of this in Note 1 about algebra. If we need, we can define multiplicative inverse. (Exercise: think of a way to define being multiplicative inverse of a) This allows us to perform all four arithmetic operations without resulting in a "non-number". We call this system rational numbers $\mathbb{Q}$. (Doing arithmetic on $\mathbb{Q}$ never results in things outside $\mathbb{Q}$)

You may think: Oh finally we have got all numbers! This was a misconception that rational numbers can represent every length, because the denominator can be arbitrarily large to make the number precise. But people soon realised that the length of diagonal of a unit square($\sqrt{2}$) cannot be expressed as $\frac{m}{n}$. Here is a simple proof:

Proof. Assume $\sqrt{2} = \frac{m}{n}$, then $2n^2 = m^2$. That means m^2 is an even number, so m must be even. (As square of odd number must be odd) Let's say $m = 2k$. Then $2n^2 = 4k^2$, which implies $n^2 = 2k^2$. So n must also be even! Say $n = 2l$. Then actually $\sqrt{2} = \frac{2k}{2l} = \frac{k}{l}$.

But based on this, we can argue similarly that k, l are even numbers, and extract another factor 2. This process never ends, we can keep extracting factor 2. What integers m, n have infinite factor 2? No! So this is impossible, $\sqrt{2}$ is not rational number. $\square$

There are many ways to define real numbers($\mathbb{R}$) to fill in the holes on $\mathbb{Q}$ like $\sqrt{2}, \sqrt{5}$ and many others. One of the easy ones is simply all decimals: $a_n a_{n-1} ... a_1 . d_1 d_2 ....$ (a_i are digits of integer part, d_i are digits of decimal part, there can be infinite d_i) For example, $0.\overline{3} = 0.3333...$ is a real number.

Someone did propose possibly enabling the operator $\sqrt{}$ extends $\mathbb{Q}$ to $\mathbb{R}$, but numbers like π, e can never be written from rational numbers simply using arithmetic and square root. And $\sqrt{}$ possibly extends some parts too far, as $\sqrt{-5}$ is not a real number.

Also, if you try to define real numbers as roots of all non-constant polynomials with rational coefficient(such numbers are $\mathbb{A}$, algebraic numbers), there are two problems:

- Some numbers, like e, π, are not solutions to any polynomial with rational coefficient.
- Roots of some polynomials, like $x^2 + 1 = 0$, are not real numbers. You may argue there is no root here! Well, there are, they are just not real. Similarly, before defining negative numbers, $x + 3 = 0$ "had no root"(just not in natural numbers $\mathbb{N}$).

Let us summarise what we have defined so far.

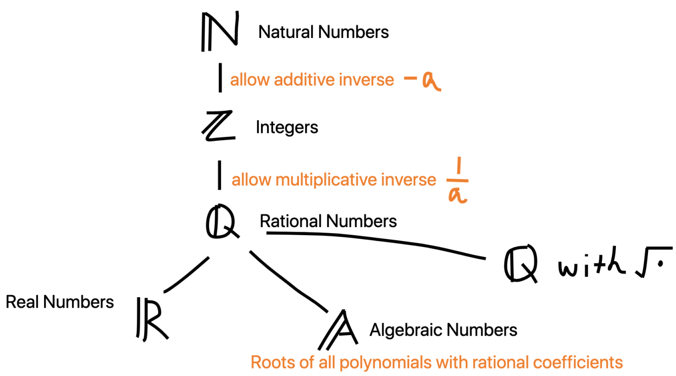

Figure 1.2: The Number System

1.3 Complex, not Imagery

Mathematicians developing the theory of complex numbers strongly oppose the name "imagery" given to them. Because it sounds that all complex numbers are just purely imaginations, just fun games. But complex numbers have very important applications, and come from very realistic scenes. Previously we saw that $\mathbb{Q}$, the rational numbers, diverges into several branches based on different extension

techniques used. Is there a way to cover all the three systems? Like a boss level of $\mathbb{R}, \mathbb{A}, \mathbb{Q}$ with $\sqrt{}$. The answer is: simply by defining $i = \sqrt{-1}$ to welcome members like $\sqrt{-3}, \sqrt{-5}$ into our number family.
Exercise. Try to write $\sqrt{-3}, \sqrt{-5}, \sqrt{-a}$ ($a > 0$ and a is a real number) into the form of ki. (Where k is a real number) This means we do not have to define a symbol for any square root of negative numbers.

Does this mean we can just say $\{ki|k$ is real number$\}$ to be the set of Complex numbers? Oops, then any real number(like $1, 2, 3, \pi, e$) is not a complex number. This is not what we want. So instead, we define $\{a + bi|a, b$ are real numbers$\}$ to be the set of complex numbers: $\mathbb{C}$.(So now, $\mathbb{R}$ is contained in $\mathbb{C}$) This form is valid because, for any two numbers of the form $a + bi$: $a_1 + b_1i, a_2 + b_2i$, the sum can also be written in the form $a + bi$. That is $(a_1 + a_2) + (b_1 + b_2)i$ in this cases.

Exercise. Prove in similar manner that the difference, product, quotient of any two numbers of the form $a + bi$ can also be written in the form $a + bi$.

Hint. You may assume that distribution law of multiplication applies to complex numbers. For the quotient part of exercise, you may simulate the "rationalisation of the denominator". That is given a quotient $\frac{4}{1+\sqrt{2}}$, we can make denominator rational by multiplying $1 - \sqrt{2}$ on top and bottom. This results in $\frac{4-4\sqrt{2}}{1^2-(\sqrt{2})^2} = \frac{4-4\sqrt{2}}{1-2}$. Similarly, if the denominator is $1 + 2i$, multiplying $1 - 2i$ results in $1^2 - (2i)^2 = 1 - 4i^2 = 1 + 4$, a rational number! (Remember, $i^2 = \sqrt{-1}$ by definition)

We state the following theorem without proof, this is one of the reasons we use complex numbers:

Theorem 1 (Fundamental Theorem of Calculus). *Root of any non-constant polynomial with complex coefficient has at least one root.*

Furthermore, it can be shown that every polynomial of degree n has n roots(some may be duplicates). This is not surprising for $n = 2$, because we know, by formulae that two roots are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If square roots of negative numbers are allowed by defining i, there must be two roots! (or one root with multiplicity 2, that is when $b^2 - 4ac = 0$) And if you happens to know that there is a general formulae for $n = 3$ just using arithmetic and square roots, this result is also trivially true. But it is surprising that this applies to polynomial of any degree. So by factor theorem, any polynomial can be factorised to $(x - c_1)(x - c_2)...(x - c_n)$ where c_i are complex numbers.

The summary of number system we have defined so far is on the next page.

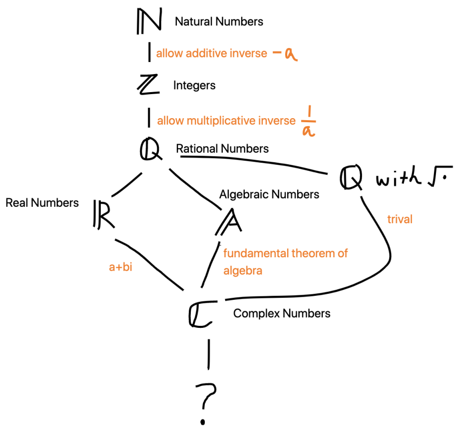

Figure 1.3: The Number System

So, is there any further expansion(above $\mathbb{C}$)? Later you will see that complex numbers are convenient to express 2-D rotations, so mathematicians guessed maybe there is an expansion that represents 3-D rotations, with j defined in some way similar to $i = \sqrt{-1}$. But they failed. Because 3-D rotations are represented by i, j, k, with some defined operation rules! Such numbers, $a + bi + cj + dk$, are defined to be quaternions.

Some may ask can we define a number $j = 1/0$ such that we can define any number $a/0$ to be $a*j$. This yields many problems like $1-1 = 0*j-0*j = j*(0-0) = j*0 = 1$. And $0j$ cannot be naturally nailed down to 0, it should be 1 from the definition $j = 1/0$, but should be 0 according to the rule "0 times every number is 0".

Another interesting thing to notice is that $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, \ldots$. So $1, i, -1, -i$ forms a loop.

1.4 Extension: Mandelbrot set

The cover image of this note is the famous Mandelbrot's set, it is defined very simply by the divergence of iteration process $z_{n+1} = z_n^2 + c$ beginning with $z_0 = 0$ for a complex number c.

The colours indicate the speed at which the iteration explodes(by which we mean the growth of modulus $|z_n|$). Black color means the process generally not blow up, brighter colours mean the process blow up very fast. For example, for $c = 0$, $z_1 = 0^2 + 0, z_2 = 0^2 + 0, \ldots$ so it is marked black. Many small numbers around the origin are marked black.

The magic about Mandelbrot set is that it has marvelous patterns, and no matter how close you zoom in each pattern, there are more patterns. You can search a video on YouTube to see how complex it is.

Chapter 2

Complex numbers and Vectors

Have you noticed the similarities between 2-D vectors and complex numbers? Every 2-D vector can be written as $a\boldsymbol{i}+b\boldsymbol{j}$, whereas every complex number z can be written as $a+bi$. (a, b are real numbers) This is not a coincidence. We can actually view complex number as vectors to better visualise them! That is, drawing complex numbers on 2-D planes. The horizontal axis represents the real part a (called $Re\, z$), with unit 1; vertical axis represents the imagery part bi(called $Im\, z$), with unit i. Such representations are called Argand's diagram, to distinguish between the regular 2-D plane.

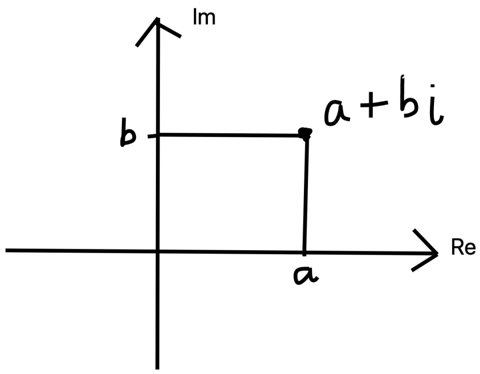

Figure 2.1: Argand's Diagram

From here, we can naturally define the modulus of $z = a+bi$, $|z| = \sqrt{a^2+b^2}$. (Because this is the definition for modulus of vectors) Furthermore, if we factor $|z|$ out from z, we get

$$z = |z|(\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}i)$$

. If we write

$$a' = \frac{a}{\sqrt{a^2+b^2}}, b' = \frac{b}{\sqrt{a^2+b^2}}$$

, notice the fact that $a'^2+b'^2 = 1$ and a', b' are numbers between -1 and 1. So we must be able to find angle θ such that $\cos\theta = a', \sin\theta = b'$. Therefore, $z = |z|(\cos\theta + i\sin\theta)$.

θ is called the argument of z. $(\arg z)$

Another concept is about conjugate. Any complex solutions(not real solutions) of polynomials with real coefficients occur in pairs of form $a \pm bi$. So if you found a solution $4+3i$, you can say $4-3i$ must also be a solution. So we define the conjugate of $z = a + bi$ to be $a - bi$, written as $\overline{z}$ or z^*.
Exercise. Recall the corresponding definitions for vectors, state the geometric meanings of: (Remember on Argand's diagram, if $z = a + bi$, actually z has "coordinates" (a, b).

- Sum $z_1 + z_2$ of two complex numbers: $z_1 = a + bi, z_2 = c + di$
- Difference $z_1 - z_2$
- $|z|$
- $\arg z$ (Draw a circle centred at origin with radius $|z|$, that is, circle passing through point z, to see the geometric meaning of $\arg z$)
- Conjugate of z

Exercise. Recall what does it mean for two 2-D vectors $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}$ to be equal. State what does it mean for two complex numbers $a + bi, c + di$ to be equal.

Of course, there are differences between complex numbers and vectors. We never do scalar product on complex numbers, and multiplication between complex numbers is not defined on vectors.

2.1 Square Roots

Every complex number has two square roots. For example, $z = 2 + i$, to find them, simply assume that $(a+bi)^2 = 2+i$. Expand left hand side(LHS) to get $a^2+2abi-b^2 = a^2 - b^2 + 2abi$. Use the fact that two complex numbers are equal if and only if real and imagery parts are the same, so $a^2 - b^2 = 2, 2abi = i$.
Exercise. Find the two square roots completely by solving the simultaneous equations above.

Chapter 3

Polar and Cartesian forms

Now we will explore the must powerful form of complex numbers: polar form. Remember that for ordinary 2-D coordinates(two real axes), we have Cartesian form (x, y) and polar form (r, θ). For complex number z, distance to origin $r = |z|$, angle made with initial line(positive real axis) $\theta = \arg z$. And $z = |z|(\cos\theta + i\sin\theta)$ based on this interpretation(this is called polar form of complex numbers). Note that usually, we take arguments in the range $-\pi < \theta \leq \pi$. (But in some cases, $0 \leq \theta < 2\pi$ is also acceptable. They are equivalent after all) And usually we assume $r > 0$. Otherwise we can write i in two ways: $r = 1, \theta = \pi/2$ or $r = -1, \theta = -\pi/2$. This is not what we want as this may cause ambiguity.

Warning. If $z = a + bi$, then $\arg z = \tan^{-1}\frac{b}{a}$.(Think of why) This is right in most cases but for angles in ranges and $(\pi/2, \pi)$, your calculator may not be able to tell the right value. And when $a = 0$ this formulae does not work.(Tell the argument directly if $a = 0$, it is $\pm\pi/2$ depending on the value of b) Remember $\tan x$ has two periods in $(-\pi, \pi)$, so some values are duplicates. Your calculator never knows which angle do you want, so calculator always pick the value closest to 0. (Principle argument) But in some cases you have to adjust it to the value you want, by moving forward or backward a period. Draw the graph of $\tan x$ to figure out why when θ is in $(\pi/2, \pi)$, $\theta = \theta_{\text{calculator}} + \pi$, and when θ is in $(-\pi, -\pi/2)$, $\theta = \theta_{\text{calculator}} - \pi$.

Example. The process of converting Cartesian form, e.g. $z = 3 + 4i$ to polar form is very simple, find $|z|$ and $\arg z$! In this case: $|z| = \sqrt{3^2 + 4^2} = 5, \arg z = \arctan\frac{4}{3} = 0.927$. (Draw a right-angled triangle with the line joining origin and z being hypotenuse to see why $\arg z$ is calculated in this way) So $z = 5(\cos 0.927 + i\sin 0.927)$.

Exercise. Find the polar form of the following complex numbers. (Keeping the argument θ between $-\pi$ and π)

- $5 + 5i$
- $3 - i$
- 9 (or equivalently, $9 + 0i$)
- $2i$ (or equivalently, $0 + 2i$)

3.1 Taylor's expansion

You may wonder what is the magic here. Because from what we have described above, polar form of complex number is nothing more than extraction of factor $|z|$. Well, it is because we have not learnt the Euler's formulae, probably one of the most amazing math formulas. But before getting to that, we have to learn a little bit about Taylor's expansion first. (Formal study of Taylor expansion is very complex, we need a lot of hypothesis here to prove it works. But just simply witness the fact that it works)

The idea is: every function (at least the continuous ones) has a polynomial estimation. Of course, for polynomials $p(x)$, they are already polynomials so this is very true. But later people realise that polynomials of finite degree cannot really estimate functions like $\sin x$ precisely. So they turned to infinite polynomials: $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$, there are infinite terms. Hey, wait, this is not even a function! How can we have infinite terms? Well, here is the point: if the coefficients form a sequence: $1, 0.1, 0.01, 0.001, 0.0001, ...$, that is, the sequence descends to 0 gradually, then the function will not grow infinitely large because large powers of x are balanced by very small coefficient. For example, when $x = 2$, $x^{40} = 1.1 \times 10^{12}$, but the coefficient is 10^{-40}. So this term is actually 1.1×10^{-28}, not blowing up right? Let us see the first few estimations of $\sin x$ below:

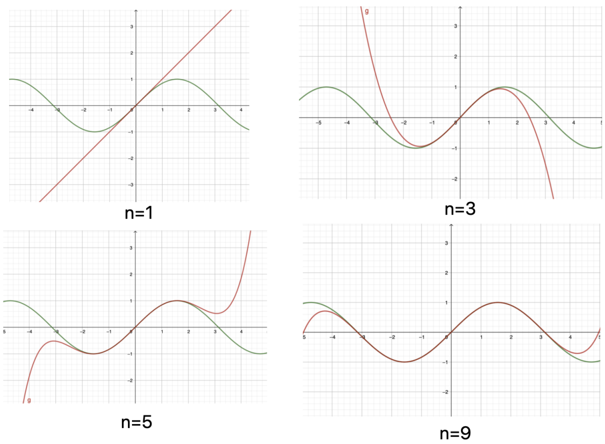

Figure 3.1: Polynomial estimations to $\sin x$

n is the number of terms(or degree of polynomial) used to estimate. We can see that as we add more terms, the polynomial estimation is becoming more precise, especially around $x = 0$. $n = 9$ already gives precise estimation between $-\pi$ and π.

So what are these coefficients? How do we find them? The method is simple: differentiation. You will see why by doing this:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4... \text{ , so } f(0) = a_0$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3... \text{ , so } f'(0) = a_1$$

Exercise. Keep differentiating (assuming that you can differentiate infinite polynomial as many times as you wish using the usual rules) to get a_2, a_3, a_4. Hence conclude that $a_k = \frac{f^{(k)}(0)}{k!}$, where $f^{(k)}$ means the k'th derivative of $f(x)$, $f^{(0)}$ means $f(x)$ itself.

This is a Taylor series(expansion) of a function $f(x)$, namely

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

The ∞ on the $\sum$ indicates that this is an infinite polynomial.

Exercise. Use the above formulae to prove Taylor series of the following three functions are:

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$

Hence prove that $\frac{d}{dx} \sin x = \cos x$, $\frac{d}{dx} \cos x = -\sin x$. (Using the usual way to differentiate polynomials)

3.2 Euler's formulae

Euler's formulae is $e^{i\theta} = \cos\theta + i\sin\theta$, so we can simply write the polar form of complex numbers to be $z = |z|e^{i\arg z} = |z|e^{i\theta}$. Isn't this a simple form?

Exercise. By doing Taylor expansion on both sides (you may assume Taylor series works for complex numbers) prove Euler's formulae. You may assume that two infinite polynomials are the same if all the coefficients are the same, so simply check the first few are correct.

3.3 Rotations

Here comes the exciting part: what happens if we multiply $e^{i\theta}$ to a complex number z? Let's first write z into polar form $|z|e^{i\phi}$.

$$ze^{i\theta} = |z|e^{i\phi}e^{i\theta} = |z|e^{i(\phi+\theta)}$$

The length(or modulus) is again $|z|$, but the argument(or angle to initial line) has increased by θ. What is this transformation? Rotation anticlockwise by angle θ! Look how simple it is to perform rotations on complex numbers, just by multiplying $e^{i\theta}$.

Now, what does it mean to multiply another complex number $z_2 = re^{i\theta}$ ($r = |z_2|, \theta = \arg z$) to z?

$$z * z_2 = |z|re^{i(\phi+\theta)}$$

This is a rotation, but with a scaling of factor r.

And look how dividing becomes very simple using polar form:

$$\frac{z}{z_2} = \frac{|z|e^{i\phi}}{re^{i\theta}} = \frac{|z|}{r}e^{i(\phi-\theta)}$$

Exercise. State the transformation represented by ”dividing by z_2” precisely (So do not say ”rotation” but ”rotation anticlockwise/clockwise by some amount”) Hint: note how the argument and modulus has changed.

Chapter 4

Loci on Argand's diagram

Sometimes we use Argand's diagram to indicate a set of solutions satisfied by an inequality or an equation. The simplest example is $|z| = 1$, or distance to origin is 1. Obviously this is the unit circle, so we simply draw a unit circle on Argand's diagram.

Exercise. Draw the loci(set of solutions) of $|z - z_0| = 3$ for $z_0 = 2 + 4i$. (Hint: you can always treat complex numbers as vectors, so $z - z_0$ is the vector from point z_0 to z)

$|z - z_0| = r$ just represent a curve(actually a circle with no area). Guess what $|z - z_0| < r$ means?

Consider $|z - z_1| = |z - z_2|$ where z_1, z_2 are two complex numbers, this says distance of z to point z_1 is the same as distance to point z_2. Obviously the midpoint of the line segment connecting z_1, z_2 must be a solution. Is there any other solution? (Hint: consider points outside that line segment)

One final example is about $\arg z$. Something like $\arg z = \frac{\pi}{2}$ is easy to interpret: all points that makes an angle $\frac{\pi}{2}$ to positive real axis. (Remember, this is signed angle, so clockwise rotation does not count) Namely, this represents all points on positive imagery axis. (Not including negative imagery axis, be careful!)

Exercise. Draw the loci(set of solutions) of $\arg(z - z_0) = \frac{\pi}{4}$ for $z_0 = 1 + i$.(Hint: what happens if you change the origin to z_0? That is using a new system with $z' = z - z_0$, so equation becomes $\arg z' = \frac{\pi}{4}$)

I will include a summary tables of common loci just in case you cannot figure out above situations. Of course, there can be many other combinations. Simply translate the equation into words to see the result.

4.1 Common loci

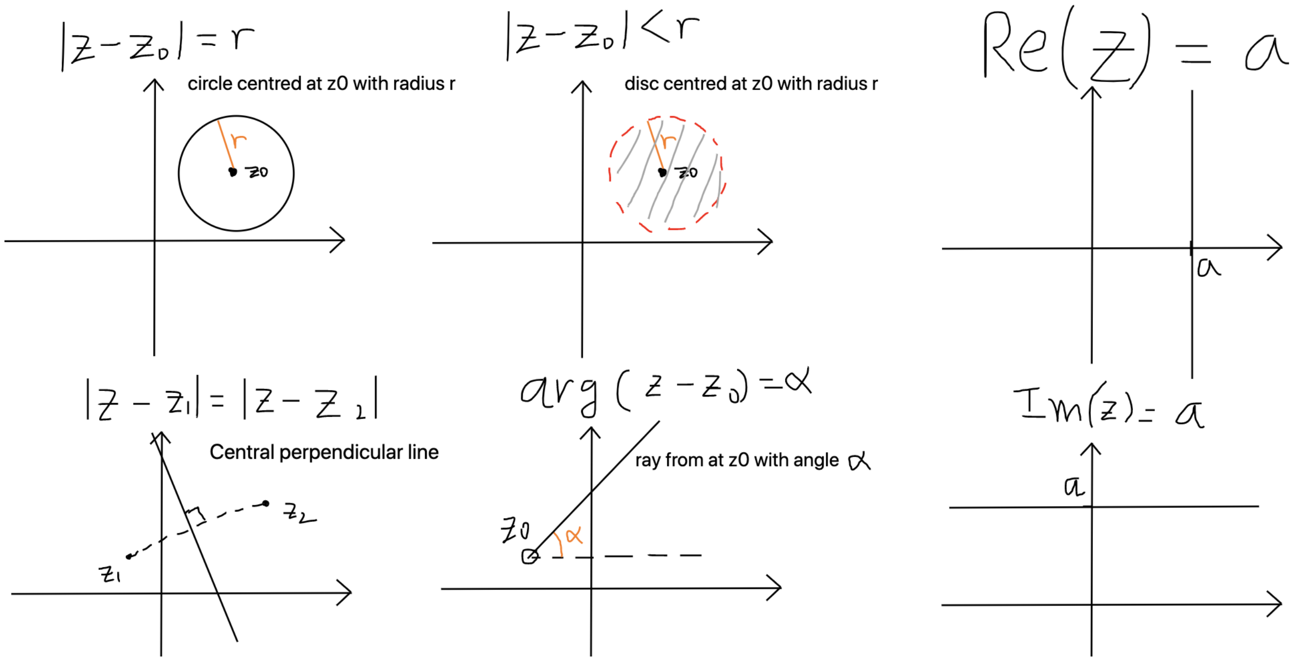

Figure 4.1: Common loci

Chapter 5

Exercises

Q1. Simplify the following expressions:

- Re$(4-3i)$
- Im$(5+i)$
- Re$(7i)$
- $|8+3i|$
- $\arg(4+3i)$
- $(3i+4)(3-4i)$
- $\frac{2+i}{4-4i}$

Q2. Use the fact that $3+2i$ is a solution to $p(x)=x^3-5x^2-3x+13$, factorise $p(x)$ completely. Hence, find all the roots to $p(x)$.

Q3. Draw the following complex numbers and the corresponding operations on Argand's diagram

- $z=3+i$, add $2+5i$
- $z=2+2i$, subtract $1+i$
- $z=5-3i$, multiply by $\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$
- $z=2-2i$, multiply by $2+3i$
- $z=3+2i$, conjugate

Q4. Prove the following identities using Polar form $z=re^{i\theta}$:

- $|z_1z_2|=|z_1||z_2|, |\frac{z_1}{z_2}|=\frac{|z_1|}{|z_2|}$

- $\arg(z_1 z_2) = \arg z_1 + \arg z_2, \arg(\frac{z_1}{z_2}) = \arg z_1 - \arg z_2$

Q5. Prove that $\arg z^* = -\arg z, |z^*| = |z|$
Q6. Find the square roots of he following complex numbers, and draw them together with z on the same Argand's diagram. Have you found anything?
(a) $z = 1 + i$
(b) $z = 3$
(c) $z = 2i$

Q7. Draw the following loci on Argand's diagram:
(a) $|z + 3 + 4i| = |z - 2 + i|$
(b*) $\frac{1}{z} = z^*$ Hint: use polar form.
(c) Re $z > 3$
(d*) Re $((3 + i)z + (-1 - i)) > 0$ Hint: change z and $3 + i$ to polar form, break the Re.
(e) $|z + 1 + i| = 3$
(f*) $|z| = Re(z) + 1$ Hint: write $z = a + bi$.

Q8. Prove compound angle formulae for $\cos(\theta + \phi), \sin(\theta + \phi)$ using polar form of complex numbers.
Hint: you have two ways to change $e^{i(\theta+\phi)}$ into Cartesian form($a + bi$). One is directly break down using Euler's formulae to get trig functions with argument $\theta + \phi$. Another way is first treat it as $e^{i\theta}e^{i\phi}$, then break each one using Euler's formulae. Two expansions must give the same result, so you can say that their real parts and imagery parts must be the same. Therefore, yields two equations, which are the compound angle formulae.

5.1 Extensional Questions

Q9*
(a) Prove the following identities: $(z_1 + z_2)^* = z_1^* + z_2^*, (z_1 - z_2)^* = z_1^* - z_2^*, (z_1 z_2)^* = z_1^* z_2^*$ and similarly for division. For multiplication and division, it is worth using result from Q5. (if $z = re^{i\theta}, z^* = re^{-i\theta}$)
(b) Prove that if the imagery part of a complex number z is 0(that is the number is purely real), $z = z^*$
(c) Use part a,b to prove that if z solves polynomial $f(x) = a_0 + a_1 x + ... + a_n x^n$ (equivalently, $f(z) = a_0 + a_1 z + ... + a_n z^n = 0$), then z^* also solves this polynomial. (Hint: take the conjugate of the whole expression $f(z)$)

Q10*
We are already very confident that solutions to $x^2 = 1$ are ± 1. What about solutions to $x^3 = 1$? $1 = 1 * e^{i*0} = 1 * e^{i*2\pi} = ...$ so if we write $x = re^{i\theta}$, $r^3 e^{3i\theta} = 1 * e^{i*2k\pi}$.
(a) Find all solutions to r, θ in the above case. (Note: θ should be in $(-\pi, \pi]$) Similarly find all solutions to $x^4 = 1$, and write the solutions of $x^2 = 1$ in complex polar

form.
(b) write down the general solution to $x^n = 1$, where n is a natural number
(c) Draw all solutions to $n = 2, 3, 4, 5$ on 4 Argand's diagrams respectively, explain what have you found.

Chapter 6

Postscript

This is the end of Pure Math 3 series. I hope you have found it entertaining and useful. Many contents of the series may never be tested, but they indeed enhance your understandings of the topic, and even mathematics in a broad picture.

If you choose to study math or math-related majors in the University, you may have to learn Mathematics-Further. Do not be frightened with the difficulty of Further Mathematics, because I have secretly put some topics there in the notes. You may be wondering what is a matrix or why some integration rules make sense while learning Further Math, please do check 3Blue1Brown's video series *Essence of Calculus* and *Essence of Linear Algebra*. For students pursuing economics, literature, language, arts or any other non-STEM majors, I strongly recommend that you learn more about statistics instead of picking Further Math. Because this subject, especially the pure math part, may hardly help you in the future. But indeed, statistics is broadly used in many majors.

If you will choose math major, it is very different from the math other STEM majors will learn. University math focus more on the rigorous constructions of theories and strict proofs of theorems. Please do read some of the following materials (in the summer of Year 11 or 12) to ensure that you are really prepared to start a math degree:

- *How to Think Like a Mathematician* – Kevin Houston
- *How to study for a maths degree* – Lara Alcock
- *Towards Higher Mathematics* – Richard Earl
- *Alan Turing, the Enigma* – A. Hodges (if you are also interested in computing)
- *The Music of the Primes* – Marcus du Sautoy (number theory)
- *To Infinity and Beyond* – Eli Maor (Easy to understand, but contains a lot of important ideas)
- *Calculus for the Ambitious* T.W. Korner

Good luck and keep enjoying the magic of math everywhere!