Theorem 7.3.5. *If B is an orthonormal basis for an n-dimensional inner product space V, then $\langle \mathbf{u}, \mathbf{v} \rangle = [\mathbf{u}]_B \cdot [\mathbf{v}]_B$ for all $\mathbf{u}, \mathbf{v} \in V$.*

Proof. Let $B = \{\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_n\}$ & $\underset{\sim}{u} = \sum_{i=1}^{n} \alpha_i \underset{\sim}{w}_i$ & $\underset{\sim}{v} = \sum_{j=1}^{n} \beta_j \underset{\sim}{w}_j$. Then:

$$\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = \left\langle \sum_{i=1}^{n} \alpha_i \underset{\sim}{w}_i, \sum_{j=1}^{n} \beta_j \underset{\sim}{w}_j \right\rangle$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} \beta_j \langle \underset{\sim}{w}_i, \underset{\sim}{w}_j \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \beta_i \qquad \text{where } [\underset{\sim}{u}]_B = (\alpha_1, \ldots, \alpha_n)$$

$$[\underset{\sim}{v}]_B = (\beta_1, \ldots, \beta_n)$$

$$= [\underset{\sim}{u}]_B \cdot [\underset{\sim}{v}]_B$$

$\square$

$\square$

Theorem 7.3.7 (Projection Theorem)**.** *Let V be an inner product space. Let W be a subspace with an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Every vector $\mathbf{u} \in V$ can be expressed uniquely as $\mathbf{u} = \mathbf{w} + \mathbf{w}_\perp$ with $\mathbf{w} \in W$ and $\mathbf{w}_\perp \in W^\perp$, and*

$$\mathbf{w} = \sum_{j=1}^{k} \frac{\langle \mathbf{u}, \mathbf{v}_j \rangle}{\|\mathbf{v}_j\|^2} \mathbf{v}_j.$$

Definition 7.3.8. *We write $\mathbf{w} = \text{proj}_W \mathbf{u}$, and call it the **orthogonal projection** of $\mathbf{u}$ on W.*

Remark 7.3.9. *If V is Euclidean n-space, then $\text{proj}_W(\mathbf{u}) = \sum_{j=1}^{k} P_j \mathbf{u}$, where $P_j = \dfrac{1}{\|\mathbf{v}_j\|^2} \mathbf{v}_j \mathbf{v}_j^T$ is the matrix of orthogonal projection along $\mathbf{v}_j$.*

Remark 7.3.10. *If the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for W is orthonormal then*

$$\text{proj}_W \mathbf{u} = \sum_{j=1}^{k} \langle \mathbf{u}, \mathbf{v}_j \rangle \mathbf{v}_j.$$

Proof of Theorem 7.3.7. Let $u \in V$ & $w = \sum_{j=1}^{k} \frac{\langle u, v_j \rangle}{\langle v_j, v_j \rangle} v_j$

& $w^\perp = u - w$. Clearly $w \in W$ & $u = w + w^\perp$.

$\langle w^\perp, v_i \rangle = \langle u - w, v_i \rangle = \langle u, v_i \rangle - \langle w, v_i \rangle$

$= \langle u, v_i \rangle - \left\langle \sum_{j=1}^{k} \frac{\langle u, v_j \rangle}{\langle v_j, v_j \rangle} v_j, v_i \right\rangle$

$= \langle u, v_i \rangle - \sum_{j=1}^{k} \frac{\langle u, v_j \rangle}{\langle v_j, v_j \rangle} \langle v_j, v_i \rangle$

$= \langle u, v_i \rangle - \frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle} \langle v_i, v_i \rangle = \langle u, v_i \rangle - \langle u, v_i \rangle = 0$

$\Rightarrow \quad \underset{\sim}{w}^{\perp} \in W^{\perp}$ via Th^{m} 7.2.12.

To prove uniqueness, suppose

$\underset{\sim}{u} = \underset{\sim}{v} + \underset{\sim}{v}^{\perp}$ with $\underset{\sim}{v} \in W$, $\underset{\sim}{v}^{\perp} \in W^{\perp}$.

Then $\underset{\sim}{v} + \underset{\sim}{v}^{\perp} = \underset{\sim}{u} = \underset{\sim}{w} + \underset{\sim}{w}^{\perp} \Rightarrow \boxed{\underset{\sim}{v} - \underset{\sim}{w} = \underset{\sim}{w}^{\perp} - \underset{\sim}{v}^{\perp}}$ $(*)$

But W is a subspace so $\underset{\sim}{v} - \underset{\sim}{w} \in W$

$W^{\perp}$ is a subspace so $\underset{\sim}{w}^{\perp} - \underset{\sim}{v}^{\perp} \in W^{\perp}$

so $\underset{\sim}{v} - \underset{\sim}{w} \in W \cap W^{\perp} = \{\underset{\sim}{0}\} \Rightarrow \underset{\sim}{v} = \underset{\sim}{w}$

$\Rightarrow \quad \underset{\sim}{w}^{\perp} = \underset{\sim}{v}^{\perp}$ by $(*)$

$\square$

$\square$

(Gram - Schmidt)

Theorem 7.3.11. *Every nonzero finite dimensional inner product space has an orthonormal basis.*

Proof. Let $B = \{\underset{\sim}{u}_1, \dots, \underset{\sim}{u}_n\}$ be any basis for the space. We begin by using B to construct an orthogonal basis $B' = \{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n\}$

1. Let $\underset{\sim}{v}_1 = \underset{\sim}{u}_1$

2. $\underset{\sim}{v}_2 = \underset{\sim}{u}_2 - \mathrm{proj}_{W_1} \underset{\sim}{u}_2$ where $W_1 = \mathrm{span}\{\underset{\sim}{v}_1\}$

$\in W_1^{\perp}$

Furthermore since $\underset{\sim}{v}_2 = \underset{\sim}{u}_2 - \dfrac{\langle \underset{\sim}{u}_2, \underset{\sim}{u}_1 \rangle}{\langle \underset{\sim}{u}_1, \underset{\sim}{u}_1 \rangle} \underset{\sim}{u}_1$

is a nontrivial linear combination of vectors in B, $\underset{\sim}{v}_2$ cannot be zero because B is L.I. Therefore $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ are orthogonal <u>non zero</u> vectors.

3. $\underset{\sim}{v}_3 = \underset{\sim}{u}_3 - \mathrm{proj}_{W_2} \underset{\sim}{u}_3$ where $W_2 = \mathrm{span}\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$

$\in W_2^{\perp}$. Furthermore:

$$\underset{\sim}{v}_3 = \underset{\sim}{u}_3 - \frac{\langle \underset{\sim}{u}_3, \underset{\sim}{v}_1 \rangle}{\langle \underset{\sim}{v}_1, \underset{\sim}{v}_1 \rangle} \underset{\sim}{u}_1 - \frac{\langle \underset{\sim}{u}_3, \underset{\sim}{v}_2 \rangle}{\langle \underset{\sim}{v}_2, \underset{\sim}{v}_2 \rangle} \left[\underset{\sim}{u}_2 - \frac{\langle \underset{\sim}{u}_2, \underset{\sim}{u}_1 \rangle}{\langle \underset{\sim}{u}_1, \underset{\sim}{u}_1 \rangle} \underset{\sim}{u}_1 \right]$$

$\Rightarrow \underset{\sim}{v}_3$ is a nontrivial linear combination of vectors in $B \Rightarrow \underset{\sim}{v}_3 \neq \underset{\sim}{0}$ because B is LI

$\vdots$

n. $\underset{\sim}{v}_n = \underset{\sim}{u}_n - \mathrm{proj}_{W_{n-1}} \underset{\sim}{u}_n \in W_{n-1}^{\perp}$

where $W_{n-1} = \mathrm{span}\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_{n-1}\}$.

We obtain an orthogonal basis

$B' = \{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n\}$.

Setting $\underset{\sim}{w}_1 = \dfrac{\underset{\sim}{v}_1}{\|\underset{\sim}{v}_1\|}$, $\underset{\sim}{w}_2 = \dfrac{\underset{\sim}{v}_2}{\|\underset{\sim}{v}_2\|}$, $\ldots$, $\underset{\sim}{w}_n = \dfrac{\underset{\sim}{v}_n}{\|\underset{\sim}{v}_n\|}$

it then follows that $B'' = \{\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_n\}$ is an orthonormal basis for V.

$\square$

$\square$

The proof of Theorem 7.3.11 justifies the following algorithm.

Algorithm 7.3.12 (Gram-Schmidt)**.** *To convert a basis* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ *into an orthogonal basis:*

(i) $\mathbf{v}_1 = \mathbf{u}_1$

(ii) $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$ $= u_2 - \mathrm{proj}_{W_1} u_2$, $W_1 = \mathrm{span}\{v_1\}$

(iii) $\mathbf{v}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$ $= u_3 - \mathrm{proj}_{W_2} u_3$

$W_2 = \mathrm{span}\{v_1, v_2\}$

(iv) $\mathbf{v}_4 = \mathbf{u}_4 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

$\vdots$

(continue for n steps) $= u_4 - \mathrm{proj}_{W_3} u_4$.

$W_3 = \mathrm{span}\{v_1, v_2, v_3\}$

To obtain an orthonormal basis, we can then simply normalize the orthogonal basis constructed by the Gram-Schmidt algorithm.