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A Note on the Probability of Casting a Decisive Vote

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Although it is widely believed that the probability that any one vote will decide an election is so small that the prospect of influencing an election cannot explain why rational people vote,[1] there have been to our knowledge few attempts to calculate the relevant probability. The calculations which we know of are neither general nor rigorous.[2]

The purpose of this note is to establish under fairly general circumstances that the probability that a voter casts the decisive ballot in an election in which $2N+1$ voter choose among two alternatives is of order N^{-1}. This is both smaller and larger than one might on first, or second thought, expect. An individual's vote matters if, and only if, the other $2N$ members of the polity split their votes evenly. The simplest way to calculate the probability of this happening is to suppose that the votes of individual members of society are independent binomial random variables while the probability that any individual votes yes is p. Then the probability that exactly N of the $2N$ members of society vote yes is $\binom{2N}{N} p^N(1-p)^N$, which we will write as $Q(N,p)$.

It is convenient to use Stirling's formula to evaluate $Q(N,p)$. Since $\binom{2N}{N} \approx 2^{2N}/\sqrt{\pi N}$, we see that for $p=1/2$,

$$Q(N, 1/2) \approx \frac{1}{\sqrt{\pi N}}.$$

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[1] See, for example, Ferejohn and Fiorina [4]. For some evidence that is consistent with the view that people are more likely to vote if the probability of their influencing the outcome is higher, see Barzel and Silverberg [1]. This evidence is, of course, not inconsistent with the position that the probability of influencing an election is so small that it cannot explain why rational men and women vote.

[2] We review this literature briefly in the Appendix.

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Since $(\sqrt{N})^{-1}$ decreases fairly slowly, the probability of having a decisive vote is, in this case, quite high. For example, in a society of 100,001 persons, the probability of having a decisive vote is approximately 0.0025; in a society of 1,000,001, this probability is about 0.0008.

However, if $p \neq 1/2$, $Q(N, p)$ decreases much more quickly.

$$\begin{aligned} Q(N,p) &\approx \frac{2^{2N}p^N(1-p)^N}{\sqrt{\pi N}} \\ &\approx Q(N,1/2) \exp N[2 \log 2 + \log p + \log(1-p)]. \end{aligned}$$

If $p \neq 1/2$, $[2 \log 2 + \log p + \log(1-p)] < 0$ and thus $Q(N, p)$ is of order e^{-cN} for some strictly positive c and goes to zero very quickly, as Table I shows.

What are we to make of these two results? One might argue that since the case $p = 1/2$ is unlikely ever to hold exactly, it must be that the probability that an individual's vote matters is in general of order e^{-cN}. It follows that it almost never makes sense to vote in any moderately large polity. We prefer to take seriously the notion that p is unknown and calculate

$$\int_0^1 \binom{2N}{N} p^N(1-p)^N \, dF(p) = \int_0^1 Q(N,p) \, dF(p). \tag{1}$$

One way of thinking about how (1) might arise is to suppose that nature picks a p at random from the distribution $F(p)$. Then $2N$ coins are tossed; each has probability p of coming up heads. For that p the probability of observing exactly N heads is $Q(N, p)$. The total probability of observing N

TABLE I

Probability of Having Decisive Vote as a Function of M (Size of Polity) and p (Probability a Polity Member Votes "Yes")

p \ M	1001	100,001	1,000,001	100,000,001
0.4	0.3×10^{-10}	*	*	*
0.45	0.2×10^{-3}	*	*	*
0.475	0.007	*	*	*
0.49	0.021	0.5×10^{-11}	*	*
0.4999	0.025	0.0025	0.0008	0.1×10^{-4}
0.5	0.025	0.0025	0.0008	0.8×10^{-4}

* Less than 10^{-35}.

heads is gotten by integrating against $F(p)$ as in (1). A model in which the probability that each person votes yes is unknown but which does *not* give the expression (1) as the probability of a tie is the following: Each individual in the society has his own probability of voting yes. When the ith voter enters the polling booth, he votes yes with probability p_i. The p_i's are independently and identically distributed in the population according to the distribution function B. In this model, the probability that an individual votes yes is simply $\mu_B = \int_0^1 p\, dB(p)$ and since voters are independent, the probability that their N of $2N$ votes are yes is $Q(N, \mu_B)$. Thus the only important aspect of B is its mean, μ_B. If μ_B is known then the model is the same as one in which each voter's probability of voting yes is μ_B. If B is not known, then this argument shows that all that we need to know about B is its mean; the only important uncertainty about B is uncertainty about μ_B. This is the model discussed above which leads to (1).

In his original memoir, Bayes calculated $\int_0^1 Q(N,p)\, dF(p)$ for the special case where $F(p)$ is a uniform distribution. In this case $\int_0^1 Q(N,p)\, dF(p)$ reduces to $\int_0^1 \binom{2N}{N} p^N(1-p)^N\, dp$. This expression is easily evaluated by noting, as Bayes did, that if $X_0, X_1, \ldots, X_M$ is a random sample of $M+1$ random variables, each of which has a uniform distribution on $[0, 1]$, then if $X_0 = p$, $\binom{M}{k} p^k(1-p)^{M-k}$ is just the probability that k of the succeeding M random variables are less than p and $M-k$ of them are greater than p. Integrate over p to obtain the probability that in the sample, the first random variable observed is the $k+1$st smallest. Symmetry implies that this probability must be equal to $(M+1)^{-1}$, independent of k, or that

$$\int_0^1 \binom{M}{k} p^k(1-p)^{M-k}\, dp = (M+1)^{-1}.$$

It follows that $\int_0^1 \binom{2N}{N} p^N(1-p)^N\, dF(p)$ is equal to $(2N+1)^{-1}$ when $F(p)$ is uniform. We now show that this result is a prototype of the general case, that if one is uncertain about the probability that others vote yes then the probability of one's vote being decisive is of order N^{-1}.

PROPOSITION 1. *If the probability that voters will vote yes is distributed according to a distribution $F(p)$ with a continuous density $f(p)$ in the neighborhood of* 1/2, *then*

$$\lim_{N\to\infty} N \int_0^1 Q(N,p)\, dF(p) = (1/2) f(1/2).$$

Proof. Note that $N\int_0^1 Q(N,p)\, dF(p) = \int_0^1 N\binom{2N}{N} p^N(1-p)^N\, dF(p)$. Let $A = \{p \in (1/2-\varepsilon, 1/2+\varepsilon)\}$ and $B = \{p \in [0, 1/2-\varepsilon] \cup [1/2+\varepsilon, 1]\}$ for

$\varepsilon > 0$. Since $\binom{2N}{N} p^N(1-p)^N$ is uniformly of order e^{-cN}, $c > 0$, for $p \in B$, $N\binom{2N}{N} p^N(1-p)^N \to 0$ uniformly in p for $p \in B$. Thus,

$$\lim_{N\to\infty} \int_B N \binom{2N}{N} p^N(1-p)^N \, dF(p) = 0.$$

For any $\delta > 0$ we can choose ε small enough that $f(1/2) - \delta \leqslant f(p) \leqslant f(1/2) + \delta$ for $p \in A$. Hence

$$\begin{aligned}\left[f\left(\frac{1}{2}\right) - \delta\right] &\int_A N \binom{2N}{N} p^N(1-p)^N \, dp \\ &\leqslant \int_A N \binom{2N}{N} p^N(1-p)^N f(p) \, dp \\ &\leqslant \left[f\left(\frac{1}{2}\right) + \delta\right] \int_A N \binom{2N}{N} p^N(1-p)^N \, dp.\end{aligned}$$

From the uniform case, we know that

$$\lim_{N\to\infty} \int_0^1 N \binom{2N}{N} p^N(1-p)^N \, dp = \lim_{N\to\infty} \int_A N \binom{2N}{N} p^N(1-p)^N \, dp = \frac{1}{2}.$$

Hence

$$[f(\tfrac{1}{2}) - \delta]\,\tfrac{1}{2} \leqslant \lim_{N\to\infty} N \int_0^1 Q(N,p)\, dF(p) \leqslant [f(\tfrac{1}{2}) + \delta]\,\tfrac{1}{2}$$

for any $\delta > 0$, and so

$$\lim_{N\to\infty} N \int_0^1 Q(N,p)\, dF(p) = \tfrac{1}{2} f(\tfrac{1}{2}).$$

Note that Proposition 1—and its proof—remains true when $f(1/2) = 0$.

The result of Proposition 1 is considerably more general than the simple model from which it comes. Suppose that the polity consists of several distinct groups, and that we believe the propensity to vote yes varies from group to group. For simplicity we consider the case of two groups, but the analysis generalizes to any arbitrary number of groups. If voters come from two distinct groups, for concreteness think of men and women, then our uncertainty about the propensity to vote yes is expressed by a joint prior density $f(p_1, p_2)$ of the propensities for men and women to vote yes.

Information about propensities of different groups to vote is most easily thought of in terms of the marginal distributions for each group;

$f_1(p_1) = \int_0^1 f(p_1, p_2)\, dp_2$ and $f_2(p_2) = \int_0^1 f(p_1, p_2)\, dp_1$, and it is natural to ask why the propensities are not independent in which case

$$f(p_1, p_2) = f_1(p_1) f_2(p_2). \tag{2}$$

We consider the more general case where (2) does not necessarily hold. It is worthwhile to digress on the meaning of independence in this case. Consider an arbitrary sequence of voters. Suppose a statistician has observed the vote of N_1 men and N_2 women. He wants to use this information, and his prior information, $f(p_1, p_2)$, to predict the next man's vote. If he uses the information about the votes of women to make this prediction, then the prior distribution is not independent. We believe that, in general, votes of men will be used to predict the votes of women and vice-versa, and thus we consider the general case where (2) does not hold. This includes the case where voting behavior is believed to be completely independent of sex so that $f(p_1, p_2) = f(p_2, p_1)$.

We now consider the probability of a tie vote in an electorate of size $2N$, in which a fraction, α, of the voters are men and a fraction $(1 - \alpha)$ are women. Different models can be used to calculate this probability. The simplest supposes that the vote is a random sample of size $2N$, where each vote is chosen as follows. Before voting begins a pair (p_1, p_2) is drawn at random from the distribution $f(p_1, p_2)$. With probability α a man is chosen to vote first; he votes yes with probability p_1 and no with probability $(1 - p_1)$. If, as will happen with probability $1 - \alpha$, a woman is chosen to vote first, she votes yes with probability p_2 and no with probability $1 - p_2$. The process, which is one of sampling with replacement, continues in this way. At each stage the probabilities of men and women voting yes are given by the (p_1, p_2) selected initially. The kth vote in this sequence is a random variable X_k which assumes the values of 0 or 1. An electorate of size k produces a random sequence $X_1, \ldots, X_k$. It is a consequence of random sampling that the probability of any observed sequence of votes is independent of the order in which they are counted or that

$$\Pr\{X_1, \ldots, X_k\} = \Pr\{X_{\pi(1)}, \ldots, X_{\pi(k)}\}, \tag{3}$$

where π is any permutation of the first k integers. Since (3) holds for any k, the random variables $X_1, \ldots, X_m$ are what is known as *infinitely exchangeable*. It follows from a theorem of de Finetti's[3] that there is a cumulative distribution function G, independent of m, such that

$$\Pr\{X_1 = x_1, \ldots, X_m = x_m\} = \int_0^1 p^j (1 - p)^{m-j}\, dG(p), \tag{4}$$

[3] For a discussion of this theorem and the idea of exchangeability, see Diaconis [3].

where $j = \sum_{i=1}^{m} x_i = S_m$. It follows that

$$\Pr\{S_m = j\} = \int_0^1 \binom{m}{j} p^j (1-p)^{m-j}\, dG(p). \tag{5}$$

Since (5) holds for any m including $m = 1$, $G(p)$ must be the cumulative distribution function of the mixture $\alpha p_1 + (1-\alpha) p_2$. The density of $G(p)$ is

$$g(p) = \frac{1}{1-\alpha} \int_0^1 f\left(p_1, \frac{p - \alpha p_1}{1-\alpha}\right) dp_1. \tag{6}$$

Proposition 1 and the representations (4) and (5) immediately imply

PROPOSITION 2. *If voting is conceived of as drawing a sample of size* $2N$ *with replacement from a population in which the ratio of men to women is* $(\alpha/(1-\alpha))$ *and the prior probability of men and women voting yes is given by the continuous density* $f(p_1, p_2)$, *then* $N \Pr\{S_{2N} = N\} \to (1/2)\, g(1/2)$, *where* $g(p)$ *is given by* (6).

Thinking of voting as sampling with replacement is reasonable if the fraction of men and women who vote is not large. If turnout is high, the approximation is inaccurate. The sequence of votes generated by sampling without replacement will not in general be infinitely exchangeable so that de Finetti's theorem does not apply.[4]

Nonetheless, the conclusions of Propositions 1 and 2 continue to hold

[4] A sequence of length N which arose from sampling (without replacement) two populations of size $N_1 + N_2 = N$ would be exchangeable of order N but it would not in general be infinitely exchangeable. Suppose for example that the population consisted of one man and one woman and that men always voted yes while women always voted no. Assuming that the probability that the man votes first is 1/2, then

$$\Pr\{X_1 = 1, X_2 = 0\} = \Pr\{X_1 = 0, X_2 = 1\} = \tfrac{1}{2} \tag{7}$$

and

$$\Pr\{X_1 = 0, X_2 = 0\} = \Pr\{X_1 = 1, X_2 = 1\}. \tag{8}$$

If this sequence, which is exchangeable, could be extended to an infinitely exchangeable sequence, then there would exist a distribution function G such that (7) and (8) could be represented as in (4). But (8) implies

$$0 = \int_0^1 p^2\, dG(p) = \int_0^1 (1-p)^2\, dG(p),$$

or that the probability measure which G represents puts mass one at the point 0 and at the point 1, which is impossible. Diaconis's paper, which gives this example, also shows that it is not pathological.

when voting is explicitly modeled as sampling without replacement. Suppose that N_1 men and N_2 women vote, where $N_1 \leqslant N_2$ and $N_1 + N_2 = 2N$. Let R_1 and R_2 be the number of men and of women voting yes. Then, if the prior distribution for p_1 and p_2 has a density $f(p_1, p_2)$, the probability of a tie is

$$\Pr\{R_1 + R_2 = N\} = \int_0^1 \int_0^1 S_N(p_1, p_2) f(p_1, p_2)\, dp_1\, dp_2, \tag{9}$$

where

$$\begin{aligned} S_N(p_1, p_2) &= \Pr\{R_1 + R_2 = N | p_1, p_2\} \\ &= \sum_{n=0}^{N_1} \binom{N_1}{n} p_1^n (1-p_1)^{N_1-n} \binom{N_2}{N-n} p_2^{N-n}(1-p_2)^{N_2-N+n}. \end{aligned} \tag{10}$$

PROPOSITION 3. *Suppose $N \to \infty$ in such a way that $N_1/2N \to \alpha$. Then if $f(p_1, p_2)$ is bounded and continuous,* $N \Pr\{R_1 + R_2 = N\} \to 1/2g(1/2)$, *where $g(p)$ is given by* (6).

Proof. Note that

$$\begin{aligned} \psi_{N,N_1}(p_1, p_2) &= (N_2 + 1)\, S_N(p_1, p_2) \\ &= \sum_{n=0}^{N_1} \frac{1}{N_1 + 1} f_\beta(p_1 | n+1, N_1 - n + 1) \\ &\quad \times f_\beta(p_2 | N - n + 1, N_2 - N + n + 1), \end{aligned} \tag{11}$$

where $f_\beta(p|A, B)$ is the density of a Beta distribution with parameters A and B. The important thing about (11) is that the right-hand side is a probability density for (p_1, p_2) with parameters N_1 and N. Since we are interested in taking limits where $N \to \infty$ and $N_1/2N \to \alpha$, we can without abuse ignore the dependence of ψ_{N,N_1} on N_1 and write $\psi_N(p_1, p_2)$. In this notation, the probability of a tie times N is

$$\begin{aligned} &N \int_0^1 \int_0^1 S_N(p_1, p_2) f(p_1, p_2)\, dp_1\, dp_2 \\ &= N \int_0^1 \int_0^1 \frac{1}{N_2 + 1} \psi_N(p_1, p_2) f(p_1, p_2)\, dp_1\, dp_2 \\ &= \frac{1}{2} \frac{2N}{N_2 + 1} \int_0^1 \int_0^1 f(p_1, p_2)\, \psi_N(p_1, p_2)\, dp_1\, dp_2 \\ &= \frac{1}{2} \frac{2N}{N_2 + 1} E_{\psi_N} f(p_1, p_2), \end{aligned}$$

where the expectation is taken with respect to the distribution of (p_1, p_2) specified by ψ_N.

Observe that if p_1 and p_2 are distributed according to $\psi_N(p_1, p_2)$, the marginal distribution of p_1 is uniform on $[0, 1]$ since

$$\begin{aligned}
h(p_1) &= \int_0^1 \psi_N(p_1, p_2)\, dp_2 \\
&= \frac{1}{N_1+1} \sum_{n=0}^{N_1} f_\beta(p_1 | n+1, N_1-n+1) \\
&\quad \times \int_0^1 f_\beta(p_2 | N-n+1, N_2-N+n+1)\, dp_2 \\
&= \frac{1}{N_1+1} \sum_{n=0}^{N_1} f_\beta(p_1 | n+1, N_1-n+1) \\
&= \frac{1}{N_1+1} \sum_{n=0}^{N_1} \frac{(N_1+1)!}{n!\,(N_1-n)!} p_1^n (1-p_1)^{N_1-n} \\
&= \sum_{n=0}^{N_1} \binom{N_1}{n} p_1^n (1-p_1)^{N_1-n} = 1.
\end{aligned}$$

Now define the random variable $Z = \alpha p_1 + (1-\alpha) p_2$. Since p_1, p_2 are distributed according to the density ψ_N, the distribution of Z is determined by N, which we indicate by writing Z_N. Of the three random variables, p_1, p_2, and Z_N, the joint distribution of any two determines the third. Let H_N be the joint distribution function for (p_1, Z_N). Then

$$\int_0^1 \int_0^1 f(p_1, p_2)\, \psi_N(p_1, p_2)\, dp_1 p_2 = \int_0^1 \int_0^1 f\left(p_1, \frac{z - \alpha p_1}{1-\alpha}\right) dH_N(p_1, z).$$

Suppose that if $N \to \infty$ (and, of course, $N_1/2N \to \alpha$), then the distribution of the random variable Z_N converges weakly to the distribution which puts probability one at $Z = 1/2$. Then $H_N(p_1, z)$ converges weakly to $H(p_1, z)$, where $H(p_1, z)$ puts probability one at $Z = 1/2$ and the marginal distribution of p_1 is uniform on $[0, 1]$. Then, since f is bounded and continuous, we have that

$$\begin{aligned}
\lim_{N\to\infty} \int_0^1 \int_0^1 f\left(p_1, \frac{z-\alpha p_1}{1-\alpha}\right) dH_N(p_1, z) &= \int_0^1 \int_0^1 f\left(p_1, \frac{z-\alpha p_1}{1-\alpha}\right) dH(p_1, z) \\
&= \int_0^1 f\left(p_1, \frac{1/2 - \alpha p_1}{1-\alpha}\right) dp_1 \\
&= (1-\alpha)\, g(1/2),
\end{aligned}$$

where g is defined as in (6). Now

$$\lim_{N\to\infty} N \Pr\{R_1 + R_2 = N\} = \lim_{N\to\infty} \frac{1}{2}\frac{2N}{N_2+1} E_{\psi_N} f(p_1, p_2)$$

$$= \frac{1}{2}\frac{1-\alpha}{1-\alpha} g\left(\frac{1}{2}\right) = \frac{1}{2} g\left(\frac{1}{2}\right).$$

Thus to complete the proof it is only necessary to show that the distribution of Z_N converges weakly to a distribution concentrated on 1/2.

Let $\mathbf{P}_N$ be the random variable (p_{1N}, p_{2N}) with density ψ_N. Then, (11) states that conditional on a parameter n which is uniformly distributed on $0, 1, \ldots, N_1$, $\mathbf{P}_N$ is distributed as two independent beta distributions.

Thus

$$E(\mathbf{P}_N) = EE(\mathbf{P}_N | n) = E\left(\frac{n+1}{N_1+2}, \frac{N-n+1}{N_2+2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

since $E(n) = N_1/2$.

Let $V(\mathbf{P}_N)$ be the variance–covariance matrix of $\mathbf{P}_N$. Then

$$\begin{aligned} V(\mathbf{P}_N) &= EV(\mathbf{P}_N | n) + VE(\mathbf{P}_N | n) \\ &= E\begin{bmatrix} V(p_{1,N} | n) & 0 \\ 0 & V(p_{2,N} | n) \end{bmatrix} \\ &\quad + \begin{bmatrix} \dfrac{1}{(N_1+2)^2} & -\dfrac{1}{N_1+2}\dfrac{1}{N_2+2} \\ -\dfrac{1}{N_1+2}\dfrac{1}{N_2+2} & \dfrac{1}{(N_2+2)^2} \end{bmatrix} V_N(n), \end{aligned}$$

where $V_N(n) = ((N_1+1)^2 - 1)/12$ is the variance of a random variable uniformly distributed on $(0, \ldots, N_1)$.

Now,

$$V(p_{1,N} | n) = \frac{n+1}{N_1+2}\left(1 - \frac{n+1}{N_1+2}\right)\frac{1}{N_1+3} \leqslant \frac{1}{4}\frac{1}{N_1+3}$$

and similarly

$$V(p_{2,N} | n) \leqslant \frac{1}{4}\frac{1}{N_2+3},$$

so that

$$\lim_{N\to\infty} V(\mathbf{P}_N) = \frac{1}{12}\begin{bmatrix} 1 & -\dfrac{\alpha}{1-\alpha} \\ -\dfrac{\alpha}{1-\alpha} & \left(\dfrac{\alpha}{1-\alpha}\right)^2 \end{bmatrix},$$

where $\alpha = \lim_{N\to\infty} N_1/2N$. Let $\boldsymbol{\alpha}' = (\alpha, 1-\alpha)$.

If $Z_N = \alpha p_{1,N} + (1-\alpha) p_{2,N} = \boldsymbol{\alpha}'\mathbf{P}_N$, then $E(Z_N) = \boldsymbol{\alpha}' E(\mathbf{P}_N) = 1/2$ and

$$\lim_{N\to\infty} V(Z_N) = \boldsymbol{\alpha}' \lim_{N\to\infty} V(\mathbf{P}_N)\,\boldsymbol{\alpha} = 0,$$

so $Z_N \to 1/2$ in quadratic mean and therefore in distribution. This completes the proof.

APPENDIX: A NOTE ON THE LITERATURE

Beck [2] made the first attempt of which we are aware to calculate the probability of having a decisive vote. Beck calculated $Q(N, p)$ for $p = 1/2$ and for p near $1/2$ and observed the "knife-edged" nature of the result and concluded that the probability of casting a deciding vote "will only be significant ... if an individual assumes all other voters are totally indifferent between the two [alternatives]." Beck also considered a model in which there are two groups with different (but certain) probabilities of voting yes. His analysis of this case is, as he recognizes, tentative and incomplete. Margolis [5] extended Beck's analysis in somewhat the same way we did. He introduced subjectivity by supposing that to any voter, the fraction of the other voters voting yes is a random variable, y, with a probability distribution with density ϕ. If $2N$ people vote, then the subjective probability of a tie is $\int_{1/2-1/4N}^{1/2+1/4N} \phi(y)\,dy$, which is approximately equal to $(1/2N)\,\phi(1/2)$. This is, of course, the same result as we obtained in Proposition 1, where ϕ simply replaces f. It is, however, hard to understand how Margolis justifies his approach. The fraction of people voting yes is, for any finite election, a discrete random variable which can only take on $N+1$ values if N people vote. Margolis argues that ϕ is a normal distribution by appealing to the central limit theorem (or the normal approximation to the binomial). However, this is not legitimate in the present context. The distribution of the fraction of successes from a mixture of binomial distributions does not approach a normal distribution but instead approaches the mixing distribution itself. This fact, proved rigorously below, shows why Margolis' reasoning produces an answer in agreement with our Proposition 1.

PROPOSITION 4. *Let* $\Pr\{R=n\}=\int_0^1 \binom{N}{n} p^n(1-p)^{N-n}\,dF(p)$. *Then if* $N\to\infty$, $\Pr\{R/N\leqslant\tau\}$ *converges weakly to* F.

Proof. $\Pr\{R/N\leqslant\tau)=\int_0^1 \Pr\{R/N\leqslant\tau\mid p)\,dF(p)$. Let

$$\begin{aligned} I_\tau(p) &= 0 \qquad \text{if} \quad \tau<p \\ &= 1 \qquad \text{if} \quad \tau\geqslant p. \end{aligned}$$

The weak law of large numbers implies

$$\lim_{N\to\infty} \Pr\left\{\frac{R}{N}\leqslant\tau\mid p\right\}=I_\tau(p) \qquad \text{if} \quad \tau\neq p.$$

Since $\Pr\{R/N\leqslant\tau\mid p\}\leqslant 1$ and $\int_0^1 1\,dF(p)=1$, dominated convergence implies

$$\lim_{N\to\infty} \Pr\{R/N\leqslant\tau\}=\int_0^1 I_\tau(p)\,dF(p)=F(\tau)$$

if F is continuous at τ. Hence $\Pr(R/N\leqslant\tau)$ converges weakly to F.

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