Contents

1 General Probability Theory

1. Using the properties of Definition 1.1.2 for a probability measure P, show the following.

   (a) If $A \in \mathcal{F}, B \in \mathcal{F}$, and $A \subseteq B$ then $P(A) \leq P(B)$.

   - Note that $A \cap B \backslash A = \emptyset$ as such

   $$P(B) = P(A) + P(B \backslash A) \geq P(A).$$

   (b) If $A \in \mathcal{F}$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$ with

   $$\lim_{n \to \infty} P(A_n) = 0$$

   and $A \subseteq A_n$ for every n, then $P(A) = 0$. (This property was used implicitly in Example 1.1.4 when we argued that the sequence of all heads, and indeed any particular sequence, mus have probability zero.)

   - This is just the sandwitch lemma:

   $$0 \leq P(A) \leq P(A_n) \to 0$$

   implies $P(A) = 0$.

2. The infinite coin-toss space Ω_∞ of example 1.1.4 is uncountable infinite. Suppose that were was a sequential list

   $$\omega^{(i)} = \omega_1^{(i)} \omega_2^{(i)} \omega_3^{(i)} \ldots$$

   where $i = 1, 2, \ldots$ of all elements of Ω_∞, i.e. such that

   $$\bigcup_i \omega^{(i)} = \Omega_\infty.$$

   An element that does not appear in this list is the sequence whose first component is H if $\omega_1^{(1)}$ is T and is T if $\omega_1^{(1)}$ is H, and so on. As such there is no such sequence $(\omega^{(i)})$.

   Now consider the set $A \subseteq \Omega_\infty$ such that for all $\omega \in A$ the elements ω_{2k-1} and ω_{2k} are the same for all $k \geq 1$.

   (a) Show that A is uncountably infinite.

   - Suppose there is a counting $\alpha^{(i)} = \alpha_1^{(i)} \alpha_2^{(i)} \alpha_3^{(i)} \ldots$ then

   $$\omega_j^{(i)} := \alpha_{2j-1}^{(i)}$$

   is a counting of Ω_∞, which is impossible.

   (b) Show that, when $0 < p < 1$, we have $P(A) = 0$.

- Let A_k be defined such that ωA_k means $\omega_1 = \omega_2, \dots, \omega_{2k-1} = \omega_{2k}$ and note that $A \subseteq A_k$ for all k. Furthermore it can inductively be shown that

$$P(A_k) = (p^2 + (1-p)^2)^k$$

which converges to 0 whenever $p \in (0, 1)$. $P(A)$ then follows by Exercise 1.1(ii).

3. Consider the set function P defined for every subset of $[0, 1]$ by $P(A) = 0$ if $|A| < \infty$ and $P(A) = \infty$ if $|A| = \infty$. Show that P satisfies (1.1.3)-(1.1.5), but P does not have the countable additivity property (1.1.2). We see then that the finite additivity property (1.1.5) does not imply the countable additivity property (1.1.2).

1.1.3 $|\emptyset| < 0$ so $P(\emptyset) = 0$.

1.1.4 If A and B are both finite then the union is also finite. If at least one of them is not finite then the union has infinite P value. In either case the identity holds.

1.1.5 Follows inductively from 1.1.4.

-1.1.2 Consider the sequence $x_n := 1/n$ then $P(\{x_n\} = 0$ for all n but $P\left(\bigcup\{x_n\}\right) = +\infty$.

4. (a) Construct a standard normal random variable Z on the probability space $(\Omega_\infty, \mathcal{F}_\infty, P)$ of example 1.1.4 under the assumption that the probability for head is $p = 1/2$. (Hint: Consider Examples 1.2.5 and 1.2.6).

   (b) Define a sequence of random variables $\{Z_n\}_{n=1}^{\infty}$ on Ω_∞ such that $Z_n \to Z$ pointwise, and, for each n, Z_n only depends on the first n coint tosses. (This gives us a procedure for approximating a standard normal random variable by random variables generated by a finite number of coin tosses, a useful algorithm for Monte Carlo simulation.)

5. When dealing with double Lebesgue integrals, just as with doubel Riemann integrals, the order of integration can be reversed. The only assumption required is that the function being integrated be either nonnegative or integrable. Here is an application of this fact.

   Let X be a nonnegative random variable with cumulative distribution function $F(x) = P(X \leq x)$. Show that

$$EX = \int_0^\infty (1 - F(x))dx$$

by showing that

$$\int_\Omega \int_0^\infty \mathbb{1}_{[0, X(\omega))}(x) dx dP(\omega)$$

is equal to both EX and

$$\int_0^\infty (1 - F(x))dx.$$

- Consider the following computation,

$$\begin{aligned}
EX &= \int_\Omega X(\omega)dP(\omega) \\
&= \int_\Omega \lambda^1([0, X(\omega)))dP(\omega) \\
&= \int_\Omega \int_0^\infty 1_{[0,X(\omega))}(x)dxdP(\omega) \\
&\stackrel{*}{=} \int_0^\infty \int_\Omega 1_{[0,X(\omega))}(x)dP(\omega) \\
&= \int_0^\infty P(X < x)dx \\
&= \int_0^\infty 1 - F(x)dx,
\end{aligned}$$

  where we have used Fubini's theorem inf $*$.

6. Let u be a fixed number in $\mathbb{R}$, and define the convex function

$$\varphi(x) = e^{ux}$$

for all $x \in \mathbb{R}$. Let X be a normal random variable with mean $\mu = EX$ and standard deviation

$$\sigma = (E(X - \mu))^{1/2}.$$

   (a) Verify that

$$Ee^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}.$$

   (b) Verify that Jensen's inequality holds (as it must):

$$E\varphi(X) \geq \varphi(EX).$$

7. For each positive integer n, define f_n to be normal density with mean zero and variance n, i.e.,

$$f_n(x) = \frac{1}{\sqrt{2n\pi}} e^{-x^2/2n}.$$

   (a) Determine the pointwise limit

$$f(x) = \lim_{n\to\infty} f_n(x).$$

   - $f_n \to 0$ pointwise.

(b) Calculate

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)dx.$$

- Note that f_n is a PDF for every n, as such the integrals are all equal to 1, as such

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)dx = +1.$$

(c) Note that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)dx \neq \int_{-\infty}^{\infty} f(x)dx.$$

Explain why this does not violate the Monotone Convergence Theorem, Theorem 1.4.5.

- f_n is not a monotone sequence.

8. Let X be a nonnegative random variable, and assume that

$$\varphi(t) = Ee^{tX}$$

is finite for every $t \in \mathbb{R}$. Assume further that $E\left[Xe^{tx}\right] < \infty$ for every $t \in \mathbb{R}$. The purpose of this exercise is to show that $\varphi'(t) = E\left[Xe^{tX}\right]$ and, in particular, $\varphi'(0) = EX$.

The Mean Value Theorem for our case can be stated as follows: Let $\omega \in \Omega$ be fixed and define $f(t) = e^{tX(\omega))}$, then this becomes

$$e^{tX(\omega)} - e^{sX(\omega)} = (t-s)X(\omega)e^{\theta(\omega)X(\omega)},$$

where $\theta(\omega)$ is a number depending on ω (i.e., a random variable lying between t and s.

(a) Use the Dominated Convergence Theorem (Theorem 1.4.9) and equation (1.9.1) to show that

$$\lim_{n\to\infty} EY_n = E\left[\lim_{n\to\infty} Y_n\right] = E\left[Xe^{tX}\right].$$

This establishes the desired formula

$$\varphi'(t) = E\left[Xe^{tX}\right].$$

(b) Suppose the random variable X can take both positive and negative values and $Ee^{tX} < \infty$ and $E[|X|e^{tX}] < \infty$ for every $t \in \mathbb{R}$. Show that once again

$$\varphi'(t) = E\left[Xe^{tX}\right].$$

Hint: Use the notation (1.3.1) to write $X = X^+ - X^-$.

9. Suppose X is a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, A is a set in $\mathcal{F}$, and for evvery Borel subset B of $\mathbb{R}$, we have

$$\int_A 1_B(X(\omega))dP(\omega) = P(A)P(X \in B).$$

Then we say that X is independent of the event A. Show that if X is independent of an event A, then

$$\int_A g(X(\omega))dP(\omega) = P(A)Eg(X)$$

for every nonnegative, Borel-measureable function g.

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2 Information and Conditioning

1. Let $(\Omega, \mathcal{F}, P)$ be a general probability space, and suppose a random variable X on this space is measureable with respect to the trivial σ-algebra $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Show that X is not random (i.e., there is a constant c such that $X(\omega) = c$ for all $\omega \in \Omega$). Such a random variable is called degenerate.

2. Independence of random variables can be affected by changes of measure. To illustrate this point, consider the space of two coin tosses $\Omega_2 = \{HH, HT, TH, TT\}$, and let stock prices be given by

$$\begin{aligned} S_0 = 4, S_1(H) = 8, S_1(T) = 2, \\ S_2(HH) = 16, S_2(HT) = S_2(TH) = 4, S_2(TT) = 1. \end{aligned}$$

Consider two probability measures given by

$$\begin{aligned} \tilde{P}(HH) = 1/4, \tilde{P}(HT) = 1/4, \tilde{P}(TH) = 1/4, \tilde{P}(HH) = 1/4, \\ P(HH) = 4/9, P(HT) = 2/9, P(TH) = 2/9, P(TT) = 1/9. \end{aligned}$$

Define the random variable

$$X = \begin{cases} 1, & \text{if } S_2 = 4, \\ 0, & \text{if } S_2 \neq 4 \end{cases}.$$

(a) List all the sets in $\sigma(X)$.

- Note that $\{X = 1\} = \{HT, TH\}$ and $\{X = 0\} = \{X = 1\}^c = \{HH, TT\}$. As such

$$\sigma(X) = \{\emptyset, \{HT, TH\}, HH, TT, \Omega_2\}.$$

(b) List all the sets in $\sigma(S_1)$.

- Note that $\{S_1 = 8\} = \{HH, HT\}$ and $\{S_2 = 2\} = \{TH, TT\}$. As such

$$\sigma(S_1) = \{\emptyset, \{HH, HT\}, TH, TT, \Omega_2\}.$$

(c) Show that $\sigma(X)$ and $\sigma(S_1)$ are independent under the probability measure $\tilde{P}$.

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$$\begin{aligned}\tilde{P}(\{HT, TH\} \cap \{HH, HT\}) &= \tilde{P}(\{HT\}) = 1/4\\ \tilde{P}(\{HT, TH\} \cap \{TH, TT\}) &= \tilde{P}(\{TH\}) = 1/4\\ \tilde{P}(\{HH, TT\} \cap \{HH, HT\}) &= \tilde{P}(\{HH\}) = 1/4\\ \tilde{P}(\{HH, TT\} \cap \{TH, TT\}) &= \tilde{P}(\{TT\}) = 1/4\end{aligned}$$

Because $\tilde{P}$ assigns sets of size 2 a probability of $1/4$ the corresponding products are always $1/4$, as such $\sigma(X)$ and $\sigma(S_1)$ are independent with respect to $\tilde{P}$.

(d) Show that $\sigma(X)$ and $\sigma(S_1)$ are not independent under the probability measure P.

- Note that $P(\{HT, TH\}) = P(HT) + P(TH) = 2/9 + 2/9$ and $P(\{HH, HT\}) = 4/9 + 2/9 = 6/9$. As such the product of those is equal to $32/81$.
- On the other hand the probability of the intersection is given by

$$P(HT) = 2/9.$$

(e) Under P, we have $P(S_1 = 8) = 2/3$ and $P(S_1 = 2) = 1/3$. Explain intuitively why, if you are told that $X = 1$, you would want to revise your estimate of the distribution of S_1.

3. Let X and Y be independent standard random variables. Let θ be a constant, and define random variables

$$V = X\cos\theta + Y\sin\theta, \quad W = -X\sin\theta + Y\cos\theta.$$

Show that V and W are independent standard normal random variables.

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5. Let (X, Y) be a pair of random variables with joint density function

$$f_{X,Y}(x,y) = \frac{2|x| + y}{\sqrt{2\pi}} \exp\left(-\frac{(2|x|+y)^2}{2}\right)$$

for $y \geq -|x|$ and $f_{X,Y}(x,y) = 0$ otherwise. How that X and Y are standard normal random variables and that they are uncorrelated but not independent.

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8. Let X and Y be integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then $Y = Y_1 + Y_2$, where $Y_1 = E[Y|X]$ os $\sigma(X)$-measurable and $Y_2 = Y - E[Y|X]$. Show that Y_2 and X are uncorrelated. More generally, show that Y_2 is uncorrelated with every $\sigma(X)$-measureable random variable.

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11. (a) Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, and let W be a nonegative $\sigma(X)$-measureable random variable. Show there exists a function g such that $W = g(X)$. (Hint: Recall that every set in $\sigma(X)$ is of the form $\{X \in B\}$ for some Borel set $B \subseteq \mathbb{R}$. Suppose first that W is the indicator of such a set, and then use the standard machine.)

    (b) Lett X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, and let Y be a nonnegative random variable on this space. We do not assume that X and Y have a joint density. Nonetheless, show there is a function g such that $E[Y|X] = g(X)$.

3 Brownian Motion

1. According to Definition 3.3.3(iii), for $0 \leq t < u$, the Brownian motion increment $W(u) - W(t)$ is independent of the σ-algebra $\mathcal{F}(t)$. Use this property and property (i) of that definition to show that, for $0 \leq t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}(t)$.

2. Let $W(t), t \geq 0$ be a Brownian motion, and let $\mathcal{F}(t), t \geq 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale.

   Hint: For $0 \leq s \leq t$, write $W^2(t)$ as $(W(t)-W(s))^2+2W(t)W(s)-W^2(s)$.

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