An Introduction to Calculus (Chapter 4,5 of Pure 3)

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Chapter 1

Basics

Ever since Newton began his investigations on motions, the idea of ”instant rate of change” plays an important role in many areas.

To avoid the problem of ”dividing 0”, we allow idea of limit to appear. If a quantity Δt is approaching 0, it means it is closer to 0 than any number you can imagine, but it is not 0.(Imagine a decreasing sequence of numbers: 0.1, 0.01, 0.001, 0.0001, ... or -0.1, -0.01, -0.001, ...) This is denoted by $\Delta t \to 0$. Therefore, we can define instant speed to be:

$$v_{\text{instant}} = \lim_{\Delta t \to 0} \frac{\Delta d}{\Delta t}$$

it means what happens to $\Delta d/\Delta t$ when Δt is approaching 0.

When it comes to a function, we pick a point $x = a$ on the curve, and travel by a distance Δx from there.(If Δx is negative, we go leftwards. Vice versa)

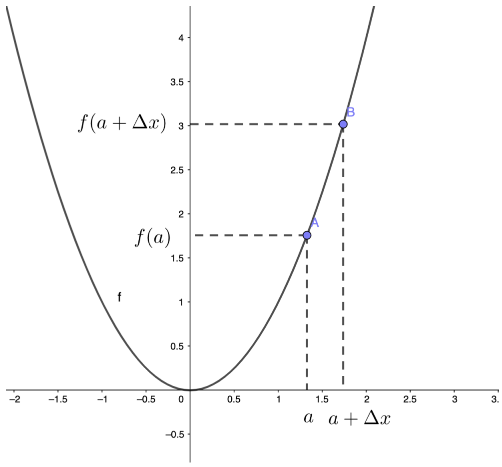

Figure 1.1: Definition of differentiation

One can easily write that slope here is:

$$\frac{f(a+\Delta x) - f(a)}{\Delta x}$$

And then we force Δx to approach 0:

$$\frac{dy}{dx}_{x=a} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

dy, dx just means Δy, Δx, but they are used in differential case ($\Delta x \to 0$). We also denote differentiation using a slash for convenience: $f'(x) = dy/dx$.

This is the definition of differentiation. Do not worry about evaluation of that fraction yet. You may have realised, for this definition to give a unique ”derivative”, the slope of approaching from left and right should be the same. On a vertex of triangle, you cannot differentiate!

1.1 Properties of differentiation

In formal mathematics, after we get a definition, we use that to prove properties.

Proposition 1 (Sum of functions). *Derivative of* $f(x) + g(x)$ *is* $f'(x) + g'(x)$.

Proof.

$$\frac{d}{dx}(f(x) + g(x)) = \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) + g(x+\Delta x)) - (f(x) + g(x))}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{(f(x+\Delta x) - f(x)) + (g(x+\Delta x) - g(x))}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = f'(x) + g'(x)$$

□

Exercise. Try to prove derivative of $f(x) - g(x)$ is $f'(x) - g'(x)$ using definition.

We will discuss more properties in the next chapter.

1.1.1 Derivatives of Basic Functions

Proof of the following requires induction and Taylor series with concerns on convergence(which are only covered in Further Mathematics or even university level) So just try to draw the graphs on graphing software, and see why one is rate of change of another.

- $\frac{d}{dx}x^n = nx^{n-1}$ true for any real number n.
- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\cos x = -\sin x$

- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}\ln x = \frac{1}{x}$

We have proved the latter two in the second note(Note2). Don't worry about $\tan x$, you can do that after learning how to differentiate quotients in chapter 2.

1.2 Integration

Integration concerns area below a curve. For example, for a v-t diagram, the area below the curve is distance travelled!

Again, idea of limit is useful here. We estimate the area using many rectangles:

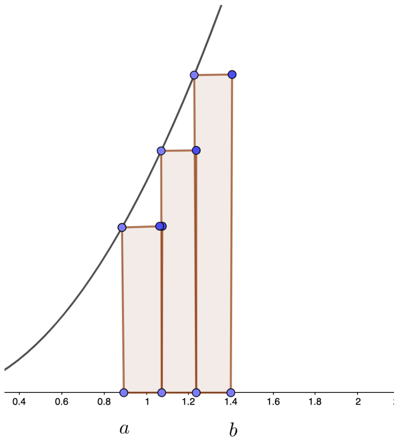

Figure 1.2: Rectangle-cutting method

One can easily write using summation that the area of all rectangles is:

$$\sum f(x)\Delta x$$

where Δx is the width of each rectangle. Now you should have figured out the next step. We make $\Delta x \to 0$!

So area below the curve is:

$$\lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x$$

in this case(differential case), we stretch the symbol of sum, S, to be come $\int$. And we change Δx to dx:

$$\int_a^b f(x)dx$$

Now can you see that integration is basically "summation", but in a differential case. So I like to call integration "differential sum".

1.3 Fundamental Theorem of Calculus

You may be confused when you look at section 1.2. Shouldn't integration be reverse of differentiation? Well, mathematicians have worked so hard to find rate of change of a function, there is no point of reversing this!

It is a fundamental property in calculus that, integration happens to be reverse process of differentiation. That is:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is a function such that $F'(x) = f(x)$.

We have a unique definition for $F(x)$, that is "indefinite integration":

$$F(x) = \int f(x)dx \text{ if } F'(x) = f(x)$$

But adding a constant to $F(x)$ also satisfies the above definition!(Think of why) So $F(x)$ is not unique. Therefore, we pick a representative from all functions that qualify, and add $+c$.
Example. $\int 2x\,dx = x^2 + c$. It is perfectly okay for you to pick $x^2 + 3$ as a representative, and say $\int 2x\,dx = x^2 + 3 + c$. But usually we pick the simplest one.

Chapter 2

Properties of differentiation

2.1 Chain rule

Let's imagine 3 gears, all driven by gear x. (Gear u and y will not move themselves!)

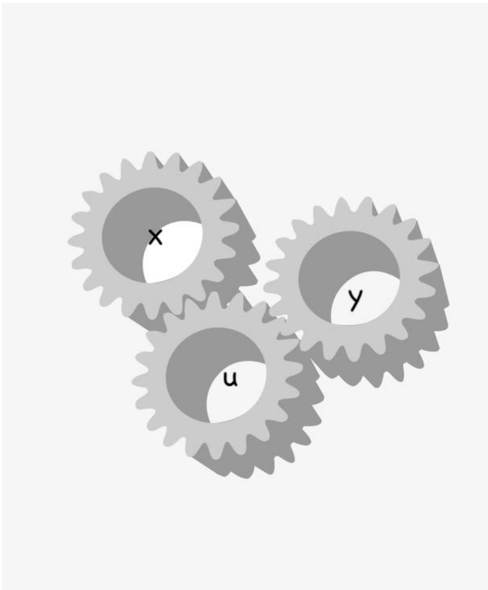

Figure 2.1: Gear model

If every unit of rotation on gear x causes 0.5 units of change on u, and every unit of change on u causes 4 units of change on y, what is the change of gear y caused by every unit of change on gear x? Simple! $0.5 \times 4 = 2$. That's it. You have learnt all about chain rule(non-strictly).

Say we want to differentiate a function $\cos x^2$, you will never be able to solve that using definition(try that, horrible idea). But now you can break this function into two parts(composition), first step: $x \to u$, where $u = x^2$. Second step: $u \to y$, where $y = \cos y$. So what is $\frac{dy}{dx}$?

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

.
Note: You cannot prove this by simply cancelling du, as they are not numbers! But it is a good way to memorise the rule.
Remark: It is a common mistake for students to just say derivative of $(2x+3)^2$ is $2(2x+3)$. You can try to expand the brackets and then differentiate term by term, you will get a different result. Please find the true solution using chain rule by yourself.

2.2 Product rule

It is perfectly okay to prove this using definition, just like what we did in 1.1. But there is a more intuitional way to understand this, I first saw it on 3b1b's series "Essence of Calculus" and this is funner than doing algebraic operations on definition. Do check out the series if you are really interested.
Imagine a rectangle with width g, height h. Then the area $A = gh$.

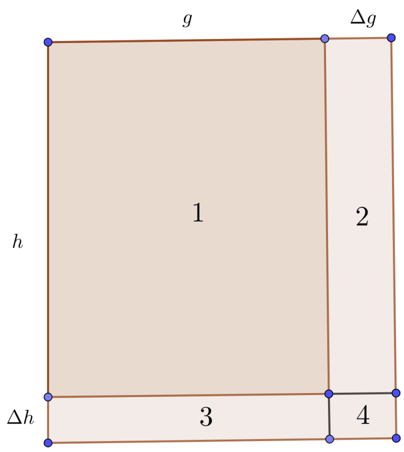

Figure 2.2: Product rule on rectangle

Now g, h are changed a little bit. What is the change in area of rectangle? Region 2, 3, 4.

$$\Delta A = g\Delta h + h\Delta g + \Delta g\Delta h$$

But say $\Delta h, \Delta g = 0.01$, who cares about the last term 0.0001, that is really negligible here. Now imagine there is a driving gear x behind g and h. We have:

$$\frac{\Delta A}{\Delta x} = g\frac{\Delta h}{\Delta x} + h\frac{\Delta g}{\Delta x}$$

.
To make it differential, we send $\Delta x \to 0$ and that is product rule:

$$\frac{dA}{dx} = g\frac{dh}{dx} + h\frac{dg}{dx}$$

Exercise. Use product rule to find derivatives of x^2, x^3 and x^5.

2.3 Quotient Rule

A quotient, $\frac{f(x)}{g(x)}$, is essentially just $f(x)\frac{1}{g(x)}$. So what we need to do now is to figure out how to do differentiation on $\frac{1}{g(x)}$. This can be done by chain rule.

Let $u = g(x)$, then $y = \frac{1}{g(x)} = \frac{1}{u} = u^{-1}$. So $\frac{dy}{du} = -u^{-2}$ and $\frac{du}{dx} = g'(x)$. By chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{g'(x)}{g^2(x)}$$

Exercise. Use the about result and product rule, prove quotient rule: derivative of $\frac{f(x)}{g(x)}$ is

$$\frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Exercise. Use quotient rule and derivatives of $\sin, \cos$, prove that $\frac{d}{dx}\tan x = \sec^2 x$

2.4 Parametric

Parametric functions basically means that x, y are all actually driven by a parameter t. For example, the x, y, z coordinates of a moving object can be described using time t. Say $x = 3t, y = t^2, z = 1$.

So it is important to understand how to differentiate parametric functions. Again using 3 gears, but this time gear t is driving gear x and y.

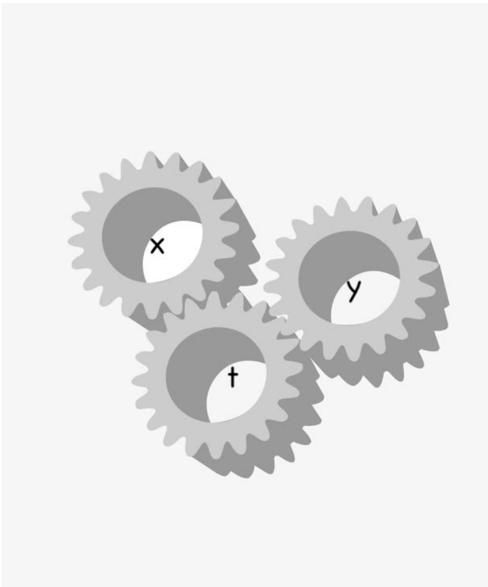

Figure 2.3: Gear model

If every unit of change on gear t causes 2-unit change on gear x and 4-unit change on gear y. How many units of change on y will be caused(though not directly) by one-unit change on gear x? Well, $4 \div 2 = 2$. That's it!

Theorem 1 (Parametric Differentiation). $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

Can you find similarities between chain rule and parametric differentiation?

2.5 Implicit Differentiation

There are, unfortunately, many equations that cannot be changed to the form $y = f(x)$ in a neat way. For example, the equation of circle: $x^2 + y^2 = 1$. What can we do if we want derivatives of these functions?

The simple idea is, we imagine there is a hidden parameter t driving x, y! And then we can use parametric differentiation on $x(t), y(t)$. And remember, $x^2 + y^2$ is kept constant. So of course when t changes, rate of change of $x(t)^2 + y(t)^2 = 0$. Using chain rule, we know rate of change of LHS is $2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt}$. What we want is $\frac{dy}{dx}$, by parametric differentiation, this is $\frac{dy}{dt} \div \frac{dx}{dt}$. So after slight rearrangements, we can prove

$$\frac{dy}{dx} = -\frac{x}{y}$$

We can actually ignore dt as it will be cancelled out anyway. This should make sense as t does not really exist. From this idea, we can also see that for circle centred at origin with any radius, the tangent of curve is given by the same expression $-\frac{x}{y}$.

So let's summarise the process of implicit differentiation:

1. Differentiate each term as usual, treating y as an independent variable, using product rule, quotient rule etc.
2. Whenever you differentiated with x, you should add a dx after that. Similarly for y.
3. Rearrange to create $\frac{dy}{dx}$. This can be done as each term after step 1, 2 is either $\ldots dx$, $\ldots dy$ or 0.

One immediate application is differentiation of $\tan^{-1} x$. We cannot do this directly. Change $y = \tan^{-1} x$ to $x = \tan y$ first, and then we can use implicit differentiation.

$$dx = \sec^2 y \, dy \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

Exercise. Use similar method to differentiate $\tan^{-1} \frac{x}{a}, \sin^{-1} \frac{x}{a}, \cos^{-1} \frac{x}{a} (a \neq 0)$. Hence figure out how to integrate

$$\frac{1}{x^2 + a^2}, \frac{1}{\sqrt{a^2 - x^2}}$$

.

Chapter 3

Integration Techniques

3.1 Reverse Thinking

From now on, you can use integration rules on multiplication by a number, sum and difference without proof. As integration is just reverse process of differentiation anyway.

Proposition 2 (Integration Rules). $\int cf(x)\,x = c\int f(x)\,dx$, $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Suppose we are asked to find $\int e^{ax+b}\,dx$ where a, b are constants. Then since derivative of e^x is e^x, we know the result is something with the form $e^{\cdots}$. Let's try to differentiate e^{ax+b}. Using chain rule, the result is ae^{ax+b}. An additional a! We should cancel that using $\frac{1}{a}$ So:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}$$

Exercise. Use similar method to integrate the following functions: (a, b are constants)

- $\frac{1}{ax+b}$
- $\sin(ax+b)$
- $\cos(ax+b)$
- $\sec^2(ax+b)$
- $\frac{1}{x^2+a^2}$

Also we know

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$$

by chain rule, so sometimes we can recognise integrand of the form $\frac{f'(x)}{f(x)}$, then we know its integral is $\ln f(x)$! For example, given $\frac{x}{x^2+1}$. You should recognise that $\frac{d}{dx}x^2 + 1 = 2x$. So integral of $\frac{2x}{x^2+1} = \ln(x^2+1)$. $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$. This is a common technique to solve rational functions(fraction of polynomials) where degree of nominator is 1 degree lower than that of denominator.

3.2 Partial Fraction

Generally speaking, integrating rational functions(fraction of polynomials) like $\frac{x}{3x^3-9}$ can be quite hard. It is very difficult to figure out its integral just by reverse thinking. But remember we have been developing the idea of factors on polynomials in the first chapter. So if somehow we can break the denominator down into many factors, we can break the fraction down into many fractions.(Called partial fraction)

Say $\frac{1}{x^2-4}$, $x^2 - 4 = (x+2)(x-2)$. So there must be a way to write $\frac{1}{x^2-4}$ as $\frac{A}{x+2} + \frac{B}{x-2}$ where A and B are polynomials that should be determined(actually A, B must be constants). Now you can use your results from section 3.1 to integrate!

Exercise. Figure out the A, B in above example, and solve the integral $\int \frac{1}{x^2-4}\,dx$. (Hint: what are the derivatives of $\ln x \pm 2$?)

What about $\frac{1}{x^3-x^2-5x-3}$. The denominator has decomposition: $(x+1)^2(x-3)$. Same idea applies! You can break it down to

$$\frac{A}{(x+1)^2} + \frac{B}{x-3}$$

. (Think of why not $\frac{A}{(x+1)} + \frac{B}{x+1} + \frac{C}{x-3}$) But this time, A, B may not be constants.
Exercise. Figure out the A, B in above example, and solve the integral $\int \frac{1}{x^3-x^2-5x-3}\,dx$. (Hint: for the fraction with quadratic denominator, you can try to use the last technique in section 3.1)

There are cases where we have to stop at $cx^2 + d$, like $x^2 + 4$. Because we cannot decompose further. If the fraction in partial fraction comes out to be something like $\frac{3x}{x^2+4}$, you can use last technique in section 3.1. What about $\frac{3}{x^2+4}$? Well, check the last part of section 2.5!

A famous theorem states that decomposition of any polynomial with real number coefficients consists of either linear polynomials or quadratics. So you should not worry about something like $\frac{1}{x^3+1}$ or even higher degree appearing in the result of partial fraction. If that is the case, you have not completed partial fraction!

3.3 Integration by parts

This result comes directly from product rule but it is a really powerful tool. Recall product rule:

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

We can integrate both left and right hand side to get another equation:

$$\int \frac{d}{dx}\left(f(x)g(x)\right)\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

LHS is just $f(x)g(x)$, as integration is reverse of differentiation. After rearranging we have

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

This is integration by part. Analogue to product rule. But here, the choice of which part to be $f'(x)$, which part to be $g(x)$ matters. As we can see, $f'(x)$ must be something that is easy to integrate, otherwise we cannot extract $f(x)$ from there. And $f(x)g'(x)$ should be easier to integrate than original integrand.

Let's say we want to integrate $x\cos x$. Try to let $f'(x) = x$. Then $f(x) = \frac{x^2}{2}$. Not a good idea as now it is even harder to integrate. Therefore, we let $f'(x) = \cos x$, $g(x) = x$. $f(x) = \sin x, g'(x) = 1$. Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x$$

. This process of course works with definite integrals, and from our derivation, the term $f(x)g(x)$ should be $f(b)g(b) - f(a)g(a)$. (As this term comes from a definite integral!)

There are cases where we have to integrate by part for twice, or even more: **Exercise.** Try to integrate $x^2\cos x$. How many times of integration by part would you need to apply for $x^5\cos x$?

This technique can even appear at integrand that doesn't even have many parts. $\int \ln x\,dx$ can be solved by treating integrand as $1 \times \ln x$.
Exercise. Try to figure out letting which one of $1, \ln x$ be $f'(x)$ can solve the integral. And solve it!
Remark. You must use integration by part with care, as this may push your integral to be more difficult to solve.

3.4 Substitution

There is an analogue for chain rule in integration: u-substitution.

If you see integrand of the form $f(u(x))u'(x)$, then the integral should be the same as integrating $f(u)$ with respect to u. (Just like derivative of $f(u(x))$ will be $f(u(x))u'(x)$ by chain rule) But remember for definite integral, you should change the bounds!

Example

$$\int_{\ln 2}^{\ln 8} \frac{e^x}{1+e^x}\,dx$$

Should make substitution $u = 1 + e^x$ to simplify the annoying denominator. $u' = e^x$. So indeed the fraction is of the form $f(u(x))u'(x)$ where $f(u) = \frac{1}{u}$. This integral should be the same as integrating $f(u)$. What are the bounds? We should change $\ln 8, \ln 2$ to u correspondingly. Upper bound is $1+e^{\ln 8} = 9$, lower bound is $1+e^{\ln 2} = 3$. So the integral becomes

$$\int_3^9 \frac{1}{u}\, du$$

. The result is $\ln 3$. (Notice that after changing to u, we do not have to care about x anymore, because this is a definite integral. But for indefinite integral, remember to substitute u back!)
Have you realised that the technique we learnt in section 3.1 is just a special case of this?

3.5 Tips for integration

You should be familiar some of the basic functions immediately as stated in section 3.1.
The first step when getting a complex integrand is to see if there is any form here that simplifies your work. Like $f(u(x))u'(x)$. If not, try to use integration by part to change integral to simpler ones. I would say if you are still stuck after applying 3 times of integration by part, you probably went into the wrong direction.

If the function is rational:

- If denominator has lower degree than nominator, you should first do long division.
- Then try to factorise denominator as far as you can.
- Do partial fraction.
- For linear terms, use $\ln$ directly; for quadratic terms, use substitution or $\tan^{-1} \frac{x}{a}$ depending on cases.

3.5.1 Numerical solution

The method of treating integrals as reverse process of differentiation, finding the original function and evaluating definite integrals is called analytical method. But we are not always able to solve everything explicitly using analytical method, for example: the simple integral

$$\int \frac{\sin x}{x}\, dx$$

There is another way to evaluate definite integral. That is using trapeziums.

The graph on the next page is a function $f(x) = xln(x)$. To evaluate the integral between 1 and 4, we can cut the interval into 4 parts, each of length 0.5. Taking

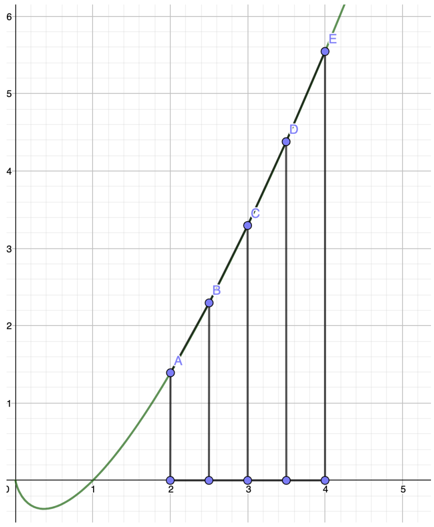

Figure 3.1: Trapezium method

the interval [1, 1.5] as an example, it is okay to construct a rectangle using height at $x = 1$ or $x = 1.5$. But first looks like an underestimation and second is an overestimation. So we can choose a compromised way: use trapezium! The more intervals you cut, the better you can be at evaluating. You can throw the function into computer and get a very precise value in few seconds. So what is the point of developing so many analytical methods to solve integrals?

First of all, analytical solution reveals the behaviour of the function. Is it a wave? Is it a polynomial? And the theory behind numerical solution(including why that works strictly, how precise?) depends largely on analytical solutions. But for our stage, do not worry to much about why numerical solutions work.

Chapter 4

Exercises

Q1. Differentiate the following functions:

- $x^3 \sin x$
- $\ln x + e^x$
- $\ln \sin x$
- $\sin^2 x cos x$
- $\frac{2x-4}{3x+2}$
- $x^2 \ln x$
- xe^{1-x^2}
- Find $\frac{dy}{dx}$ for $x = t - e^{2t}, y = t + e^{2t}$.
- Find $\frac{dy}{dx}$ for $x^2 + y^2 = xy + 7$
- $\cot x$
- $\sec x$
- $\csc x$

Q2.
Integrate the following functions:

- $\frac{1}{2+3x^2}$
- $x \sin 2x$
- $3 \cos x^2 \sin x$
- $x^2 e^{-x}$

- $x\tan^{-1}x$
- $\frac{4x+6}{x^2+3x}$
- $\frac{1}{x^2-1}$
- $\frac{2x}{3x^3+3x^2+7x+1}$

Q3.
Evaluate the following definite integrals:

- $\int_0^{\pi}\cos 2x$
- $\int_0^{\pi/2}\sin x\cos^4 x$
- $\int_1^5 e^{3x+4}$
- $\int_1^5 e^{3x^2+4}$
- $\int_1^4 \ln x/x^2$
- $\int_{\pi}^{2\pi} x^2\sin x$

Q4. Calculus has become much more difficult compared to Pure1, so please please really make sure you practice differentiation and integration as much as possible. As this is really important and appear in many other areas like electrics, engineering, chemistry and differential equations. And exercises embedded within the note can enhance your understanding, so please try them out.