Summation

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Summation for the many who are not well acquainted with it, as old friends are with each other, do not much enjoy the sight of the, as some might say, horrendous notation given to it. I'm talking about the notorious *sigma* notation Σ, which is meant to—if learned and used correctly—make handling ugly sums a walk in the park.

Such people often invoke the "path of least resistence" arguement by which they say there is no need to introduce new notation maintaining the existing ellipsis notation, i.e. $a_1 + a_2 + \ldots + a_n$, works just fine since it is less mechanical, easier to manipulate, and therefore faster to use. The point being: if it's not broke, and it gets the job done, why fix it?

It is not the author's intention to turn this observation into a debate. Instead, the intention of this article is to make refinements in understanding the notation and why it exists (other than as a shortform of the horizontally longer ellipsis representation), as well as demonstrating the power it holds (most notably with finding closed forms for multiple sums).

1 Sigma notation

We will be working with what is called the generalized sigma notation. It turns out that there are two varieties which are worth introducing since they will both help a great deal in manipulating sums. The first provides us with a set-theoretic perspective, and the second handles the notation with the idea of properties.

So to begin, the following notation:

$$\sum_{k \in I} a_k$$

is simply an abbreviation for the sum of all terms a_k such that k is an *integer* that belongs $\in$ to the set I, where I is called the *index set* and is a finite subset of the integers $\mathbb{Z}$. Just to clarify, the Σ part of the sigma notation is called the *summand*.

When forming the index set, one may simply collect integers together, even if it's for no apparent reason. But one can also collect the necessary integers together in a systematic way by means of some known property $P(k)$. In that case, as in the book "Concrete Mathematics" for which this article is inspired (and is otherwise referenced as the bracketed index [1]), we write:

$$\sum_{P(k)} a_k$$

and add terms a_k to our sum whenever k is an integer satisfying the given property $P(k)$. The mindful reader at this point might ask what happens when no integers satisfy the property $P(k)$? Well, since we aren't adding anything to the sum, we will again borrow from [1] and define such a structure to be *zero*.

The same thing happens when we sum terms over an empty index set, so we should name this peculiarity the *empty sum*.

For concreteness, the following example is provided. Let $a_k = 1$ and $I = \{-2, -1, 0, 1, 2, 3\}$, then

$$\sum_{k \in I} 1 = 1 + 1 + 1 + 1 + 1 + 1 = \sum_{-2 \leq k \leq 3} 1$$

We now take this idea a bit further to introduce double sums. These types of sums are actually pretty simple, they arise when we consider the possibility of the term of a series $\sum_{j \in H} b_j$ being equal to a sum itself. With this comes a new consideration though, the term b_j as a sum will have its own term $a_{j,k}$ and its own index set I, both of which depend on k, but *can* also depend on j.

$$\sum_{j \in H} \sum_{k \in I_j} a_{j,k} \quad \text{or} \quad \sum_{Q(j)} \sum_{P(j,k)} a_{j,k}$$

Noting one more thing before we move on, I will actually use the better known delimited form of the sigma notation in this article, ie.

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \ldots + a_{n-1} + a_n$$

but only to *display* equations in their finished form.

1.1 Manipulation

The goal of our notation now that we have it, is to be able to manipulate the complicated structure of summation in a concise and easy way. With this in mind, let's see what we can do.

We begin actually with double sums. The essential thing to realize here, is that a double sum is a sum of sums, which in the end is just another sum (say that ten times fast). Once this is thought out, one would expect that there is a reasonable *single summand* representation of certain (infact many) double sums. This of course is true.

We approach this idea, of a single summand representation of a double sum, by extending the original definitions, by which we extend the term a_j to $a_{j,k}$ and the index set I of *integers* to an index set I of *ordered pairs of integers* (or a property $P(k)$ to a property $P(j,k)$). Here, an example of this alternate approach would be of some help. How about letting $a_{j,k} = j + k$ and

$$I = \{(0,0), (0,1), (0,2), (1,1), (1,2), (2,2)\}$$

so then

$$\sum_{(j,k) \in I} (j+k) = \sum_{0 \leq j \leq k \leq 2} (j+k)$$
$$= (0+0) + (0+1) + (0+2) + (1+1) + (1+2) + (2+2)$$

See! easy as π.

With this extension in hand, we are led to:

$$\begin{aligned} \sum_{(m \leq j \leq k \leq n)} a_{j,k} &= \sum_{(m \leq k \leq n)} \sum_{(m \leq j \leq k)} a_{j,k} \\ &= \sum_{(m \leq j \leq n)} \sum_{(j \leq k \leq n)} a_{j,k} \end{aligned}$$

for some integer constants m and n. This handy little 3-way equation is called the *summand switch law*. There are infact broader summand switch laws, but for our applications, the one presented here is sufficient.

The truth of this law isn't exactly "obvious," so it helps if you look at it this way: since all three formulae are sums of the terms $a_{j,k}$, we only have to show that whenever $a_{j,k}$ belongs to any of the above sums for particular values j, k they also belong to the two others (the same number of times of course).

We can actually "streamline" this proof, not by showing that for each sum containing any given term the other two sums contain that given term, but by showing that the properties cummulated within each sum, for each sum (that need to be satisfied in order for the term to belong), are equivalent. Let's see, how do we do this?... Ah, that's easy! It is first worthwhile to note that in general

$$(a \le b \le c)$$

is shortform for

$$(a \le b) \ and \ (b \le c)$$

With this we observe that

$$(m \le k \le n) \ and \ (m \le j \le k)$$

is true if and only if

$$(m \le j) \ and \ (j \le k) \ and \ (k \le n)$$

is true, or equivalently

$$(m \le j \le k \le n)$$

which working backwards is the same as

$$(m \le j \le n) \ and \ (j \le k \le n)$$

A symmetric arguement may be given in the case of the properties being false to show their complete, and total logical equivalence.

Applying this now proven law to our above example we see that

$$\begin{aligned}
& \sum_{0 \le j \le k \le 2} (j+k) \\
= & \sum_{(0 \le k \le 2)} \sum_{(0 \le j \le k)} (j+k) \\
= & \ [\ (0+0)\ ] \ + \ [\ (0+1) + (1+1)\ ] \\
& + [\ (0+2) + (1+2) + (2+2)\ ] \\
= & \ [\ (0+0) + (0+1) + (0+2)\ ] \\
& + [\ (1+1) + (1+2)\ ] \ + \ [\ (2+2)\ ] \\
= & \sum_{(0 \le j \le 2)} \sum_{(j \le k \le 2)} (j+k)
\end{aligned}$$

Moving on with our manipulation laws, it turns out that there are three major ones to know when looking at single summand sums. The first two are straight-forward, so I won't spend time with any explanations, they work out to be:

$$\sum_{k \in I} (a_k + b_k) = \sum_{k \in I} a_k + \sum_{k \in I} b_k \tag{1}$$

$$\sum_{k \in I} c a_k = c \sum_{k \in I} a_k \tag{2}$$

Both of these equations have names, (1) is called the *associative law*, and (2) is called the *distributive law*. The use of the phrase "single summand sums" (above) should be noted since it implies that the sums in these equations may infact be double sums in disguise. This works as long as we consider the possibility of the index variable k being shortform for $k = (k_1, k_2)$. The truth of these extended laws may be verified with the set theoretic arguements applied to the index set.

Hmmm... what could it possibly mean if I said there were three laws and only included two as being straight-forward? Yes! that's right, the third law, called the *commutative law*, will need some clarification. It takes the form:

$$\sum_{k\in I} a_k \quad = \quad \sum_{p(k)\in I} a_{p(k)} \tag{3}$$

Wait a minute though, isn't this just substituting one variable for another? Nope! consider:

$$\sum_{0\le k\le 5} a_k \stackrel{?}{=} \sum_{0\le k^2\le 5} a_{k^2}$$

the former (on the left) equals

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5$$

while the latter (on the right) is equal to

$$a_4 + a_1 + a_0 + a_1 + a_4$$

The point is that not just any substitution will do, there needs to be a one-to-one correspondence, i.e. for every integer (or pair of integers) k_1 belonging to the index set of summation, there is *exactly* one integer (or pair) k_2 such that $k_1 = p(k_2)$, otherwise you could add the same term more than once or not at all. A good example of a "proper" substitution, is setting $p(k) = n - k$ for the sum

$$\sum_{0\le k\le n} a_k$$

since for each $k_1 \in I$ (the index set) there exists only one k_2 such that $k_1 = n - k_2$, or equivalently, there is only one corresponding $n - k_2 \in I$. Substituting this in, we have

$$\sum_{0\le k_1\le n} a_{k_1} = \sum_{0\le n-k_2\le n} a_{n-k_2}$$

So to restate the obvious, the term a_{n-k_2} is added to the sum if and only if

$$(0 \le n - k_2 \le n) \text{ is true,}$$

now looking more closely at this particular property

$$(0 \le n - k_2 \le n)$$

we see that

$$(0 \le n - k_2 \le n) \iff (0 \le k_2 \le n)$$

which follows from again using the fact that

$$(0 \le n - k_2 \le n)$$

is shortform for

$$(0 \le n - k_2) \ \mathit{and} \ (n - k_2 \le n)$$

but also

$$(0 \le n - k_2) \iff (k_2 \le n)$$
$$(n - k_2 \le n) \iff (0 \le k_2)$$

What all of this then means is that

$$\sum_{0\le k_1\le n} a_{k_1} = \sum_{0\le k_2\le n} a_{n-k_2}$$

or equivalently:

$$\sum_{0\le k\le n} a_k = \sum_{0\le k\le n} a_{n-k}$$

The important thing to remember in this chain of reasoning is the *method of derivation* rather than the specific result itself, because with it you can derive many other specific commutative laws, for example setting $p(k) = k + c$ will always work, as well as using the extensions $p(j,k) = (j + c, k + c)$ or $p(j,k) = (n - j, n - k)$ (for some integer constants c and n). Having this said, I will from now on entirely skip any other similar derivation when proving an equation.

2 Practice

We are finally ready to demonstrate the value of our now furnished sigma notation. It should be noted that the two famous series

$$\sum_{1\le k\le n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{0\le k\le n} x^k = \frac{1-x^{n+1}}{1-x}$$

are taken for granted since proofs can be supplied very easily from very many sources.

Let's start off nice and simple with

$$\begin{aligned}
&\sum_{0\le j\le k\le n} 1 \\
&= \sum_{(0\le k\le n)} \sum_{(0\le j\le k)} 1 \\
&= \sum_{(0\le k\le n)} (k+1) \\
&= \sum_{(0\le k\le n)} k + \sum_{(0\le k\le n)} 1 \\
&= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

As our second example, let us find a closed form for

$$\sum_{0\le j\le k\le n} (j+k)$$

We could solve this old friend of ours[1] one sum at a time (as in the previous example), but there is a more clever method.

We start with the permutation on $\mathbb{Z}^2$

$$p(j,k) = (n-j, n-k)$$

which after implementing the extended commutative law gives us

$$\begin{aligned}
&\sum_{0\le n-j\le n-k\le n} [(n-j)+(n-k)] \\
&= \sum_{0\le k\le j\le n} [2n-(j+k)] \\
&= 2n\sum_{0\le k\le j\le n} 1 - \sum_{0\le k\le j\le n} (j+k)
\end{aligned}$$

From here we observe, for the purpose of substitution, that

$$\begin{aligned}
\sum_{0\le k\le j\le n} (j+k) &= \sum_{0\le k\le j\le n} (k+j) \\
&= \sum_{0\le j\le k\le n} (j+k)
\end{aligned}$$

leading us (with a little bit of old-fashioned algebra) to

[1] As far as articles are concerned.

$$\begin{aligned}\sum_{0\le j\le k\le n}(j+k) &= n\sum_{0\le k\le j\le n}1\\ &= \frac{n(n+1)(n+2)}{2}\end{aligned}$$

Interestingly, this solution may be applied towards the well known series $\sum_{k=1}^{n}k^2$:

$$\begin{aligned}&\frac{n(n+1)(n+2)}{2}\\ =\;& \sum_{0\le j\le k\le n}(j+k)\\ =\;& \sum_{(0\le k\le n)}\sum_{(0\le j\le k)}(j+k)\\ =\;& \sum_{(0\le k\le n)}\sum_{(0\le j\le k)}j+\sum_{(0\le k\le n)}\sum_{(0\le j\le k)}k\\ =\;& \sum_{(0\le k\le n)}\frac{k(k+1)}{2}+\sum_{(0\le k\le n)}k(k+1)\\ =\;& \frac{3}{2}\sum_{(0\le k\le n)}(k^2+k)\\ =\;& \frac{3}{2}\sum_{(0\le k\le n)}k^2+\frac{3n(n+1)}{4}\end{aligned}$$

$$\Longrightarrow \sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

For our next example, let's try something a little different:

$$S_n=\sum_{0\le k\le n}(k+1)x^k$$

A good start is to relegate and replace this series with its alter ego:

$$\sum_{0\le j\le k\le n}x^k$$

this claim holding of course because

$$S_n=\sum_{(0\le k\le n)}\sum_{(0\le j\le k)}x^k$$

Venturing towards something new, we will partition this sum into two sums in two ways (why you ask?... be patient). We do this by partitioning the terms of the sum, effectively dividing the sum into two. The first division occurs when we use the fact that the index variable k of the general term x^k will either be $k=n$ or $k<n$:

$$\begin{aligned}\sum_{0\le j\le k\le n}x^k &= \sum_{0\le j\le k=n}x^n+\sum_{0\le j\le k<n}x^k\\ &= x^n\sum_{0\le j\le n}1+\sum_{0\le j\le k\le n-1}x^k\end{aligned}$$

the second way to partition the terms is to realize that the index variable j will either be $0 = j$ or $0 < j$ so that

$$\begin{aligned}\sum_{0\le j\le k\le n} x^k &= \sum_{0=j\le k\le n} x^k + \sum_{0<j\le k\le n} x^k \\ &= \sum_{0\le k\le n} x^k + \sum_{1\le j\le k\le n} x^k\end{aligned}$$

We then convert

$$\sum_{1\le j\le k\le n} x^k \quad \text{to} \quad \sum_{0\le j\le k\le n-1} x^k$$

with

$$p(j,k) = (j+1, k+1)$$

which is a permutation on $\mathbb{Z}^2$. This is carried out as follows:

$$\begin{aligned}\sum_{1\le j\le k\le n} x^k &= \sum_{1\le j+1\le k+1\le n} x^{k+1} \\ &= x\sum_{0\le j\le k\le n-1} x^k\end{aligned}$$

which after substituting back in and equating the two derived sides, leaves us with

$$x^n\sum_{j=0}^{n} 1 + S_{n-1} = \sum_{k=0}^{n} x^k + xS_{n-1}$$

reducing to

$$\sum_{k=0}^{n}(k+1)x^k = \frac{1-x^{n+2}}{(1-x)^2} - \frac{(n+2)x^{n+1}}{1-x}$$

As our last example, we have:

$$T_n = \sum_{0\le j\le k\le n} x^{j+k}$$

which will tell us as to whether or not our adventurous partitioning method of the last example was just luck or if it has some worth. Repeating the process provides us with the equation

$$x^n\sum_{j=0}^{n} x^j + T_{n-1} = \sum_{k=0}^{n} x^k + x^2T_{n-1}$$

so it appears this method is of use after all.

$$\sum_{0\le j\le k\le n} x^{j+k} = \frac{1-x^{n+1}}{1-x}\cdot\frac{1-x^{n+2}}{1-x^2}$$

3 Challenge Problems

Find closed forms for the following:

1.
$$\sum_{k=-3}^{n} [(k+4)^2 + (k-2) + 3]$$

2.
$$\sum_{k=2}^{n} (x+1)^{k+1}$$

3.
$$\underbrace{\sqrt{2\sqrt{2\sqrt{2\sqrt{\ldots\sqrt{2}}}}}}_{n^{\#} \text{ of times}}$$

4.
$$\sum_{k=1}^{n} kx^k$$

5.
$$\sum_{k=1}^{n} k^3$$

   hint: Commute with $p(k) = n - k$, and expand the term.

6.
$$\sum_{0 \le j,k \le n} 1$$

   hint: $(0 \le j, k \le n)$
   $\iff (0 \le j \le n)$ *and* $(0 \le k \le n)$

7.
$$\sum_{0 \le j \le k \le \ell \le n} x^{j+k+\ell}$$

   hint: Use the partitioning method.

8. If H_n is notation such that

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

   then prove the following:

$$\sum_{n=1}^{m} H_n = (m+1)H_m - m$$

   hint: Apply the summand switch law.

References

[1] R.L. Graham, D.E. Knuth, O. Patashnik. Concrete Mathematics. Addison-Wesley Publishing (1994).