Relationships between Probability Distributions

Note: The gamma distribution mentioned in this notes uses parameters shape and rate: i.e. Gamma(α,β) where α is shape, β is rate. There is a different parametrisation using shape and scale

DISCRETE DISTRIBUTIONS							
	range $\mathbb{X}$	*parameters*	*pmf* f_X	*cdf* F_X	$\mathrm{E}[X]$	$\mathrm{Var}[X]$	*mgf* M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\frac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\frac{1}{\theta}$	$\frac{(1-\theta)}{\theta^2}$	$\frac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\frac{n}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$	$\left(\frac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\frac{n(1-\theta)}{\theta}$	$\frac{n(1-\theta)}{\theta^2}$	$\left(\frac{\theta}{1-e^t(1-\theta)}\right)^n$

CONTINUOUS DISTRIBUTIONS							
		parameters	*pdf*	*cdf*	$\mathrm{E}[X]$	$\mathrm{Var}[X]$	*mgf*
$Uniform(\alpha,\beta)$ (stand. model $\alpha=0,\beta=1$)	(α,β)	$\alpha < \beta \in \mathbb{R}$	$\frac{1}{\beta-\alpha}$	$\frac{x-\alpha}{\beta-\alpha}$	$\frac{(\alpha+\beta)}{2}$	$\frac{(\beta-\alpha)^2}{12}$	$\frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (stand. model $\lambda=1$)	$\mathbb{R}^+$	$\lambda \in \mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta \in \mathbb{R}^+$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (stand. model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta \in \mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\frac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\frac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (stand. model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu \in \mathbb{R}^+$	$\frac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		$0 \quad (\text{if } \nu>1)$	$\frac{\nu}{\nu-2} \quad (\text{if } \nu>2)$	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha \in \mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\frac{\theta}{\theta+x}\right)^\alpha$	$\frac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta \in \mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

Binomial distribution is sum of iid Bernoulli

$$X \sim \text{Binomial}(n,\theta) \Longrightarrow X = \sum^{n} X_i \textit{ where } X_i \sim \text{Bernoulli}(\theta)$$

Poisson distribution can be cut into n iid small fragments

$$X \sim \text{Poisson}(\lambda) \Longrightarrow X = \sum_{i=1}^{n} X_i \textit{ where } X_i \sim \text{Poisson}\,(\lambda/n)\textit{, so that } \mu = E(X_i) = \lambda/n \textit{ and } \sigma^2 = Var(X_i) = \lambda/n$$

Negative Binomial is sum of iid geometric distributions

$$X \sim \text{Negative Binomial}(n,\theta) \Longrightarrow X = \sum_{i=1}^{n} X_i \textit{ where } X_i \sim \text{Geometric}(\theta)\textit{, so that } \mu = E(X_i) = 1/\theta \textit{ and } \sigma^2 = Var(X_i) = (1-\theta)/\theta^2\textit{, and hence}$$

Linearity of Normal distribution

if $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, then
$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
$X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

if $X \sim N(\mu, \sigma^2)$, then $kX \sim N(k\mu, k^2\sigma^2)$

Student t-distribution

If $X_1, X_2, \ldots, X_n \sim N(0, \sigma^2)$ where σ^2 is unknown, then
$(\bar{x}_n - \mu) / (S/\sqrt{n}) \sim t_{n-1}$ where $\bar{x}_n$ is mean of $X_1, X_2, \ldots, X_n$

Chi-squared ·

$\chi^2_n = \sum Z_i^2$ where Z_i are iid $N(0, 1)$
Therefore, for integer k, sum of k iid χ^2_n is χ^2_{kn}

Exp and gamma

Exp(λ) = Gamma(1, λ)
sum of n iid Gamma(1, λ) ~ Gamma(n, λ), sum of k iid Gamma(λ, 1) ~ Gamma(λ, k)
so sum of n iid Exp(λ) is Gamma(n, λ)

$\chi^2_n \sim$ Gamma(n/2, 1/2)
so Exp(1/2) = χ^2_2

Gamma(a, λ) = Gamma(a, 1) / λ

so 2λ Gamma(n, λ) = Gamma(n, 1/2) = χ^2_{2n}

Gamma and F-distribution

If U ~ Gamma(α_1, β_1), V ~ Gamma(α_2, β_2), then

$\alpha_2 \beta_1 U / \alpha_1 \beta_2 V \sim F_{2\alpha_1, 2\alpha_2}$

Dirichlet Distribution：

A general result is that if for $i = 1, 2, \ldots k$, $U_i \sim \Gamma(\alpha_i, \beta)$ independent, then

$$\frac{1}{\sum_{i=1}^{k} U_i}(U_1, U_2, \ldots U_k) \sim \text{DIRICHLET}(\alpha_1, \alpha_2, \ldots \alpha_k),$$