Theorem 8.2.3. *If A is a symmetric matrix, then eigenvectors from distinct eigenvalues are orthogonal.*

Proof. Let $\underline{A\underset{\sim}{v}_1 = \lambda_1 \underset{\sim}{v}_1}$ & $\underline{A\underset{\sim}{v}_2 = \lambda_2 \underset{\sim}{v}_2}$

$$(A\underset{\sim}{v}_1)\cdot\underset{\sim}{v}_2 = (A\underset{\sim}{v}_1)^T\underset{\sim}{v}_2 = \underset{\sim}{v}_1^T A^T \underset{\sim}{v}_2 = \underset{\sim}{v}_1^T(A\underset{\sim}{v}_2) = \underset{\sim}{v}_1\cdot(A\underset{\sim}{v}_2)$$

$$\Rightarrow \boxed{A\underset{\sim}{v}_1\cdot\underset{\sim}{v}_2 = \underset{\sim}{v}_1\cdot A\underset{\sim}{v}_2}$$

$$\Rightarrow \lambda_1\underset{\sim}{v}_1\cdot\underset{\sim}{v}_2 = \underset{\sim}{v}_1\cdot\lambda_2\underset{\sim}{v}_2$$

$$\Rightarrow \lambda_1\underset{\sim}{v}_1\cdot\underset{\sim}{v}_2 - \lambda_2\underset{\sim}{v}_1\cdot\underset{\sim}{v}_2 = 0$$

$$\Rightarrow (\lambda_1-\lambda_2)\underset{\sim}{v}_1\cdot\underset{\sim}{v}_2 = 0$$

So if $\lambda_1 \neq \lambda_2$ then $\underset{\sim}{v}_1\cdot\underset{\sim}{v}_2 = 0$

□

□

Theorems 8.2.3 and 7.3.11 imply we have the following algorithm for orthogonally diagonalizing a symmetric matrix.

Algorithm 8.2.4. *Let A be a symmetric matrix.*

(i) Find a basis for each eigenspace of A

(ii) For each eigenspace, use the Gram-Schmidt process to convert the given basis into an orthonormal basis

(iii) Form the matrix P whose columns are the union of all the basis vectors found in (ii)

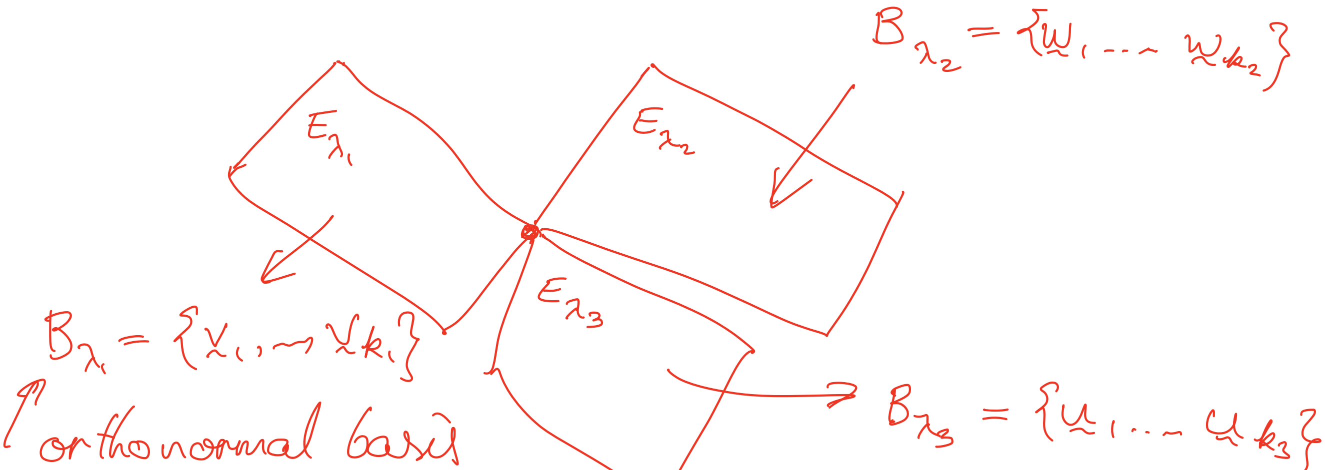
The matrix P will orthogonally diagonalize A, and the eigenvalues on the diagonal of $D = P^T AP$ will be in the same order as their corresponding eigenvectors in P.

E.g. suppose A has 3 distinct eigenvalues

Let E_{λ_i} denote eigenspace of λ_i, $i=1,2,3$

$E_{\lambda_i} \perp E_{\lambda_j} \quad \forall i \neq j$

$P = [\underset{\sim}{v}_1 \dots \underset{\sim}{v}_{k_1} \ \underset{\sim}{w}_1 \dots \underset{\sim}{w}_{k_2} \ \underset{\sim}{u}_1 \dots \underset{\sim}{u}_{k_3}]$



E.g. 8.2.5. *Find an orthogonal matrix P that diagonalizes*

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Since A is symmetric, it is orthogonally diagonalizable. $\det(\lambda I - A) = (\lambda-4)(\lambda-2) \Rightarrow$ eigenvalues are $\lambda = 2, 4$

A basis for eigenspace of $\lambda=2$ is $\{\underset{\sim}{v}_1\}$ w/ $\underset{\sim}{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

A basis for eigenspace of $\lambda=4$ is $\{\underset{\sim}{v}_2\}$ w/ $\underset{\sim}{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Setting $\underset{\sim}{w}_1 = \frac{\underset{\sim}{v}_1}{\|\underset{\sim}{v}_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\underset{\sim}{w}_2 = \frac{\underset{\sim}{v}_2}{\|\underset{\sim}{v}_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

we obtain an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors of A. Let

$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ & $D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ then

$A = PDP^T$

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~~**Theorem 8.2.6**~~ (Spectral decomposition)**.** *Let A be a symmetric $n \times n$ matrix, let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be an orthonormal basis of eigenvectors of A, and let $\lambda_1 \ldots, \lambda_n$ be the corresponding eigenvalues. Then*

$$A = \sum_{i=1}^{n} \lambda_i P_i$$

where $P_i = \mathbf{u}_i \, \mathbf{u}_i^T$ is the matrix of orthogonal projection in the direction of $\mathbf{u}_i$.

Proof.

$\square$

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E.g. 8.2.7. *Find a* 2×2 *matrix with eigenvalues* $\lambda_1 = 3$ *and* $\lambda_2 = -2$ *and corresponding eigenvectors* $\mathbf{v}_1 = (3, 4)$, $\mathbf{v}_2 = (-4, 3)$.

8.3 Singular value decomposition (A&R §9.4)

Theorem 8.3.1. *If A is an $m \times n$ matrix then:*

(a) $A^T A$ is orthogonally diagonalizable.

(b) The eigenvalues of $A^T A$ are nonnegative.

Proof. (a) $(A^TA)^T = A^T(A^T)^T = A^TA \Rightarrow A^TA$ is symmetric

$\Rightarrow$ its orthogonally diagonalizable.

(b) Let $\{\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n\}$ be orthonormal eigenvectors of A^TA with eigenvalues $\lambda_1, \dots, \lambda_n$.

Then
$$
\begin{aligned}
0 \le A\underset{\sim}{v}_i \cdot A\underset{\sim}{v}_i &= (A\underset{\sim}{v}_i)^T(A\underset{\sim}{v}_i) \\
&= \underset{\sim}{v}_i^T(A^TA)\underset{\sim}{v}_i \\
&= \underset{\sim}{v}_i^T(\lambda_i \underset{\sim}{v}_i) \\
&= \lambda_i \, \underset{\sim}{v}_i^T \underset{\sim}{v}_i \\
&= \lambda_i \, \underset{\sim}{v}_i \cdot \underset{\sim}{v}_i \\
&= \lambda_i \qquad \text{since } \|\underset{\sim}{v}_i\| = 1
\end{aligned}
$$

So $\lambda_i \geqslant 0$.

$\square$

$\square$

Definition 8.3.2. *If A is an $m \times n$ matrix, and if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $A^T A$, then the numbers*

$$\sigma_1 = \sqrt{\lambda_1}, \quad \sigma_2 = \sqrt{\lambda_2}, \dots, \quad \sigma_n = \sqrt{\lambda_n}$$

are the ***singular values*** *of A.*

Theorem 8.3.3. *Let A be an $m \times n$ matrix with rank r. Then A has precisely r positive singular values.*

Proof. $r = \text{rank}(A) = \text{rank}(A^T A) \Rightarrow$

$\boxed{r = n - \dim(\text{null}(A^T A))}$ $(*)$

Now suppose A has precisely k positive singular values.

$\sigma_1 \geq \dots \geq \sigma_k > 0$. The nonzero eigenvalues of $A^T A$ are $\sigma_1^2 \geq \dots \geq \sigma_k^2 > 0$.

The algebraic multiplicity of $\lambda = 0$ for $A^T A$ is $n-k$

$\Leftrightarrow$ (Since $A^T A$ is diagonalizable)

The geometric multiplicity of $\lambda = 0$ for $A^T A$ is $n-k$

$\Leftrightarrow$ $\dim(\text{null}(A^T A)) = n-k$

So $(*)$ implies $r = n-(n-k) = k$.

$\square$

$\square$

E.g. 8.3.4. *Find the singular values of*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$A^TA = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

From E.g. 6.1.7 we know eigenvalues of A^TA are 1 & 3 $\Rightarrow$

the singular values of A are 1 & $\sqrt{3}$.

Definition 8.3.5. *Let A be an $m \times n$ matrix. A **Singular Value Decomposition**(SVD) of A is a factorization $A = U\Sigma V^T$, where U is an $m \times m$ orthogonal matrix, Σ is a diagonal $m \times n$ echelon matrix with nonnegative entries, and V is an orthogonal $n \times n$ matrix.*

Theorem 8.3.6 (Singular value decomposition)**.** *Let A be an $m \times n$ matrix of rank $r > 0$. If $A = U\Sigma V^T$ is an SVD of A then:*

Prob. Set

(a) *The nonzero entries of Σ are $(\Sigma)_{ii} = \sigma_i$ for $1 \leq i \leq r$, where $\sigma_1, \ldots, \sigma_r$ are the nonzero singular values of A*

(b) *For each $1 \leq i \leq n$, the ith column $\mathbf{v}_i$ of V is an eigenvector of A^TA with eigenvalue σ_i^2* $\Rightarrow A\underset{\sim}{v}_i = \sigma_i \underset{\sim}{u}_i$

(c) *For each $1 \leq i \leq r$, the ith column of U is $\mathbf{u}_i = \dfrac{1}{\sigma_i} A\mathbf{v}_i$*

(d) *The set $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is an orthonormal basis for* $\mathrm{col}(A)$

(e) *If $r < m$, $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m\}$ is an orthonormal basis for* $\mathrm{null}(A^T)$

(f) *If $r < n$, $\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for* $\mathrm{null}(A)$

(g) *The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for* $\mathrm{row}(A)$

Remark 8.3.7. *One simple consequence of Theorem 8.3.3 is that if A has SVD $A = U\Sigma V^T$, then* $\mathrm{rank}(A) = \mathrm{rank}(\Sigma)$.