This short article provides a quick informal look at the idea that inequality expressions—or intervals—can be viewed as objects in their own right with their own algebra. As this idea occurred to me recently (at the time of this writing: September 3rd, 2015) I have additionally googled it and it seems it is a *topological algebra*, something I very much plan on looking into in the future but as of yet have only heard about. The difference here as stated is this is an informal treatment, deriving the simplest of results needed for other projects.

Our general inequality expressions are of the following nature:

$$a \leq x \leq b$$

In particular, I reserve the lower end of the alphabet (in the expression as: a, b) for constant real numbers $\mathbb{R}$, while the upper end of the alphabet (in the expression as: x) is for real number variables. Although we could represent the same construct with interval notation:

$$x \in [a, b]$$

for the purpose of this article the interval representation flows better within the narrative of a vector style object.

Informally, the inequality $a \leq x \leq b$ has a truth-value: *true* or *false*. If you had need, you could also give it a partial truth-value ($a \leq x : true$, but $x \leq b$ false for example), but such complicated logics aren't of use here. I mention this to emphasize the point that if the whole inequality expression is not *true*, it defaults to *false*. Another point to bring up is, although an expression can be false, we are only interested in ones which are true for our algebra. This is to say, we view the expression to *exist* as an *object* if it is true.

With this in mind, here is our first lemma regarding *componentwise addition*:

Lemma 0.1

$$a \leq b \quad and \quad c \leq d \qquad \Longrightarrow \qquad a + c \leq b + d$$

Proof First we note that

$$\begin{aligned} a \leq b \quad &\Longleftrightarrow \quad 0 \leq b - a \\ &\Longleftrightarrow \quad 0 = b - a \quad \text{or} \quad 0 < b - a \\ c \leq d \quad &\Longleftrightarrow \quad 0 \leq d - c \\ &\Longleftrightarrow \quad 0 = d - c \quad \text{or} \quad 0 < d - c \end{aligned}$$

If $0 = b - a$ then $0 \leq (d - c) = (d - c) + (b - a)$ which is equivalent to $a + c \leq b + d$. This symmetrically holds for when $0 = d - c$. As such, this covers all cases except when $0 < b - a$ and $0 < d - c$. We take it for granted that a *positive* number plus a *positive* number equals a *positive* number, so it follows $0 < (d - c) + (b - a)$ which is equivalent to $a + c < b + d$. Hence, regardless of case, we have at worst $a + c \leq b + d$. ∎

Moving onto our second lemma, we take a look at subtraction (more or less):

Lemma 0.2

$$a \leq b \qquad \Longleftrightarrow \qquad -b \leq -a$$

Proof The proof is as follows:

$$\begin{aligned} a \leq b \quad &\Longleftrightarrow \quad 0 \leq b - a \\ &\Longleftrightarrow \quad 0 \leq (-a) - (-b) \\ &\Longleftrightarrow \quad -b \leq -a \end{aligned} \qquad ∎$$

As for our next lemma, we move onto multiplication:

Lemma 0.3

$$a \le b \quad \textit{and} \quad 0 \le c \qquad \Longrightarrow \qquad ac \le bc$$

Proof Similar to the addition lemma, we note that

$$\begin{array}{rcllll} a \le b & \Longleftrightarrow & 0 \le b - a & & & \\ & \Longleftrightarrow & 0 = b - a & \text{or} & 0 < b - a \\ 0 \le c & \Longleftrightarrow & 0 = c & \text{or} & 0 < c \end{array}$$

If $0 = b - a$ then $0 \le 0 = c(b-a) = cb - ca$ which is equivalent to $ac \le bc$. This similarly holds for $0 = c$. As such the only case left is when $0 < b - a$ and $0 < c$. We take it for granted that a *positive* number times a *positive* number is equal to a *positive* number. Hence $0 < c(b-a)$ which is equivalent to $ac < bc$. Thus, regardless of case, we have at worst $ac \le bc$. ∎

For our final lemma, we have division:

Lemma 0.4

$$0 < a \le b \qquad \Longleftrightarrow \qquad 0 < \frac{1}{b} \le \frac{1}{a}$$

Proof We take for granted that the multiplicative inverse of a *positive* number is a *positive* number.

case: "$\Longrightarrow$" Since $0 < a \le b$ we have both $0 < a$ and $0 < b$, and so $0 < a^{-1}, b^{-1}$. As such, by the multiplication lemma:

$$\begin{array}{rcrcr} a \le b & \Longrightarrow & aa^{-1} & \le & ba^{-1} \\ & \Longrightarrow & 1 & \le & ba^{-1} \\ & \Longrightarrow & b^{-1} & \le & b^{-1}ba^{-1} \\ & \Longrightarrow & b^{-1} & \le & a^{-1} \end{array}$$

case: "$\Longleftarrow$" Since $0 < b^{-1} \le a^{-1}$ we have both $0 < a^{-1}$ and $0 < b^{-1}$, and so $0 < a, b$. As such, by the multiplication lemma:

$$\begin{array}{rcrcr} b^{-1} \le a^{-1} & \Longrightarrow & b^{-1}b & \le & a^{-1}b \\ & \Longrightarrow & 1 & \le & a^{-1}b \\ & \Longrightarrow & a & \le & aa^{-1}b \\ & \Longrightarrow & a & \le & b \end{array}$$

∎

We have enough now for our theorems:

Theorem 0.1

$$a \le x \le b \quad \textit{and} \quad c \le y \le d \qquad \Longrightarrow \qquad a + c \le x + y \le b + d$$

Proof

$$\begin{array}{rcrcr} a \le x \le b & \Longleftrightarrow & a \le x & \text{and} & x \le b \\ c \le y \le d & \Longleftrightarrow & c \le y & \text{and} & y \le d \\ & \Longrightarrow & a + c \le x + y & \text{and} & x + y \le b + d \\ & \Longleftrightarrow & a + c \le x + y \le b + d & & \end{array}$$

∎

Theorem 0.2

$$a \le x \le b \qquad \Longleftrightarrow \qquad -b \le -x \le -a$$

Proof

$$\begin{array}{rcrcr} a \le x \le b & \Longleftrightarrow & a \le x & \text{and} & x \le b \\ & \Longleftrightarrow & -x \le -a & \text{and} & -b \le -x \\ & \Longleftrightarrow & -b \le -x & \text{and} & -x \le -a \\ & \Longleftrightarrow & -b \le -x \le -a & & \end{array}$$

∎

Theorem 0.3

$$0 \le a \le x \le b \quad \textit{and} \quad 0 \le c \le y \le d \qquad \Longrightarrow \qquad 0 \le ac \le xy \le bd$$

Proof

$$\begin{array}{lclcl}
0 \le a & \text{and} & 0 \le c & \Longrightarrow & 0c \le ac \\
c \le y & \text{and} & 0 \le a & \Longrightarrow & ac \le ay \\
a \le x & \text{and} & 0 \le y & \Longrightarrow & ay \le xy \\
y \le d & \text{and} & 0 \le x & \Longrightarrow & xy \le xd \\
x \le b & \text{and} & 0 \le d & \Longrightarrow & xd \le bd
\end{array}$$

The transitive law assures us that $0 \le ac \le xy \le bd$ ∎

Theorem 0.4

$$0 < a \le x \le b \qquad \Longleftrightarrow \qquad 0 < \frac{1}{b} \le \frac{1}{x} \le \frac{1}{a}$$

Proof

$$\begin{aligned}
0 < a \le x \le b \quad &\Longleftrightarrow \quad 0 < a \le x \quad \text{and} \quad 0 < x \le b \\
&\Longleftrightarrow \quad 0 < \tfrac{1}{b} \le \tfrac{1}{x} \quad \text{and} \quad 0 < \tfrac{1}{x} \le \tfrac{1}{a} \\
&\Longleftrightarrow \quad 0 < \tfrac{1}{b} \le \tfrac{1}{x} \le \tfrac{1}{a} \qquad \blacksquare
\end{aligned}$$

We finally have enough for some constructive object definitions:

Definition (additive object): Let

$$(a \le x \le b) \quad , \quad (c \le y \le d)$$

be objects, then

$$(a \le x \le b) + (c \le y \le d) \qquad := \qquad (a + c \le x + y \le b + d)$$

Definition (additive inverse object): Let

$$(a \le x \le b)$$

be an object, then

$$-(a \le x \le b) \qquad := \qquad (-b \le -x \le -a)$$

Definition (multiplicative object): Let

$$0 \le (a \le x \le b) \quad , \quad 0 \le (c \le y \le d)$$

be objects, then

$$(a \le x \le b)(c \le y \le d) \qquad := \qquad (ac \le xy \le bd)$$

Notice here I used the notation $0 \le (a \le x \le b)$, though not formalized I mean to intuitively say the object is "positive", which in particular translates as $0 \le a \le x \le b$.

Definition (multiplicative inverse object): Let

$$0 < (a \le x \le b)$$

be an object, then

$$\frac{1}{(a \le x \le b)} \qquad := \qquad (\frac{1}{b} \le \frac{1}{x} \le \frac{1}{a})$$

A note about these definitions: To be fair, it seems likely with the above definitions you have to take an equivalence relation to stabilize the objects, but such finer considerations are for an actual theory of intervals as topological algebra.

As mentioned in the opening, my intention here is much more the theorems to use for other projects than anything else. Regarding such other projects: generally *numerical analysis* where one needs to find inequality bounds for error terms within approximations. The thing is, *real analysis* at its lower levels makes heavy use of inequalities as well, and although the nature of such "arithmetic yoga"—as a topology professor of mine once referred to them—is subtly different, this discrete style of inequality manipulation may be of use there as well.

I'll leave you with a final thought-provoking example. Consider the inequality:

$$1 \leq a \leq x \leq b$$

Let's say we wanted a bound on $(x-1)^2$ instead, we could do this as follows:

$$\begin{array}{rcccccccc} 1 \leq a \leq x \leq b & \Longrightarrow & 0 & \leq & a-1 & \leq & x-1 & \leq & b-1 \\ & \Longrightarrow & 0 & \leq & (a-1)^2 & \leq & (x-1)^2 & \leq & (b-1)^2 \end{array}$$

and so we have our bound already. The thing is, we also know $(x-1)^2 = x^2 - 2x + 1$, so we could have also derived our bound in the additional following way:

$$\begin{array}{rcrclcl} 1 \leq a \leq x \leq b & \Longrightarrow & a^2 & \leq & x^2 & \leq & b^2 \\ & \Longrightarrow & -2b & \leq & -2x & \leq & -2a \\ & \Longrightarrow & 1 & \leq & 1 & \leq & 1 \\ \hline & \Longrightarrow & a^2 - 2b + 1 & \leq & x^2 - 2x + 1 & \leq & b^2 - 2a + 1 \\ & \Longrightarrow & a^2 - 2b + 1 & \leq & (x-1)^2 & \leq & b^2 - 2a + 1 \end{array}$$

where we took the top three inequalities and added them together to get the bottom two.

First of all, this alone is thought provoking if you've never really studied it or considered it before—*identity manipulations don't preserve inequality manipulations!* I mean it makes sense once you really think about it, but when you spend all your time with identity manipulations you might form the bad habit of expecting preservation. What's more thought provoking though is this that we have now two differing bounds, so which is better?

To figure this out we note that $a \leq b$ and so $-2b \leq -2a$. In this case then we have

$$a^2 - 2b + 1 \leq a^2 - 2a + 1 = (a-1)^2$$

similarly we also know

$$(b-1)^2 = b^2 - 2b + 1 \leq b^2 - 2a + 1$$

and so it appears using the factored form $(x-1)^2$ gives tighter bounds than the expanded form $x^2 - 2x + 1$. It's also worth noting that if we had instead wondered about $(x+1)^2$ our two methods would not have diverged in the bounds offered (subtraction makes the difference).

What is the hidden pattern here? What are there larger patterns for *best practice* inequality manipulation? Is there a logical mathematical system of it all? Something to think about...