Contents

1 General Probability Theory

1. Using the properties of Definition 1.1.2 for a probability mea- measure P, show the following.

   (a) If $A \in \mathcal{F}, B \in \mathcal{F}$, and $A \subseteq B$ then $P(A) \leq P(B)$.

   (b) If $A \in \mathcal{F}$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$ with

   $$\lim_{n \to \infty} P(A_n) = 0$$

   and $A \subseteq A - n$ for every n, then $P(A) = 0$. (This property was used implicitly in Example 1.1.4 when we argued that the sequence of all heads, and indeed any particular sequence, mus have probability zero.)

2. The infinite coin-toss space Ω_∞ of example 1.1.4 is uncountable infinite. Suppose that were was a sequential list

   $$\omega^{(i)} = \omega_1^{(i)} \omega_2^{(i)} \omega_3^{(i)} \ldots$$

   where $i = 1, 2, \ldots$ of all elements of Ω_∞, i.e. such that

   $$\bigcup_i \omega^{(i)} = \Omega_\infty.$$

   An element that does not appear in this list is the sequence whose first component is H if $\omega_1^{(1)}$ is T and is T if $\omega_1^{(1)}$ is H, and so on. As such there is no such sequence $(\omega^{(i)})$.

   Now consider the set $A \subseteq \Omega_\infty$ such that for all $omega \in A$ the elements ω_{2k-1} and ω_{2k} are the same for all $k \geq 0$.

   - Show that A is uncountably infinite.
   - Show that, when $0 < p < 1$, we have $P(A) = 0$.

3. Consider the set function P defined for every subset of $[0, 1]$ by $P(A) = 0$ if $|A| < \infty$ and $P(A) = \infty$ if $|A| = \infty$. Show that P satisfies (1.1.3)-(1.1.5), but P does not have the countable additivity property (1.1.2). We see then that the finite additivity property (1.1.5) does nto imply the countable additivity property (1.1.2).

4. (a) Construct a standard normal random variable Z on the probability space $(\Omega_\infty, \mathcal{F}_\infty, P)$ of example 1.1.4 under the assumptionm that the probability for head is $p = 1/2$. (Hint: Consider Examples 1.2.5 and 1.2.6).

   (b) Define a sequencec of random variables $\{Z_n\}_{n=1}^{\infty}$ on Ω_∞ such that $Z_n \to Z$ pointwise, and, for each n, Z_n only depends on the first n coint tosses. (This gives us a procedure for approximating a standard normal random variable by random variables generated by a finite number of coin tosses, a useful algorithm for Monte Carlo simulation.)

5. When dealing with double Lebesgue integrals, just as with doubel Riemann integrals, the order of integration can be reversed. The only assumption required is that the function being integrated be either nonnegative or integrable. Here is an application of this fact.

   Let X be a nonnegative random variable with cumulative distribution function $F(x) = P(X \le x)$. Show that

   $$EX = \int_0^\infty (1 - F(x))dx$$

   by showing that

   $$\int_\Omega \int_0^\infty 1_{[0,X(\omega))}(x)dxdP(\omega)$$

   is equal to both EX and

   $$\int_0^\infty (1 - F(x))dx.$$

6. Let u be a fixed number in $\mathbb{R}$, and define the convex function

   $$\varphi(x) = e^{ux}$$

   for all $x \in \mathbb{R}$. Let X be a normal random variable with mean $\mu = EX$ and standard deviation

   $$\sigma = (E(X - \mu))^{1/2}.$$

   (a) Verify that

   $$Ee^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}.$$

   (b) Verify that Jensen's inequality holds (as it must):

   $$E\varphi(X) \ge \varphi(EX).$$

7. For each positive integer n, deinfe f_n to be normal density with mean zero and variance n, i.e.,

   $$f_n(x) = \frac{1}{\sqrt{2n\pi}} e^{-x^2/2n}.$$

   (a) Determine the pointwise limit

   $$f(x) = \lim_{n\to\infty} f_n(x).$$

   (b) Calculate

   $$\lim_{n\to\infty} \int_{-\infty}^\infty f_n(x)dx.$$

(c) Note that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} f(x)dx.$$

Explain why this does not violate the Monotone Convergence Theorem, Theorem 1.4.5.

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2 Information and Conditioning

1. Let $(\Omega, \mathcal{F}, P)$ be a general probability space, and suppose a random variable X on this space is measureable with respect to the trivial σ-algebra $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Show that X is not random (i.e., there is a constant c such that $X(\omega) = c$ for all $\omega \in \Omega$). Such a random variable is called degenerate.

2.

3. Let X and Y be independent standard random variables. Let θ be a constant, and define random variables

$$V = X\cos\theta + Y\sin\theta, \quad W = -X\sin\theta + Y\cos\theta.$$

Show that V and W are independent standard normal random variables.

4.

5. Let (X, Y) be a pair of random variables with joint density function

$$f_{X,Y}(x,y) = \frac{2|x|+y}{\sqrt{2\pi}}\exp\left(-\frac{(2|x|+y)^2}{2}\right)$$

for $y \geq -|x|$ and $f_{X,Y}(x,y) = 0$ otherwise. How that X and Y are standard normal random variables and that they are uncorrelated but not independent.

6.

7.

8. Let X and Y be integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then $Y = Y_1 + Y_2$, where $Y_1 = E[Y|X]$ os $\sigma(X)$-measurable and $Y_2 = Y - E[Y|X]$. Show that Y_2 and X are uncorrelated. More generally, show that Y_2 is uncorrelated with every $\sigma(X)$-measureable random variable.

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11. (a) Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, and let W be a nonegative $\sigma(X)$-measureable random variable. Show there exists a function g such that $W = g(X)$. (Hint: Recall that every set in $\sigma(X)$ is of the form $\{X \in B\}$ for some Borel set $B \subseteq \mathbb{R}$. Suppose first that W is the indicator of such a set, and then use the standard machine.)

    (b) Lett X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, and let Y be a nonnegative random variable on this space. We do not assume that X and Y have a joint density. Nonetheless, show there is a function g such that $E[Y|X] = g(X)$.

3 Brownian Motion

4 Stochastic Calculus

5 Risk-Neutral Pricing

6 Connections with Partial Differential Equations

7 Exotic Options

8 American Derivative Securities

9 Change of Numéraire

10 Term-Structure Models

11 Introduction to Jump Processes