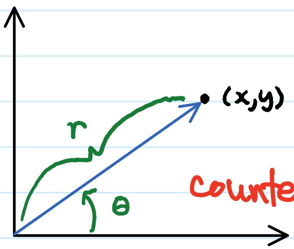
Lecture 15

Double Integrals in Polar Coordinates § 15.3

Polar coordinates

$(x,y) = (r\cos\theta,\ r\sin\theta)$



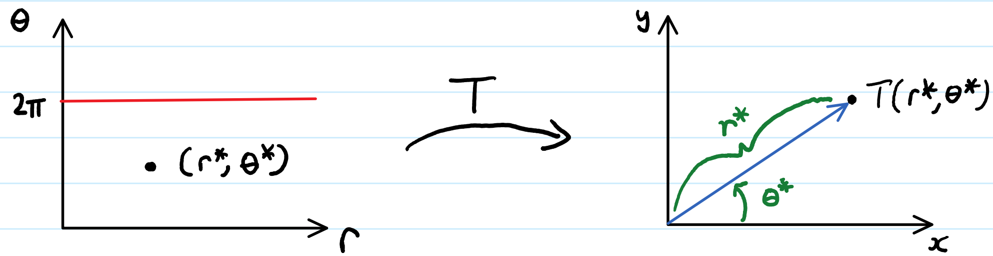
counterclockwise
radian

We can view polar coordinates as a map

$$T : [0,\infty) \times [0, 2\pi) \longrightarrow \mathbb{R}^2$$

defined by

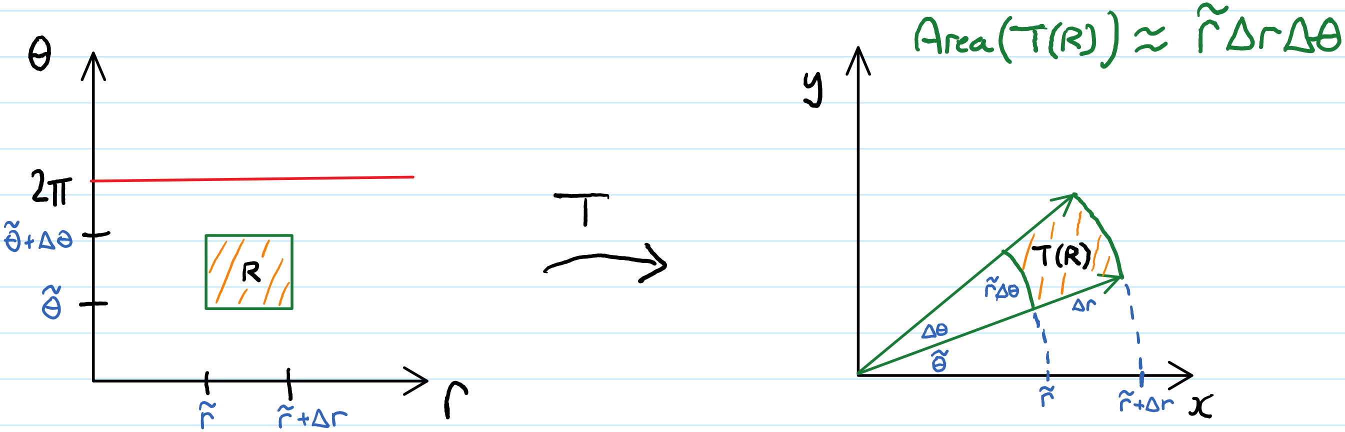
$$T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$



Integration in Polar Coordinates

Area Element

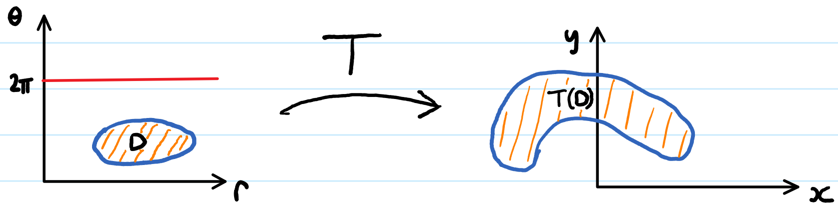
$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = r\,dr\,d\theta}$$



Integration Formula

$$\iint_{T(D)} f(x,y)\,dx\,dy = \iint_{D} f(T(r,\theta))\,\underbrace{r}_{|J(T)|}\,dr\,d\theta$$

where $T(r,\theta) = (r\cos\theta, r\sin\theta)$



Mapping regions using T

Q Question

If $D = [r_1, r_2] \times [\theta_1, \theta_2]$ then what does

$$T(D) = \{T(r,\theta) \mid (r,\theta) \in D\}$$

look like?

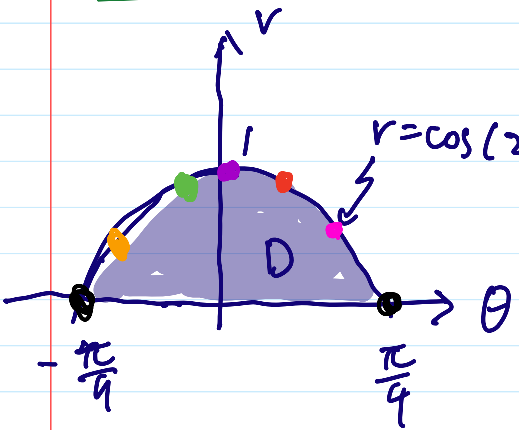
Answer

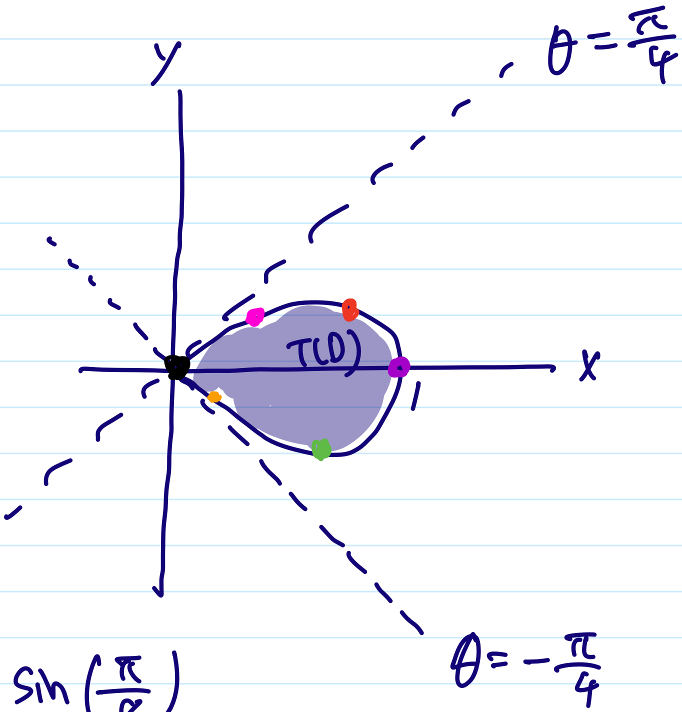
Example

Describe the region $T(D)$ where

$$D = \left\{ (r,\theta) \;\middle|\; -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \quad 0 \le r \le \cos(2\theta) \right\}$$

Answer





e.g. $\theta = \frac{\pi}{8}$, $r = \cos\left(\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}$

$x = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{8}\right) \quad y = \frac{1}{\sqrt{2}} \sin\left(\frac{\pi}{8}\right)$

$$r = \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$

$$x^2 - y^2 = r^3 = (x^2+y^2)^{3/2}$$

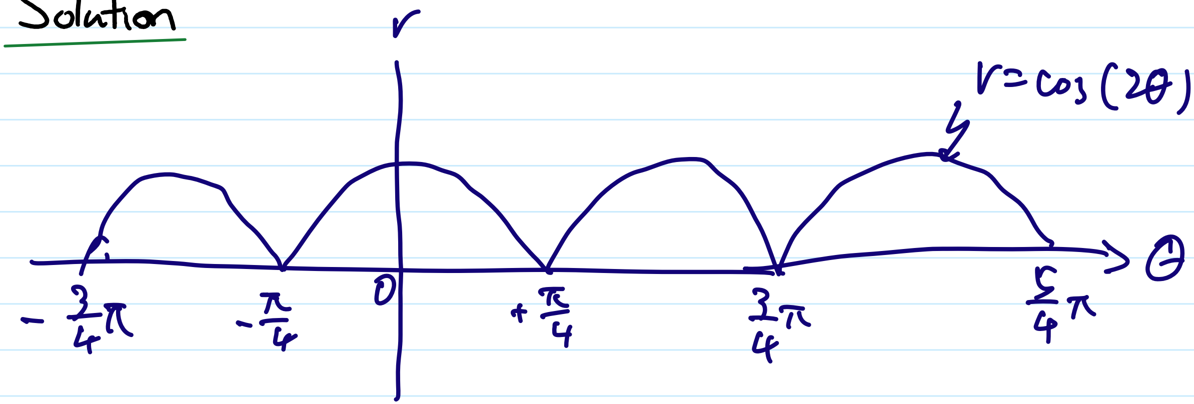
Computing Integral in Polar Coordinates

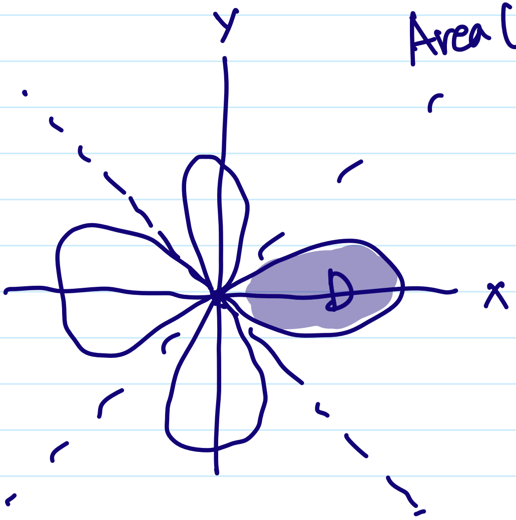
Example

Find the area of the region D enclosed by one loop of the four leaved rose $r = |\cos 2\theta|$

The D in the previous example.

Solution



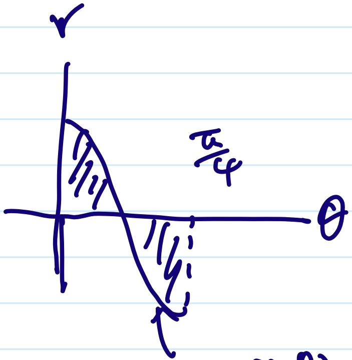


$$\text{Area}(D) = \iint_D dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos(2\theta)} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\cos^2(2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1+\cos(4\theta)}{2}\,d\theta$$

$$= \frac{\pi}{8}.$$



$r = \cos(4\theta)$

Example

Evaluate

$$\iint_{B_R(\underline{0})} \sin(x^2+y^2)\,dx\,dy \,.$$

$B_R(\underline{0})$ ← disc with radius R centred at $\vec{0}$

$$= \int_{-\sqrt{R}}^{\sqrt{R}} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sin(x^2+y^2)\,dy\,dx$$

↑ almost impos. to calculate.

Solution

$$\int_0^{2\pi}\int_0^R \sin(r^2)\; r\,dr\,d\theta$$

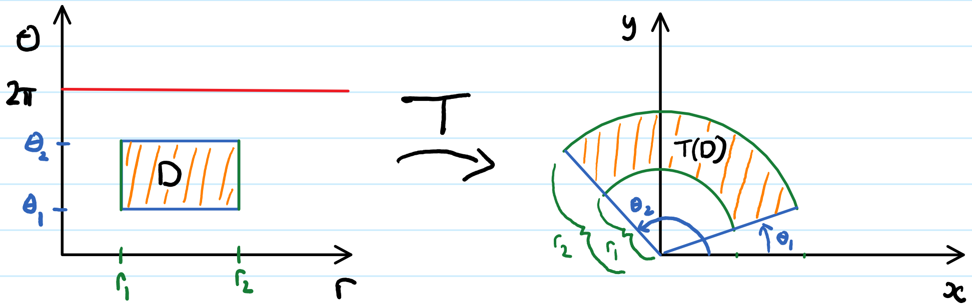
$$= \int_0^{2\pi} \left[-\frac{1}{2}\cos(r^2)\right]_{r=0}^{R} d\theta$$

$$= 2\pi \cdot \left(\frac{1}{2} - \frac{1}{2}\cos(R^2)\right)$$

When to use polar coordinates for integration

(I) It simplifies the domain of integration

Eg.



(II) It simplifies the function being integrated

Eg. If $f(x,y) = \ln(1+x^2+y^2)$, then

$$\iint_{T(D)} f(x,y)\,dx\,dy = \iint_{D} f(T(r,\theta))\,r\,dr\,d\theta = \iint_{D} \ln(1+r^2)\,r\,dr\,d\theta$$

$$T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

Question

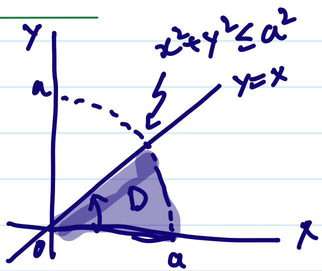
Use polar coordinates to evaluate the integral

$$\iint_D (x+y)\,dx\,dy$$

where D is the is the region in the first quadrant lying inside the disk $x^2+y^2 \le a^2$ $(a>0)$ and under the line $y=x$.

$x = r\cos\theta \qquad y = r\sin\theta$

Answer



$$\iint_D (x+y)\,dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^a (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^a r^2 (\cos\theta + \sin\theta)\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta) \cdot \underbrace{\int_0^a r^2 dr}_{\frac{1}{3}a^3} \, d\theta$$

$$= \frac{1}{3}a^3 \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta) \, d\theta$$

$$= \frac{1}{3}a^3 \left[+\sin\theta - \cos\theta \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{3}a^3 \left(0 - (0 - 1) \right) = \frac{1}{3}a^3$$