Calculus Cheat Sheet

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Based on notes by Dr A.G. Walton

1 Basics

Formulae relating permutation and Kronecker delta:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta jl$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Dot product

$$\boldsymbol{a}.\boldsymbol{b} = a_i b_i$$

Cross product

$$[\boldsymbol{a} \times \boldsymbol{b}]_i = \epsilon_{ijk} a_j b_k$$

Triple scalar product

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = \epsilon_{ijk} a_i b_j c_k$$

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = 0 \Leftrightarrow a, b, c \text{ are coplanar.}$$

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c}$$

Triple vector product

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}$$

2 Gradient, divergence and curl

Gradient & directional derivative

$$\nabla\phi = \frac{\partial\phi}{\partial x}\widehat{\boldsymbol{i}} + \frac{\partial\phi}{\partial y}\widehat{\boldsymbol{j}} + \frac{\partial\phi}{\partial z}\widehat{\boldsymbol{k}}, \frac{\partial\phi}{\partial \boldsymbol{s}} = \widehat{\boldsymbol{s}}.\nabla\phi$$

note that $\widehat{\boldsymbol{s}}$ here must be unit vector.
Formal definition of gradient, S is a surface, τ is the volume of region enclosed by S:

$$\nabla\phi = \lim_{\tau\to 0}\frac{1}{\tau}\int_S \widehat{\boldsymbol{n}}\,\phi\,dS$$

Laplace

$$\nabla^2\phi := \nabla.\nabla\phi = \frac{\partial^2}{\partial x_i^2}\phi$$

$$[\nabla^2\boldsymbol{A}]_i = \frac{\partial^2}{\partial x_j^2}\boldsymbol{A_i}$$

Finding tangent plane at point P:

1. Change the equation of surface to $\phi(x, y, z) = c$ where c is a constant. Try to use addition instead of multiplication so that ϕ is more linear.
2. $$\left.\frac{\partial\phi}{\partial x}\right|_P (x - x_P) + \left.\frac{\partial\phi}{\partial y}\right|_P (y - y_P) + \left.\frac{\partial\phi}{\partial z}\right|_P (z - z_P)$$
   is the equation of tangent plane.

For a vector field A

$$div(\boldsymbol{A}) = \nabla.\boldsymbol{A}, \quad curl(\boldsymbol{A}) = \nabla \times \boldsymbol{A}$$

Note: 2D curl is defined assuming it is on $x - y$ plane of 3-D space, so

$$curl(\boldsymbol{A}) = \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\widehat{\boldsymbol{k}}$$

Strict definitions:

$$div(\boldsymbol{A}) = \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\widehat{\boldsymbol{n}}.\boldsymbol{A})\,dS \qquad curl(\boldsymbol{A}) = \lim_{\tau\to 0}\frac{1}{\tau}\int_S (\widehat{\boldsymbol{n}} \times \boldsymbol{A})\,dS$$

Properties:

- $\nabla(\phi_1 + \phi_2) = \nabla\phi_1 + \nabla\phi_2$
- $div(\boldsymbol{A} + \boldsymbol{B}) = div(\boldsymbol{A}) + div(\boldsymbol{B}),\ curl(\boldsymbol{A} + \boldsymbol{B}) = curl(\boldsymbol{A}) + curl(\boldsymbol{B})$
- $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$
- $div(\phi\boldsymbol{A}) = \phi div(\boldsymbol{A}) + \nabla\phi.\boldsymbol{A},\ curl(\phi\boldsymbol{A}) = \phi curl(\boldsymbol{A}) + \nabla\phi \times \boldsymbol{A}$
- $div(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B}.curl(\boldsymbol{A}) - \boldsymbol{A}.curl(\boldsymbol{B})$
- $curl(\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B}.\nabla)\boldsymbol{A} - (\boldsymbol{A}.\nabla)\boldsymbol{B} + div(\boldsymbol{B})\boldsymbol{A} - div(\boldsymbol{A})\boldsymbol{B}$
- $\nabla(\boldsymbol{A}.\boldsymbol{B}) = (\boldsymbol{B}.\nabla)\boldsymbol{A} + (\boldsymbol{A}.\nabla)\boldsymbol{B} + \boldsymbol{B} \times curl(\boldsymbol{A}) + \boldsymbol{A} \times curl(\boldsymbol{B})$
- $curl(\nabla\phi) = 0$
- $div(curl(\boldsymbol{A})) = 0$
- $curl(curl(\boldsymbol{A})) = \nabla(div(\boldsymbol{A})) - \nabla^2\boldsymbol{A}$

Irrotational vector field: $curl(\boldsymbol{A}) = 0$
solenoid vector field: $div(\boldsymbol{A}) = 0$

3 Integrals

3.1 Line Integral

$$\int_\gamma f\ ds := \lim_{N\to\infty, \max(\delta s_n)\to 0} \sum_{n=1}^{N} f_n \delta s_n$$

where δs_n are lengths of segments on the path γ(must be smooth or piece-wise smooth). Function f is usually a scalar field, if it is vector field, integral is calculated element-wise.

Length of path γ can be calculated using integration $l = \int_\gamma\ ds$. If γ is $y = y(x)$, then the arc length between $(a, f(a)), (b, f(b))$ is

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

ds - Change of arc length, $\widehat{\boldsymbol{t}} = \frac{d\boldsymbol{r}}{ds}$ - tangent vector, $d\boldsymbol{r} = \widehat{\boldsymbol{t}}ds$ - path element. So we have another line integral defined for vector field $\boldsymbol{F}$:

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{r} := \int_\gamma (\boldsymbol{F} \cdot \hat{t})\ ds$$

Circulation: when the path is closed (same beginning and end point) denoted as $\oint_\gamma \boldsymbol{F}.d\boldsymbol{r}$
Field $\boldsymbol{F}$ is conservative (circulation around any closed path γ is 0) $\Leftrightarrow \boldsymbol{F} = \nabla\phi$ for some function ϕ. (ϕ is called potential) $\Leftrightarrow curl(\boldsymbol{F}) = 0$.
In this case, given any path γ joining point A, B, we have

$$\int_\gamma f\ ds = \phi(B) - \phi(A)$$

Steps to find potential ϕ:

- $\frac{\partial\phi}{\partial x} = F_1$ so integrate F_1 w.r.t. x to find ϕ. The integration constant should be $C(y, z)$ (a function depending on y, z only)
- Then differentiation ϕ w.r.t y and compare with F_2 to solve for $C(y, z)$. It should be of the form $g(y, z) + C(z)$.
- Finally differentiation ϕ w.r.t. z. You may leave the constant c there as there is no way to ger rid of it.

Another practical result is if there is a vector field $\boldsymbol{B}$ s.t. $curl(\boldsymbol{B}) = \boldsymbol{A}$, then $div(\boldsymbol{A}) = 0$

Evaluation Line Integral
With parameterisation $x = x(t), y = y(t), z = z(t) t_0 \le t \le t_1$, we have

$$\int_\gamma \boldsymbol{F}.d\boldsymbol{r} = \int_{t_0}^{t_1} \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)\ dt$$

3.2 Surface Integral

Some definitions: Convex surface: crossed by any straight line at most twice.
Closed surface: can divide the space into two non-connected regions. (interior & exterior)
Simply Connected Region: Any curve inside the region can be shrunk to a point (any point) without leaving the region. Otherwise the region is called multiply-connected.

Similar to line integral, we define surface integral on surface S as below

$$\int_S f \; dS := \lim_{N\to\infty, \max(\delta S_n)\to 0} \sum_{n=1}^{N} f_n \delta S_n$$

dS - area element (infinite small area of surface), $d\boldsymbol{S} = \widehat{\boldsymbol{n}} dS$ - vector areal element ($\widehat{\boldsymbol{n}}$ is normal to dS). Note area of S can be calculated using $\int_S 1 \; dS$.

Evaluating surface integral

$$I = \int_S f(P) dS$$

where P is a general point on surface S.
1. (From our notes) Projection
Choose a plane to project to (say the x, y plane) and find projection area Σ and normal vector $\widehat{\boldsymbol{m}} = \widehat{\boldsymbol{k}}$ to projected plane. Change variables in $f(x, y, z)$ if necessary to get rid of z (using the equation of surface S, NOT equation of Σ)

$$I = \int_\Sigma f(P) \frac{dx \, dy}{|\widehat{\boldsymbol{n}}.\widehat{\boldsymbol{k}}|}$$

where $\widehat{\boldsymbol{n}}$ is normal to S and it may depend on x, y, z. If the surface is given by $g(x, y, z) = c$ for some constant c, then $\widehat{\boldsymbol{n}} = \nabla g / |\nabla g|$(ALWAYS remember to check $\widehat{\boldsymbol{n}}$ is pointing to exterior of region). Projection to other planes can be done similarly.

2. Parameterisation
Any surface can be parameterised by two parameters say $\boldsymbol{r} = \boldsymbol{r}(\theta, \phi)$.
For scalar function f:

$$\int_S f(\boldsymbol{r}) dS = \iint_A f(\boldsymbol{r}(\theta, \phi)) \left| \frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \phi} \right| d\phi \, d\theta$$

where A is the corresponding area on $\theta - \phi$ plane.
For vector field $\boldsymbol{F}$:

$$\int_S \boldsymbol{F}(\boldsymbol{r}).d\boldsymbol{S} = \iint_A (\boldsymbol{F}(\boldsymbol{r}(\theta, \phi)).\widehat{\boldsymbol{n}}) \left| \frac{\partial \boldsymbol{r}}{\partial \theta} \times \frac{\partial \boldsymbol{r}}{\partial \phi} \right| d\phi \, d\theta$$

where $\widehat{\boldsymbol{n}}$ is unit normal to surface S.

3.3 Volume Integral

If τ is a region in 3D space, $\int_\tau f d\tau$ (defined similar to above) is volume integral where $d\tau = dx\,dy\,dz$ is volume element.

3.4 Some useful Integrals

Given curve $y = y(x), x \in [a, b]$, the arc length is

$$\int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

surface area of surface generated by revolving $y = y(x)$ between $x = a, b$ about x-axis is:

$$2\pi \int_a^b y\sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

for revolution about y-aixs:

$$2\pi \int_a^b x\sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

volume of revolution about x-axis is

$$\int_a^b \pi y^2\ dx$$

volume of revolution about y-axis is

$$\int_a^b \pi x^2\ dy$$

4 Green's theorem, divergence theorem and Stokes theorem

Green's theorem gives an important connection between line integral and surface integral. If R is a closed plane region bounded by simple closed convex curve C (anti-clockwise) and $\boldsymbol{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$:

$$\oint_C \boldsymbol{F}.d\boldsymbol{r} = \oint_C (P\ dx + Q\ dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dS$$

Green's theorem is proved directly for simply connected convex R but can be generalised to non-convex/multiply connected regions using suitable breakdown. For example, given a region R is bounded by an interior boundary C_0(clockwise) and an exterior boundary C_1(anti-clockwise):

$$\oint_{C_1} \boldsymbol{F}.d\boldsymbol{r} - \oint_{C_0} \boldsymbol{F}.d\boldsymbol{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dS$$

If $\boldsymbol{F}$ is undefined at some point P in region R, you should dig a circle with radius r around point P, calculate the integral and then let $r \to 0$.

Area of a region If a closed region has boundary curve C, then its area is given by

$$\frac{1}{2} \oint_C (x\,dy - y\,dx)$$

This integral can be calculated by parametrisation.

Flux

Flux of a surface S and vector field A is

$$\int_S \boldsymbol{A}.\widehat{\boldsymbol{n}} dS$$

where $\widehat{\boldsymbol{n}}$ is normal to surface S.

Divergence theorem

For closed convex surface S with normal $\widehat{\boldsymbol{n}}$ and region τ enclosed by S that is simply connected. $\boldsymbol{A}$ is a vector field with continuous derivative on τ:

$$\int_S \boldsymbol{A}.d\boldsymbol{S} = \int_S \boldsymbol{A}.\widehat{\boldsymbol{n}}\, dS = \int_\tau div(\boldsymbol{A}) d\tau$$

This can be generalised to non-convex surfaces and multiply-connected regions just like Green's theorem.

You can use divergence theorem to evaluate $\int_\tau f(\boldsymbol{r})\, d\tau$ but you have to first find function $\boldsymbol{A}(\boldsymbol{r})$ s.t. $div\boldsymbol{A} = f$. Try to pick $\boldsymbol{A}$ as simple as possible (say, $\boldsymbol{A} = x\boldsymbol{i}$)

Green's identities: (ϕ, ψ are scalar functions, other symbols have the same meaning as in Divergence theorem)

$$\iint_S \phi \frac{\partial \psi}{\partial \boldsymbol{n}}\, dS = \int_\tau \phi \nabla^2 \psi + \nabla\phi . \nabla\psi\, d\tau$$

$$\iint_S \phi \frac{\partial \psi}{\partial \boldsymbol{n}} - \psi \frac{\partial \phi}{\partial \boldsymbol{n}}\, dS = \int_\tau \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right)\, d\tau$$

Generalised integration by part: (R is region enclosed by curve C)

$$\iint_R \phi \nabla^2 \psi\, dx\, dy = \oint_C \phi \frac{\partial \psi}{\partial \boldsymbol{n}} dS - \iint_R \nabla\phi . \nabla\psi\, dx\, dy$$

Gauss's flux theorem: S is closed surface with normal $\widehat{\boldsymbol{n}}$ and O means origin.

$$\int_S \frac{\widehat{\boldsymbol{n}}.\boldsymbol{r}}{|\boldsymbol{r}|^3}\, dS = \begin{cases} 0 & \text{if O is exterior} \\ 4\pi & \text{if O is interior} \end{cases}$$

Stokes theorem S is open surface with boundary curve γ. $\boldsymbol{A}$ is continuously differentiable on S,

$$\oint_\gamma \boldsymbol{A}.d\boldsymbol{r} = \int_S curl(\boldsymbol{A}).\widehat{\boldsymbol{n}}\, dS$$

Note LHS is independent of the surface S chosen.

5 Curvillinear system

For coordinate system $\boldsymbol{x} = (x_1, x_2, x_3)$, transformation to new coordinate system $\boldsymbol{u} = (u_1, u_2, u_3)$ is possible if

$$\det(J(u_x)) = \det(\begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix}) \neq 0$$

Also remember that $J(x_u)J(u_x) = I$, so $\det(J(x_u))\det(J(u_x)) = 1$. This means if $\det(J(x_u)) \neq 0$, the above condition is satisfied.

For a given point $P = (x_1, x_2, x_3)$, by finding $\bigtriangledown u_i(P)$ and normalising (change to a unit vector). We have $\widehat{\boldsymbol{u}}_i$, the unit normal vector to the surface $u_i = u_i(P)$ (where $u_i(P)$ is a constant). So

$$\widehat{\boldsymbol{u}}_i = \frac{\bigtriangledown u_i(P)}{|\bigtriangledown u_i(P)|}$$

system is orthogonal of $\widehat{\boldsymbol{u}}_i$ are mutually orthogonal.

In general, tangential vectors(to the line where only u_i varies) $\widehat{e_i}$ are used more. In Cartesian coordinates, $\widehat{\boldsymbol{e}}_i$ are $\hat{\imath}, \hat{\jmath}, \hat{k}$. To find $\widehat{\boldsymbol{e}}_i$, use this formulae:

$$\frac{\partial \boldsymbol{r}}{\partial u_i} = h_i \widehat{e_i}, \text{ where } h_i = \left|\frac{\partial \boldsymbol{r}}{\partial u_i}\right|, \boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

h_i are called length scale. Showing $\widehat{\boldsymbol{e}}_i$ are mutually orthogonal also proves orthogonality of the coordinate system.

In orthogonal system:

$$\widehat{\boldsymbol{e}}_i = \widehat{\boldsymbol{u}}_i$$

Path element

$$d\boldsymbol{r} = \sum h_i du_i \widehat{e_i}$$

(for orthogonal system)

$$(ds)^2 = (d\boldsymbol{r}.d\boldsymbol{r}) = \sum h_i^2 du_i^2$$

Volume element

$$dV = h_1 h_2 h_3 du_1 du_2 du_3$$

Area element (on the surface where u_1 is constant)

$$dS = h_2 h_3 du_2 du_3$$

Gradient

$$\bigtriangledown = \sum \frac{1}{h_i} \widehat{e_i} \frac{\partial}{\partial u_i}$$

from this we have $\widehat{\boldsymbol{e}}_i = \nabla \boldsymbol{u}_i h_i$

Divergence

$$div(\boldsymbol{A}) = \frac{1}{h_1 h_2 h_3} \left\{ \sum \frac{\partial}{\partial u_i} (A_i h_j h_k) \right\}$$

Curl

$$curl(\boldsymbol{A}) = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \widehat{e_1} & h_2 \widehat{e_2} & h_3 \widehat{e_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Laplacian

$$\bigtriangledown^2 \Phi = \frac{1}{h_1 h_2 h_3} \left\{ \sum \frac{\partial}{\partial u_i} (\frac{h_j h_k}{h_i} \frac{\partial \Phi}{\partial u_i}) \right\}$$

5.1 Cartesian, Cylindrical and Spherical

Three systems are all orthogonal.

Definition of Cylindrical coordinates:

$$x = r \cos \theta, y = r \sin \theta, z = z$$

Definition of Spherical coordinates:

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

where $r \geq 0$ is radius, $\theta \in [0, \pi]$ is rotation above x-y plane and $\phi \in [0, 2\pi]$ is rotation with in x-y plane.

Tables below are ordered by (x, y, z), (r, θ, z), (r, θ, ϕ)

Coordinate system	h_i	$\widehat{e_i}$
Cartesian	1,1,1	$\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$
Cylindrical	1	$\cos\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{j}}$
	r	$-\sin\theta\hat{\boldsymbol{i}} + \cos\theta\hat{\boldsymbol{j}}$
	1	$\hat{\boldsymbol{k}}$
Spherical	1	$\sin\theta\cos\phi\hat{\boldsymbol{i}} + \sin\theta\sin\phi\hat{\boldsymbol{j}} + \cos\theta\hat{\boldsymbol{k}}$
	r	$\cos\theta\cos\phi\hat{\boldsymbol{i}} + \cos\theta\sin\phi\hat{\boldsymbol{j}} - \sin\theta\hat{\boldsymbol{k}}$
	$r\sin\theta$	$-\sin\theta\sin\phi\hat{\boldsymbol{i}} + \sin\theta\cos\phi\hat{\boldsymbol{j}}$

Table 1: Table of tangential vectors and length scales

Coordinate system	gradient	div	Laplacian
Cartesian	$\hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}$	$\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$	$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$
Cylindrical	$\hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}$	$\frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r}\frac{\partial A_2}{\partial \theta} + \frac{\partial A_3}{\partial z}$	$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta} + \frac{\partial^2 \Phi}{\partial z}$
Spherical	$\hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial \phi}$	$\frac{1}{r^2\sin\theta}\{\frac{\partial}{\partial r}(r^2\sin\theta A_1) + \frac{\partial}{\partial \theta}(r\sin\theta A_2) + \frac{\partial}{\partial \phi}(rA_3)\}$	$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi}$

Table 2: Table of div, gradient and Laplacian for vector valued function $\boldsymbol{A}$ or scalar valued function Φ

5.2 Change of variables

If we parameterise surface S by (u_1, u_2),

$$dS = |J| du_1 du_2, \text{ where } J = \frac{\partial \boldsymbol{r}}{\partial u_1} \times \frac{\partial \boldsymbol{r}}{\partial u_2}$$

If S is on x-y plane

$$J = J(\boldsymbol{x_u}), \text{ where } \boldsymbol{x} = (x, y)$$

If S is $z = f(x, y)$ and we parameterise using x, y:

$$|J| = \sqrt{1 + |\bigtriangledown f|^2}$$

So surface area of any surface S is

$$\int_\Sigma \sqrt{1 + |\bigtriangledown f|^2} \, dxdy$$

where Σ is projection of S onto x-y plane.

6 Calculus of Variation

Vanishing lemma If g is a continuous function s.t. for every smooth function $\eta(x)$ with $\eta(x_1) = \eta(x_2) = 0$,

$$\int_{x_1}^{x_2} g(x)\eta(x) \, dx = 0$$

then $g \equiv 0$

Target of this section: find function $y = y(x)$ that minimises the integral

$$I := \int_{x_1}^{x_2} L(x, y, y') \, dx$$

L is a functional.

1D E-L equation and special cases
Full equation

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left\{\frac{\partial L}{\partial y'}\right\}$$

L independent of y

$$\frac{\partial L}{\partial y'} = \text{constant}$$

L independent of y'

$$\frac{\partial L}{\partial y} = 0$$

L independent of x:

$$L - y' \frac{\partial L}{\partial y'} = \text{constant}$$

Finding extrema of $I = \int_{x_1}^{x_2} L(x, y, y')\, dx$ where L is a functional

1. Use E-L equation to obtain a differential equation.
2. Solve it, remember to include the integration constants.
3. Check that $I''(0) > 0$ or $I''(0) < 0$
4. use boundary conditions to determine integration constants

Multivariate E-L If we are finding extrema of

$$I = \int_{t_1}^{t_2} L(t, x_1(t), x_1'(t), ..., x_n(t), x_n'(t))\, dt$$

E-L equation becomes a set of equations

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt} \left\{ \frac{\partial L}{\partial x_i'} \right\}$$

With constraint

$$J = \int_{t_1}^{t_2} g(t, x_1(t), x_1'(t), ..., x_n(t), x_n'(t))\, dt = \text{constant}$$

E-L becomes

$$\frac{\partial}{\partial x_i}(L + \lambda g) - \frac{d}{dt} \frac{\partial}{\partial x_i'}(L + \lambda g) = 0$$

Leave λ there and after finishing regular steps, plug in integral J to determine λ. However, the order you determine the constants is not restricted, you may as well find λ in terms of an integration constant c first, then solve c using the restriction.

E-L for surface integral The function $f(\boldsymbol{r})$ where $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ that maximises

$$I = \int_R L(\boldsymbol{r}, f(\boldsymbol{r}), \nabla f(\boldsymbol{r}))\, dx\, dy$$

can be found by solving the following equation

$$\frac{\partial L}{\partial f} = div(\nabla_{\nabla f} L) = div(\hat{\boldsymbol{\imath}} \frac{\partial L}{\partial f_x} + \hat{\boldsymbol{\jmath}} \frac{\partial L}{\partial f_y})$$

where f_x, f_y are partial derivatives of f.

Isoperimetric inequality For any simple curve with area A and perimeter l,

$$4\pi A \leq l^2$$

equality holds iff curve is circle.

7 Integration Techniques

Ways of showing $f(x) \equiv 0$ using integration:

1. Use vanishing lemma
2. Prove $\int_{x_1}^{x_2} f^2 \, dx = 0$. Or if you know $f(x) \geq 0$, prove $\int_{x_1}^{x_2} f \, dx = 0$

Some inverse trig integrals

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(\frac{x}{a}) + c$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}(\frac{|x|}{a}) + c$$

Some inverse hyperbolic integrals

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}(\frac{x}{a}) + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}(\frac{x}{a}) + c$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + c \quad (x^2 < a^2)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\coth^{-1}(\frac{x}{a}) + c \quad (x^2 > a^2)$$

Derivatives of trigs:

$$(\sin(x))' = \cos(x) \quad (\cos(x))' = -\sin(x) \quad (\tan(x))' = \sec^2(x)$$

$$(\csc(x))' = -\cot(x)\csc(x) \quad (\sec(x))' = \tan(x)\sec(x) \quad (\cot(x))' = -\csc^2(x)$$

Reduction formulas:

$$I_n = \int \cos^n(x) \, dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}I_{n-2}$$

$$I_n = \int \sin^n(x) \, dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}I_{n-2}$$

$$I_n = \int x^n e^{ax} \, dx = \frac{x^n e^a x}{a} - \frac{1}{n}I_{n-1}$$

Product to sum rules for trigs:

$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin(x)\sin(y) = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$$