This short article provides a notation for the positional (radix) representation of natural numbers with the intention of formula manipulation regarding numeric analytic as well as number theoretic proofs.

I'll get straight to it. Let $\mathbb{N}^{\mathbb{Z}}$ be the bidirectional sequences of natural numbers:

$$\boldsymbol{u} := < \ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots >$$

Given we're working with sequences as our data structures, we will want to be able to *filter* as means of navigating and accessing the data. As such, we begin with subsequences:

$$\underset{\mathbb{P}_2(k)}{<\mathbb{P}_1(\boldsymbol{u})>}, \quad k \in \mathbb{Z}$$

Currently this notation is too vague and might not make much sense. As an example to help clarify, take the property:

$$<a \leq \boldsymbol{u} \leq b> \qquad :\Longleftrightarrow \qquad < u_k \mid a \leq u_k \leq b,\ k \in \mathbb{Z} >$$

to mean the subsequence of $\boldsymbol{u}$ satisfying the given bounds. With such a property $\mathbb{P}_1(\boldsymbol{u}) = a \leq \boldsymbol{u} \leq b$, we've constrained and thus *filtered* the possible sequences in a *weak* sense, meaning we haven't isolated a unique case but still have many sequences satisfying the filtering condition.[1] In addition to this type of filtering, we can further constrain possible sequences by filtering on the index set instead:

$$\underset{c \leq k \leq d}{<\boldsymbol{u}>}, \quad k \in \mathbb{Z}$$

which means the subsequence of $\boldsymbol{u}$ with indices $[c \ldots d]$.

It is also possible to filter sequences using both filtering styles:

$$<a \leq \underset{c \leq k \leq d}{\boldsymbol{u}} \leq b>, \quad k \in \mathbb{Z}$$

though in practice this exact notation will seldom be used—it is powerful, but a bit ugly. In the very special but important case where $c \leq k \leq c$ (or $k = c$), we simplify our notation:

$$<\boldsymbol{u}>_c \quad := \quad u_c, \quad c \in \mathbb{Z}$$

to mean the c^{th} ordered/indexed element of the sequence $\boldsymbol{u}$.

As the filtering properties can be anything, they can also be constructive in nature:

$$\underset{c \leq k \leq d}{<b^k>}$$

which would be interpreted as:

$$< b^c, b^{c+1}, \ldots, b^d >$$

the finite subsequence of powers of b between c, d.

You can also have constants:

$$<\mathbf{1}> := < \ldots, 1, 1, 1, \ldots >$$

Given our sequences can be interpreted as *vectors*, we also have *componentwise operators* and the *dot-product*.

Finally, as our *stream* sequence is infinite in both directions, we are also able to *shift*:

$$\begin{aligned} <(\boldsymbol{u} \ll s)> \quad &:= \quad < u_{k-s} \mid k \in \mathbb{Z} > \\ <(\boldsymbol{u} \gg s)> \quad &:= \quad < u_{k+s} \mid k \in \mathbb{Z} > \end{aligned}$$

[1] As a way of thinking, weak specifications are in fact just fine—and used all the time. This relates to a way of thinking called concurrent logic.

This brings us to our main notation:

$$\underset{0\le k\le n}{(\boldsymbol{u})^b} \quad := \quad \underset{0\le k\le n}{<0 \le \boldsymbol{u} < b>} \cdot \underset{0\le k\le n}{<b^k>}$$

where $(\cdot)$ is the dot-product. Note, when expanded out this is our positional notation:

$$\underset{0\le k\le n}{(\boldsymbol{u})^b} \quad = \quad u_0 + u_1 b + u_2 b^2 + \ldots + u_n b^n$$

when the context is clear (and frankly it often is), we will omit the base b:

$$\underset{0\le k\le n}{(\boldsymbol{u})}$$

Finally, keep in mind it is implicit in the above definition that $b > 0$.

We will prove a few basic manipulation properties of this notation. The first thing to realize is our notation holds true under weaker constraints:

$$\underset{m\le k\le n}{(\boldsymbol{u})^b} \quad := \quad \underset{m\le k\le n}{<0 \le \boldsymbol{u} < b>} \cdot \underset{m\le k\le n}{<b^k>}$$

which follows simply from the notational expansion above.

Lemma 0.1

$$\underset{j\le k\le \ell}{(\boldsymbol{u})^b} \quad = \quad \underset{0\le k\le \ell-j}{(\boldsymbol{u} \ll j)^b}\, b^j$$

Proof

$$\begin{aligned}
\underset{j\le k\le \ell}{(\boldsymbol{u})^b} &= \underset{j\le k\le \ell}{<\boldsymbol{u}>} \cdot \underset{j\le k\le \ell}{<b^k>} \\
&= \underset{0\le k\le \ell-j}{<\boldsymbol{u} \ll j>} \cdot \underset{0\le k\le \ell-j}{<b^{k+j}>} \\
&= \underset{0\le k\le \ell-j}{<\boldsymbol{u} \ll j>} \cdot \underset{0\le k\le \ell-j}{<b^k>}\ b^j \\
&= \underset{0\le k\le \ell-j}{(\boldsymbol{u} \ll j)^b}\, b^j \qquad \blacksquare
\end{aligned}$$

Lemma 0.2

$$\underset{0\le k\le \ell}{(\boldsymbol{u})^b} \quad < \quad b^{\ell+1}$$

Proof

$$\begin{aligned}
\underset{0\le k\le \ell}{(\boldsymbol{u})^b} &= \underset{0\le k\le \ell}{<0 \le \boldsymbol{u} \le b-1>} \cdot \underset{0\le k\le \ell}{<b^k>} \\
&\le (b-1) \underset{0\le k\le \ell}{<b^k>} \\
&= (b-1)\frac{b^{\ell+1}-1}{b-1} \\
&= b^{\ell+1} - 1 \\
&< b^{\ell+1} \qquad \blacksquare
\end{aligned}$$

Lemma 0.3

$$\underset{j\leq k\leq \ell}{(\boldsymbol{u})^b} \quad\leq\quad \underset{0\leq k\leq \ell}{(\boldsymbol{u})^b} \quad<\quad \underset{j\leq k\leq \ell}{(\boldsymbol{u})^b} + b^j$$

Proof From the lemma above, the fact that $b > 0$ as well as the constraint that $0 \leq \boldsymbol{u} < b$, we have:

$$0 \quad\leq\quad \underset{0\leq k<j}{(\boldsymbol{u})^b} \quad<\quad b^j$$

Adding $\underset{j\leq k\leq \ell}{(\boldsymbol{u})^b}$ componentwise to all three sides leaves us with:

$$\begin{aligned}
\underset{j\leq k\leq \ell}{(\boldsymbol{u})^b} \quad&\leq\quad \underset{0\leq k<j}{(\boldsymbol{u})^b} + \underset{j\leq k\leq \ell}{(\boldsymbol{u})^b} \quad<\quad \underset{j\leq k\leq \ell}{(\boldsymbol{u})^b} + b^j \\
\underset{j\leq k\leq \ell}{(\boldsymbol{u})^b} \quad&\leq\quad \underset{0\leq k\leq \ell}{(\boldsymbol{u})^b} \quad<\quad \underset{j\leq k\leq \ell}{(\boldsymbol{u})^b} + b^j \qquad \blacksquare
\end{aligned}$$