# Cheat Sheet
# Applied Probability

Daniel Lin; Professor: Almut E. D. Veraart

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This is only a cheat sheet with key definitions, formulas, theorems and propositions. Please use the official note or recommended reading to study this course properly.

## 1 Preliminaries

### 1.1 Exchanging operators

**Tonelli's theorem**: The following operations are commutative: Integration, countable summation and expectation (requires non-negativity), and they can exchange with each other.

Limit $\leftrightarrow$ expectation:

$$\lim_{n\to\infty} E(Z_n) = E(\lim_{n\to\infty} Z_n)$$

if $Z_n$ is monotonic sequence, or dominated by some random variable $Y$ with finite $E|Y|$.

Limit $\leftrightarrow$ summation: if $\mathcal{I}$ is finite,

$$\lim_{n\to\infty} \sum_{i\in\mathcal{I}} f_i(n) = \sum_{i\in\mathcal{I}} \lim_{n\to\infty} f_i(n)$$

but you should be careful when $\mathcal{I}$ is infinite.

You must be extremely careful when exchanging infinite sum with: limit, integration, differentiation.

### 1.2 Basics of probability

Probability generating function:

$$G_X(u) = E(u^X)$$

Moment generating function:

$$M_X(u) = E(e^{uX})$$

Laplace transformation

$$L_X(u) = E(e^{-uX})$$

Characteristic function:

$$\phi_X(u) = E(e^{iuX})$$

all the functions are unique for each distribution.
Property about Laplace transform: if $E(e^{-X}) = 0$, then $P(X = \infty) = 1$.

Joint density from CDF:

$$\frac{\partial^2}{\partial x \partial y} P(X \le x, Y \le y) = \frac{\partial^2}{\partial x \partial y} P(X > x, Y > y) = f_{X,Y}(x, y)$$

**Theorem 1.1** (Dominated convergence theorem)**.** *Given countable index set $\mathcal{I}$. If $\forall n$, $\sum_{i\in\mathcal{I}} a_i(n)$ is absolutely convergent, $\lim_{n\to\infty} a_i(n) =: a_i$ exists and there is sequence $b_i \ge 0$ s.t. $\sum_{i\in\mathcal{I}} b_i < \infty$, and $|a_i(n)| \le b_i$. Then $\sum_{i\in\mathcal{I}} |a_i| < \infty$ and*

$$\sum_{i\in\mathcal{I}} \lim_{n\to\infty} a_i(n) = \lim_{n\to\infty} \sum_{i\in\mathcal{I}} a_i(n)$$

**Definition 1** (Convergence in probability). $X_n \xrightarrow{P} X$ if for all $\epsilon > 0$

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

**Definition 2** (Convergence in distribution). $X_n \xrightarrow{d} X$ if

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \text{for all } x \text{ s.t. } F(x) \text{ is continuous}$$

where $F_n, F$ are CDFs of $X_n, X$.

**Theorem 1.2** (Slutsky's theorem). *If $X_n \xrightarrow{d} X$, $A_n \xrightarrow{P} a$, $B_n \xrightarrow{P} b$, then*

$$A_n X_n + B_n \xrightarrow{d} aX + b$$

**Theorem 1.3** (Central limit theorem). *If $Z_i$ are i.i.d. with FINITE mean $\mu$, FINITE variance $\sigma^2$, then*

$$\frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{n} Z_i - n\mu\right) \to N(0,1)$$

.

**Inequalities**:

$$\text{(Markov)} \quad P(X \geq a) \leq \frac{E(X)}{a} \quad X \geq 0, a > 0$$

$$\text{(Chebyshev)} \quad P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$log(1+x) > \frac{x}{x+1} \quad \text{if } x > -1$$

.

**Sequential Independence**:
If $(X_n), (Y_n)$ are two finite independent sequences of random variables with $X_i, Y_i$ independent, then $(X_n + Y_n)$ is an independent sequence.
If $(Z_n)$ is an independent sequence of random variables that can be decomposed by $Z_n = X_n + Y_n$, then the independence of $(X_n), (Y_n)$ inherits from sequence $(Z_n)$. Here $X_n, Y_n$ must be child sequences of $Z_n$ in the sense that $X_n = PZ_n, Y_n = (1-P)Z_n$ where $P$ is random variable taking values on $[0,1]$ and $P$ is independent from $Z_n$ and $n$. The same applies to any finite decomposition.

**Law of total ...**
Assume here $\{B_i\}$ is a partition of $\Omega$

- Probability:

$$P(A) = \sum P(A|B_i)P(B_i)$$

- Probability (continuous):

$$P(A) = \int P(A|X = x) f_X(x)\, dx$$

- Probability (continuous random variable):

$$P(Y > y) = \int P(Y > y|X = x) f_X(x)\, dx$$

- Probability with condition: assume $P(B_i \cap E) > 0$,

$$P(A|E) = \sum P(A|B_i \cap E)P(B_i|E)$$

- Expectation:

$$E(X) = E(E(X|Y))$$

- Variance:

$$\text{Var}[X] = \text{Var}[E(X|Y)] + E(\text{Var}[X|Y])$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

### 1.3 Others

Some important infinite sums to remember:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \quad \text{for } |x| < \frac{1}{4}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

If $\sum_{n=1}^{\infty} a_n < \infty$, then $a_n \to 0$ as $n \to \infty$.

**Poisson Approximation to Binomial**
If $X \sim \text{Binomial}(n, p)$ where $n$ is sufficiently large, $p$ is sufficiently small, then $X$ can be approximated by $\text{Pois}(np)$.

IF $T$ is non-negative integer-valued random variable and $P(A) > 0$,

$$E(T \mid A) = \sum_{n=1}^{\infty} P(T \geq n \mid A)$$

## 2 Discrete Markov Chain

A discrete stochastic process is simply an infinite series of random variables $(X_n)_{n \in \mathbb{N}_0}$ where $n$ represents the current time stamp.

### 2.1 Some Remarks on key definitions

Question: "Prove $(X_n)$ is Markov chain": only Markov property is required.
"Prove $(X_n)$ is a time-homogeneous Markov chain": Markov property + Time homogeneity.
"Prove $(X_n)$ is Markov chain with transition probabilities $p_{ij} = \cdots$": Markov property + proving $p_{ij} = \cdots$ + verify probabilities are positive and sum to 1

Proving a matrix is stochastic must include the argument: all entries $\geq 0$.
You must also verify entries are non-negative and sum to 1 for stationary distribution.

### 2.2 Definitions

Below is a table of key definitions and their symbols for the study of discrete Markov chains.

| Symbol | Name | Side note/Definition |
|---|---|---|
| $E$ | state space | finite/countably infinite |
| $K$ | State space size | $K = \text{card}(E)$ |
| $p_{ij}$ | transition probability | $P(X_1 = j \mid X_0 = i)$ |
| $P$ | transition matrix | $P_{ij} = p_{ij}$ |
| $p_{ij}(n)$ | $n$-step transition probability | $P(X_n = j \mid X_0 = i)$ |
| $P_n$ | $n$-step transition matrix | |
| $\nu_i^{(n)}$ | pmf of $X_n$ | $\nu_i^{(n)} = P(X_n = i)$ |
| $\nu^{(n)}$ | marginal distribution | a vector of length $K$ with entries $\nu_i^{(n)}$ |
| $\nu^{(0)}$ | initial distribution | |
| $T_j$ | first hitting time | $T_j = \min\{n \in \mathbb{N} : X_n = j\}$<br>$T_j = \infty$ if not hit |
| $f_{ij}(n)$ | first passage probability | $f_{ij}(n) = P(T_j = n \mid X_0 = i)$<br>NOTE: $f_{ij}(0) \equiv 0$ |
| $f_{ij}$ | | $f_{ij} = P(T_j < \infty \mid X_0 = i)$ |
| $f_{ii}$ | return probability | |
| $\mathcal{S}_X$ | finite expectation set | $\{s \in \mathbb{R} : \sum_{x=0}^{\infty} \|s\|^x P(X = x) < \infty\}$ |
| $G_X$ | PGF | $G_X : \mathcal{S}_X \to \mathbb{R}$<br>$G_X(s) = E(s^X)$ |
| $G_{(a_n)}(s)$ | generating function | $G_{(a_n)}(s) = \sum_{n=0}^{\infty} a_n s^n$<br>defined on $\{s : \sum_{n=0}^{\infty} \|a_n s^n\| < \infty\}$ |
| | recurrent state | $P(X_n = j \text{ for some } n \in \mathbb{N} \mid X_0 = j) = 1$ |
| | transient state | $P(X_n = j \text{ for some } n \in \mathbb{N} \mid X_0 = j) < 1$ |
| $N_j$ | Total time spent on $j$ | $N_j = \sum_{n=0}^{\infty} I_n^{(j)}$<br>$I_n^{(j)} = 1$ if $X_n = j$, 0 otherwise |
| $N_i(j)$ | number of visits to $i$ before reaching $j$ | $N_i(j) = \sum_{n=1}^{T_j} I_n^{(i)}$ |
| $V_i(n)$ | number of visits to $i$ before time $n$ | $V_i(n) = \sum_{k=1}^{n} I_k^{(i)}$ |
| $\rho_i(j)$ | Expected visits to $i$ between two visits to $j$ | $\rho_i(j) = E(N_i(j) \mid X_0 = j)$ |
| | Holding time | time spent on a state before first leaving it |
| $\mu_i$ | mean recurrence time | $\mu_i = E(T_i \| X_0 = i)$ |
| | null recurrent state | $\mu_i = \infty$ |
| | positive recurrent state | $\mu_i < \infty$ |
| $d(i)$ | period | $d(i) = gcd\,\{n \in \mathbb{N} : p_{ii}(n) > 0\}$ |
| | ergodic | positive recurrent + periodic |
| $i \to j$ | $j$ accessible from $i$ | exists $m \in \mathbb{N}_0$ s.t. $p_{ij}(m) > 0$ |
| $i \leftrightarrow j$ | communicate | $i \to j$, $j \to i$ |

| **Symbol** | **Name** | **Side note/Definition** |
|---|---|---|
| | closed set $C$ | $p_{ij} = 0$ whenever $i \in C, j \notin C$ |
| | irreducible set $C$ | $i \leftrightarrow j$ for all $i, j \in C$ |
| | absorbing state | $\{i\}$ is closed set |
| | irreducible chain | $E$ is closed set |
| $P_C$ | restriction of $P$ to set $C$ | only take rows and columns for states in $C$ |
| $\boldsymbol{\lambda}$ | Distribution | $\sum_{j \in E} \lambda_j = 1$ |
| | invariant | $\lambda P = \lambda$ |
| $\boldsymbol{\pi}$ | Stationary distribution | an invariant distribution |
| $l_{ji}(n), i \neq j$ | probability of reaching $i$ in $n$ steps without return to $j$ | $P(X_n = i, T_j \geq n \vert X_0 = j)$ |
| | Limiting distribution | $(\lim_{n\to\infty} p_{ij}(n))_{j \in E}$ should be the same value for all $i$ |

## 2.3 Important formulae/theorems

(**) means this is not proved in this course, but is proved in the course *Probability for statistics* or can be deduced easily.

**Recurrence relations**

- Chapman-Kolmogorov
$$p_{ij}(m+n) = \sum_{l\in E} p_{il}(m)p_{lj}(n)$$
- transition probabilities vs first passage probabilities
$$p_{ij}(n) = \sum_{l=1}^{n} f_{ij}(l)p_{jj}(n-l)$$
- (**) equations on first passages: fix $i$, let $\eta_j = P(T_i < \infty | X_0 = j)$, then
$$\eta_j = \sum_{k\in E} p_{jk}\eta_k$$
and $\boldsymbol{\eta} := (\eta_k)_{k\in E}$ is the minimum solution to this equation.
- (**) equation on mean first passages: let $\rho_j := E(T_i|X_0 = j)$,
$$\rho_j = 1 + \sum_{k\in E} p_{jk}\rho_k$$
and $\boldsymbol{\rho} := (\rho_k)_{k\in E}$ is the minimum solution to this equation.
- first passage probabilities and first passage without returning probabilities
$$f_{jj}(m+n) = \sum_{i\in E, i\neq j} l_{ji}(m)f_{ij}(n)$$
- Recursive formulae for $l_{ij}$
$$l_{ji}(n) = \sum_{r\in E, r\neq j} p_{ri}l_{jr}(n-1)$$
and $l_{ji}(1) = p_{ji}$.

**Summations**

- sum of first passage probabilities
$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$
- sum of $N_i(j)$
$$T_j = \sum_{i\in E} N_i(j)$$
- sum of $\rho_i(j)$
$$\mu_j = \sum_{i\in E} \rho_i(j)$$
- sum of $l_{ij}(n)$
$$\rho_i(j) = \sum_{n=1}^{\infty} l_{ji}(n)$$

**Recurrent, transient criterion**

- using $p_{jj}$

$$\sum_{n=1}^{\infty} p_{jj}(n) = \infty \quad j \text{ is recurrent.}$$

$$\sum_{n=1}^{\infty} p_{jj}(n) < \infty \quad j \text{ is transient.}$$

- mean recurrence time for the recurrent state:

$$\mu_i = \sum_{n=1}^{\infty} n f_{ii}(n)$$

- mean recurrence time for the transient state:

$$i \text{ is transient} \Rightarrow P(T_i = \infty | X_0 = i) > 0 \Rightarrow \mu_i = \infty$$

- transition probabilities for the transient state: if $j$ is transient/null recurrent

$$p_{ij}(n) \to 0 \text{ as } n \to \infty \text{ for all } i \in E$$

- criterion for null/positive recurrence: if $i$ is recurrent,

$$\lim_{n\to\infty} p_{ii}(n) = 0 \Leftrightarrow i \text{ is null recurrent}$$

**Stationarity**

- stationary property of $\boldsymbol{\rho}(j) = (\rho_i(j))_{i\in E}$: if chain is irreducible, recurrent, then $\rho_i(j) < \infty\ \forall i, j$, and

$$\boldsymbol{\rho}(j)P = \boldsymbol{\rho}(j)$$

- Any irreducible recurrent chain has unique invariant vector(up to multiplicative constant) given by

$$\boldsymbol{x} := (\rho_i(j))_{i\in E}$$

  where $j$ is any state in $E$.
  if $\sum_i x_i < \infty$, chain is positive recurrent; $\sum_i x_i = \infty$, chain is null recurrent.

- Any irreducible positive recurrent chain has stationary distribution given by

$$\pi_i := \frac{\rho_i(j)}{\mu_j}$$

  where $j$ is any state in $E$.

- For irreducible chain, if stationary distribution exists, then the chain is positive recurrent and the stationary distribution is uniquely given by $\pi_i = \mu_i^{-1}$.

- For irreducible transient/null recurrent chains, there is no stationary distribution.

- If $\boldsymbol{\pi}$ is stationary distribution, for transient/null recurrent states $i$, $\pi_i = 0$.

- On finite space, $\boldsymbol{\pi}$ is unique $\Leftrightarrow$ there is unique closed communicating class $\Leftrightarrow$ there is a unique positive recurrent communicating class (recall there is at least one positive recurrent communicating class on finite space)

- On finite space, stationary distribution always exists and can be written as a linear combination of the unique stationary distributions of the closed communicating classes.

**Limiting Distribution**

- Limiting distribution independent of the initial point

$$\lim_{n\to\infty} p_{ij}(n) = \lim_{n\to\infty} P(X_n = j)$$

- irreducible, **aperiodic** chain:

$$\lim_{n\to\infty} p_{ij}(n) = \frac{1}{\mu_j}$$

- irreducible transient/null recurrent chain:

$$\lim_{n\to\infty} p_{ij}(n) = 0$$

- irreducible, **aperiodic** and positive recurrent chain:

$$\lim_{n\to\infty} p_{ij}(n) = \pi_j$$

  which is the unique stationary distribution

- Ergodic theorem: if the chain is irreducible,

$$P\left(\lim_{n\to\infty} \frac{V_i(n)}{n} = \frac{1}{\mu_i}\right) = 1$$

- On finite state space, limiting distributions are all stationary distributions.

**Reverse Chain**:
If chain $\{X_n\}_{n=1,\cdots,N}$ is positive recurrent, and marginal distributions are all stationary distributions. The chain can be reversed by $Y_n := X_{N-n}$. If transition matrix of $Y, X$ are the same, $X$ is called *time-reversible*

- Reversed transition probabilities

$$q_{ij} = \frac{\pi_j}{\pi_i} p_{ji}$$

- detailed balance: $X$ is time-reversible

$$\Leftrightarrow \pi_i p_{ij} = \pi_j p_{ji}$$

  For an irreducible chain if some $\boldsymbol{\pi}$ satisfies this equation, then the chain is time-reversible and positive recurrent with stationary distribution $\boldsymbol{\pi}$.

**Others**

- Determination of distribution by the initial distribution

$$P(X_{n_1} = x_1, X_{n_2} = x_2, \cdots X_{n_k} = x_k) = (\nu^0 P^{n_1})_{x_1} p_{x_! x_2}(n_2 - n_1) \cdots p_{x_{k-1} x_k}(n_k - n_{k-1})$$

- time spent on $j$ and returning probability

$$P(N_j = n \mid X_0 = j) = f_{jj}^{n-1}(1 - f_{jj})$$

$$P(N_j = n \mid X_0 = i) = \begin{cases} 1 - f_{ij} & n = 0 \\ f_{ij} f_{jj}^{n-1}(1 - f_{jj}) & n > 0 \end{cases}$$

- expectation of total visit time

$$E(N_j | X_0 = j) = \frac{1}{1 - f_{jj}}$$

$$E(N_j | X_0 = i) = \frac{f_{ij}}{1 - f_{jj}}$$

## 2.4 Communicating classes

Communicating classes are equivalent classes of the relation $i \leftrightarrow j$.

The following properties are shared within communicating classes:

- period
- transient
- recurrent
- null recurrent

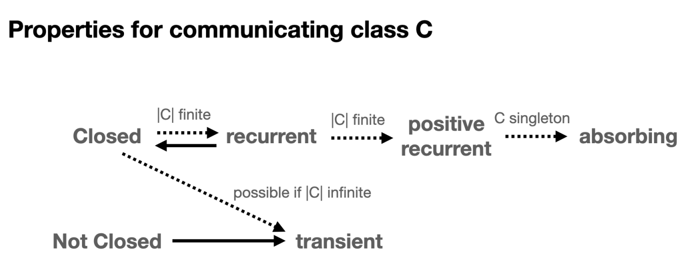
For any closed communicating class $C$, $P_C$ is stochastic.
For finite state space, there is at least one recurrent state, and all recurrent states are positive.

**Decomposition Theorem**
There is a unique partition of $E$

$$E = T \cup \left( \bigcup_i C_i \right)$$

$C_i$ are irreducible closed sets of recurrent states, $T$ are transient states.

Relationship between properties of communicating classes:




# 3 Exponential Distribution

Exponential distribution is closely associated to gamma functions and gamma distributions:
if $Y \sim \text{Gamma}(n, \lambda)$

$$f_Y(t) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \quad t > 0$$

where the gamma function $\Gamma(n) := \int_0^\infty e^{-x} x^{n-1} \, dx$.

Note $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, and $\Gamma(1/2) = \sqrt{\pi}$. $\Gamma(z+1) = z\Gamma(z)$.

If $X_i \sim \text{Gamma}(1, \lambda)$ i.i.d., then $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \lambda)$. And $\text{Exp}(\lambda) = \text{Gamma}(1, \lambda)$.

## 3.1 Laplace Transform

Given a function $f$ defined on $(0, \infty)$, Laplace transform is defined as

$$\mathscr{L}[f(t)](u) := \int_0^\infty e^{-ut} f(t) \, dt$$

And for continuous random variable $X$, the Laplace transform is performed on PDF. In this case, it is equivalent to $E(e^{-uX})$ by the law of the unconscious statistician. This definition is generalised to all random variables,

$\mathscr{L}[X](u) := E(e^{-uX})$

Laplace of $X \sim \text{Poi}(\lambda)$:

$$\mathscr{L}[X](u) = \exp\{\lambda(e^{-u} - 1)\}$$

Properties of Laplace transform

- $\mathscr{L}$ is linear transformation. i.e.

$$\mathscr{L}[\alpha f] = \alpha \mathscr{L}[f], \mathscr{L}[f_1 + f_2] = \mathscr{L}[f_1] + \mathscr{L}[f_2]$$

- $\mathscr{L}[f(t-\alpha)] = e^{-\alpha s}\mathscr{L}[f(t)]$
- $\mathscr{L}[e^{\alpha t} f(t)](u) = \mathscr{L}[f](u-\alpha)$
- $\mathscr{L}[f'](u) = u\mathscr{L}[f] - f(0)$
- Most importantly, the Laplace transform of a random variable is unique.

Find expectations using Laplace transform:

$$E(X) = -\frac{d}{ds}(\log(E(e^{-sX})))\Big|_{s=0}$$

## 3.2 Properties of Exponential distribution

Minimum of independent exponential:

$$\min_i\{\text{Exp}(\lambda_i)\} \sim \text{Exp}\left(\sum_{i=1}^n \lambda_i\right)$$

Further, for any $k$,

$$P(\text{minimum attained at } i = k) = \frac{\lambda_k}{\sum_{i=1}^n \lambda_i}$$

applies to any countable index set $I$ if $\sum_{i\in I} \lambda_i < \infty$, by replacing min by $\inf_{i\in I}$.

Positive-valued random variable(no need to be continuous) has exponential distribution $\Leftrightarrow$ it satisfies lack of memory property

$$P(X > x + y \,|\, X > x) = P(X > y) \quad \forall x, y > 0$$

whenever $P(X > x) > 0$.

Sum of infinite exponential distributions: If $H_i \sim \text{Exp}(\lambda_i)$,

$$\sum_{i=1}^\infty \frac{1}{\lambda_i} < \infty \Rightarrow P(\sum_{i=1}^\infty H_i < \infty) = 1$$

$$\sum_{i=1}^\infty \frac{1}{\lambda_i} = \infty \Rightarrow P(\sum_{i=1}^\infty H_i = \infty) = 1$$

# 4 Poisson Process

The jump process of a continuous time Markov chain $(X_t)_{t\geq 0}$ on countable state space $E$ is defined as $Z_n := X_{J_n}$ where $J_n$ is the time of n'th jump. Here $X_t$ are assumed to be *right continuous*, i.e. $\forall \omega \in \Omega$, $\exists \epsilon$ s.t. $\forall s \in (t, t+\epsilon)$,

$$X_t(\omega) = X_s(\omega)$$

**Minimal process**:
$\infty$ is appended to $E$, and for $t > J_\infty := \lim_{n\to\infty} J_n$, $X_t$ is defined as $\infty$.

**Definition of Poisson process**: Setup: $(N_t)_{t\geq 0}$ with rate $\lambda > 0$ and range of $N_t$ is in $\mathbb{N}_0$. Conditions: (DEF1)

- Almost surely start at 0: $P(N_0 = 0) = 1$
- Increments are independent: for any $n \in \mathbb{N}$ and $0 \le t_1 < t_2 < \cdots < t_n$, the increment times $N_{t_n} - N_{t_{n-1}}, N_{t_{n-1}} - N_{t_{n-2}}, \cdots, N_{t_2} - N_{t_1}, N_{t_1}$ are mutually independent.
- Stationary increment: $P(N_t - N_s = k) = P(N_{t-s} = k)$
- $N_t \sim \text{Pois}(\lambda t) \Leftrightarrow$ single arrival conditions:

$$P(N_{t+\delta} - N_t = 1) = \lambda\delta + o(\delta), \quad P(N_{t+\delta} - N_t = 0) = 1 - \lambda\delta + o(\delta)$$
$$P(N_{t+\delta} - N_t \ge 2) = o(\delta)$$

*Remark.* Though mutual independent increments are required, when proving this property, you only need to prove pairwise independence i.e. any pair of increments are independent. Because $N_t$ is a class of infinite divisible(I.D.) random variables. i.e. for any $t \ge 0$, $n \in \mathbb{N}$, $N_t = \sum_{i=1}^n N_t^{(i)}$ for some i.i.d. random variables $N_t^{(i)}$. For any class of I.D. random variables, pairwise independence implies mutual independence. (See Infinitely Divisible Distributions, Conditions for Independence, and Central Limit Theorems for details)

But for pairwise independence, you do not need to worry about independence between non-consecutive increments, e.g. $N_{t_3} - N_{t_2}, N_{t_1}$. Because conditioning on $N_{t_2} - N_{t_1}$ and using the independence of consecutive increments proves the result. Therefore, checking for any $s, t > 0$,

$$P(N_{t+s} - N_s = i, N_s = j) = P(N_t = i)P(N_s = j)$$

is enough.

Another equivalent definition: (DEF2)

- Inter-arrival times $H_i$ are iid $\text{Exp}(\lambda)$
- $J_0 := 0, J_n := \sum_{i=1}^n H_i$, define
$$N_t := \sup\{n \in \mathbb{N}_0 \,:\, J_n \le t\}$$

.
Meaning of $N_t = k$ under the second definition: $P(J_k \le t < J_{k+1})$.

**Inter-arrival time**: $H_i$ follows $\text{Exp}(\lambda)$.
**Time of $n$'th event**: $J_n := \sum_{i=1}^n H_i$ satisfies $\text{Gamma}(n, \lambda)$.
**Joint Distribution of** $(J_1, \cdots, J_n)$:

$$f(J_1 = t_1, \cdots, J_n = t_n | N_t = n) = \begin{cases} \frac{n!}{t^n} & \text{if } 0 < t_1 < \cdots < t_n \le t \\ 0 & \text{otherwise} \end{cases}$$

**k'th item of order statistics over** $[0, t]$:

$$P(J_i = x | N_t = n) = \frac{n!}{(k-1)!\,(n-k)!}\frac{1}{t}(x/t)^{k-1}(1 - x/t)^{n-k} \quad x \in [0, t]$$

so

$$E(J_i | N_t = n) = \frac{it}{n+1} \quad 1 \le i \le n$$

**Superposition**:
If $N_t^{(1)}, \quad, N_t^{(n)}$ are independent Poisson processes with rates $\lambda_1, \cdots, \lambda_n > 0$ then $N_t := \sum_i N_t^{(i)}$ is Poisson process with rate $\lambda := \sum_i \lambda_i$.

**Thinning**:
If $\{N_t\}$ has rate $\lambda > 0$ and each arrival is marked as type $k$ with probability $p_k$, then $N_t^{(k)}$, number of type $k$ arrivals in $[0, t]$ are Poisson processes with rate $\lambda p_k$.

## 4.1 List of Key Symbols

| Symbol | Meaning |
|---|---|
| $N_t$ | number of arrivals in $[0, t]$ |
| $\lambda$ | rate of inter-arrival time (exponential) |
| $J_n$ | time of $n$'th arrival |
| $H_i$ | inter-arrival time: $J_i - J_{i-1}$ |

## 4.2 Non-homogeneous Poisson

Same as Poisson, but with rate $\lambda$ replaced by the continuous function $\lambda(t)$. Distribution of $N_t$:

$$N_t \sim \text{Poi}(m(t)) \quad m(t) := \int_0^t \lambda(s)\, ds$$

Distribution of $N_t - N_s$:

$$N_t - N_s \sim \text{Poi}(m(t) - m(s))$$

Warning: increment not have stationary distribution!

## 4.3 Compound Poisson Process

Given $\{Y_i\}$ sequence of i.i.d. independent from Poisson process $\{N_t\}_{t\geq 0}$,

$$\{S_t\}_{t\geq 0} = \left\{ \sum_{i=1}^{N_t} Y_i \right\}_{t\geq 0}$$

is called the compound Poisson process.
Generating function: (derived by conditioning on $N_t$)

$$G_{S_t}(u) = G_{N_t}(G_{Y_1}(u)) = \exp\{\lambda u(G_{Y_1}(u) - 1)\}$$

$$M_{S_t}(u) = G_{N_t}(M_{Y_1}(u)) = \exp\{\lambda u(M_{Y_1}(u) - 1)\}$$

# 5 Continuous Markov Chain

## 5.1 Definitions

A continuous-time process $\{X_t\}_{t\geq 0}$ is called continuous-time Markov chain(CTMC) if

$$P(X_{t_n} = j \,|\, X_{t_1} = i_1, \cdots, X_{t_{n-1}} = i_{n-1}) = P(X_{t_n} = j \,|\, X_{t_{n-1}} = i_{n-1})$$

for all $j, i_1, \cdots, i_{n-1} \in E$ and any $0 \leq t_1 < \cdots < t_n < \infty$ with $n \in \mathbb{N}$.

Below is a table summarising symbols and definitions:

| Symbol | Name | Side note/Definition |
|---|---|---|
| $E$ | state space | finite/countably infinite |
| $K$ | State space size | $K = \mathrm{card}(E)$ |
| $p_{ij}(s,t)$ | transition probability | $p_{ij}(s,t) = P(X_t = j \mid X_s = i)$ |
| $p_{ij}(t)$ | transition probability for homogeneous chain | $p_{ij}(t) = P(X_t = j \mid X_0 = i)$<br>$= P(X_{s+t} = j \mid X_s = i)$ |
| $X_{t_+}$ | right limit of $X$ at $t$ | Note when $t = J_n$,<br>$X_{t_+} \neq X_{t_-}$ |
| $\{P_t : t \geq 0\}$ | stochastic semi-group | $P_0 = I_{K \times K}$<br>$P_t$ is stochastic<br>$P_{s+t} = P_s P_t \ \forall s,t \geq 0$ |
| | standard stochastic semi-group | $\lim_{t \downarrow 0} P_t = I$ |
| | uniform stochastic semi-group | $P_t \to I$ uniformly as $t \downarrow 0$ |
| $H_{\mid i}$ | holding time of state $i$ | Given $X_t = i$,<br>$H_{\mid i} := \inf\{s \geq 0 : X_{t+s} \neq i\}$<br>$H_{\mid i}$ always exponentially distributed |
| $q_i$ | | parameter of exponential distribution associated to $H_{\mid i}$ |
| | explosion | when infinite jumps occur in finite time interval |
| | absorbing state $i$ | $q_i = 0$ |
| | instantaneous state $i$ | $q_i = \infty$ |
| $p_{ij}^Z$ | transition probability of jump chain | $p_{ij}^Z = \lim_{\delta \downarrow 0} P(X_\delta = j \mid X_0 = i, X_\delta \neq i)$ |
| $(Z_n)_{n \in \mathbb{N}_0}$ | Jump chain | $Z_n$ is n'th state visited by CTMC $C$ |
| $P^Z$ | jump chain transition matrix | $[P^Z]_{ij} = p_{ij}^Z$ |
| $n_i$ | number of exponential clocks | number of states reachable from state $i$ |
| $q_{ij}$ | transition rates | at state $i$, rate of exponential alarm clock set for state $j$<br>note $q_{ii} = 0$ |
| $G$ | generator | $\lim_{\delta \downarrow 0} \frac{1}{\delta}(P_\delta - P_0)$<br>requires $P_t$ differentiable at $t = 0$ |
| | instantaneous transition rates from $i$ to $j \neq i$ | $\lim_{\delta \downarrow 0} \frac{\mathrm{E}(\text{transitions to } j \text{ in } (t,t+\delta] \mid X_t = i)}{\delta}$<br>actually equals $p'_{ij}(0) = g_{ij}$ |
| $g_i$ | parameter of exponential distribution for $H_{\mid i}$ | equals $-g_{ii}$ |
| | irreducible chain | $\forall i,j \in E$, $\exists t$ s.t. $p_{ij}(t) > 0$ |
| $\boldsymbol{\pi}$ | limiting distribution | $\forall i,j \in E$,<br>$\lim_{t \to \infty} p_{ij}(t) = \pi_j$ |
| $\boldsymbol{\pi}$ | stationary distribution | $\forall t \geq 0$, $\boldsymbol{\pi} = \boldsymbol{\pi} P_t$ |
| $\boldsymbol{v}^{(t)}$ | marginal distribution | $\boldsymbol{v}^{(t)} = \boldsymbol{v}^{(0)} P_t$ |
| $Y_n$ | skeleton | $Y_n := X_{\delta n}$ for fixed $\delta > 0$ |
| | Recurrent state $i$ | $P(\{t \geq 0 : X_t = i\}\text{unbounded} \mid X_0 = i) = 1$ |
| | Transient state $i$ | $P(\{t \geq 0 : X_t = i\}\text{unbounded} \mid X_0 = i) = 0$ |

| Symbol | Name | Side note/Definition |
|---|---|---|
| $J_n$ | jump times | $J_0 = 0$<br>$J_{n+1} = \inf\{t \geq J_n : N_t \neq N_{J_n}\}$ |
| $J_\infty$ | explosion time | $\lim_{n\to\infty} J_n$ |
| | explosion of chain is possible | $P(J_\infty < \infty) > 0$ |

## 5.2 Important Formulae and Theorems

**Relations between various characterisations of CTMC**
Remember there are three ways to describe the dynamics of CTMC (dynamics with initial distribution can derive the whole chain): full matrix $(P_t)_{t\geq 0}$, jump chain and holding time (which can be summarised to generator $G$), exponential alarm clocks with rates $q_{ij}$

- Exponential alarm clock and transition probabilities, rates: if $0 < q_i < \infty$,

$$q_i = \sum_j q_{ij}, \quad p_{ij}^Z = \frac{q_{ij}}{q_i} = -\frac{g_{ij}}{g_{ii}}$$

  if $q_i = 0$:

$$p_{ii}^Z = 1, p_{ij}^Z = 0 \text{ for } j \neq i$$

- Generator and instantaneous transition rate:

$$G = P_0' = \frac{d}{dt}(P_t)\bigg|_{t=0}$$

  Breakdown: For $i \neq j$,

$$\begin{aligned} g_{ij} = q_{ij} = p_{ij}'(0) \quad &\text{i.e. } p_{ij}(\delta) \approx g_{ij}\delta \text{ for small } \delta > 0 \\ g_{ii} = p_{ii}'(0) = -q_i \quad &p_{ii}(\delta) \approx 1 + g_{ii}\delta \text{ for small } \delta > 0 \end{aligned}$$

**Continuous, discrete**

- There is always a unique discrete Markov chain corresponding to CTMC, namely the jump chain $(Z_n)$ with

$$p_{ij}^Z = \frac{g_{ij}}{-g_{ii}}$$

- Given discrete Markov chain $Z$ with transition matrix $P^Z$, pick any set of non-negative constants $\{g_i \geq 0\}_{i\in E}$,

$$g_{ij} = \begin{cases} g_i p_{ij}^Z & \text{if } i \neq j \\ -g_i & \text{if } i = j \end{cases}$$

  defines a CTMC (i.e. $G = (g_{ij})$ is the generator of the new CTMC $X$) with jump chain $Z$. These $g_i$ will be parameters for exponential distributions corresponding to holding time $H_{|i}$.

- Jump chain to CTMC with a possible explosion

$$X_t = \begin{cases} Z_n & \text{if } J_n \leq t < J_{n+1} \text{ for some } n \\ \infty & t \geq J_\infty \end{cases}$$

- Conditions for the explosion to not happen: if any one of the following holds
  1. state space $E$ is finite,
  2. $\sup_{i\in E} g_i < \infty$
  3. $X_0 = i$ where $i$ is recurrent in the jump chain $Z$.

**Characterisation of key properties**

- Relationship between the holding time and holding time of state $i$
  If $X_{J_{n-1}+} = i$, $H_n = H_{|i}$. Or equivalently,

$$H_n | X_{J_{n-1}+} = H_{|X_{J_{n-1}+}}$$

- Characterisation of standard stochastic semigroup: $\{P_t\}$ is standard $\Leftrightarrow$ $p_{ij}(t)$ are continuous in $t$ for all $i, j \in E$.

- Characterisation of Holding Time

$$\{H_{|i} > x\} = \{X_t = i, \ \forall t \text{ with } 0 \leq t \leq x\}$$

- Markov Property
  For continuous-time Markov chain $\{X_t\}_{t \geq 0}$:

$$P(X_t = f(t), \text{ for } x < t \leq x + y \,|\, X_t = g(t) \text{ for } 0 \leq t \leq x) = P(X_t = f(t), \text{ for } x < t \leq x + y \,|\, X_x = g(x))$$

  $\forall x, y \geq 0$, functions $f : [x, x+y] \to E, \ g : [0, x] \to E$.

**Forward and backward equations**

- Kolmogorov forward and backward equation (direct result from C-K equation)

$$\text{(forward) } P_t' = P_t G, \quad \text{(backward) } P_t' = G P_t$$

- finding $e^{tG}$:
  Diagonalise $G$, i.e. $G = S\text{diag}(\lambda_1, \cdots, \lambda_K)S^{-1}$, then

$$e^{tG} = S\text{diag}(e^{\lambda_1 t}, \cdots, e^{\lambda_K t})S^{-1}$$

- semigroup is uniform iff $\sup_i(g_{ii}) < \infty$

- If $\{P_t\}$ is a uniform semigroup with generator $G$, then $\{P_t\}$ is the unique solution to both forward, and backward equations with boundary condition $P_0 = I$.

$$P_t = e^{tG}, \quad G\mathbf{1} = \mathbf{0}$$

**Properties of CTMC**

- marginal distribution and initial distribution

$$\boldsymbol{\nu}^{(t)} = \boldsymbol{\nu}^{(0)} P_t$$

- Aperiodicity of CTMC
  If $p_{ij}(t) > 0$ for some $t > 0$, then it is true for all $t > 0$.

- Stationary distribution
  $\boldsymbol{\pi}$ is stationary distribution $\Leftrightarrow$ $\boldsymbol{\pi} G = \mathbf{0}$

- Ergodic Theorem for CTMC
  Given **irreducible** chain $X$ with **standard** semigroup $\{P_t\}_{t \geq 0}$, if stationary distribution $\boldsymbol{\pi}$ exists, it is unique and is given by limiting distribution.
  If no stationary distribution exists,

$$\lim_{t \to \infty} p_{ij}(t) = 0 \ \forall i, j \in E$$

**Recurrent and Transient states**

- $i$ is recurrent(transient)) in jump chain $(Z_n)$ $\Leftrightarrow$ $i$ is recurrent(transient) for $(X_t)$

- Every state is either recurrent or transient

- recurrence and transience are class properties.

## 5.3 Birth Processes

**Definition**: Birth process with intensities $\{\lambda_i \geq 0\}_{i\in\mathbb{N}_0}$ is stochastic process $\{N_t\}_{t\geq 0}$ s.t. $N_t \in \mathbb{N}_0$ and

- $N_t$ is non-decreasing (w.r.t $t$)
- single arrival property is satisfied: if $t \geq 0, \delta > 0, n, m \in \mathbb{N}_0$

$$P(N_{t+\delta} = n+m \mid N_t = n) = \begin{cases} 1-\lambda_n\delta + o(\delta) & \text{if } m = 0 \\ \lambda_n\delta + o(\delta) & \text{if } m = 1 \\ o(\delta) & \text{if } m > 1 \end{cases}$$

- conditional on $N_s$, increment $N_t - N_s$ is independent of all arrivals before $s$. For all $k, l, x(r) \in \mathbb{N}_0$,

$$P(N_t - N_s = k \mid N_s = l, N_r = x(r) \text{ for } 0 \leq r < s) = P(N_t - N_s = k \mid N_s = l)$$

**Properties of birth process**

- It is CTMC with a generator
$$g_{ii} = -\lambda_i, g_{i,i+1} = \lambda_i, g_{i,j} = 0 \text{ otherwise}$$
- Forward equations: if $i < j$
$$\frac{dp_{ij}(t)}{dt} = -\lambda_j p_{ij}(t) + \lambda_{j-1}p_{i,j-1}(t)$$
backward:
$$\frac{dp_{ij}(t)}{dt} = -\lambda_i p_{ij}(t) + \lambda_{i+1}p_{i+1,j}(t)$$
subjected to boundary condition $p_{ij}(0) = \delta_{ij}$.
Forward equations have a unique solution, and it satisfies the backward equations.
- Explosion time: if birth process starts at $k \in \mathbb{N}_0$,
$\sum_{i=k}^{\infty} \frac{1}{\lambda_i} < \infty \Rightarrow P(J_\infty < \infty) = 1$ explode with probability 1
$\sum_{i=k}^{\infty} \frac{1}{\lambda_i} = \infty \Rightarrow P(J_\infty < \infty) = 0$ explode with probability 0

## 5.4 Birth-Death Process

**Definition**: Birth-death process with birth rates $\{\lambda_i \geq 0\}_{i\in\mathbb{N}_0}$, death rates $\{\mu_i \geq 0\}_{i\in\mathbb{N}_0}$ ($\mu_0 = 0$) is Markov chain $\{X_t\}$ on state space $E = \mathbb{N}_0$ s.t. for all $t \geq 0, \delta > 0, n \in \mathbb{N}_0, m \in \mathbb{Z}$

$$P(X_{t+\delta} = n+m \mid X_t = n) = \begin{cases} 1-(\lambda_n+\mu_n)\delta + o(\delta) & \text{if } m = 0 \\ \lambda_n\delta + o(\delta) & \text{if } m = 1 \\ \mu_n\delta + o(\delta) & \text{if } m = -1 \\ o(\delta) & \text{otherwise} \end{cases}$$

**Generator**:

$$G = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

**Asymptotic behaviour**: Given $\lambda_i > 0$, chain settles to equilibrium iff

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n} < \infty$$

and the limiting distribution/stationary distribution is uniquely given by

$$\pi_0 := \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n}\right)^{-1}, \quad \text{for } n \in \mathbb{N},\ \pi_n := \frac{\lambda_0 \times \cdots \times \lambda_{n-1}}{\mu_1 \times \cdots \times \mu_n}\pi_0$$

(use $\boldsymbol{\pi} G = \mathbf{0}$ to derive this)

some conventions:

$$\lambda_{-1} := 0,\ \mu_0 := 0,\ \prod_{\emptyset} \text{anythings} = 1$$

# 6 Brownian Motion

**Definition 3** (Standard Brownian Motion)**.** $B = \{B_t\}_{t \geq 0}$ is standard Brownian motion if

- $B_0 = 0$ almost surely(i.e. probability is 1)
- $B$ has independent, stationary increments
- for $0 \leq s < t$, $B_t - B_s \sim N(0, (t-s))$
- $t \mapsto B_t$ is almost surely continuous in $t$. (i.e. the set of points at which $t \mapsto B_t$ is not continuous has zero measure in $\mathbb{R}_+$)

Drift parameter $\mu$ and variance parameter $\sigma^2$ can be added ($\sigma > 0$),

$$Y_t := \sigma B_t + \mu t$$

is also called a Brownian motion.

Formulae for general increment:

$$Y_t - Y_s \sim N(\mu(t-s), \sigma^2(t-s))$$

**Theorem 6.1** (Donsker's Theorem)**.** *Let $X_n$ be simple random walk, $B_t$ be the standard Brownian motion, then*

$$\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \xrightarrow{d} B_t$$

also, using CLT,

$$\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \xrightarrow{d} N(0, t)$$

.

**Finite Dimensional Distributions(FDD)**:
For continuous time random process $X_t(\omega)$ where $X : \mathcal{T} \times \Omega \to E$, FDD is

$$P(X_{t_1} \leq x_1, \cdots, X_{t_n} \leq x_n)$$

with $0 \leq t_1 < \cdots < t_n$.

FDD of (standard) Brownian motion:

$$\begin{aligned} f_{(B_{t_1}, \cdots B_{t_n}}(x_1, \cdots, x_n) &= f_{B_{t_1}}(x_1) f_{B_{t_2} - B_{t_1}}(x_2 - x_1) \cdots f_{B_{t_n} - B_{t_{n-1}}}(x_n - x_{n-1}) \\ &= \frac{\exp\left(-\frac{1}{2}\left\{\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right\}\right)}{\sqrt{(2\pi)^n t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})}} \end{aligned}$$

Transition density (or Gauss kernel) of standard Brownian motion

$$p_t(y|x) := f_{B_{t+s}|B_s}(y|x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(y-x)^2\right)$$

**Theorem 6.2** (Solution to heat equation)**.** *Given $f : \mathbb{R} \to \mathbb{R}$, the unique solution to initial value problem*

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u_t(x)$$
$$u_0(x) = f(x)$$

*is given by*

$$u_t(x) = E(f(W_t^x)) = \int_{-\infty}^{\infty} p_t(x, y) f(y)\, dy$$

*where $W_t^x$ is Brownian motion starting at $x$.*

**Proposition 6.3** (Symmetry laws). *If $\{B_t\}_{t\geq 0}$ is standard Brownian motion, everything below is a standard Brownian motion:*
$\{B_t\}_{t\geq 0}$*(reflection)*
*fix* $s \geq 0$, $\{B_{t+s} - B_s\}_{t\geq 0}$*(translation)*
$\{aB_{t/a^2}\}_{t\geq 0}$*(rescaling,* $a > 0$*)*
$\{tB_{1/t}\}_{t\geq 0}$ *(inversion)*

Maximum and minimum processes:

$$M_t^+ := \max\{B_s \,:\, 0 \leq s \leq t\}$$

$$M_t^- := \max\{B_s \,:\, 0 \leq s \leq t\}$$

(well defined as $[0,t]$ is compact, $s \mapsto B_s$ is continuous).
Properties of maximum and minimum processes:

- $M_t^+, M_t^-$ have the same distribution
- for any $a > 0$, $M_t^+, aM_{t/a^2}^+$ have the same distribution (so the sample path of Brownian motion is nowhere differentiable with probability one)
- Given $x > 0$, $P(M_t^+ \geq x) = 2P(B_t > x) = 2 - 2\Phi(x/\sqrt{t})$
- (Reflection principle) Given $x > 0$, $\tau := \min\{s \,:\, B_s \geq x\}$, define

$$B_t'' := \begin{cases} B_t, & t \leq \tau \\ x - (B_t - x), & t > \tau \end{cases}$$

  $\{B_t''\}_{t\geq 0}$ is also Brownian motion. The second part of the definition is a reflection about level $x$.
  Further,

$$P(\tau \leq t, B_t \geq x) = P(\tau \leq t, B_t \leq x), \quad F_\tau(t) = P(M_t^+ \geq x)$$

  and distribution of $\tau$ is given by

$$f_\tau(t) = \frac{x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{x^2}{2t}\right\}$$

# 7 Application: Models appeared in Lecture Notes

## 7.1 Gambler's Ruin

$N \geq 2$, initial fortune $i \in \{0, 1, \cdots, N\}$, Gambler's fortune $\{X_n\}_{n\in\mathbb{N}_0}$ is a Markov chain with

$$p_{00} = p_{NN} = 1, \qquad p_{i(i+1)} = p = 1 - p_{i(i-1)}, \qquad \text{for } i \in \{1, \cdots, N-1\}$$

Define $V_i := \min\{n \in \mathbb{N}_0 \,:\, X_n = i\}$, the probability winning is

$$h_i := P(V_N < V_0 \,|\, X_0 = i)$$

and ruin probability is $1 - h_i(N)$.

It can be shown that

$$h_i(N) = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq 1/2 \\ \frac{i}{N}, & \text{if } p = 1/2 \end{cases}$$

## 7.2 Cramér-Lundberg Model

The model consists of two sets of random variables: *claim sizes* $(Y_k)_{k\in\mathbb{N}}$ which are i.i.d. with mean $\mu$, variance $\sigma^2 \leq \infty$. ($Y_k$ represents amount claimed by $k$th customer), *claim arrival process* $(N_t)_{t\geq 0}$ (this represents number of arrivals in $[0,t]$).

- Claim times $J_k$ are s.t. $0 < J_1 < J_2 < \dots$
- $N_t := \sup\{n \in \mathbb{N} : J_n \leq t\}$
- Inter-arrival times $H_1 := J_1$ $H_k := J_k - J_{k-1}$ are i.i.d. $\text{Exp}(\lambda)$
- $(Y_k), (H_k)$ are independent of each other

**Total claim amount**: Stochastic process $(S_t)_{t\geq 0}$ defined by

$$S_t := \begin{cases} \sum_{i=1}^{N_t} Y_i, & N_t > 0 \\ 0, & N_t = 0 \end{cases}$$

Cumulative distribution: (can be found by conditioning onf $N_t$)

$$P(S_t \leq x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} P\left(\sum_{i=1}^{n} Y_i \leq x\right)$$

**Risk Process**: Stochastic process $(U_t)_{t\geq 0}$ defined by

$$U_t := u + ct - S_t$$

where $u \geq 0$ is *initial capital*, $c > 0$ is *premium income rate*.
$E(U_t) = u + (c - \lambda\mu)t$. **Net profit condition**: $c > \lambda\mu$.

Ruin probabilities:

$$\phi(u, T) := P(U_t < 0 \text{ for some } t \leq T)$$

ruin probability in infinite time: $\phi(u) := \phi(u, \infty)$. Note $u$ represents initial capital of $(U_t)$.

## 7.3 The Coalescent Process

Initially $n$ individuals, each pair coalesce according to independent $Exp(\lambda)$. Each event is two individuals coalescent to one. The process continues until there is only one individual, so there are $n-1$ coalescent events.

If $H_k$ is the time of $k$th coalescence:

$$H_k \sim \text{Exp}\left((n - (k-1)2)\right)$$

## 7.4 Models for Asset Prices

Asset price: $\{S_t\}_{0\leq t\leq T}$ Model 1:

$$S_t := S_0 \exp\left\{(\mu - \sigma^2/2)t + \sigma B_t\right\}$$

$\mu$: risk-free interest rate, $\sigma$: volatility, $B_t$: Brownian motion.

Model 2: make $\sigma$ into a stochastic process $(\sigma_t)$,

$$S_t := S_0 \exp\left\{\left(\mu t - \frac{1}{2}\int_0^t \sigma_s^2\, ds\right) + \int_0^t \sigma_s dB_s\right\} \qquad \text{where } \sigma_t = \sigma_0 \exp\{\gamma t + \eta W_t\}$$

where $W_t$ is independent Brownian motion.

# 8 Table of Useful Distributions

| Distribution | PDF | CDF | $E(x)$ | $\mathrm{Var}[X]$ | $M(t) = E(e^{tX})$ | $G(z) = E(z^X)$ |
|---|---|---|---|---|---|---|
| $\mathrm{Exp}(\lambda)$ | $\lambda e^{-\lambda x}$ | $1 - e^{-\lambda x}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | $\frac{\lambda}{\lambda - t}$ | $\frac{\lambda}{\lambda - \ln z}$ |
| $\mathrm{Pois}(\lambda)$ | $\lambda^x e^{-\lambda}/x!\ (x \in \mathbb{N})$ | | $\lambda$ | $\lambda$ | $\exp\{\lambda(e^t - 1)\}$ | $\exp\{\lambda(z - 1)\}$ |
| $\mathrm{Bern}(p)$ | $p^x(1-p)^x$ | | $p$ | $p(1-p)$ | $1 - p + pe^t$ | $(1-p) + pz$ |