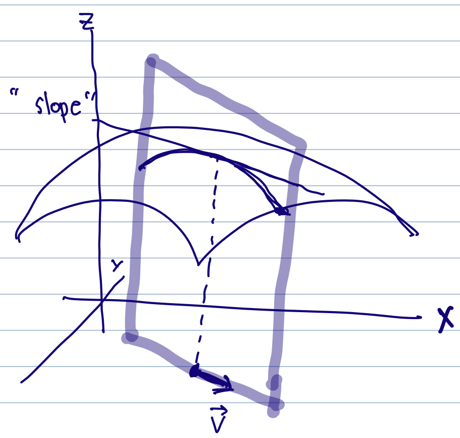
Partial derivatives: motivation



For multivariable ftn, the natural definition of "derivative" would be the directional derivative

$$D_{\vec{v}} f(x,y) := \lim_{h \to 0} \frac{f((x,y)+h\vec{v}) - f(x,y)}{h}$$

There is a formula

$$D_{\vec{v}} f(x,y) = \left(\underbrace{D_{(1,0)} f(x,y)}_{\displaystyle \overset{\shortparallel}{\frac{\partial f}{\partial x}}},\ \underbrace{D_{(0,1)} f(x,y)}_{\displaystyle \overset{\shortparallel}{\frac{\partial f}{\partial y}}} \right) \cdot \vec{v}$$

partial derivative

Partial Derivatives

Definition The first partial derivative of a two variable function $f(x,y)$ with respect to the variables x and y at the point (a,b) are the limits defined by

$$\frac{\partial f}{\partial x}(a,b) := \lim_{h \to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

and

$$\frac{\partial f}{\partial y}(a,b) := \lim_{h \to 0} \frac{f(a,b+h)-f(a,b)}{h},$$

respectively.

when y is a function of x,

$$\frac{df}{dx} = \underbrace{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}}_{\text{chain rule we will learn later.}} \neq \frac{\partial f}{\partial x}$$

$$d \neq \partial$$

Remarks

(i) There is an obvious generalization to functions

$$f(x_1, x_2, \dots, x_n)$$

of n-variables given by

$$\frac{\partial f}{\partial x_i}(a_1, a_2, \dots, a_n) = \lim_{h \to 0} \frac{1}{h}\Big[f(a_1, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_n)$$
$$- f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \Big]$$

for $i = 1, 2, \dots, n$.

(ii) There exists many notations for partial derivatives

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \partial_x f(a,b) = D_1 f(a,b) = \partial_1 f(a,b)$$

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \partial_y f(a,b) = D_2 f(a,b) = \partial_2 f(a,b)$$

Relation to the derivative from single variable Calculus

Given a function $f(x,y)$ of two variables and a point $(a,b) \in \mathbb{R}^2$, define two single variable functions by

$$g(x) = f(x,b) \quad \text{and} \quad h(y) = f(a,y).$$

Then

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \frac{\partial f}{\partial x}(a,b)$$

and similarly
$$h'(b) = \frac{\partial f}{\partial y}(a,b).$$

This implies that we reduce the problem of computing partial derivatives to that of computing derivatives of functions of a single variable.

Example

Let

$$f(x,y) = \cos(x)\sin(y).$$

Compute

$$\frac{\partial f}{\partial x}(x,y).$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\cos x \sin y) = \sin y \frac{\partial}{\partial x}(\cos x)$$

$$= \sin y \cdot \frac{d}{dx}\cos x = -\sin y \sin x \quad \blacksquare$$

Example

Compute the partial derivative

$$\frac{\partial}{\partial x}\left(y^{yx^2}\right).$$

Solution

chain rule

$$\frac{\partial}{\partial x}\left(y^{yx^2}\right) = (\log y)\, y^{yx^2} \cdot \underbrace{\frac{\partial}{\partial x}(yx^2)}_{2xy}$$

$$= 2xy\,(\log y)\, y^{yx^2}$$

review $\quad \frac{d}{dx} a^x = \frac{d}{dx}\left(e^{x \log a}\right) = (\log a)\, e^{x \log a}$

$a = e^{\log(a)}$ $\qquad = (\log a)\, a^x.$

Q Question

Let

$$f(x,y) = \frac{x}{x^2+y^2}.$$

Then what is $\frac{\partial f}{\partial x}(x,y)$?

Answer

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

review

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{|g(x)|^2}$$

Q

Question

Let

$$z = \sin^2(x) + xy.$$

Then what is

$$\left.\frac{\partial z}{\partial y}\right|_{\left(\frac{\pi}{2},1\right)} ?$$

Answer

$$\frac{\partial z}{\partial y} = x$$

$$\left.\frac{\partial z}{\partial y}\right|_{\left(\frac{\pi}{2},1\right)} = \frac{\pi}{2}$$