Lemma 8.1.5. *If A and B are orthogonal $n \times n$ matrices then AB is orthogonal.*

Proof. $(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$

$\Rightarrow$ AB is orthogonal.

□

□

Lemma 8.1.6. *If A is orthogonal then* $\det(A) = \pm 1$.

Proof. If A is orthogonal then $\underline{A^T A} = \underline{I}$

$\Rightarrow \det(A^T A) = \det(I)$

$\Rightarrow \det(A^T)\det(A) = 1$

$\Rightarrow \det(A)^2 = 1 \Rightarrow \det(A) = \pm 1$

□

□

Theorem 8.1.7. *If A is an $n \times n$ matrix, then the following are equivalent.*

(a) *A is orthogonal*

(b) *$\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$*

(c) *$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*] (a) $\Leftrightarrow$ (c) Prob. Set

Proof. (a) $\Rightarrow$ (b): If A is orthogonal then $A = [\underset{\sim}{a}_1 \dots \underset{\sim}{a}_n]$ with $\{\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_n\}$ an orthonormal set.

$$\begin{aligned}
\Rightarrow \|A\underset{\sim}{x}\|^2 &= \|x_1 \underset{\sim}{a}_1 + \dots + x_n \underset{\sim}{a}_n\|^2 \\
&= \|x_1 \underset{\sim}{a}_1\|^2 + \dots + \|x_n \underset{\sim}{a}_n\|^2 \quad \text{(by Pythagoras)} \\
&= x_1^2 \|\underset{\sim}{a}_1\|^2 + \dots + x_n^2 \|\underset{\sim}{a}_n\|^2 \\
&= x_1^2 + \dots + x_n^2 \\
&= \|\underset{\sim}{x}\|^2
\end{aligned}$$

$\Rightarrow \|A\underset{\sim}{x}\| = \|\underset{\sim}{x}\|$.

(b) $\Rightarrow$ (a): Suppose $\|A\underset{\sim}{x}\| = \|\underset{\sim}{x}\| \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^n$.

Choose $\underset{\sim}{x} = \underset{\sim}{e}_i \Rightarrow \|A\underset{\sim}{e}_i\| = \|\underset{\sim}{e}_i\|$

$\Rightarrow \|\underset{\sim}{a}_i\| = 1 \quad \forall\, 1 \le i \le n$. So all columns are unit vectors.

Choosing $\underset{\sim}{x} = \underset{\sim}{e}_i + \underset{\sim}{e}_j$ for $1 \le i, j \le n$.

$\|A(\underset{\sim}{e}_i + \underset{\sim}{e}_j)\|^2 = \|\underset{\sim}{e}_i + \underset{\sim}{e}_j\|^2 = \|\underset{\sim}{e}_i\|^2 + \|\underset{\sim}{e}_j\|^2 = 1 + 1 = 2$

$\Rightarrow \|A\underset{\sim}{e}_i + A\underset{\sim}{e}_j\|^2 = 2 \Rightarrow \|\underset{\sim}{a}_i + \underset{\sim}{a}_j\|^2 = 2$

$$\langle \underset{\sim}{a}_i + \underset{\sim}{a}_j, \underset{\sim}{a}_i + \underset{\sim}{a}_j \rangle = 2 \quad \Longrightarrow$$

$$\langle \underset{\sim}{a}_i, \underset{\sim}{a}_i \rangle + \langle \underset{\sim}{a}_j, \underset{\sim}{a}_j \rangle + 2\langle \underset{\sim}{a}_i, \underset{\sim}{a}_j \rangle = 2$$

$$1 + 1 + 2\langle \underset{\sim}{a}_i, \underset{\sim}{a}_j \rangle = 2 \quad \Longrightarrow \quad 2\langle \underset{\sim}{a}_i, \underset{\sim}{a}_j \rangle = 0$$

$$\Longrightarrow \langle \underset{\sim}{a}_i, \underset{\sim}{a}_j \rangle = 0$$

So the columns form an orthonormal set

So A is orthogonal.

$\square$

E.g. 8.1.8. *Verify, by direct matrix multiplication, that the rotation matrix*

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

preserves the Euclidean inner product and length.

Let $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ & $\underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ & let $c = \cos\theta$, $s = \sin\theta$

Then $R_\theta \underset{\sim}{x} = \begin{pmatrix} c x_1 - s x_2 \\ s x_1 + c x_2 \end{pmatrix}$ so

$R_\theta \underset{\sim}{x} \cdot R_\theta \underset{\sim}{y} = (c^2+s^2) x_1 y_1 + (c^2+s^2) x_2 y_2 = \underset{\sim}{x} \cdot \underset{\sim}{y}$

So R_θ preserves dot product, & therefore lengths too.

Theorem 8.1.9. *Let V be a finite dimensional inner product space. If P is the change-of-basis matrix from one orthonormal basis for V to another orthonormal basis for V, then P is orthogonal.*

Prob_Set

E.g. 8.1.10. *Consider the two orthonormal bases for $\mathbb{R}^2$, $B = \{(1,0),(0,1)\}$ and $B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$. Find the change-of-basis matrix from B to B' and verify that it is orthogonal.*

$$P_{B' \to B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R_\theta$$

$$\Longrightarrow \quad P_{B \to B'} = P^{-1}_{B' \to B} = R_{-\theta}$$

which is orthogonal

8.2 Orthogonal diagonalization (A&R §7.2)

Definition 8.2.1. *If A and B are square matrices, then we say that A and B are* ***orthogonally similar*** *if there exists an orthogonal matrix P for which $P^T A P = B$. If A is orthogonally similar to a diagonal matrix D*

$$P^T A P = D$$

we say that A is ***orthogonally diagonalizable*** *and that P* ***orthogonally diagonalizes*** *A.*

Given that orthogonal matrices have such nice properties, it is natural to ask which matrices can be orthogonally diagonalized.

Theorem 8.2.2. *If A is an $n \times n$ matrix, then the following are equivalent*

(a) A is orthogonally diagonalizable

(b) A has an orthonormal set of n eigenvectors

(c) A is symmetric

$(a) \Rightarrow (b)$ Alg. 6.2.6 $\Rightarrow$ $A = PDP^{-1}$

Proof. where the columns of P are eigenvectors of A.
But if $P^{-1} = P^T$ then the columns of P are orthonormal (Thm 8.1.4). Therefore if $A = PDP^T$ then the columns of P are an orthonormal set of n eigenvectors of A.

$(b) \Rightarrow (a)$ & $(a) \Rightarrow (c)$ on Prob. Set.

$(c) \Rightarrow (a)$ (in MTH2025 Workshop) ∎