Monotone Summation Identities

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1 Terminology

Monotone sums are of the following form:

$$\sum_{0 \leq u_1 \leq u_2 \leq u_3 \leq \ldots \leq u_{k-1} \leq u_k \leq n} f(u_1, u_2, u_3, \ldots, u_{k-1}, u_k)$$

As is, this is quite tedious to parse so we shorten our notation as follows:

$$\sum_{(0 \leq \mathbf{u} \leq n)}^{k} f(\mathbf{u})$$

where

$$\mathbf{u} \ := \ (u_1, u_2, \ldots, u_{k-1}, u_k)$$

belongs to the montone k-space: The set of monotonically increasing sequences of integers of length k.

This inspires further notation: In particular, if you wish to refer the j^{th} coordinate of $\mathbf{u}$, you simply write:

$$\mathbf{u}_{/j}$$

Such notation is analogous to being able to access the *real* and *imaginary* components of a complex number:

$$\mathcal{R}e(z), \ \mathcal{I}m(z)$$

We will also want a *concatenation* operator:

$$\mathbf{u} \mid \mathbf{v} \ := \ (\mathbf{u}_{/1}, \ldots, \mathbf{u}_{/k}, \mathbf{v}_{/1}, \ldots, \mathbf{v}_{/\ell})$$

We can now express our summand switch laws:

$$\begin{aligned}
\sum_{(0 \leq \mathbf{u} | \mathbf{v} \leq n)}^{j,k} f(\mathbf{u} \mid \mathbf{v}) \quad &= \quad \sum_{(0 \leq \mathbf{u} \leq n)}^{j} \ \sum_{(\mathbf{u} \leq \mathbf{v} \leq n)}^{k} f(\mathbf{u} \mid \mathbf{v}) \\
&= \quad \sum_{(0 \leq \mathbf{v} \leq n)}^{k} \ \sum_{(0 \leq \mathbf{u} \leq \mathbf{v})}^{j} f(\mathbf{u} \mid \mathbf{v})
\end{aligned}$$

2 Identities

These sums are able to provide discrete closed forms for well known combinatorial numbers:

$$\begin{aligned}
\binom{k+n}{n} &= \sum_{(0\le\mathbf{u}\le n)}^{k} 1 \\
\left\{ {k+n+1 \atop n+1} \right\} &= \sum_{(0\le\mathbf{u}\le n)}^{k} (\mathbf{u}_{/1}+1)(\mathbf{u}_{/2}+1)\ldots(\mathbf{u}_{/k}+1) \\
\left[{k+n+1 \atop n+1} \right] &= \sum_{(0\le\mathbf{u}\le n)}^{k} (\mathbf{u}_{/1}+1)(\mathbf{u}_{/2}+2)\ldots(\mathbf{u}_{/k}+k)
\end{aligned}$$

As for general identities, if we denote $1\cdot\mathbf{u} := \mathbf{u}_{/1}+\ldots+\mathbf{u}_{/k}$, then:

$$\begin{aligned}
\sum_{(0\le\mathbf{u}\le n)}^{k} \mathbf{u}_{/s} &= s\binom{k+n}{k+1} \\
\sum_{(0\le\mathbf{u}\le n)}^{k} 1\cdot\mathbf{u} &= \frac{kn}{2}\binom{k+n}{k} \\
\sum_{(0\le\mathbf{u}\le n)}^{k} t^{\mathbf{u}_{/1}} &= \sum_{0\le j\le n} \binom{k+n}{k+j}(t-1)^j \\
\sum_{(0\le\mathbf{u}\le n)}^{k} t^{\mathbf{u}_{/k}} &= \sum_{0\le j\le n} \binom{k+n}{k+j} t^{n-j}(t-1)^j \\
\sum_{(0\le\mathbf{u}\le n)}^{k} t^{1\cdot\mathbf{u}} &= \prod_{1\le j\le k} \frac{1-t^{n+j}}{1-t^j} \\
\sum_{(0\le\mathbf{u}\le n)}^{k} \mathbf{u}_{/s+1}^{m} &= \sum_{1\le j\le m} j!\left\{ {m \atop j} \right\}\binom{s+j}{j}\binom{k+n}{k+j} \\
\sum_{(0\le\mathbf{u}\le n)}^{k} (\mathbf{u}_{/1}^{m}+\ldots+\mathbf{u}_{/k}^{m}) &= \sum_{1\le j\le m} j!\left\{ {m \atop j} \right\}\binom{k+j}{1+j}\binom{k+n}{k+j} \\
\sum_{(0\le\mathbf{u}\le n)}^{k} t^{\mathbf{u}_{/s+1}} &= \sum_{0\le j\le n} \binom{s+j}{j}\binom{k+n}{k+j}(t-1)^j
\end{aligned}$$

References

[1] R.L. Graham, D.E. Knuth, O. Patashnik. Concrete Mathematics. Addison-Wesley Publishing (1994).