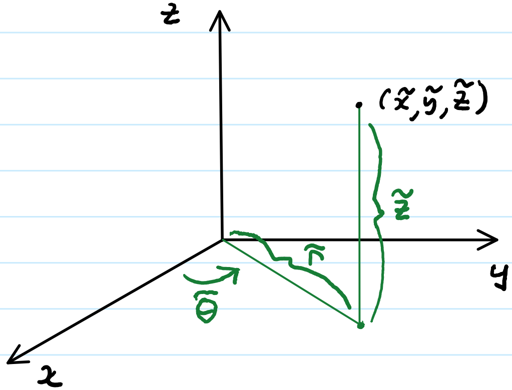
Lecture 17

Triple Integrals in Cylindrical Coordinates § 15.7

Cylindrical Coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

We can view cylindrical coordinates as a map

$$T : [0,\infty) \times [0, 2\pi) \times (-\infty, \infty) \longrightarrow \mathbb{R}^3$$

defined by

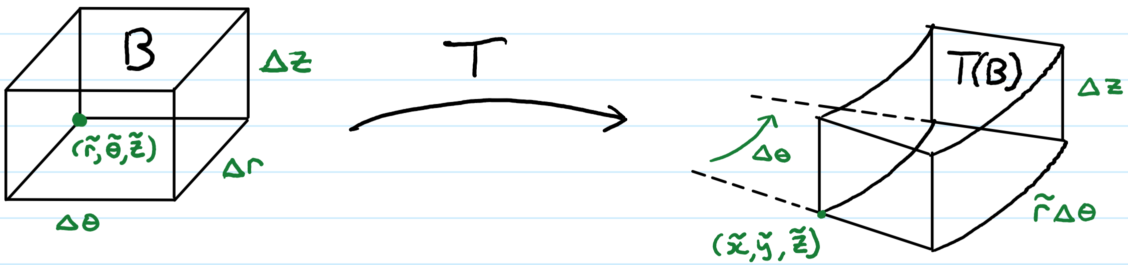
$$T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$

Integration in Cylindrical Coordinates

Volume Element

$$\boxed{dV = \frac{\partial(x,y,z)}{\partial(r,\theta,z)}\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz}$$

$$\mathrm{Vol}(T(B)) \approx \tilde{r}\, \Delta r\, \Delta\theta\, \Delta z$$



$$B = [\tilde{r}, \tilde{r}+\Delta r] \times [\tilde{\theta}, \tilde{\theta}+\Delta\theta] \times [\tilde{z}, \tilde{z}+\Delta z]$$

Integration Formula

$$\iiint_{T(E)} f(x,y,z)\,dV = \iiint_{E} f(T(r,\theta,z))\,\underbrace{r}_{|J(T)|}\,dr\,d\theta\,dz$$

where

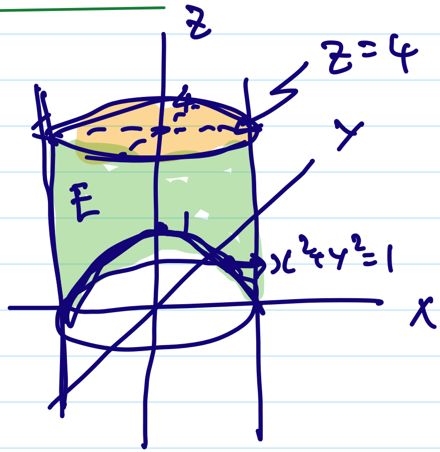
$$T(r,\theta,z) = (r\cos\theta,\ r\sin\theta,\ z)$$

Example

Let $E \subset \mathbb{R}^3$ be the region bounded by $x^2+y^2=1$, $z=4$ and $z=1-x^2-y^2$.

Suppose E represents a solid will mass density proportional to the distance from the z-axis. Find the mass of E.

Solution



$$\iiint_E \underbrace{c \cdot \sqrt{x^2+y^2}}_{\text{mass density}} \, dV$$

$$= \int_0^4 \int_0^{2\pi} \int \; c\,r \quad r\,dr\,d\theta\,dz$$

$r \in [\sqrt{1-z}, 1]$ if $z \le 1$

$r \in [0, 1]$ if $z > 1$

$$= \int_0^1 \int_0^{2\pi} \int_{\sqrt{1-z}}^1 c\,r \quad r\,dr\,d\theta\,dz$$

$$+\int_1^4\int_0^{2\pi}\int_0^1 cr\; r\,dr\,d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\underbrace{\int_{\sqrt{1-z}}^1 cr^2\,dr}_{\left[\frac{c}{3}r^3\right]_{\sqrt{1-z}}^1 \;=\; \frac{c}{3}\left(1-(1-z)^{\frac{3}{2}}\right)}\,d\theta\,dz + \underbrace{\int_1^4\int_0^{2\pi}\underbrace{\int_0^1 cr^2\,dr}_{\frac{c}{3}}\,d\theta\,dz}_{\frac{c}{3}\cdot 2\pi\cdot(4-1)}$$

$$= \int_0^1 \frac{c}{3}\left(1-(1-z)^{\frac{3}{2}}\right)\cdot\underbrace{\int_0^{2\pi}d\theta}_{=\,2\pi}\;dz + 2c\pi$$

$$= \frac{2\pi c}{3}\left(1 - \left[\frac{2}{5}(1-z)^{\frac{5}{2}}\right]_{z=0}^{z=1}\right) + 2c\pi = \frac{2}{\cancel{3}}\pi c\,\frac{\cancel{3}}{5} + 2c\pi$$

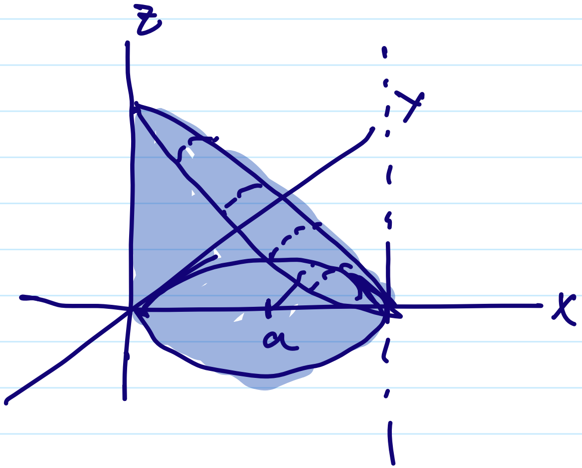
$$= \frac{12}{5}\pi c$$

Example

Find the volume bounded by $z=0$, the cone $z=2a-\sqrt{x^2+y^2}$, and the cylinder $x^2+y^2=2ay$.

Solution

$$x^2+y^2=2ay \iff x^2+(y-a)^2=a^2$$



$$\iiint_E dV = \iiint_{\tilde{E}} r \, dr \, d\theta \, dz$$

$$\tilde{E} = \{ (r, \theta, z) : \quad 0 \le z \le 2a - r$$
$$0 \le r^2 \le 2ay = 2ar\sin\theta \}$$

$$\int_0^{\pi} \int_0^{2a\sin\theta} \int_0^{2a-r} r \, dz \, dr \, d\theta \qquad r \le 2a\sin\theta$$

$$= \int_0^{\pi} \int_0^{2a\sin\theta} r(2a-r) \, dr \, d\theta.$$

$$= \int_0^{\pi} \left[ar^2 - \frac{1}{3} r^3 \right]_{r=0}^{r=2a\sin\theta} d\theta$$

$$= 2a^3 \int_0^\pi \underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}} \, d\theta - \frac{8a^3}{3} \int_0^\pi \sin\theta \, \underbrace{\sin^2\theta}_{(1-\cos^2\theta)} \, d\theta$$

$u = \cos\theta \quad du = -\sin\theta \, d\theta$

θ	$0 \to \pi$
u	$1 \to -1$

$$= 2a^2 \cdot \frac{1}{2}\pi - \frac{8a^3}{3} \left(-\int_1^{-1} (1-u^2)\, du \right)$$

$$= \pi a^2 - \frac{8a^3}{3} \underbrace{\int_{-1}^{1} (1-u^2)\, du}_{\left[u - \frac{1}{3}u^3\right]_{-1}^{1} = \frac{2}{3} - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}}$$

$$= \pi a^2 - \frac{8a^3}{3} \cdot \frac{4}{3} \quad \blacksquare$$

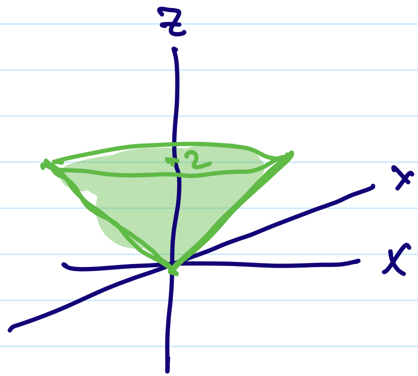
Question

Let $E \subset \mathbb{R}^3$ be the region bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$.

What does the integral

$$\iiint_E x^2+y^2 \, dV$$

evaluate to?



Answer

$$\iiint_E (x^2+y^2)\, dV = \int_0^2 \int_0^{2\pi} \int_0^z r^2 \cdot r \, dr\, d\theta\, dz$$

$$= \frac{1}{4} \int_0^2 \int_0^{2\pi} z^4 \, d\theta\, dz$$

$$= \frac{1}{4}\int_0^2 z^4 \underbrace{\int_0^{2\pi} d\theta}_{2\pi}\, dz$$

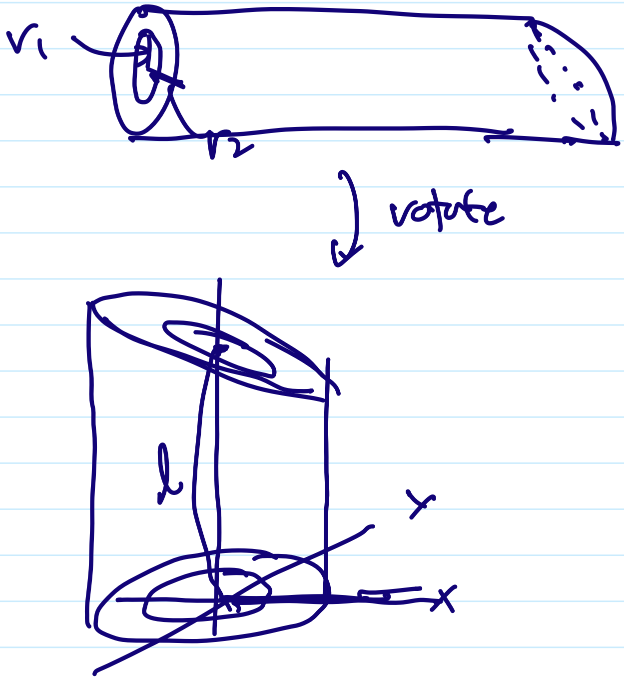
$$= \frac{\pi}{2}\cdot\left[\frac{1}{5}z^5\right]_{z=0}^{2}$$

$$= \frac{\pi}{5}\cdot 16 \quad \blacksquare$$

Example

Find the volume of a pipe with one end cut at $45°$ and the other at $90°$. The average length of the pipe is l and its inner and outer radii are r_1 and r_2.

Solution



$$E = \{(x,y,z) : \quad r_1^2 \le x^2+y^2 \le r_2^2, \quad 0 \le z \le l - x\}$$

$$\tilde{E} = \{(r,\theta,z) : \quad r_1 \le r \le r_2, \quad 0 \le \theta \le 2\pi, \; 0 \le z \le l - r\cos\theta\}$$

$$\mathrm{vol}(E) = \iiint_E dV$$

$$= \int_{r_1}^{r_2}\int_0^{2\pi}\int_0^{l-r\cos\theta} r \, dz \, d\theta \, dr$$

$$= \int_{r_1}^{r_2}\int_0^{2\pi} r\,(l - r\cos\theta)\, d\theta \, dr$$

$$= \int_{r_1}^{r_2}\left(rl \cdot 2\pi - r^2 \underbrace{\int_0^{2\pi}\cos\theta \, d\theta}_{0}\right) dr$$

$$= 2\pi l \cdot \frac{1}{2}\left(r_2^2 - r_1^2\right) = \pi l \left(r_2^2 - r_1^2\right)$$