An Introduction to Trigonometry (Chapter 3 of Pure 3)

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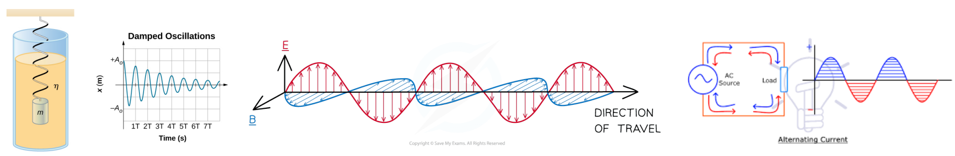
Chapter 1

Why do we care about trigonometry?

Trigonometry began with astronomy, when ancient explorers are looking for the constellations, they realised that triangles are helpful for drawing constellation graphs. Then due to occurrences in other fields, people began to study triangles and treat that as a separate subject: ”trigonometry”.

Figure 1.1: Constellations

What about generalised trig functions(argument expanded to all numbers, not just from 0 to 90)? It sounds like a meaningless idea until physicists found that trig functions appear in alternating currents(AC), waves, oscillations and many other places! There are many shapes of wave that we can imagine, but many waves happen to be sine wave! We can say that without trigonometry, we will not have phones, laundry machines nor aeroplanes.



(a) Oscillation (b) electromagnetic wave (c) AC

Figure 1.2: Trig functions in physics

Chapter 2

Three hidden trigonometric functions

We have learnt $\sin, \cos, \tan$ till now. Just a reminder here of what you should know about a function:

- Domain, range
- x and y interceptions
- Intervals where function value(y) is positive/negative
- Intervals with (monotonic) increasing, decreasing and constant feature.
- Maxima, Minima
- Does it belong to a family of functions? (Like trig function, quadratic function etc.)
- Any singular point? (Singular points are points where the function cannot be defined, $\tan$ has singular points)
- Behaviours around singular points. (Does it go to positive/negative infinity? Or approach to other values?) You should separate left and right cases.

If you cannot answer any of the above questions for any of $\sin, \cos, \tan$, you should review function, trigonometry and differentiation of Pure Math 1.
Remember that though we can extend the trig functions beyond a rectangular triangle, they were originally defined from triangles. Instead of saying ”base”, ”height”, ”hypotenuse”, I will use a, b, c for simplicity.

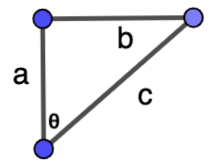


Figure 2.1: triangle

How many pairs of a, b, c can you find to form a fraction like $\frac{a}{b}$? Well, 9 if you consider the three boring $\frac{a}{a}, \frac{b}{b}, \frac{c}{c}$. There are 6 meaningful fractions. And we can spot that

$$\frac{b}{a} = \tan\theta, \frac{a}{c} = \cos\theta, \frac{b}{c} = \sin\theta$$

. Other three fractions are just reciprocals of these three trig functions, but why we have to pick these three fractions to be representatives? I can define my own trig functions: $dan(\theta) = \frac{a}{b}, cas(\theta) = \frac{c}{a}, san(\theta) = \frac{c}{b}$ and say let's make them the three main trig functions we will use!

There is nothing stopping you to do that in a triangle, but when it comes to general case (like $\theta = 10\ rad$), you will see the problem.

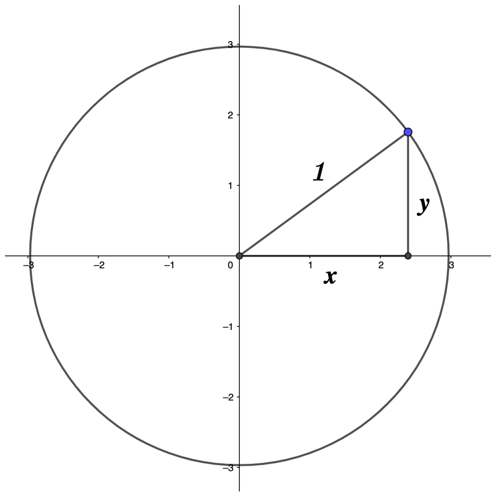


Figure 2.2: Definition of general trig functions

We defined general trig functions using unit circle. $x = \cos\theta, y = \sin\theta$, they are defined for any angle. But what if I use my $cas(\theta), san(\theta)$? Are they always defined? Certainly when x or $y = 0$, it is troublesome! We do not like singular points. For tangent it is just a convention to use $\frac{y}{x}$ instead of $\frac{x}{y}$, both fractions may become undefined.

For convenience of writing, there are indeed definitions for other three fractions:

$$\cot\theta = \frac{a}{b}, \sec\theta = \frac{c}{a}, \csc\theta = \frac{c}{b}$$

but remember they are just reciprocals of three basic trig functions:

$$\cot\theta = \frac{1}{\tan\theta}, \sec\theta = \frac{1}{\cos\theta}, \csc\theta = \frac{1}{\sin\theta}$$

2.1 Graphs

We have to learn to graph three new trig functions now. But it should be easy as we know how to graph their reciprocals! I will just guide you through one of them and you should graph other two by yourself!

$\csc x$ is the reciprocal of $\sin x$, remember $\sin x$ has period 2π so $\csc x$ must also have period 2π(Think of why? Definition of periodicity is $f(x+T) = f(x)$ for any x in domain) It should be enough to just consider what happens within a period.

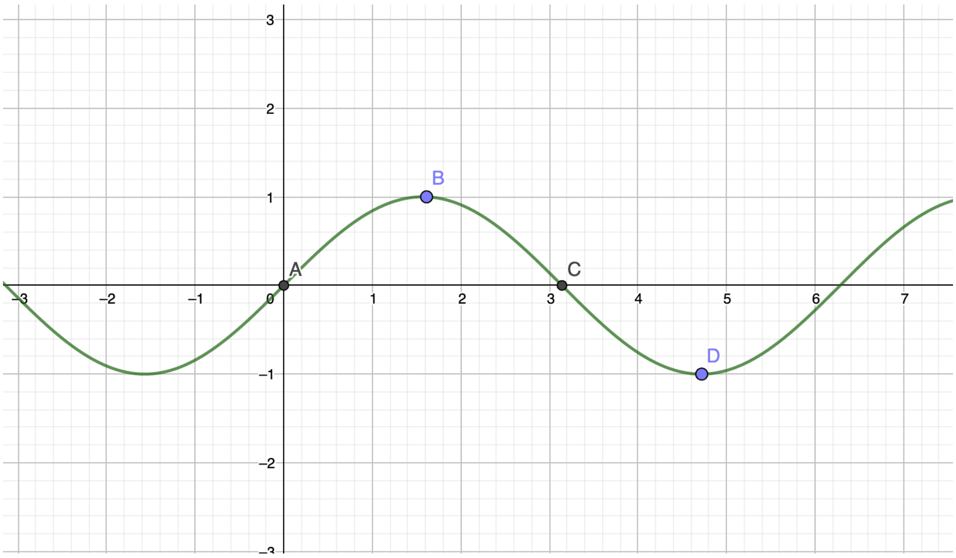


Figure 2.3: Graph of sine

A, B, C, D are 4 critical points here, we can draw $\csc x$ after we found where they are in csc. $\frac{1}{1} = 1, \frac{1}{-1} = -1$, so points B and C stay in their positions. For point A, $\frac{1}{0}$ is undefined! So we know this is a singular point and we should investigate how $\csc x$ behave around this point. This is simple, just think of reciprocals of $0.1, 0.01, 0.001, 0.0001, \ldots$, you will find that on the right of y-axis, csc goes to $+\infty$. Then try $-0.1, -0.01, -0.001, \ldots$ for left hand side of y-axis. Similar tactics apply to point C.

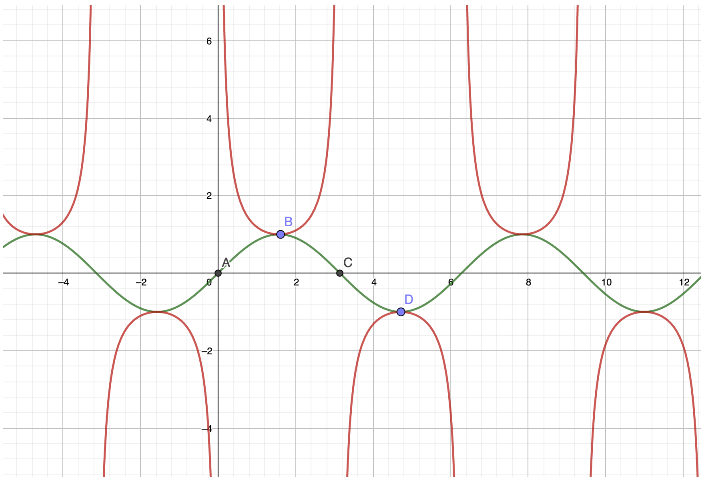


Figure 2.4: Green: sine, red: csc

Graphs of $\sec x, \cot x$ are left as exercises.

Chapter 3

Trig identities

3.1 Pairs

We have learnt that

$$\sin^2 x + \cos^2 x \equiv 1$$

($\equiv$ means this is true for any value x) Do we have similar equations for other four trig functions? Sure! Divide through the equation by $\cos^2 x$, we have

$$\frac{\sin^2 x}{\cos^2 x} + 1 \equiv \frac{1}{\cos^2 x}$$

which is equivalent to:

$$\tan^2 x + 1 \equiv \sec^2 x$$

Please try yourself to derive the third equation

$$1 + \cot^2 x = \csc^2 x$$

And recall that $\sin, \cos$ are very intimate as derivative of $\sin x$ is $\cos x$, derivative of $\cos x$ is $-\sin x$. Though we cannot derive derivatives of $\tan, \sec, \csc, \cot$ before learning product and quotient rules (they are in the next chapter). I will put the results here:

$$\frac{d}{dx}\tan x = \sec^2 x, \frac{d}{dx}\sec x = \tan x \sec x$$

$$\frac{d}{dx}\cot x = -\csc^2 x, \frac{d}{dx}\csc x = -\cot x \csc x$$

So $\tan, \sec$ is an intimate pair and so does $\csc, \cot$.

3.2 Symmetry

We will learn an important concept here: symmetry. It basically means walking from a point to two sides for the same distance, the results are the same. For example, the function $f(x) = |x|$ is symmetric about the line $x = 0$ (or y-axis) as walking

from $x = 0$ towards left or right with the same distance, values of $f(x)$ are the same! That is $f(0 + x) = f(0 - x)$ for any x. There is another type of symmetry: $f(0 + x) = -f(0 - x)$. This means a symmetry around origin, a simple example is $f(x) = x$.
Exercise1 What symmetries can you find for $f(x) = |x - 3|$ and $f(x) = x - 3$?

Exercise2 Observe graphs of $\sin x, \cos x$, understand the following identities:

- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$
- $\sin(\pi - x) = \sin x$
- $\cos(\pi - x) = -\cos x$

There are many other symmetries on $\sin, \cos$ that you can find, and also there are symmetries on other 4 trig functions. It is fun to play around these identities.

3.3 Breaking down sums

In many situations we may encounter expressions like $\sin a + b$ or $\cos 4x$. How to break them down into simpler forms? We have two powerful tools:

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

You will learn why expansion of $\sin$ has different trig functions multiplied together but expansion of $\cos$ has same trig functions multiplied after knowing complex numbers. And the minus term in expansion of $\cos$ comes from multiplying i with itself. (As complex number i is defined as $\sqrt{-1}$)
Exercise. Prove the following two equations:

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

Hint: Use the identities $\sin -x = -\sin x$ and $\cos -x = \cos x$.
Now, we can easily get

$$\sin 2x = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x$$

$$\cos 2x = \cos x \cos x - \sin x \sin x$$

Think of how you can prove that $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

3.4 A little trick

If I have $a\sin x + b\cos x$, is there a way to simplify that to the form $R\sin(x+\alpha)$?

$$R\sin(x+\alpha) = R\sin\alpha\cos x + R\cos\alpha\sin x$$

so $R\sin\alpha = b, R\cos\alpha = a$. Not finished yet, we need to write R and α individually in terms of a, b.
Exercise. Try to use trig identities to prove,

$$\tan\alpha = \frac{b}{a}, R = \sqrt{a^2+b^2}$$

.
Notice that we have two parameters: a, b in the first form, and two parameters R, α in the second form. This is expected. If your new form has different number of parameters, you should check your work.
You may ask hey why we do this? We have two trig functions to deal with, but now they are combined into one single trig function which is much easier to plot and analyse.

Chapter 4

Exercises

Q1. Draw the graphs of $\cot x, \sec x$ for all number x. It is suggested that you use radian instead of degree. Remember $\cot x = \frac{1}{\tan x}, \sec x = \frac{1}{\cos x}$. Hint: $\cos x$ is just $\sin x$ shifted to left for $\frac{1}{2}\pi$, what about the relationship of graphs of $\sec, \csc$?

Q2.
(a) Try to write $\sin^2 x, \cos^2 x$ in terms of $\cos 2x$ using identities we have learnt before. Is it easier to integrate $\cos 2x$ or $\sin^2 x, \cos^2 x$?
(b) Integrate $\sin^2 x, \cos^2 x$.(You may have to look at chain rule in next chapter to do this)

Q3. Express $\sin 3x$, $\sin 4x$ in terms of $\cos x, \sin x$. Can you guess a general formulae for $\sin nx$ where n is a natural number? Try the same for $\cos$. (But you should keep in mind that guessing from existing examples is not a proof!)

Q4. Can you also try to write $a \sin x + b \cos x$ in the form of $R \cos (x + \alpha)$?

Q5.
(a) Find the maximum value of $5 \cos x + 3 \sin x + 7$. Can you use the trick from Q4 to simplify this expression?
(b) Find a way to write $\tan a + b$ using $\tan a, \tan b$
(c) Solve $\tan 2x = 5 \sin 2x$ for $0 \le x \le \pi$. Hint: first write into the form $(1 - 5 \cos 2x) \sin 2x = 0$.
Q6. Solve the following equations for $0 \le x \le \pi$:
(a) $\tan x + \cot x = 4$ (b) $2 \sec^2 x - \tan x = 5$ (c) $3 \cos x + 2 \sin x = 1$