$\lambda = (b/a)^{1/4}$. So set $\lambda^2 = \sqrt{\dfrac{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle}{\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle}}$ in $\circledast$

$$2\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle \leq \sqrt{\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} + \sqrt{\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle}$$

$$\Rightarrow \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle \leq \sqrt{\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle \langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} = \|\underset{\sim}{u}\| \|\underset{\sim}{v}\|$$

$$\Rightarrow \boxed{\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle \leq \|\underset{\sim}{u}\| \|\underset{\sim}{v}\| \quad (\ast\ast) \quad \forall\ \underset{\sim}{u}, \underset{\sim}{v} \in V.}$$

If $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle \geq 0$ we are done.

If $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle < 0$ apply $(\ast\ast)$ to $\underset{\sim}{u}$ & $-\underset{\sim}{v}$ to get:

$$|\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle| = -\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = \langle \underset{\sim}{u}, -\underset{\sim}{v} \rangle \leq \|\underset{\sim}{u}\| \|-\underset{\sim}{v}\| = \|\underset{\sim}{u}\| \|\underset{\sim}{v}\|$$

So in either cas

$$|\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle| \leq \|\underset{\sim}{u}\| \|\underset{\sim}{v}\|$$

$\square$

$\square$

It follows immediately from the Cauchy-Schwarz inequality that

$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \leq 1. \tag{7.4}$$

This then allows us to define the **angle** between pairs of vectors in real inner product spaces, by analogy with the usual definition for the Euclidean inner product.

Definition 7.2.2. *Let V be a real inner product space & let $\mathbf{u}, \mathbf{v} \in V$. The **angle** between $\mathbf{u}$ and $\mathbf{v}$ is*

$$\theta = \arccos\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right).$$

Note that (7.4) guarantees that $\arccos$ is only invoked on its domain, so θ is always well-defined.

E.g. 7.2.3. *Consider $C[0, 2\pi]$ endowed with the inner product given in E.g. 7.1.7. Find the angle between $\sin(x)$ and $\cos(x)$.*

$$\langle \sin, \cos \rangle = \int_0^{2\pi} \sin(x)\cos(x)\,dx = 0$$

$$\Rightarrow \quad \theta = \arccos(0) = \pi/2.$$

7.2.2 Orthogonality

The angle between two elements of a real inner product space is $\pi/2$ iff their inner product is 0.

Definition 7.2.4. *If V is a real inner product space and $\mathbf{u}, \mathbf{v} \in V$ then we say $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.*

E.g. 7.2.3 shows that $\cos$ and $\sin$ are orthogonal, on the space $C[0, 2\pi]$ with inner product (7.3).

E.g. 7.2.5. *Consider* $\mathbf{u} = (1, 1)$ *and* $\mathbf{v} = (1, -1)$.

(a) *Are* $\mathbf{u}$ *and* $\mathbf{v}$ *orthogonal with respect to the Euclidean inner product?*

(b) *Are* $\mathbf{u}$ *and* $\mathbf{v}$ *orthogonal with respect to the weighted inner product* $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$*?*

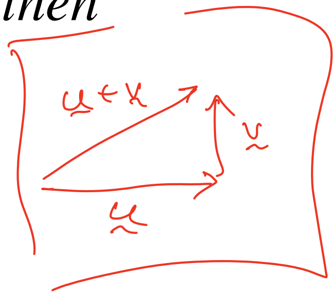
(a) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0 \implies \underset{\sim}{u}$ & $\underset{\sim}{v}$ are orthogonal

(b) $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = 3 - 2 = 1$

$\implies \underset{\sim}{u}$ & $\underset{\sim}{v}$ are not orthogonal

Theorem 7.2.6 (Generalized Pythagorean theorem)**.** *If V is a real inner product space and $\mathbf{u}, \mathbf{v} \in V$ are orthogonal, then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$



Proof.

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$

$$= \langle u+v, u \rangle + \langle u+v, v \rangle$$

$$= \langle u, u \rangle + \underbrace{\langle v, u \rangle}_{=0} + \underbrace{\langle u, v \rangle}_{=0} + \langle v, v \rangle$$

$$= \|u\|^2 + \|v\|^2$$

□

Definition 7.2.7. *Let U and W be subspaces of an inner product space V. Then U and W are* ***orthogonal*** *if every vector of U is orthogonal to every vector of W. This is denoted $U \perp W$.*

Lemma 7.2.8. *Let U and W be subspaces of an inner product space V. If $U \perp W$, then $U \cap W = \{\mathbf{0}\}$.*

Proof. Identical proof to lemma 4.7.4 □

Corollary 7.2.9. *If U and W are orthogonal subspaces of an inner product space V, then their sum is direct, i.e. $U + W = U \oplus W$.*

Definition 7.2.10. *Let U be a subspace of an inner product space V. The* ***orthogonal complement*** *of U is the set $U^\perp$ of all vectors in V that are orthogonal to every vector of U.*

Lemma 7.2.11. *Let U be a subspace of an inner product space V. Then $U^\perp$ is a subspace of V.*

Proof. Identical proof to Lemma 4.7.6 □

Lemma 7.2.12. *Let $U = \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} \subset V$, where V is an inner product space. Then*

$$U^\perp = \{\mathbf{v} \in V \text{ such that } \langle \mathbf{v}, \mathbf{u}_i \rangle = 0 \text{ for all } i = 1, \ldots, r\}.$$

Proof. Prob. Set. □

7.3 Gram-Schmidt Algorithm (A&R §6.3)

In an inner product space, certain bases are especially convenient to work with – orthonormal bases. The Gram-Schmidt algorithm shows us how to construct such bases.

Definition 7.3.1. *A set of two or more vectors in a real inner product space are **orthogonal** if all pairs of vectors in the set are orthogonal. If, in addition, each vector has length 1, the set is **orthonormal**.*

If $\mathbf{v}$ is a nonzero vector, then $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector. The process of dividing a vector by its length is called **normalization**. Any orthogonal set of nonzero vectors can be normalized to obtain an orthonormal set.

E.g. 7.3.2. *Verify that the set $S = \{(1,1,1), (0,1,-1), (2,-1,-1)\}$ is orthogonal in Euclidean 3-space. Is S orthonormal? If not, normalize each element of S to obtain an orthonormal set.*

$$\left.\begin{array}{l}(1,1,1)\cdot(0,1,-1) = 0 \\ (1,1,1)\cdot(2,-1,-1) = 0 \\ (0,1,-1)\cdot(2,-1,-1) = 0\end{array}\right\} \text{ S is orthogonal}$$

$$\left.\begin{array}{l}\|(1,1,1)\| = \sqrt{3} \\ \|(0,1,-1)\| = \sqrt{2} \\ \|(2,-1,-1)\| = \sqrt{6}\end{array}\right\} \text{ S is \underline{not} orthonormal}$$

But $\left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(0,1,-1), \frac{1}{\sqrt{6}}(2,-1,-1)\right\}$ is orthonormal

Theorem 7.3.3. *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of non-zero vectors in an inner product space, then S is linearly independent.*

Proof. Let $\sum_{j=1}^{n} c_j \underset{\sim}{v}_j = \underset{\sim}{0}$ & let $1 \le i \le n$.

Then $\langle \underset{\sim}{v}_i, \sum_{j=1}^{n} c_j \underset{\sim}{v}_j \rangle = \langle \underset{\sim}{v}_i, \underset{\sim}{0} \rangle$

But $\langle \underset{\sim}{v}_i, \underset{\sim}{0} \rangle = 0$ (Prob Set), so linearity $\Rightarrow$

$\sum_{j=1}^{n} c_j \langle \underset{\sim}{v}_i, \underset{\sim}{v}_j \rangle = 0$

But $\langle \underset{\sim}{v}_i, \underset{\sim}{v}_j \rangle = 0 \ \forall j \neq i \Rightarrow \boxed{c_i \langle \underset{\sim}{v}_i, \underset{\sim}{v}_i \rangle = 0}$

Since $\underset{\sim}{v}_i \neq \underset{\sim}{0}$ by assumption, $\langle \underset{\sim}{v}_i, \underset{\sim}{v}_i \rangle > 0$

$\Rightarrow c_i = 0$. Since this holds $\forall\ 1 \le i \le n$, S is L.I.

□

Corollary 7.3.4. *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal subset of an inner product space, then S is linearly independent.*

Proof. If S is orthonormal, then $\|\underset{\sim}{v}_i\| = 1$ $\forall\ 1 \le i \le n \Rightarrow \underset{\sim}{v}_i \neq \underset{\sim}{0}$ $\forall\ 1 \le i \le n$ so S is an orthogonal set of non zero vectors

□

It is particularly easy to express a vector $\mathbf{u}$ in an inner product space as a linear combination of an orthonormal basis.

Theorem 7.3.6. *If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then for any $\mathbf{u} \in V$ we have*

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Proof. Let $\underset{\sim}{u} = \sum_{j=1}^{n} c_j \underset{\sim}{v}_j$ & $1 \leq i \leq n$. Then:

$$\langle \underset{\sim}{u}, \underset{\sim}{v}_i \rangle = \left\langle \sum_{j=1}^{n} c_j \underset{\sim}{v}_j , \underset{\sim}{v}_i \right\rangle$$

$$= \sum_{j=1}^{n} c_j \langle \underset{\sim}{v}_j, \underset{\sim}{v}_i \rangle \quad \text{by linearity}$$

$$= c_i \quad \text{since } \langle \underset{\sim}{v}_j, \underset{\sim}{v}_i \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

So $\langle \underset{\sim}{u}, \underset{\sim}{v}_i \rangle = c_i \quad \forall\ 1 \leq i \leq n$

$$\Rightarrow \quad \underset{\sim}{u} = \sum_{j=1}^{n} \langle \underset{\sim}{u}, \underset{\sim}{v}_j \rangle \underset{\sim}{v}_j \qquad \square$$

$\square$