7 Inner Product Spaces

7.1 Inner Products (A&R §6.1)

Based on our experience with the familiar dot product on $\mathbb{R}^n$, we now generalize the notion of dot product to "inner product". This gives us a way to define analogues of the dot product on real vector spaces other than $\mathbb{R}^n$.

Definition 7.1.1. *An inner product on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $k \in \mathbb{R}$.*

1. ***Symmetry:*** $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. ***Linearity:*** $\langle \mathbf{u}, \mathbf{v} + k\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + k\langle \mathbf{u}, \mathbf{w} \rangle$
3. ***Positive definiteness:*** $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ *with* $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ *iff* $\mathbf{v} = \mathbf{0}$

Definition 7.1.2. *A real vector space endowed with an inner product is a* ***real inner product space****.*

Lemma 7.1.3. $\langle \mathbf{u} + k\mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + k\langle \mathbf{w}, \mathbf{v} \rangle$

E.g. 7.1.4. *The dot product on $\mathbb{R}^n$ satisfies all the axioms of an inner product. (This is no coincidence – the axioms were chosen to provide a natural generalization of the dot product to arbitrary real vector spaces.) $\mathbb{R}^n$ endowed with this inner product is called Euclidean n-space.*

E.g. 7.1.5. *If $w_1, w_2, \ldots, w_n$ are any positive real numbers, then*

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n \tag{7.1}$$

defines an inner product on $\mathbb{R}^n$.

Verify that this definition satisfies the inner product axioms.

Let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{x} \in \mathbb{R}^n$ & $k \in \mathbb{R}$.

$$\begin{aligned}\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle &= w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots w_n u_n v_n \\ &= w_1 v_1 u_1 + w_2 v_2 u_2 + \ldots + w_n v_n u_n \\ &= \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle \Rightarrow \text{symmetry is satisfied.}\end{aligned}$$

$$\begin{aligned}\langle \underset{\sim}{u}, \underset{\sim}{v} + k\underset{\sim}{x} \rangle &= w_1 u_1 (v_1 + k x_1) + \ldots + w_n u_n (v_n + k x_n) \\ &= w_1 u_1 v_1 + \ldots + w_n u_n v_n \\ &\quad + k(w_1 u_1 x_1 + \ldots + w_n u_n x_n) \\ &= \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle + k\langle \underset{\sim}{u}, \underset{\sim}{x} \rangle \Rightarrow \text{linearity.}\end{aligned}$$

$\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = \underline{w_1 v_1^2} + \ldots + \underline{w_n v_n^2} \geq 0$ since $w_i > 0$

If $\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = 0$ then $w_i v_i^2 = 0 \quad \forall\ 1 \leq i \leq n$

$\Rightarrow v_i^2 = 0$ since $w_i > 0$

$\Rightarrow v_i = 0 \Rightarrow \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow$ positive definiteness

$\square$

E.g. 7.1.6 (Matrix inner product). *Let A be an invertible $n \times n$ matrix and define*

Prob. Set

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A^T A \mathbf{v} = (A\underset{\sim}{u}) \cdot (A\underset{\sim}{v}) \tag{7.2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. This definition satisfies the inner product axioms.

E.g. 7.1.7 (Function space inner product). *Let f and g be two functions in $C[a, b]$ and define*

Prob Set

$$\langle f, g \rangle = \int_a^b f(x)\, g(x) dx. \tag{7.3}$$

This definition satisfies the inner product axioms.

Remark 7.1.8. *Note that the standard Euclidean dot product is a special case of a matrix inner product* (7.2) *in which $A = I$. More generally, the weighted inner product* (7.1) *corresponds to taking A to be diagonal, with diagonal entries $\sqrt{w_i}$.*

Remark 7.1.9. *The inner product given in E.g. 7.1.7 is the analogue of the dot product on the function space $C[a, b]$.*

If $w(x)$ is a continuous and positive function then one can define a weighted generalization of this inner product:

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, w(x)\, dx.$$

This is the analogue of (7.1) *on $C[a, b]$.*

E.g. 7.1.10. *Let V be an inner product space and let $\mathbf{v}$ be any fixed vector in V. Show that the transformation $T : V \to \mathbb{R}$ defined by*

$$T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v} \rangle, \qquad \mathbf{x} \in V$$

*is linear. T is referred to as a **linear functional**.*

Let $\underset{\sim}{w}, \underset{\sim}{x} \in V$ & $k \in \mathbb{R}$. Then

$$T(\underset{\sim}{w} + k\underset{\sim}{x}) = \langle \underset{\sim}{w} + k\underset{\sim}{x}, \underset{\sim}{v} \rangle$$
$$= \langle \underset{\sim}{w}, \underset{\sim}{v} \rangle + k\langle \underset{\sim}{x}, \underset{\sim}{v} \rangle$$
$$= T(\underset{\sim}{w}) + k\,T(\underset{\sim}{x})$$

$\Longrightarrow$ T is linear.

Inner products can be used to define length and distance on general inner product spaces, just as we used the dot product to define length and distance on Euclidean spaces.

Definition 7.1.11. *If V is a real inner product space, then the **length** (or **norm**) of $\mathbf{v} \in V$ is*

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

*The **distance** between two vectors is*

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}.$$

*(This distance function is an example of a **metric**.)*

The following properties of the norm and distance follow from their definitions and the inner product axioms:

Theorem 7.1.12. *If V is a real inner product space, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and $k \in \mathbb{R}$, then*

(a) $\|\mathbf{v}\| \geq 0$, *with* $\|\mathbf{v}\| = 0$ *iff* $\mathbf{v} = \mathbf{0}$

(b) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

(c) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

(d) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

(e) $d(\mathbf{u}, \mathbf{v}) \geq 0$, *with* $d(\mathbf{u}, \mathbf{v}) = 0$ *iff* $\mathbf{u} = \mathbf{v}$

(f) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$

E.g. 7.1.13. *Suppose we have an inner product on* $\mathbb{R}^2$ *given by*

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1 v_1 + 5u_2 v_2.$$

If $\mathbf{u} = (1, 2)$ *and* $\mathbf{v} = (2, 3)$*, then find* $\|\mathbf{u}\|$ *and* $d(\mathbf{u}, \mathbf{v})$*, and compare them with the values obtained if we instead use the dot product.*

$\|\underset{\sim}{u}\| = \sqrt{\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle} = \sqrt{2u_1^2 + 5u_2^2} = \sqrt{22}$

whereas $\|\underset{\sim}{u}\|_E = \sqrt{\underset{\sim}{u} \cdot \underset{\sim}{u}} = \sqrt{5}$

$d(\underset{\sim}{u}, \underset{\sim}{v}) = \sqrt{\langle \underset{\sim}{v} - \underset{\sim}{u}, \underset{\sim}{v} - \underset{\sim}{u} \rangle} = \sqrt{7}$

whereas $d_E(\underset{\sim}{u}, \underset{\sim}{v}) = \sqrt{2}$

E.g. 7.1.14. *Consider* $C[0, 2\pi]$ *endowed with the inner product given in E.g. 7.1.7. Compute the norm of* $\sin(x)$ *and* $\cos(x)$*.*

$\|\sin\|^2 = \langle \sin, \sin \rangle = \int_0^{2\pi} \sin^2(x)\, dx = \pi$

$\Rightarrow \|\sin\| = \sqrt{\pi}$

Likewise $\|\cos\| = \sqrt{\pi}$.

7.2 Angles & Orthogonality (A&R §6.2)

7.2.1 Angles

One of the most fundamental results concerning inner products is:

Theorem 7.2.1 (Cauchy-Schwartz Inequality)**.** *If V is a real inner product space and $\mathbf{u}, \mathbf{v} \in V$, then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

Proof. If $u = 0$ then $\langle 0, v \rangle = 0 = \|0\| \|v\|$. So assume $u, v \neq 0$. Positive definiteness $\Rightarrow$

$$\boxed{\langle u - v, u - v \rangle \geq 0 \iff}$$

$$\Downarrow$$

$$\boxed{2 \langle u, v \rangle \leq \langle u, u \rangle + \langle v, v \rangle}$$

. But for any $\lambda > 0$,

$$\boxed{\left\langle \lambda u, \frac{1}{\lambda} v \right\rangle = \langle u, v \rangle}$$

and so

$$2 \langle u, v \rangle = 2 \left\langle \lambda u, \frac{1}{\lambda} v \right\rangle \leq \langle \lambda u, \lambda u \rangle + \left\langle \frac{1}{\lambda} v, \frac{1}{\lambda} v \right\rangle$$

$$\Rightarrow \boxed{2 \langle u, v \rangle \leq \lambda^2 \langle u, u \rangle + \frac{1}{\lambda^2} \langle v, v \rangle \quad (*)}$$

Choose λ to minimize RHS:

$f(\lambda) = a\lambda^2 + b\lambda^{-2}$ has a minimum at . . .