Contents

1 Probability

1.1 Probability

1. Fill in the details of the proof of Theorem 1.8. Also, prove the monotone decreasing case.

   - For the readers' convenience we restate the Continuity of Probabilities theorem. If $A_n \to A$ then $P(A_n) \to P(A)$ as $n \to \infty$. Here $A_n \to A_n$ means that either A_n is monotone increasing ($A_n \subseteq A_{n+1}$) and we define $A = \bigcup_{n=1}^{\infty} A_n$, or, A_n is monotone decreasing ($A_n \supseteq A_{n+1}$) and we define $A = \bigcap_{n=1}^{\infty} A_n$.
   - We fill in the details now. First of all we want to show that $B_i \cap B_j = \emptyset$ for all $i \neq j$. Suppose without loss of generality that $i < j$ and note that $B_i \subseteq A_i$ then
   $$B_j = A_j \setminus \bigcup_{k=1}^{j-1} A_k$$
   and $B_i \subseteq A_i \subseteq \bigcup_{k=1}^{j-1}$ as such $B_i \cap B_j = \emptyset$.
   - To see that $A_n = \bigcup_{i=1}^{n} B_i$ let $x \in A_n$ then there exists a minimal $k = k(x)$ such that $x \in A_k$, i.e., for all $k' < k : x \notin A_{k'}$. Then $x \notin \bigcup_{i=1}^{k-1} A_i$ and therefore $x \in B_k$. Because x are arbitrary it folows that $A_n \subseteq \bigcup_{i=1}^{n} B_i$. On the other hand
   $$\bigcup_{i=1}^{n} \underbrace{B_i}_{\subseteq A_i} \subseteq \bigcup_{i=1}^{n} A_i = A_n,$$
   where we have used that A_n is monotone increasing. The property that
   $$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$
   holds is identical to the finite case, an element is part of the (countably) infinite union if there exists some minimal i such that ...
   - For the monotone decreasing case we instead want to define
   $$B_n := A_n \setminus \bigcup_{i>n} A_i.$$

2. Prove the statements in equation (1.1).

   - This can immediately be seen by noting that $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$ is a disjoint union.

3. Let Ω be a sample space and let $A_1, A_2, \ldots,$ be events. Define $B_n = \bigcup_{i=n}^{\infty} A_i$ and $C_n = \bigcap_{i=n}^{\infty} A_i$

(a) Show that (B_i) is monotone decreasing and that (C_i) is monotone increasing.

(b) Show that $\omega \in \bigcap_{n=1}^{\infty} B_n$ if and only if ω belongs to an infinite number of events $A_1, A_2, \ldots$.

- Let $\omega \in \bigcap_{n=1}^{\infty} B_n$ be such that ω does not belong to an infinite number of events $A_1, A_2, \ldots$, i.e., there exists $N \in \mathbb{N}$ such that for all $k > N$ it follows that $\omega \notin A_k$. But then $\omega \notin B_k$ for all $k > N$ and as such does not lie in the intersection over all the B_k, which is a contradiction.
- Suppose ω lies in infinitely many $A_1, A_2, \ldots$. Then there exists a sequence $(A_{n_i})_{i \in \mathbb{N}}$ such that $n_i < n_j$ for all $i < j$ and such that $\omega \in A_{n_i}$ for all $i \in \mathbb{N}$. In that case

$$\omega \in B_{n_i}$$

for all $i \in \mathbb{N}$, in particular $\omega \in \cap_{i=\mathbb{N}} B_{n_i}$. Because B_n is monotone decreasing the statement follows.

(c) Show that $\omega \in \bigcup_{n=1}^{\infty} C_n$ if and only if ω belongs to all the events $A_1, A_2, \ldots$ except possibly a finite number of those events.

- Let $\omega \in \bigcup_{n=1}^{\infty} B_n$ then there exists $n \geq 1$ such that $\omega \in B_n$. This means that $\omega \in \bigcap_{i=n}^{\infty}$, i.e., ω belongs to all A_i where $i \geq n$.
- Now suppose ω belongs to all events $A_1, A_2, \ldots$ except for a finite number of events. In that case there exist $N \in \mathbb{N}$ such that $\forall k \geq N : \omega \in A_k$. This means that

$$\omega \in \bigcap_{i \geq k} A_i,$$

i.e., $\omega \in C_k$. In particular this shows

$$\omega \in C_k \subseteq \bigcup_{r \geq 1} C_r.$$

4. Let $\{A_i : i \in I\}$ be a collection of events where I is an arbitrary index set. Show that

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c, \quad \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c$$

holds.

Hint: First prove this for $I = \{1, 2, \ldots, n\}$.

- I guess they want us to show it using induction in the hint, I'll just do it directly.

$$
\begin{aligned}
\bigcap_i A_i^c &= \{x : \forall i : x \notin A_i\} \\
&= \{x : \neg\,(\exists i : x \in A_i)\} \\
&= \{x : \exists i : x \in A_i\}^c = \left(\bigcup_{i \in I} A_i\right)^c
\end{aligned}
$$

The other equality can be proven analogously.

5. Suppose we toss a fair coin until we get exactly two heads. Describe the sample space S. What is the probability that exactly k tosses are required?

6. Let $\Omega = \{0, 1, \dots\}$. Prove that there does not exist a uniform distribution on Ω (i.e., if $P(A) = P(B)$ whenever $|A| = |B|$, then P cannot satisfy the axioms of probability.)

   - Note that $P(\{0\}) \neq 0$ because otherwise $P = 0$, which would mean P is not a probability function. But then it follows for all $n \in \mathbb{N}$:

   $$
   \begin{aligned}
   P(\{0, \dots, n\}) &= P\left(\bigcup_{k=0}^{n} \{k\}\right) \\
   &= \sum_{k=0}^{n} P(\{k\}) \\
   &= (n+1)P(\{0\}) \to \infty
   \end{aligned}
   $$

   for $n \to \infty$, which is a contradiction to $P(\Omega) = +1$.

7. Let $A_1, A_2, \dots$ be events. Show that

   $$\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

   Hint: Define $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$. Then show that the B_n are disjoint and that

   $$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

   - This was basically already shown in the first exercise, the only thing we mention here is that we only have to consider the case where the series on the right hand side converges, because the left hand side is always ≤ 1.

8. Suppose that $P(A_i) = 1$ for each i. Prove that

   $$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1.$$

- Note that $P(A_i^c) = 1 - P(A_i) = 0$ and that

  $$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right).$$

  The latter can be calculated as follows:

  $$\begin{aligned} 0 &\leq P\left(\bigcup_{i=1}^{\infty} A_i^c\right) \\ &\leq \sum_{i=1}^{\infty} P(A_i^c) = 0 \end{aligned}$$

  where we have used that probability measures are subadditive. As such $P\left(\bigcup_{i=1}^{\infty} A_i^c\right) = 0$ and therefore

  $$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1.$$

9. For fixed B such that $P(B) > 0$, show that $P(\cdot|B)$ satisfies the axioms of probability.

Axiom 1: Let A be arbitrary then

$$P(A|B) = \frac{P(A,B)}{P(B)} \geq P(A,B) \geq 0.$$

Axiom 2:

$$P(\Omega|B) = \frac{P(\Omega, B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

Axiom 3:

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) &= \frac{1}{P(B)} P\left(\bigcup_{i=1}^{\infty} A_i,\right) \\ &= \frac{1}{P(B)} \sum_{i=1}^{\infty} P(A_i, B) \\ &= \sum_{i=1}^{\infty} P(A_i|B). \end{aligned}$$

10. You have probably heard it before. Now you can solve it rigorously. It is called the "Monty Hall Problem." A prize is placed at random behind one of three doors. You pick a door. To be concrete, let's suppose you always pick door 1. Now Monty Hall chooses one of the other two doors, opens it and shows you that it is empty. He then gives you the opportunity to keep your door or switch to the other unopened door.Should you stay or switch? Intuition suggests it doesn't matter. The correct answer is that

you should switch. Prove it. It will help to specify he sample space and the relevant events carefully. Thus write $\Omega = \{(\omega_1, \omega_2) : \omega_i \in \{1, 2, 3\}\}$ where ω_1 is where the prize is and ω_2 is the door Monty opens.

- Note that $\{(1,1), (2,2), (3,3)\}$ are invalid because he'll never open the door with the price. The staying strategy wins on $\{(1,2), (1,3)\}$, the switching strategy wins on $\{(2,3), (3,2)\}$. Note that this means, if $\omega_1 \neq 1$, then we are guaranteed to win. As such we win whenever $\omega_1 \in \{2, 3\}$, as such the winning probability is $2/3$ for the switching strategy.
- This is a pretty subtle problem, the intuition is that him opening a door does not grant new information. If $\omega_1 = 1$ then his reveal is arbitrary and we lose on switching. If $\omega_1 = 2, 3$ then he's forced to reveal the nonempty door, as such either the remaining door is the price or $\omega_1 = 1$. As such when we switch we are guaranteed to win whenever $\omega_1 = 2, 3$.

11. Suppose that A and B are independent events. Show that A^c and B^c are independent events.

    - $$\begin{aligned} P(A^c, B^c) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A, B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^C). \end{aligned}$$

12. There are three cards. The first is green on both sides, the second is red on both sides and the third is green on one side and red on the other. We choose a card at random and we see one side (also chosen at random) . If the side we see is green, what is the probability that the other side is also green? Many people intuitively answer $\frac{1}{2}$. Show that the correct answer is $\frac{2}{3}$.

13. Suppose that a fair coin is tossed repeatedly until both a head and tail have appeared at least once.

    (a) Describe the sample space Ω.

    (b) What is the probability that three tosses will be required?

14. Show that if $P(A) = 0$ or $p(A) = 1$ then A is independent of every other event. Show that if A is independent of itself then $P(A)$ is either 0 or 1.

    - $P(A) = P(A, A) = P(A)P(A) = P(A)^2 \iff P(A)(P(A) - 1).$

15. The probability that a child has blue eyes is $1/4$. assume independence between children. Consider a family with 3 children.

(a) If it is known that at least one child has blue eyes, what ist he probability that at least two children have blue eyes?

- Assuming that having blue eyes is independent from the other children then the chance that none of the siblings have blue eyes is $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$. As such the probability is $7/16$.

(b) If it is known that the youngest child has blue eyes, what is the probability that at least two children have blue eyes?

- While this doesn't seem intuitive this actually changes the probabilities. To see this note that in the first problem the arrangements of having at least 2 blue eyed children were BBB, BBG, BGB, GBB where the children are ordered by age. But now we disregard the case GBB as such the probability becomes smaller.
- The probability is then $\left(\frac{1}{4}\right)^2 + \frac{1}{4}\frac{3}{4} + \frac{3}{4}\frac{1}{4}$

16. Prove Lemma 1.14.

- $$P(A|B) = \frac{P(A,B)}{P(B)} = \frac{P(A)P(B))}{P(B)} = P(A).$$
- $$\begin{aligned} P(AB) &= P(A,B)\frac{P(B)}{P(B)} \\ &= P(A|B)P(B) \\ &= P(B|A)\frac{P(A)}{P(B)}P(B) \\ &= P(B|A)P(A). \end{aligned}$$

17. Show that
$$P(A,B,C) = P(A|BC)P(B|C)P(C).$$

18. Suppose k events form a partition of the sample space Ω, i.e., they are disjoint and
$$\bigcup_{i=1}^{k} A_i = \Omega.$$
Assume that $P(B) > O$. Prove that if $P(A_1|B) < P(A_1)$ then $P(A_i|B) > P(A_i)$ for some $i = 2, \ldots, k$.

- Suppose this statement is wrong, i.e. $P(A_1|B) < P(A_1)$ and for all $i = 2, \ldots, k$ we have $P(A_i|B) \leq P(A_i)$. It then follows that
$$\begin{aligned} \sum_{i=1}^{k} P(A_i|B) &= \underbrace{P(A_1|B)}_{<P(A_1)} + \sum_{i=2}^{k} \underbrace{P(A_i|B)}_{\leq P(A_i)} \\ &< P(A_1) + \sum_{i=1}^{k} P(A_i) = 1. \end{aligned}$$

Multiplying $\sum_{i=1}^{n} P(A_i|B) < 1$ by $P(B) > 0$ yields

$$\sum_{i=1}^{n} P(A_i|B)P(B) < P(B).$$

This is a contradiction because

$$\begin{aligned}\sum_{i=1}^{n} P(A_i|B)P(B) &= \sum_{i=1}^{n} P(A_i, B)\\ &= P\left(\bigcup_{i=1}^{n} A_i, B\right)\\ &= P(\Omega, B) = P(B).\end{aligned}$$

19. Suppose that 30% of computer owners use a Macintosh, 50% use Windows, and 20% use Linux. Suppose that 65% of the Mac users have succumbed to a computer virus, 82% of the Windows users get the virus, and 50% of the Linux users get the virus. We select a person at random and learn that her system was infected with the virus. What is the probability that she is a Windows user?

    - Denote the operating systems by M, W, L and having the virus by V. Using this notation we obtain $P(M) = 0.3$, $P(W) = 0.5$ and $P(L) = 0.2$. Furthermore $P(V|M) = 0.65$, $P(V|W) = 0.82$ and $P(V|L) = 0.5$ We can use the law of total probability to calculate the probability of a random person having a virus as being

    $$\begin{aligned}P(V) &= P(V, M) + P(V, W) + P(V, L)\\ &= P(V|M)P(M) + P(V|W)P(W) + P(V|L)P(L)\\ &= 0.65 \cdot 0.3 + 0.82 \cdot 0.5 + 0.5 \cdot 0.2\\ &= 0.705 = 70.5\%.\end{aligned}$$

    Now we can use Bayes' Theorem to calculate

    $$P(W|V) = P(V|W)\frac{P(W)}{P(V)} = 0.82\frac{0.5}{0.705} = 0.5816 = 58.16\%$$

20. A box contains 5 coins and each has a different probability of showing heads. Let P_1, P_2, P_3, P_4, P_5 denote the probability of heads on each coin. Suppose that

    $$p_1 = 0, \quad p_2 = 1/4, \quad p_3 = 1/2, \quad p_4 = 3/4, \quad p_5 = 1.$$

    Let H denote "heads is obtained" and let C_i denote the event that coin i is selected.

    (a) Select a coin at random and toss it. Suppose a head is obtained. What is the posterior probability that coin i was selected ($i = 1, ..., 5$)? In other words, find $P(C_i|H)$ for $i = 1, ..., 5$.

- Note that $P(C_i) = \frac{1}{5}$ and $P(H|C_i) = p_i$. The total probability of gettings heads is

$$
\begin{aligned}
P(H) &= \sum_{i=1}^{5} P(H|C_i)P(C_i) \\
&= \frac{1}{5}\sum_{i=1}^{5} P(H|C_i) \\
&= \frac{1}{5}\sum_{i=1}^{5} p_i \\
&= \frac{1}{5}(0 + 1/4 + 1/2 + 3/4 + 1) \\
&= 0.5 = 50\%.
\end{aligned}
$$

We can use Bayes' theorem to calculate

$$
\begin{aligned}
P(C_i|H) &= P(H|C_i)\frac{P(C_i)}{P(H)} \\
&= P(H|C_i)\frac{\frac{1}{5}}{0.5} \\
&= P(H|C_i)\frac{2}{5}.
\end{aligned}
$$

Using this we obtain

i	1	2	3	4	5
$P(C_i\|H)$	0	1/10	1/5	3/10	2/5

(b) Toss the coin again. What is the probability of another head? In other words find $P(H_2|H_1)$ where H_j = "heads on toss j."

- Suppose we have chosen the *ith* coin then the probability of getting heads is $P(H|C_i)$. The probabilities of the individual coins have been calculated before and we just have to take the weighted sum over those.

```
import numpy as np

mC = np.array([0, 1/10, 1/5, 3/10, 2/5])
mP = np.array([0, 0.25, 0.5, 0.75, 1])
np.dot(mC, mP)

> 0.75
```

(c) Find $P(C_i|B_4)$ where B_4 = "first head is obtained on toss 4."

- First note that $P(C_i) = P(C_i, H_1) + P(C_i, H_1^c)$ as such

$$
\begin{aligned}
P(C_i, H_1^c) &= P(C_i) - P(C_i, H_1) \\
&= P(C_i) - P(C_i|H_1)P(H_1) \\
&= \frac{1}{5} - P(C_i|H_1)\frac{1}{2}
\end{aligned}
$$

as such

i	1	2	3	4	5
$P(C_i \mid H_i^c)$	0.2	0.15	0.1	0.05	0

$P(H_1, H_2) = P(H_2|H_1)P(H_1) = \frac{3}{4}$

21. (Computer Experiment.) Suppose a coin has probability p of falling heads up. If we flip the coin many times, we would expect the proportion of heads to be near p. We will make this formal later. Take $p = 0.3$ and $n = 1000$ and simulate n coin flips. Plot the proportion of heads as a function of n. Repeat for $p = 0.03$.

- ```python
  import numpy as np
  import matplotlib.pyplot as plt
  import math

  def Ex1_21(p = 0.3, n = 1000, draw = True, save = False):
      rand = np.random.random(size = 1000)
      _range = np.arange(1, n + 1)

      result = np.cumsum(rand < p) / _range

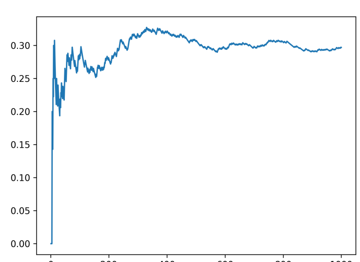
      fig, ax = plt.subplots()

      ax.plot(_range, result)

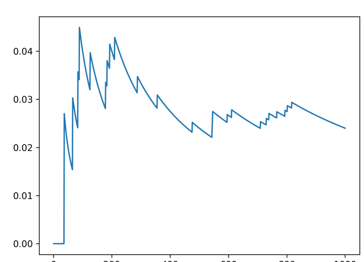
      if draw:
          plt.show()

      if save:
          # p = 0.213, n = 1000 -> 21pct_1000
          str_form = f"{math.floor(p * 100)}pct_{n}"
          fig.savefig(f"Ex1_21-{str_form}.png")

  Ex1_21(p = 0.3, draw = True, save = True)
  Ex1_21(p = 0.03, draw = True, save = True)
  ```



(a) 30% Probability



(b) 3% Probability
  ```

22. (Computer Experiment.) Suppose we flip a coin n times and let P denote the probability of heads. Let X be the number of heads. We call X a binomial random variable, which is discussed in the next chapter. Intuition suggests that X will be close to n p. To see if this is true, we can repeat this experiment many times and average the X values. Carry out a simulation and compare the average of the X's to n p . Try this for $p = 0.3$ and $n = 10$, $n = 100$, and $n = 1000$.

    - ```
      import numpy as np

      def Ex1_22(p = 0.3, n = 1000, num_iter = 100):
          sum = 0
          for i in range(num_iter):
              rand = np.random.random(size = n)
              sum += np.sum(rand < p)

          mean = sum / num_iter
          expected = n * p
          delta = abs(mean - expected) / (expected)
          print(f"mean = {mean}, delta = {delta * 100:.2f}%")

      for n in [10, 100, 1000]:
          Ex1_22(n = n)

      > mean = 2.93, delta = 2.33%
      > mean = 30.2, delta = 0.67%
      > mean = 297.83, delta = 0.72%
      ```

23. (Computer Experiment.) Here we will get some experience simulating conditional probabilities. Consider tossing a fair die. Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$. Then, $P(A) = 1/2$, $P(B) = 2/3$ and $P(AB) = 1/3$. Since $P(AB) = P(A)P(B)$, the events A and B are independent. Simulate draws from the sample space and verify that $\hat{P}(AB) = \hat{P}(A)\hat{P}(B)$ where $\hat{P}(A)$ is the proportion of times A occurred in the simulation and similarly for $\hat{P}(AB)$ and $\hat{P}(B)$. Now find two events A and B that are not independent. Compute $\hat{P}(A)$,$\hat{P}(B)$ and $\hat{P}(AB)$. Compare the calculated values to their theoretical values. Report your results and interpret.

    - ```
      import numpy as np

      labels = ['P(A)', 'P(B)', 'P(A, B)', 'P(A)P(B)',
                'P(A, B) - P(A)P(B)']
      pct_str = lambda x : f"{x * 100:.2f}%"

      def Ex1_23_helper(probabilities):
          for label, prob in zip(labels, probabilities):
              print(f"\t{label}: {pct_str(prob)}")
      ```

```
def Ex1_23(A, B, n = 10000):
    cap = A.intersection(B)

    # rand does [) so I need to increment high to get 6
    rand = np.random.randint(low = 1, high = 6 + 1, size = n)
    prob_A = sum(np.isin(rand, list(A))) / n
    prob_B = sum(np.isin(rand, list(B))) / n
    prob_cap = sum(np.isin(rand, list(cap))) / n

    prob_prod = prob_A * prob_B

    probabilities = [prob_A, prob_B, prob_cap, prob_prod,
                     prob_cap - prob_prod]

    print("Experimental Values")
    Ex1_23_helper(probabilities)

A, B = {2, 4, 6}, {1, 2, 3, 4}
print(f"A = {A}, B = {B}, AB = {A.intersection(B)}")
print("Theoretical Values")
Ex1_23_helper([1/2, 2/3, 1/3, 1/3, 0])
Ex1_23(A, B)
print()

A, B = {1, 2, 3}, {3, 4, 5, 6}
print(f"A = {A}, B = {B}, AB = {A.intersection(B)}")
print("Theoretical Values")
Ex1_23_helper([1/2, 2/3, 1/6, 1/3, -1/6])
Ex1_23(A, B)

> A = {2, 4, 6}, B = {1, 2, 3, 4}, AB = {2, 4}
> Theoretical Values
>         P(A): 50.00%
>         P(B): 66.67%
>         P(A, B): 33.33%
>         P(A)P(B): 33.33%
>         P(A, B) - P(A)P(B): 0.00%
> Experimental Values
>         P(A): 50.18%
>         P(B): 67.42%
>         P(A, B): 33.70%
>         P(A)P(B): 33.83%
>         P(A, B) - P(A)P(B): -0.13%
>
> A = {1, 2, 3}, B = {3, 4, 5, 6}, AB = {3}
```

```
> Theoretical Values
>         P(A): 50.00%
>         P(B): 66.67%
>         P(A, B): 16.67%
>         P(A)P(B): 33.33%
>         P(A, B) - P(A)P(B): -16.67%
> Experimental Values
>         P(A): 50.20%
>         P(B): 66.30%
>         P(A, B): 16.50%
>         P(A)P(B): 33.28%
>         P(A, B) - P(A)P(B): -16.78%
```

1.2 Random Variables

1. Show that
$$P(X = x) = F(x^+) - F(x^-).$$

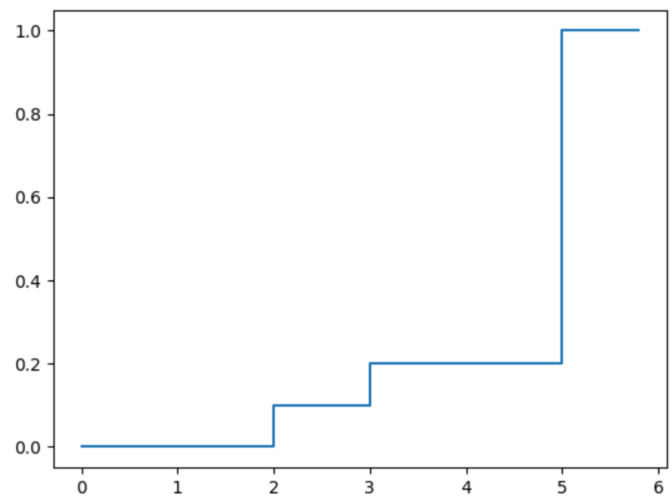
   - Note that (∞, x) and $\{x\}$ are disjoint as such
   $$P(X \leq x) = P(X < x) + P(X = x).$$
   This allows us to calculate:
   $$\begin{aligned} P(X = x) &= P(X \leq x) - P(X < x) \\ &= F(x) - P(X < x) \\ &= F(x) - \lim_{\varepsilon \searrow 0} P(X \leq x - \varepsilon) \\ &= F(x) - \lim_{y \searrow x} P(X \leq y) \\ &= F(x) - \lim_{y \searrow x} F(y) \\ &= F(x^+) - F(x^-), \end{aligned}$$
   where we have used that $F(x) = F^+(x)$ by definition of CDFs.

2. Let X be such that $P(X = 2) = P(X = 3) = 1/10$ and $P(X = 5) = 8/10$. Plot the CDF F. Use F to find $P(2 < X \leq 4.8)$ and $P(2 \leq X \leq 4.8)$.

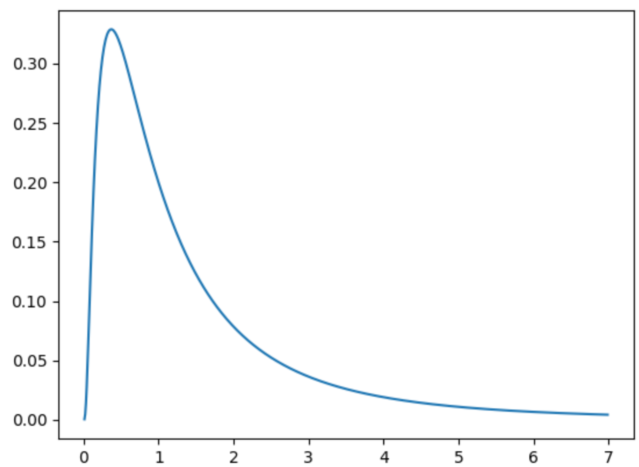
   - ```python
     import numpy as np
     import matplotlib.pyplot as plt

     def Ex2_2(draw = True, save = False):
         x = np.arange(start=0, stop = 6, step = 0.2)
         F = (x >= 2) * 0.1 + (x >= 3) * 0.1 + (x >= 5) * 0.8
         plt.step(x, F, where='post')

         if save:
     ```
     ```

```
        plt.savefig("Ex2_2.png")
    if draw:
        plt.show()

    right = np.where(abs(x - 4.8) < 0.1)[0]
    left = np.where(abs(x - 2) < 0.1)[0]
    print("P(2 <  X <= 4.8) = ", (F[right] - F[left])[0])
    print("P(2 <= X <= 4.8) = ", (F[right] - F[left - 1])[0])

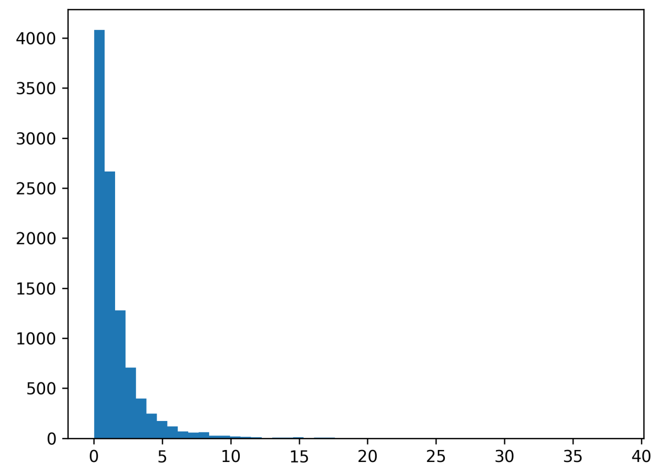
Ex2_2(draw = False, save = False)

> P(2 <  X <= 4.8) =  0.1
> P(2 <= X <= 4.8) =  0.2
```



3. Prove Lemma 2.15.

4. Let X have probability density function

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } 0 < x < 1 \\ \frac{3}{8} & \text{if } 3 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the cumulative distribution function of X.

   - We can obtain the CDF F of f by integration:

$$F(t) := \int_{-\infty}^{t} f(x)dx,$$

For $t \leq 0$ we have $F(t) = 0$. Let $t \in (0,1)$ then

$$F(t) = \int_0^t \frac{1}{4} dx = \frac{t}{4}.$$

For $t \in [1,3]$ we have

$$F(t) = \int_0^1 f(x) dx = \frac{1}{4}.$$

Now let $t \in (3,5)$ then

$$\begin{aligned} F(t) &= \int_0^1 f(x) dx + \int_3^t f(x) dx \\ &= \frac{1}{4} + (t-3)\frac{3}{8}. \end{aligned}$$

And for $t \geq 5 : F(t) = 1$. In total this shows that

$$F(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \frac{t}{4}, & \text{if } t \in (0,1), \\ \frac{1}{4}, & \text{if } t \in [1,3], \\ \frac{1}{4} + \frac{3}{8}(t-3), & \text{if } t \in (3,5), \\ 1, & \text{if } t \geq 5. \end{cases}$$

(b) Let $Y = 1/X$. Find the probability density function $f_Y(y)$ for Y. Hint. Consider three cases: $\frac{1}{5} \leq y \leq \frac{1}{3}$, $\frac{1}{3} \leq y \leq 1$, and $y \geq 1$.

- First of all, note that $0 < X < 5$, as such $\frac{1}{5} < Y$. This means that

$$P(Y \leq y) = P\left(\frac{1}{5} < Y \leq y\right)$$

Now suppose $y \in \left(\frac{1}{5}, \frac{1}{3}\right)$, then

$$P\left(Y \leq y\right) = P\left(\frac{1}{5} < Y \leq y\right)$$

By definition

$$\frac{1}{5} < Y \leq y \iff \frac{1}{y} \leq \frac{1}{Y} < 5 \iff \frac{1}{y} \leq X < 5.$$

Then

$$P\left(Y \leq y\right) = P\left(\frac{1}{y} \leq X < 5\right) = F(5) - F\left(\frac{1}{y}\right).$$

This can then be computed by noting that $F(5) = 1$ and $F(1/y) = \frac{1}{4} + \frac{3}{8}(\frac{1}{y} - 3)$.

For $\frac{1}{3} \leq y \leq 1$ we have

$$P(Y \leq y) = F(3) - F\left(\frac{1}{y}\right)$$

and this is equal to $\frac{1}{4}$. For $y \geq 1$ we have $\frac{1}{y} \leq 1$ and

$$\frac{1}{4} - \frac{1}{4y}.$$

In total this means [TODO: Finish this].

5. Let X and Y be discrete random variables. Show that X and Y are independent if and only if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x and y.

6. Let X have distribution F and density function f and let A be a subset of $\mathbb{R}$. Denote by $I_A(x)$ the indicator function for A. Let $Y = I_A(X)$. Find an expression for the cumulative distribution of Y. Hint: first find the probability mass function for Y.

   - Denote the CDF of Y by G. Note that Y either takes 0 or 1 as its value. The PDF g of Y is $g(t) = 1$ if $X(t) \in A$ and 0 else. This means that the CDF for $y < 0$ is just 0, for $0 \leq y < 1$, $G(y) = P(X \in A)$ and $G(y) = 1$ for all $y \geq 1$. Note that

     $$P(X \notin A) = 1 - P(X \in A) = 1 - \int_A f(x)dx.$$

7. Let X and Y be independent and suppose that each has a Uniform$(0,1)$ distribution. Let $Z = \min\{X, Y\}$. Find the density $f_Z(z)$ for Z. Hint: It might be easier to first find $P(Z > z)$.

   - By definition we have

     $$P(Z > z) = P(\min X, Y > z) = P(X > z, Y > z),$$

     because X and Y are independent it follows that this is equal to $P(X > z)P(Y > z)$, because both of those have the same distribution those two probabilities are the same:

     $$P(X > z)P(Y > z) = P(X > z)^2 = (1 - P(X \leq z))^2 = (1 - z)^2.$$

     As such

     $$P(Z \leq z) = 1 - p(Z > z) = 1 - (1 - z)^2 = 2z - z^2.$$

     Differentiating this yields the PDF

     $$f_Z(z) = 2 - 2z.$$

8. Let X have CDF F. Find the CDF of $X^+ = \max 0, X$.

   - Note that for $y \leq 0$ we have $P(X^+ \leq y) = F(0)$. For $x > 0$ we have

$$
\begin{aligned}
F(X^+ \leq x) &= F(X^+ \leq 0) + P(0 < X^+ \leq x) \\
&= F(0) + P(0 < X \leq x) \\
&= F(0) + F(x) - F(0) \\
&= F(x).
\end{aligned}
$$

9. Let $X \sim \text{Exp}(\beta)$. Find $F(x)$ and $F^{-1}(q)$.

   - The PDF of f is given by $f(x) = \frac{1}{\beta} e^{-x/\beta}$ for $x > 0$ and 0 for $x \leq 0$. We obtain the CDF of X by integrating. For $t \leq 0$ we have F(t) and for $t > 0$

$$F(t) = \int_{-\infty}^{t} f(x)dx = \int_{0}^{t} \frac{1}{\beta} e^{-x/\beta} dx.$$

   Recall that the antiderivative of e^{Ax} is $\frac{1}{A} e^{Ax}$ as such

$$
\begin{aligned}
F(t) &= \frac{1}{\beta} \frac{1}{-\frac{1}{\beta}} e^{-x/\beta} \Bigg|_{x=0}^{x=t} \\
&= -e^{-t/\beta} + e^{-0/\beta}.
\end{aligned}
$$

   As such

$$F(t) = 1 - e^{-t/\beta}.$$

   Per definition $F^{-1}(q) = \inf\{x : F(x) > q\}$ for $q \in [0, 1]$. For a continuous and monotone increasing function F (which is the case for the exponential distribution) this is simply

$$x = F^{-1}(q).$$

   As such we want to solve $F(x) = q$ for x, this is a straightforward computation:

$$
\begin{aligned}
q &= 1 - e^{-x/\beta} \\
\iff \quad e^{-t/\beta} &= 1 - q \\
\iff \quad -t/\beta &= \log(1 - q) \\
\iff \quad t &= -\beta \log(1 - q).
\end{aligned}
$$

10. Let X and Y be independent. Show that $g(X)$ is independent of $h(Y)$ where g and h are functions.

11. Suppose we toss a coin once and let p be the probability of heads. Let X denote the number of heads and let Y denote the number of tails.

    (a) Prove that X and Y are dependent.

- Suppose $p \in (0,1)$ then $P(X=1) = p$ and $P(Y=1) = 1-p$ but $P(X=1, Y=1) = 0$, as such X and Y can't be independent.

(b) Let $N \sim \text{Poisson}(\lambda)$ and suppose we toss a coin N times. Let X and Y be the number of heads and tails. Show that X and Y are independent.

12. Prove Theorem 2.33

13. Let $X \sim \mathcal{N}(0,1)$ and let $Y = e^X$.

(a) Find the PDF for Y. Plot it.

- Note that $P(Y \leq t) = P(e^X \leq t) = P(X \leq \log(t))$. We can obtain the PDF g of Y by differentiation:

$$\frac{d}{dt} F(\log(t)) = \frac{1}{\sqrt{2\pi} t} e^{-\frac{\log(t)^2}{2}}.$$

```
import numpy as np
import matplotlib.pyplot as plt

x = np.arange(start = 0.01, stop = 7, step = 0.01)
y = np.exp(- np.log(x)**2 * 0.5) * 0.5 / x
y = y / np.sqrt(2 * np.pi)
plt.plot(x, y)

plt.savefig('Ex2_13a.png')
```



(b) (Computer Experiment) Generate a vector $x = (x_1, \ldots, x_{10000})$ consisting of 10000 random standard Normals. Let $y = (y_1, \ldots, y_{10000})$ where $y_i = e^{x_i}$. Draw a histogram of y and compare it to the PDF you found in part (a).

- ```
  import numpy as np
  import matplotlib.pyplot as plt

  x = np.random.randn(10000)
  y = np.exp(x)

  plt.hist(y, bins=50)
  plt.savefig('Ex2_13b.png', dpi=300)
  ```



14. Let (X, Y) be uniformly distributed on the unit disk $\{(x, y) : x^2+y^2 \leq 1\}$. Let $R = \sqrt{X^2 + Y^2}$. Find the CFD and PDF of R.

15. Let X have a continuous, strictly increasing CDF F. Let $Y = F(X)$. Find the density of Y. This is called the probability integral transform. Now, let $U \sim \text{Uniform}(0, 1)$ and let $X = F^{-1}(U)$. Show that $X \sim F$. Now write a program that takes Uniform $(0, 1)$ random variables and generates random variables from an Exponential (λ) distribution.

16. Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ and assume that X and Y are independent. Show that the dstribution of X given that $X + Y = n$ is $\text{Binomial}(n, \pi)$ where $\pi = \frac{\lambda}{\lambda+\mu}$. Hint 1: You may use the following fact: If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$, and X and Y are independent, then $X + Y \sim \text{Poisson}(\lambda + \mu)$. Hint 2: Note that $\{X = x, X + Y = n\} = \{X = x, Y = n - x\}$.

    - Recall that

    $$P(X = x|X + Y = n) = \frac{P(X = x, X + Y = n)}{P(X + Y = n)}.$$
  ```

By the first hint $P(X+Y=n) = \frac{(\lambda+\mu)^n}{n!}e^{-\lambda-\mu}$. Using the second hint we obtain

$$P(X=x, X+Y=n) = P(X=x, Y=n-x),$$

using the independence of X and Y this becomes

$$P(X=x)P(Y=n-x) = \frac{\lambda^x}{x!}e^{-\lambda}\frac{\mu^{n-x}}{(n-x)!}e^{-\mu}.$$

Combining everything then yields

$$\begin{aligned} P(X=x|X+Y=n) &= \frac{\frac{\lambda^x}{x!}e^{-\lambda}\frac{\mu^{n-x}}{(n-x)!}e^{-\mu}}{\frac{(\lambda+\mu)^n}{n!}e^{-\lambda-\mu}} \\ &= \frac{\lambda^x\mu^{n-x}}{(\lambda+\mu)^n}\frac{n!}{x!(n-x)!} \quad = \frac{\lambda^x\mu^{n-x}}{(\lambda+\mu)^n}\binom{n}{x}. \end{aligned}$$

Note that if $Z \sim \text{Bin}\left(n, \frac{\lambda}{\lambda+\mu}\right)$ then

$$\begin{aligned} P(Z=k) &= \binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(1-\frac{\lambda}{\lambda+\mu}\right)^{n-k} \\ &= \binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k} \\ &= \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k}. \end{aligned}$$

As such we have shown that $X|X+Y=n \sim \text{Bin}\left(n, \frac{\lambda}{\lambda+\mu}\right)$.

17.

18.

19. Prove formula (2.12)

20. Let $X, Y \sim \text{Uniform}(0,1)$ be independent. Find the PDF for $X-Y$ and X/Y.

21. Let $X_1, \ldots, X_n \sim \text{Exp}(\beta)$ be iid. Let $Y = \max\{X_1, \ldots, X_n\}$. Find the PDF of Y. Hint: $Y \leq y$ if and only if $X_i \leq y$ for all $i = 1, \ldots, n$.

   - To compute the PDF of Y, we first determine the CDF of Y:

$$\begin{aligned} P(Y \leq y) &= P(X_1 \leq y, \ldots, X_n \leq y) \\ &\stackrel{1}{=} \prod_{i=1}^{n} P(X_i \leq y) \\ &\stackrel{2}{=} P(X_1 \leq y)^n \\ &= \left(1 - e^{-\frac{-y}{\beta}}\right), \end{aligned}$$

where we have used (1) that the X_i are independent and (2) that they are identically distributed. Differentiating this with respect to y yields

$$\frac{d}{dy}P(Y \leq y) = n(1 - e^{-y/\beta})^{n-1}\frac{1}{\beta}e^{-y/\beta}.$$

1.3 Expectation

1. Suppose we playa game where we start with c dollars. On each play of the game you either double or halve your money, with equal probability. What is your expected fortune after n trials?

   - Let X_n be your money after n trials. Note that

     $$E(X_n|X_{n-1} = k) = \frac{1}{2}(2k + k/2) = 1.25k,$$

     as such

     $$E(X_n) = 1.25^n c.$$

2. Show that $Var(X) = 0$ if and only if there is a constant c such that

   $$P(X = c) = 1.$$

3. Let $X_1, \ldots, X_n \sim$ Uniform$(0, 1)$ and let $Y_n = \max\{X_1, \ldots, X_n\}$. Find $E(Y_n)$.

   - We begin by calculating the CDF of Y:

     $$\begin{aligned} F_Y(y) &= P(\max\{X_1, \ldots, X_n\} \leq y) \\ &= P(X_1 \leq s, \ldots, X_n \leq x) \\ &\stackrel{1}{=} \prod i = 1^n P(X_i \leq s) \\ &\stackrel{2}{=} P(X_1 \leq s)^n \\ &= s^n, \end{aligned}$$

     where we have used (1) that the $X_1, \ldots, X_n$ are independent and (2) that they are identically distributed. To obtain the PDF of Y we differentiate this function:

     $$f(y) = F_Y'(y) = ns^{n-1}.$$

     Now we can compute the expected value of Y as follows:

     $$\int_0^1 xf(x)dx = \int_0^1 xnx^{n-1}dx = n\int_0^1 x^n dx = \frac{n}{n+1}.$$

4. A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is $1-p$ that the particle will jump one unit to the right. Let X_n be the position of the particle after n units. Find $E(X_n)$ and $V(X_n)$. (This is known as a random walk.)

   - Note that
     $$\begin{aligned} E(X_n|X_{n-1}=k) &= p(k-1)+(1-p)(k+1) \\ &= (pk-p+k+1-pk-p) \\ &= k+(1-2p), \end{aligned}$$
     as such
     $$E(X_n)=n(1-2p).$$
     [TODO: Do the Variance calculation]

5. A fair coin is tossed until a head is obtained. What is the expected number of tosses that will be required?

   - Note this follows a geometric distribution.
   - Let $X=k$ denote that the first head appeared on the kth flip. Note that every flip itself is a Bernoulli experiment with $p=1-p=1/2$ as such the probability of heads is always $1/2$. This means that
     $$P(X=k)=(1-1/2)^{k-1}(1/2)=2^{-k}.$$
     With that we can calculate
     $$E(X)=\sum_{k\geq 1} kP(X=k)=\sum_{k\geq 1} k2^{-k}=2.$$

6. Prove Theorem 3.6 for discrete random variables

7. Let X be a continuous random variable with CDF F. Suppose that $P(X>0)=1$ and that $E(X)$ exists. Show that
   $$E(X)=\int_0^\infty P(X>x)dx.$$
   Hint: Consider integrating by parts. The following fact is helpful: if $E(X)$ exists then
   $$\lim_{x\to\infty} x[1-F(x)]=0.$$

8. Prove Theorem 3.17

9. (Computer Experiment) Let $X_1, X_2, \ldots, X_n$ be $\mathcal{N}(0,1)$ random variables and let
   $$\overline{X}_n=\frac{1}{n}\sum_{i=1}^n X_i.$$
   Plot $\overline{X}_n$ versus n for $n=1,\ldots,10000$. Repeat for $X_1, X_2, \ldots, X_n \sim$ Cauchy. Explain why there is such a difference.

- ```python
  import numpy as np
  import matplotlib.pyplot as plt

  n = 10000

  samples = np.random.normal(size = n)
  samples_cauchy = np.random.standard_cauchy(size = n)

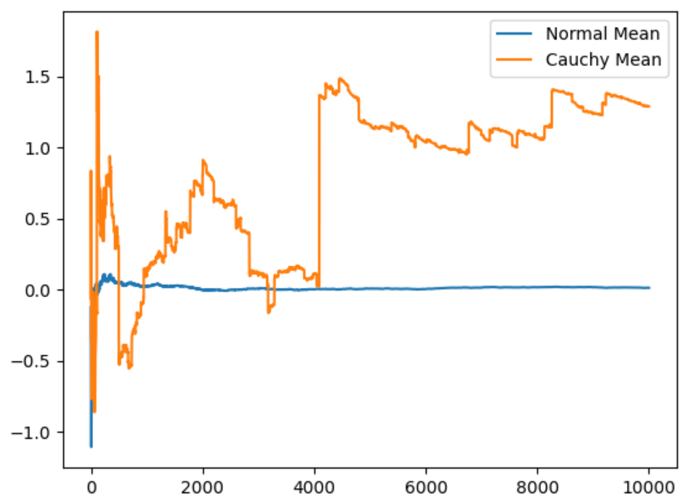
  x = (np.arange(n) + 1)

  mean = np.cumsum(samples) / x
  mean_cauchy = np.cumsum(samples_cauchy) / x

  plt.plot(x, mean, label='Normal Mean')
  plt.plot(x, mean_cauchy, label='Cauchy Mean')

  plt.legend(loc='upper right')

  plt.savefig("Ex3_9.png")
  ```



10.

11. (Computer Experiment: Simulating the Stock Market). Let $Y_1, Y_2, \ldots$ be independent random variables such that $P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2}$. Let $X_n = \sum_{i=1}^{n} Y_i$. Think of $Y_i = 1$ as "the stock market increased by one dollar", $Y_i = -1$ as "the stock market decreased by one dollar", and X_n as the value of the stock on day n.
  ```

(a) Find $E(X_n)$ and $Var(X_n)$.

- By linearity $E(X_n) = \sum_{i=1}^n E(Y_i) = \sum_{i=1}^n (-1)\frac{1}{2} + (1)\frac{1}{2} = 0$. Note that $E(X_n)^2 = 0$, as such $Var(X_n) = E(X_n^2)$. To calculate this note that

$$X_n^2 = \left(\sum_{i=1}^n Y_i\right)^2 = \sum_{i=1}^n Y_i^2 + 2\sum_{i<j} Y_i Y_j.$$

Because $Y_i \in \{-1, 1\}$ it follows that $E(Y_i^2) = 1$ and as such

$$E(X_n^2) = n+2\sum_{i<j} E(Y_i Y_j) = n+2\sum_{i<j} P(Y_i Y_j = +1)-P(Y_i Y_j = -1).$$

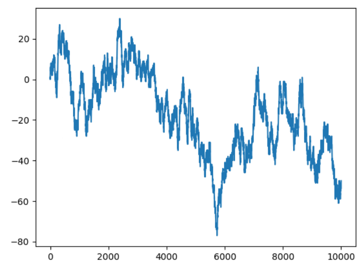
Note that $Y_i Y_j = 1$ if and only if $(Y_i, Y_j) \in \{(1,1), (-1,-1)\}$ and $Y_i Y_j = -1$ if and only if $(Y_i, Y_j) \in \{(-1,1), (1,-1)\}$, as such both of those are equally liked. In total this means that

$$V(X_n) = E(X_n^2) = n.$$

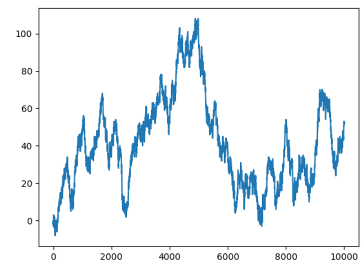
(b) Simulate X_n and plot X_n versus n for $n = 1, 2, \ldots, 10000$. Repeat the whole simulation several times. Notice two things. First, it's easy to "see" patterns in the sequence even though it is random. Second, you will find that the four runs look very different even though they were generated the same way. How do the calculations in (a) explain the second observation?

-
```python
import numpy as np
import matplotlib.pyplot as plt

def Ex3_11(n = 10000, draw = True, save = False, i = 0):
    # 2*0 - 1 = -1, 2 * 1 -1 = 1
    rand = 2 * np.random.randint(low = 0, high = 2, size = n) - 1

    fig, ax = plt.subplots()
    ax.plot(range(0, n), np.cumsum(rand))

    if draw:
        plt.show()

    if save:
        fig.savefig(f"Ex3_11-{i}.png")

for i in range(4):
    Ex3_11(draw = True, save = True, i = i)
```

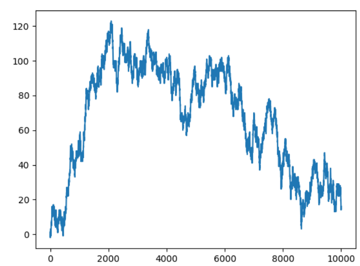
12. Prove the formulas given in the table at the beginning of Section 3.4 for the Bernoulli, Poisson, Uniform, Exponential, Gamma, and Beta. Here



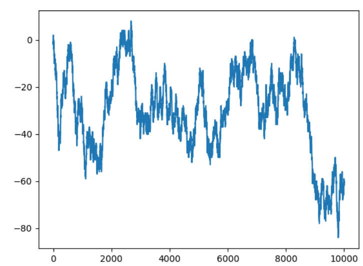
(a)



(b)



(c)



(d)

are some hints. For the mean of the Poisson, use the fact that

$$e^a = \sum_{x=0}^{\infty} \frac{a^x}{x!}.$$

To compute the variance, first compute $E(X(X-1))$. For the mean of the Gamma, it will help to multiply and divide by$\Gamma(\alpha+1)/\beta^{\alpha+1}$ and use the fact that a Gamma density integrates to 1. For the Beta, multiply and divide by

$$\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}.$$

13.

14. Let $X_1, \ldots, X_m$ and $Y_i, \ldots, Y_n$ be random variables and let $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ be constants. Show that

$$\text{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \,\text{Cov}(X_i, Y_j)$$

15. Let

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \le 0 \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $V(2X - 3Y + 8)$.

16. Let $r(x)$ be a function of x and let $s(y)$ be a function of y. Show that

$$E(r(X)s(Y)|X) = r(X)E(s(Y)|X).$$

Also, show that $E(r(X)|X) = r(X)$.

17.

18. Show that if $E(X|Y = y) = c$ for some constant c, then X and Y are uncorrelated.

19.

20. Prove Lemma 3.21.

21. Let X and Y be random variables. Suppose that $E(Y|X) = X$. Show that $\mathrm{Cov}(X, Y) = V(x)$.

22.

23. Find the moment generating function for the Poisson, Normal, and Gamma distributions.

24. Let $X_1, \ldots, X_n \sim \mathrm{Exp}(\beta)$. Find the moment generating function of X_i. Prove that

$$\sum_{i=1}^{n} X_i \sim \mathrm{Gamma}(n, \beta).$$

1.4 Inequalities

1. Let $X \sim \mathrm{Exp}(\beta)$. Find $P(|X - \mu_X| \geq k\sigma_X)$ for $k > 1$. Compare this to the bound you get from Chebyshev's inequality.

2. Let $X \sim \mathrm{Po}(\lambda)$. Use Chebyshev's inequality to show that $P(X \geq 2\lambda) \leq \frac{1}{\lambda}$.

3. Let $X_1, \ldots, X_n \sim \mathrm{Bernoulli}(p)$ (note this is just $\mathrm{Bin}(1, p)$) and

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Bound $P(|\overline{X}_n - p| > \varepsilon)$ using Chebychev's inequality and using Hoeffding's inequality. Show that, when n is large, the bound from Hoeffding's inequality is smaller than the bound from Chebychev's inequality.

4. Let $X_1, \ldots, X_n \sim \mathrm{Bernoulli}(p)$.

(a) Let $\alpha > 0$ be fixed and define

$$\varepsilon_n = \sqrt{\frac{1}{2n} \log\left(\frac{2}{\alpha}\right)}.$$

Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Define $C_n = (\overline{X}_n - \varepsilon, \overline{X}_n + \varepsilon)$. Use Hoeffding's inequality to show that

$$P(C_n \text{ contains } p) \geq 1 - \alpha.$$

In practice, we truncate the interval so it does not go below 0 or above 1.

- Let

$$\varepsilon_n = \sqrt{\frac{1}{2n} \log\left(\frac{2}{\alpha}\right)},$$

then

$$\varepsilon_n^2 = \frac{1}{2n} \log\left(\frac{2}{\alpha}\right).$$

With that we can then calculate

$$\begin{aligned} e^{-2n\varepsilon^2} &= e^{-2n\frac{1}{2n}\log\left(\frac{2}{\alpha}\right)} \\ &= e^{-\log\left(\frac{2}{\alpha}\right)} \\ &= e^{\log\left(\frac{\alpha}{2}\right)} \\ &= \frac{\alpha}{2}. \end{aligned}$$

Recall that Hoeffding's inequality (the bernoulli case) states that

$$P(|\overline{X}_n - p| > \varepsilon_n) \leq 2e^{-2n\varepsilon_n^2}.$$

As such

$$P(|\overline{X} - p| > \varepsilon_n) \leq \alpha.$$

The left hand side is

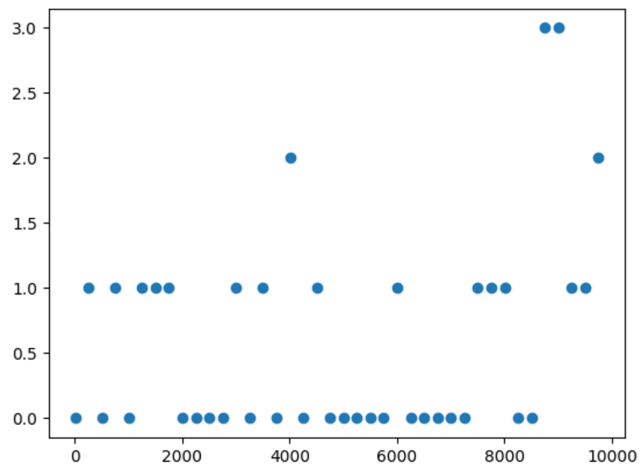
$$\begin{aligned} &1 - P(|\overline{X}_n - p| \leq \varepsilon_n) \\ &\qquad = 1 - P(\overline{X}_n - \varepsilon \leq p \leq \overline{X}_n + \varepsilon) \\ &\qquad = 1 - P(p \in C_n) = 1 - P(C_n \text{ contains } p). \end{aligned}$$

Putting everything together yields

$$1 - \alpha \leq P(C_n \text{ contains } p).$$

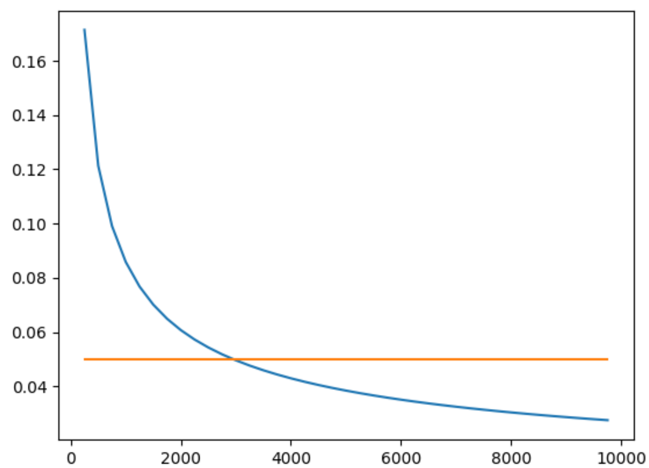
(b) (Computer Experiment) Let's examine the properties of this confidence interval. Let $\alpha = 0.05$ and $p = 0.4$. Conduct a simulation study to see how often the interval contains p (called the coverage). Do this for various values of n between 1 and 10000. Plot the coverage versus n.

- For code see the next subproblem.



(c) Plot the length of the interval versus n. Suppose we want the length of the interval to be no more than 0.05. How large should n be?

- ```
  import numpy as np
  import matplotlib.pyplot as plt

  n_list = np.arange(start=1, step=250, stop=10000, dtype=int)
  alpha = 0.05
  p = 0.4

  coverages, lengths = {}, {}
  for n in n_list:
      eps = np.sqrt(np.log(2 / alpha) / (2 * n))

      coverage, length = 0, []
      for i in range(100):
          X = np.random.binomial(n = 1, p = p, size = n)
          mean = np.mean(X)

          if (p < (mean - eps)) or (p > (mean + eps)):
              coverage += 1

          length.append(2 * eps)

      coverages[n] = coverage
      lengths[n] = np.mean(length)
  ```
  ```

```
x = list(coverages.keys())
y_cov = list(coverages.values())
y_len = list(lengths.values())

plt.scatter(x, y)
plt.savefig("Ex4_4b.png")
plt.cla()
plt.plot(x[1:], y_len[1:]) # First length skews the plot heavily
plt.plot(x[1:], [0.05]*(len(x[1:]))) # Horizontal Line
plt.savefig("Ex4_4c.png")
```



5. Prove Mill's Inequality, Theorem 4.7. Hint. Note that $P(|Z| > t) = 2P(Z > t)$. Now write out what $P(Z > t)$ means and note that $x/t > 1$ whenever $x > t$.

6. Let $Z \sim N(0,1)$. Find $P(|Z| > t)$ and plot this as a function of t. From Markov's inequality, we have the bound
$$P(|Z| > t) \leq \frac{E(|Z|^k)}{t^k}$$
for any $k > 0$. Plot these bounds for $k = 1, \ldots, 5$ and compare them to the true value of $P(|Z| > t)$. Also, plot the bound from Mill's inequality.

7. Let $X_1, \ldots, X_n \sim N(0,1)$. Bound $P(|\overline{X}_n| > t)$ using Mill's inequality, where
$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$
Compare to teh Chebyshev bound.

1.5 Convergence of Random Variable

1.

2. Let $X_1, X_2, \dots$ be a sequence of random variables.Show that $X_n \xrightarrow{qm} b$ if and only if
$$\lim_{n\to\infty} E(X_n) = b, \quad \lim_{n\to\infty} V(X_n) = 0.$$

3. Let $X_1, \dots, X_n$ be IID and let $\mu := E(X_1)$. Suppose that the variance is finite. Show that $\overline{X}_n \xrightarrow{qm} \mu$

4. Let $X_1, X_2, \dots$ be a sequence of random variables such that
$$P\left(X_n = \frac{1}{n}\right) = 1 - \frac{1}{n^2}, \quad P(X_n = n)\frac{1}{n^2}$$
Does X_n converge in probability? Does X_n converge in quadratic mean?

5. Let $X_1, \dots, X_n \sim \text{Bernoulli}(p)$. Prove that
$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} p, \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{qm} p.$$

6. Suppose that the height of men has mean 68 inches and standard deviation 2.6 inches. We draw 100 men at random. Find (approximately) the probability that the average height of men in our sample will be at least 68 inches.

7. Let $\lambda_n = 1/n$ for $n = 1, 2, \dots$. Let $X_n \sim \text{Poisson}(\lambda_n)$.
   (a) Show that $X_n \xrightarrow{P} 0$.
   (b) Let $Y_n = nX_n$. Show that $Y_n \xrightarrow{P} 0$.

8. Suppose we have a computer program consisting of $n = 100$ pages of code. Let X_i be the number of errors on the ith page of code. Suppose that the X_i are Poisson with mean 1 and that they are independent. Let
$$Y = \sum_{i=1}^{n} X_i$$
be the total number of errors. Use the central limit theorem to approximate
$$P(Y < 90).$$

9.

10. Let $Z \sim N(0,1)$. Let $t > 0$. Show that, for any $k > 0$,

$$P(|Z| > t) \leq \frac{E|Z|^k}{t^k}.$$

Compare this to Mill's inequality in Chapter 4.

11. Suppose that $X_n \sim N(0, 1/n)$ and let X be a random variable with distribution $F(x) = 0$ if $x < 0$ and $F(x) = 1$ if $x \geq 1$. Does X_n converge to X in probability? (Prove or disprove). Does X_n converge to X in distribution? (Prove or disprove).

12. Let $X, X_1, X_2, X_3, \ldots$ be random variables that are positive and integer valued. Show that $X_n \rightsquigarrow X$ if and only if

$$\lim_{n\to\infty} P(X_n = k) = P(X = k)$$

for every integer k.

13.

14. Let $X_1, \ldots, X_n \sim \text{Uniform}(0,1)$. Let $Y_n = \overline{X}_n^2$. Find the limiting distribution of Y_n.

15.

16. Construct an example where $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow Y$ but $X_n + Y_n$ does not converge in distribution to $X + Y$.

2 Statistics

2.1 Models, Statistical Inference and Learning

1. Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$ and let $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^n$. Find the bias, se, and mse of this estimator.

   - Recall that E_λ is linear and that the expected value of $X \sim \text{Po}(\lambda)$ is λ. As such
$$E_\lambda(\hat{\lambda}_n) = \frac{1}{n}\sum_{i=1}^n E(X_i) = \frac{1}{n}n\lambda = \lambda.$$
     This means that bias $= 0$, i.e., this estimator is unbiased.
   - Similiarly, the variance of $X \sim \text{Po}(\lambda)$ is equal to λ, as such $se^2 = \lambda/n$. To see this note that $Var(aX) = a^2 Var(X)$, i.e.,
$$se = \sqrt{\lambda/n}.$$
   - By Theorem 6.9 the mean square error is equal to
$$\text{mse} = \text{bias}^2(\hat{\lambda}_n) + se^2(\hat{\lambda}_n) = \frac{\lambda}{n}.$$

2. Let $X_1, \ldots, X_n \sim \text{Uniform}(0, \theta)$ and let $\hat{\theta} = \max\{X_1, \ldots, X_n\}$. Find the bias, se, and mse of this estimator.

   - We begin by determining the CDF of $\hat{\theta}$.

$$\begin{aligned}
F_{\hat{\theta}}(x) &= F(\hat{\theta} \leq x) \\
&= F(\max\{X_1, \ldots, X_n\} \leq x) \\
&= F(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x) \\
&\stackrel{1}{=} \prod_{i=1}^{n} F(X_i \leq x) \\
&\stackrel{2}{=} (F(X_i \leq x))^n \\
&= \left(\frac{x}{\theta}\right)^n \\
&= \frac{x^n}{\theta^n},
\end{aligned}$$

     where we have used (1) the independence of the random variables and (2) the fact that they are identically distributed. To obtain the PDF we differentiate with respect to x to obtain

$$\begin{aligned}
f_{\hat{\theta}}(x) &= \frac{\partial}{\partial x} F_{\hat{\theta}}(x) \\
&= \frac{\partial}{\partial x} \frac{x^n}{\theta^n} \\
&= n \frac{x^{n-1}}{\theta^n}.
\end{aligned}$$

     With that we can now calculate the expected value

$$\begin{aligned}
E_\theta(\hat{\theta}) &= \int_0^\theta x(nx^{n-1}/\theta^n)dx \\
&= n \int_0^\theta \frac{x^n}{\theta^n} dx \\
&= \frac{n}{\theta^n} \int_0^\theta x^n dx \\
&= \frac{n}{\theta^n} \left[\frac{x^{n+1}}{n+1}\right]_0^\theta \\
&= \frac{n}{n+1}\theta,
\end{aligned}$$

and

$$\begin{aligned}
\text{bias} &= E_\theta(\hat\theta) - \theta \\
&= \frac{n}{n+1}\theta - \theta \\
&= \theta\left(\frac{n}{n+1} - 1\right) \\
&= -\frac{\theta}{n+1}.
\end{aligned}$$

- To compute the standard error we first calculate the second moment of $\hat\theta$:

$$\begin{aligned}
E_\theta(\hat\theta^2) &= \int_0^\theta x^2 \frac{nx^{n-1}}{\theta^n} dx \\
&= \frac{n}{\theta^n}\int_0^\theta x^{n+1} dx \\
&= \frac{n}{\theta^n}\ \frac{1}{n+2} x^{n+2}\Big|_{x=0}^{x=\theta} \\
&= \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} \\
&= \frac{n}{n+2}\theta^2.
\end{aligned}$$

As such

$$V(\hat\theta) = E_\theta(\hat\theta^2) - E_\theta(\hat\theta)^2 = \frac{n}{n+2}\theta^2 - \frac{\theta^2}{(n+1)^2}$$

and

$$\text{se}(\hat\theta) = \sqrt{V(\hat\theta)} = \sqrt{\frac{n}{n+2} - \frac{1}{(n+1)^2}}\theta.$$

- By Theorem 6.9 it follows

$$\begin{aligned}
\text{mse}(\hat\theta) &= \text{bias}^2(\hat\theta) + V(\hat\theta) \\
&= \frac{\theta^2}{(n+1)^2} + \frac{n}{n+2}\theta^2 - \frac{\theta^2}{(n+1)^2} \\
&= \frac{n}{n+2}\theta^2.
\end{aligned}$$

3. Let $X_1, \ldots, X_n \sim \text{Uniform}(0, \theta)$ and let $\hat\theta = 2\overline{X}_n$. Find the bias, se, and mse of this estimator.

2.2 Estimating the CDF and Statistical Functionals

1. Prove Theorem 7.3.

- First of all note that

$$E(I(X_i \le x)) = 1 \cdot P(X_i \le x) + 0 \cdot P(X_i > 0) = F(x)$$

holds, with that we can calculate

$$\begin{aligned} E(\hat{F}_n(x)) &= \frac{1}{n}\sum_{i=1}^{n} E(I(X_i \le x)) \\ &= \frac{1}{n}\sum_{i=1}^{n} F(x) = F(x). \end{aligned}$$

- To compute the variance we first want to calculate the second moment of $\hat{F}_n(X)$. First of all note that $I(X_i \le x)^2 = I(X_i \le x) = F(x)$. Using this we now calculate the second moment:

$$\begin{aligned} E(\hat{F}_n^2(X)) &= E\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} I(x_i \le x)\right)\right) \\ &= E\left(\frac{1}{n^2}\sum_{i=1}^{n} I(x_i \le x)^2 + \frac{1}{n^2}\sum_{i<j} I(X_i \le x)I(X_j \le x)\right) \\ &= \frac{1}{n}E\left(\frac{1}{n}\sum_{i=1}^{n} I(X_i \le x))\right) + \frac{1}{n^2}\sum_{i<j} E(I(X_i \le x)I(X_j \le x)) \\ &= \frac{F(x)}{n} + \frac{1}{n^2}\sum_{i<j} P(I(X_i \le x)I(X_j \le x) = 1) \\ &= \frac{F(x)}{n} + \frac{1}{n^2}\sum_{i<j} P(I(X_i \le x) = 1, P(I(X_j \le x) = 1) \\ &= \frac{F(x)}{n} + \frac{1}{n^2}\sum_{i<j} P(X_i \le x)P(X_j \le x) \\ &= \frac{F(x)}{n} + \frac{1}{n^2}\sum_{i<j} F(x)^2 \\ &= \frac{F(x)}{n} + \frac{1}{n^2}\frac{n(n-1)}{2}F(x)^2 \\ &= \frac{F(x)}{n}\left(1 + \frac{n-1}{2}F(x)^2\right). \end{aligned}$$

We have already computed the expected value of $\hat{F}_n$. In total the variance then is

$$\begin{aligned} V(\hat{F}_n(x)) &= E(\hat{F}_n^2(x)) - E(\hat{F}_n(x))^2 \\ &= \frac{F(x)}{n}\left(1 + \frac{n-1}{2}F(x)^2\right) - F(x)^2 \\ &= \frac{F(x)(1-F(x))}{n}. \end{aligned}$$

- Note that the bias is equal to 0, as such by Theorem 6.9

$$\text{mse}(\hat{F}_n(x)) = V(\hat{F}_n(x)).$$

- We want to show that for all $\varepsilon > 0$

$$P(|\hat{F}_n(x) - F(x)| > \varepsilon) \to 0$$

as $n \to \infty$. Note that $F(x) = E(\hat{F}_n(x))$. Applying Theorem 4.2 (Chebyshev's Inequality) we get

$$P(|\hat{F}_n(x) - F(x)| > \varepsilon) \leq \frac{V(\hat{F}_n(x)}{\varepsilon^2}$$

and by including the variance

$$P(|\hat{F}_n(x) - F(x)| > \varepsilon) \leq \frac{\frac{F(x)(1-F(x))}{n}}{\varepsilon^2},$$

which is equivalent to

$$P(|\hat{F}_n(x) - F(x)| > \varepsilon) \leq \frac{1}{n}\frac{F(x)(1-F(x))}{\varepsilon^2} \to 0$$

as $n \to \infty$. As such $\hat{F}_n(x) \xrightarrow{P} F(x)$.

2. Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$ and let $Y_1, ..., Y_m \sim \text{Bernoulli}(q)$. Find the plug-in estimator and estimated standard error for p. Find an approximate 90 percent confidence interval for p. Find the plug-in estimator and estimated standard error for $p - q$. Find an approximate 90 percent confidence interval for $p - q$.

- Note that $Y_n := \sum_{i=1}^{n} X_i \sim \text{Bin}(n, p)$ and therefore $E(Y_n) = np$ and $V(Y_n) = np(1-p)$. Furthermore note that the mean

$$\overline{X}_n := \frac{1}{n} Y_n$$

satisfies $E(\overline{X}_n) = \frac{1}{n}E(Y_n) = p$ by linearity. This is our plug-in estimator. The variance is

$$V(\overline{X}_n) = \frac{p(1-p)}{n},$$

as such the standard error of this estimator is

$$\text{se}(\overline{X}_n) = \sqrt{\frac{p(1-p)}{n}}.$$

- This allows us to define the (Normal-based) confidence interval by using (7.6):
$$(p - z_{\alpha/2}\,\mathrm{se}, p + z_{\alpha/2}\,\mathrm{se})$$
and observing that for a 95% confidence interval $z_{\alpha/2} = z_{0.05/2} = 1.96 \approx 2$ we get
$$\left(p - 2\sqrt{\frac{p(1-p)}{n}}, p + 2\sqrt{\frac{p(1-p)}{n}}\right).$$

- ```
import numpy as np
import matplotlib.pyplot as plt
from Scripts.plots import *

def generateConfidence(n, p):
    se = np.sqrt((p * (1 - p)) / n)
    conf = [p - 2 * se, p + 2 * se]
    inConf = lambda x : (x > conf[0]) & (x < conf[1])
    return conf, inConf

def countSamples(q_list, n = 50, p = 0.3, num_samples = 100000):
    y = []
    for q in q_list:
        mu = np.random.binomial(n=n, p=q, size=num_samples) / n
        count = np.sum(inConf(mu))
        y.append(count / num_samples)
    return y

num_samples = 100000
step_size = 0.02
p_list = [0.1, 0.3, 0.5, 0.7, 0.95]
for p in p_list:
    n = 50
    conf, inConf = generateConfidence(n = n, p = p)

    q_list = np.arange(start=0, stop=1, step=step_size)
    y = countSamples(n = n, p = 0, num_samples = num_samples, q_list = q_list)
    plt.plot(q_list, y, label=f'p = {p}')

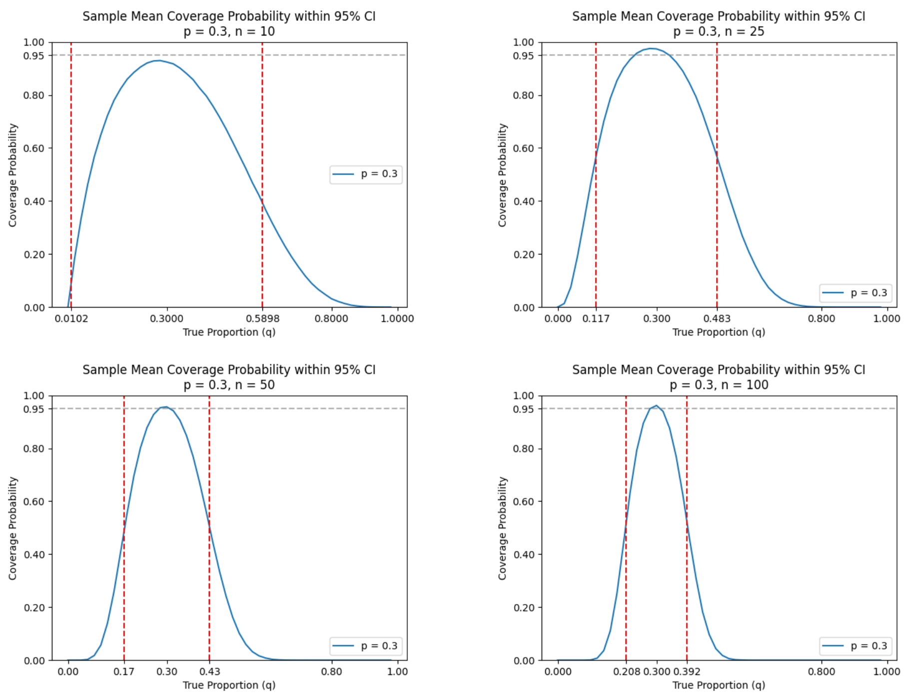
plot7_2_1(save = True)

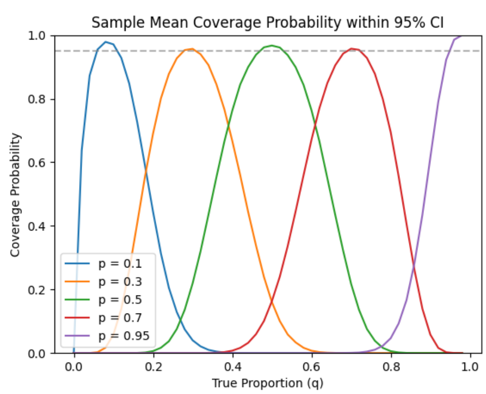
n_list = [10, 25, 50, 100]
p = 0.3
for n in n_list:
    conf, inConf = generateConfidence(n = n, p = p)
```
```

```python
q_list = np.arange(start=0, stop=1, step=step_size)
y = countSamples(n = n, p = 0, num_samples = num_samples, q_list = q_list)

plt.plot(q_list, y, label=f'p = {p}')

plot7_2_2(save = True, n = n, p = p, conf = conf)
```

- 



3. (Computer Experiment) Generate 100 observations from a $\mathcal{N}(0,1)$ distribution. Compute a 95 percent confidence band for the CDF F (as

described in the appendix). Repeat this 1000 times and see how often the confidence band contains the true distribution function. Repeat using data from a Cauchy distribution.

- ```
  import numpy as np
  import scipy
  import matplotlib.pyplot as plt
  # Allows to use Latex Code
  plt.rcParams.update({
      "text.usetex": True
  })

  rng = np.random.default_rng()

  n, alpha = 100, 0.05
  samples = rng.normal(size = n)
  eps = np.sqrt(np.log(2 / alpha)  / (2 * n))

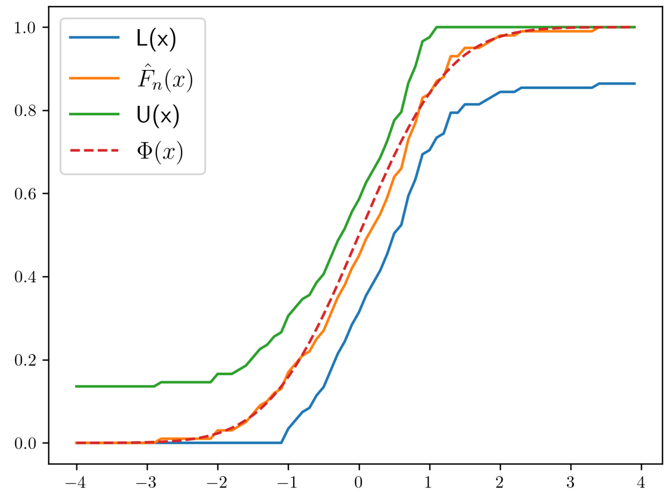
  F = lambda x : (sum(samples < x) / 100)
  L = lambda x : np.maximum(F(x) - eps, 0)
  U = lambda x : np.minimum(F(x) + eps, 1)
  N = scipy.stats.norm.cdf

  X = np.arange(start = -4, stop = 4, step = 0.1)
  yF = np.vectorize(F)(X)
  yL = np.vectorize(L)(X)
  yU = np.vectorize(U)(X)
  yN = np.vectorize(N)(X)

  plt.plot(X, yL, label = "L(x)")
  plt.plot(X, yF, label = r"$\hat{F}_n(x)$")
  plt.plot(X, yU, label = "U(x)")
  plt.plot(X, yN, linestyle='--', label = r"$\Phi(x)$")

  plt.legend()
  plt.legend(fontsize=14)

  plt.savefig("Ex7_3.png", dpi = 300)
  plt.show()
  ```
  ```




```
k, count, count_cauchy = 1000, 0, 0
for i in range(k):
    samples = rng.normal(size = n)
    samples_cauchy = rng.standard_cauchy(size = n)

    F = lambda x : (sum(samples < x) / 100)
    F_cauchy = lambda x : (sum(samples_cauchy < x) / 100)
    L = lambda x : np.maximum(F(x) - eps, 0)
    U = lambda x : np.minimum(F(x) + eps, 1)

    C = scipy.stats.cauchy.cdf

    yF = np.vectorize(F)(X)
    yC = np.vectorize(C)(X)
    yL = np.vectorize(L)(X)
    yU = np.vectorize(U)(X)

    cmp = lambda x : (N(x) >= L(x) and N(x) <= U(x))
    cmp_cauchy = lambda x : (C(x) >= L(x) and C(x) <= U(x))
    contains = np.vectorize(cmp)(X)
    contains_cauchy = np.vectorize(cmp_cauchy)(X)
    if not contains.all():
        count += 1

    if not contains_cauchy.all():
        count_cauchy += 1
```

```python
print(f"Normal : {count / k * 100:.2f}%")
print(f"Cauchy : {count_cauchy / k * 100:.2f}%")

> Normal : 2.30%
> Cauchy : 84.60%
```

4. Let $X_1, \ldots, X_n \sim F$ and let $\hat{F}(x)$ be the empirical distribution function. For a fixed x, use the central limit theorem to find the limiting distribution of $\hat{F}_n(x)$.

5. Let x and y be two distinct points. Find $\text{Cov}(\hat{F}_n(x), \hat{F}_n((y)))$.

6. Let $X_1, \ldots, X_n \sim F$ and let $\hat{F}$ be the empirical distribution function. Let $a < b$ be fixed numbers and define

$$\theta = T(F) = F(b) - F(a).$$

Let $\hat{\theta} = T(\hat{F}_n) = \hat{F}_n(b) - \hat{F}_n(a))$. Find the estimated standard error of $\hat{\theta}$. Find an expression for an approximate $1 - \alpha$ confidence interval for θ.

7. Data on the magnitudes of earthquakes near Fiji are available on the website for this book. Estimate the CDF F(x). Compute and plot a 95 percent confidence envelope for F (as described in the appendix). Find an approximate 95 percent confidence interval for F(4.9) - F(4.3).

8. Get the data on eruption times and waiting times between eruptions of the Old Faithful geyser from the website. Estimate the mean waiting time and give a error for the estimate. Also, give a 90 percent confidence interval for the mean waiting time. Now estimate the median waiting time. In the next chapter we will see how to get the standard error for the median.

   - https://www.stat.cmu.edu/ larry/all-of-statistics/=data/faithful.dat
   - I modified the text file, removing the text above the table before importing it.
   -
```python
import numpy as np
import pandas as pd
import scipy

file_path = "Data/faithful.dat"

column_names = ['eruptions', 'waiting']
df = pd.DataFrame(columns=column_names)

with open(file_path, 'r') as file:
    next(file)  # Skips the header row
    for line in file:
        _, eruptions, waiting = line.strip().split()
```

```
            data_dict = {'eruptions': float(eruptions), 'waiting': int(waiting)}
            df = df.append(data_dict, ignore_index=True)


    X = df["waiting"].values
    n = len(X)
    mu = np.mean(X)
    variance = np.dot(X - mu, X - mu) / (2 * n)
    sigma = np.sqrt(variance)

    print(f"mu = {mu:.2f}\nse = {sigma:.2f}")

    z = scipy.stats.norm.ppf(0.95)
    print(f"(mu-z*se, mu+z*se) = ({mu - z * sigma:.2f}, {mu + z * sigma:.2f})")
```

9. 100 people are given a standard antibiotic to treat an infection and another 100 are given a new antibiotic. In the first group, 90 people recover; in the second group, 85 people recover. Let p_1 be the probability of recovery under the standard treatment and let P2 be the probability of recovery under the new treatment. We are interested in estimating $\theta = p_1 - p_2$. Provide an estimate, standard error, an 80 percent confidence interval, and a 95 percent confidence interval for B.

10.