Lectures 8/9

Implicit Differentiation §14.5

In first-year calculus, functions are defined explicitly,

$$y(x) = x^3 + 2x + 1.$$

For such functions, to find the slope of tangent line to the graph of $(x, y(x))$, it suffices to differentiate:

$$y'(x) = 3x^2 + 2.$$

At a point $(x_0, y(x_0))$, the slope of the tangent line is simply

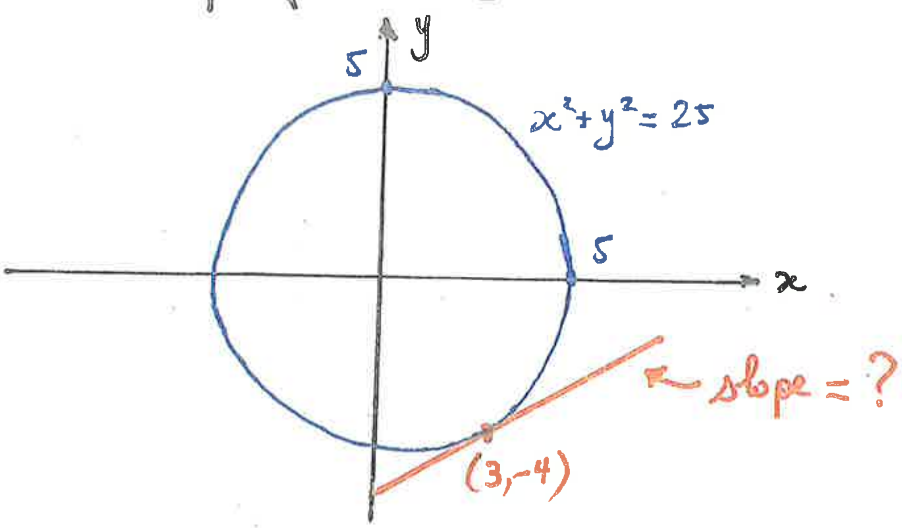
$$y'(x_0) = 3x_0^2 + 2.$$

However, some functions are defined implicitly. A famous example is:

$$x^2 + y^2 = 25,$$

which represents the circle of radius 5 centered on $(0,0)$.

How to find the slope of the tangent line at the point $(3,-4)$



One way to proceed is to note that $(+3,-4)$ lies on the graph of $y(x) = -\sqrt{25-x^2}$. Then the slope is:

$$0 = f(x,y) = x^2+y^2-25$$

$$\downarrow \frac{d}{dx}$$

$$y'(3) = \left.\frac{x}{\sqrt{25-x^2}}\right|_{x=3} = \frac{3}{4}.$$

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 2x + 2y\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Unfortunately, this is not always possible. For example, the equation

$$x^2y^3 + x^3y^2 - 2y = 0 \qquad (*).$$

cannot be solved explicitely for y in terms of x.

The point $(1,1)$ satisfies $(*)$. How to find dy/dx at the point $(1,1)$? The idea is to apply $\frac{d}{dx}$ to both sides of $(*)$:

not $\frac{\partial}{\partial x}$

$$\frac{d}{dx}\left(x^2y^3 + x^3y^2 - 2y\right) = \frac{d}{dx}(0)$$

$$\Rightarrow \quad y^3\frac{dx^2}{dx} + x^2\frac{dy^3}{dx} + y^2\frac{dx^3}{dx} + x^3\frac{dy^2}{dx} - 2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2xy^3 + 3x^2y^2\frac{dy}{dx} + 3x^2y^2 + 2x^3y\frac{dy}{dx} - 2\frac{dy}{dx} = 0$$

Solving for dy/dx gives: $\quad \dfrac{dy}{dx} = \dfrac{-2xy^3 - 3x^2y^2}{2+3x^2y^2+2x^3y}.$

At the point $(1,1)$, we find $\quad \left.\dfrac{dy}{dx}\right|_{(1,1)} = \dfrac{-2-3}{2+3+2} = \dfrac{-5}{7}.$

More generally, if y and x satisfy $F(x,y)=0$, then at a point (x_0,y_0) such that $F(x_0,y_0)=0$, we find the slope dy/dx at (x_0,y_0) by using the chain-rule:

$$dF(x,y) = d0 = 0$$

So: $$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

$$\Longrightarrow \quad \boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}}$$ implicit differentiation.

Example: Find dy/dx for $e^{xy} = e^{4x} - e^{5y}$.

Set $F(x,y) = e^{xy} - e^{4x} + e^{5y}$, so $F(x,y)=0$.

Then: $$\begin{cases} \partial F/\partial x = ye^{xy} - 4e^{4x} \\ \partial F/\partial y = xe^{xy} + 5e^{5y}. \end{cases}$$

Therefore: $$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{ye^{xy} - 4e^{4x}}{xe^{xy} + 5e^{5y}}.$$

$\blacksquare$

Q) **Question:** Find the slope $\left(\frac{dy}{dx}\right)$ of the tangent line to the ~~graph of~~ curve given by $xy^2 + y^4 = 4 - 2x$ at $(1,-1)$.

Answer:

Define $F(x,y) = xy^2 + y^4 - 4 + 2x$.

Then, our curve is defined by

$$F(x,y) = 0$$

$$0 = \frac{d0}{dx} = \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx}$$

$$= (y^2 + 2) + (2xy + 4y^3) \cdot \frac{dy}{dx}$$

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = -\left.\frac{y^2+2}{2xy+4y^3}\right|_{(1,-1)} = -\frac{3}{-2-4}$$

$$= \frac{1}{2} \qquad \square$$

Directional Derivative and Gradient.

Let's say we have

$$y = f(\vec{x})$$

A naive way to define a derivative is "directional" derivative with respect to direction $\vec{u}$

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$$

Problem!!

$$\vec{v} = 2\vec{u}$$

$$D_{\vec{v}} f = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{v}) - f(\vec{x})}{h}$$

$$= \lim_{2h \to 0} \frac{f(\vec{x} + h \cdot 2\vec{u}) - f(\vec{x})}{2h} \cdot 2$$

$$= 2 \cdot D_{\vec{u}} f$$

Definition: The directional derivative of a differentiable function $f(x,y)$ at the point (x_0,y_0) in the direction of the vector $u=(u_1,u_2)$ is:

$$D_u f(x_0,y_0) = \frac{d}{ds}\Big(f(\gamma(s))\Big)\Big|_{s=0}$$

where $\gamma(s) = (x_0,y_0)+s\hat{u}$ is the curve through (x_0,y_0) in the direction of $\hat{u} = u/|u|$.

Notation: We use ^ for a unit vector

The directional derivative $D_u f(x_0,y_0)$ measures the rate of change of the function $f(x,y)$ at the point (x_0,y_0) in the direction of the vector u.

- $D_u f(x_0,y_0) > 0 \Rightarrow f$ increasing at (x_0,y_0) in the direction of u
- $D_u f(x_0,y_0) < 0 \Rightarrow f$ decreasing at (x_0,y_0) in the direction of u

We call $|D_u f(x_0,y_0)|$ the rate of change (ROC) of f at (x_0,y_0) in the direction of the vector u.

Definition: The gradient of a differentiable function $f(x,y)$ at the point (x_0,y_0) is the VECTOR:

$$\nabla f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0) \, , \, \frac{\partial f}{\partial y}(x_0,y_0) \right).$$

Theorem: If $f(x,y)$ is differentiable at (x_0,y_0) and if u is a vector in $\mathbb{R}^2$, then:

$$\boxed{D_u f(x_0,y_0) = \hat{u} \underset{\text{dot product}}{\cdot} \nabla f(x_0,y_0)} = \overset{1}{|\hat{u}|}\, |\nabla f| \cos\theta \qquad \left(\hat{u} = u/|u| \right)$$

θ the angle bet $\hat{u}$ and ∇f

Proof: Write $\hat{u} = (\hat{u}_1, \hat{u}_2)$. By definition:

$s = s$

$$D_u f(x_0,y_0) = \left[\frac{d}{ds} f\big(\underbrace{x_0 + s\hat{u}_1 \, , \, y_0 + s\hat{u}_2}_{(x_0,y_0) + s(\hat{u}_1, \hat{u}_2)}\big) \right]_{s=0}$$

By the chain-rule:

$$\begin{aligned} D_u f(x_0,y_0) &= \Big[\frac{\partial f}{\partial x}(x_0 + s\hat{u}_1 \, , \, y_0 + s\hat{u}_2) \frac{d}{ds}(x_0 + s\hat{u}_1) \\ &\qquad + \frac{\partial f}{\partial y}(x_0 + s\hat{u}_1 \, , \, y_0 + s\hat{u}_2) \frac{d}{ds}(y_0 + s\hat{u}_2) \Big]_{s=0} \\ &= \Big[\hat{u}_1 \frac{\partial f}{\partial x}(x_0 + s\hat{u}_1, y_0 + s\hat{u}_2) + \hat{u}_2 \frac{\partial f}{\partial y}(x_0 + s\hat{u}_1, y_0 + s\hat{u}_2) \Big]_{s=0} \\ &= \hat{u}_1 \frac{\partial f}{\partial x}(x_0,y_0) + \hat{u}_2 \frac{\partial f}{\partial y}(x_0,y_0) \\ &= \hat{u} \cdot \nabla f(x_0,y_0) \, , \text{ as announced.} \end{aligned}$$

■

Example: Find the rate of change of $f(x,y) = x^2 \sin y$ at $(1,\pi)$ in the direction $u = (1,3)$.

Answer: Since f is differentiable, we have
$$D_u f(1,\pi) = \hat{u} \cdot \nabla f(1,\pi) \quad (*).$$

We have: $$\hat{u} = \vec{u}/|\vec{u}| = \frac{(1,3)}{\sqrt{1^2+3^2}} = \frac{1}{\sqrt{10}}(1,3).$$

and $$\nabla f = \Big(\overset{\frac{\partial f}{\partial x}}{2x \sin y},\ \overset{\frac{\partial f}{\partial y}}{x^2 \cos y}\Big), \text{ so } \nabla f(1,\pi) = (0,-1).$$

Putting this into $(*)$ shows:
$$D_u f(1,\pi) = \frac{1}{\sqrt{10}}(1,3)\cdot(0,-1) = -3/\sqrt{10}.$$

Therefore, ROC is $\frac{3}{\sqrt{10}}$.

Question: What is the meaning of the result of the previous example?

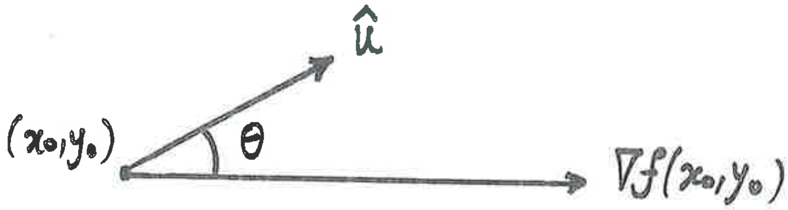
Answer: At $(1, \pi)$, the function decreases $\frac{3}{\sqrt{10}}$ per unit travel in the direction of $(1,3)$.

Theorem: The maximum rate of change of a function $f(x,y)$ at a point (x_0, y_0) is the magnitude $|\nabla f(x_0,y_0)|$, with $f(x,y)$ increasing [decreasing] at the maximum rate in the direction $\nabla f(x_0,y_0)$ $[-\nabla f(x_0,y_0)]$.

Proof: Choose a direction u. Since $\hat{u}$ has magnitude 1, we have:

$$\hat{u} \cdot \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|} = \frac{\hat{u}\cdot \nabla f(x_0,y_0)}{|\hat{u}|\,|\nabla f(x_0,y_0)|} = \cos\theta ,$$

where θ is the angle between $\hat{u}$ and $\nabla f(x_0,y_0)$:



$0 \leq \theta \leq \pi$

Since $D_{\hat{u}} f(x_0,y_0) = \hat{u}\cdot \nabla f(x_0,y_0)$, we have:

$$\frac{D_{\hat{u}} f(x_0,y_0)}{|\nabla f(x_0,y_0)|} = \cos\theta .$$

this is the direction $\nabla f(x_0,y_0)$

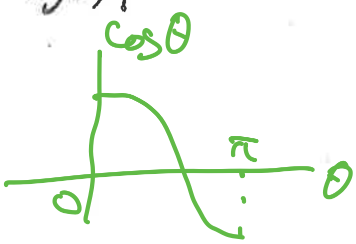
To maximize $D_{\hat{u}} f(x_0,y_0)$, take $\theta = 0$ to find:

$$D_{\hat{u}} f(x_0,y_0) = |\nabla f(x_0,y_0)|$$

this is the direction of $-\nabla f(x_0,y_0)$

To minimize $D_{\hat{u}} f(x_0,y_0)$, take $\theta = \pi$ to find:

$$D_{\hat{u}} f(x_0,y_0) = -|\nabla f(x_0,y_0)|$$



Q) Question : The temperature at a point $(x,y) \in \mathbb{R}^2$ is given by :

$$T(x,y) = x^2 - 2y^2$$

At the point $(2,-1)$, in what direction must one move to cool off at the fastest rate possible ?

Answer :

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right) = (2x, -4y)$$

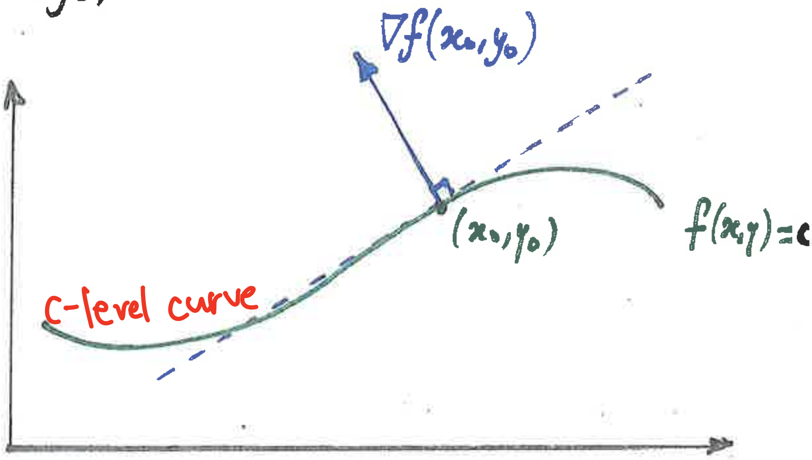
Since the gradient is the direction where the temperature increases the fastest,

the direction where the temperature decreases fastest is $-\nabla T\big|_{(2,-1)} = -(4, 4)$.

Ans $-(4,4)$

An important fact:

Theorem: If $f(x,y)$ is differentiable at (x_0,y_0) and the gradient $\nabla f(x_0,y_0) \neq (0,0)$, then $\nabla f(x_0,y_0)$ is orthogonal to the level curve of $f(x,y)$ that passes through the point (x_0,y_0).



Proof: Suppose the point (x_0,y_0) lies on the level curve

$$f(x,y)=c$$

for some constant c, so that $f(x_0,y_0)=c$. We parametrize the level curve by $\begin{cases} r(0)=(x_0,y_0) \\ r(t)=(r_1(t),r_2(t)) \end{cases}$, so that $f(r_1(t),r_2(t))=c$. Differentiating in t on both sides:

$$\frac{d}{dt} f(r_1(t),r_2(t)) = \frac{dc}{dt} = 0$$

i.e.

$$r_1' \frac{\partial f}{\partial x}(r_1(t),r_2(t)) + r_2' \frac{\partial f}{\partial y}(r_1(t),r_2(t)) = 0$$

$$\Rightarrow \quad \frac{dr}{dt} \cdot \nabla f(r(t)) = 0$$

At $t=0$, $dr/dt\big|_{t=0}$ is orthogonal to $\nabla f(\overbrace{x_0,y_0}^{r(0)})$. Since $dr/dt\big|_{t=0}$ is tangent to the curve $r(t)$ at (x_0,y_0), we're done. ■

The previous theorem has an obvious extension to higher dimension. For example, if $F(x,y,z)$ is differentiable at (x_0,y_0,z_0), the gradient $\nabla F(x_0,y_0,z_0) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_{(x_0,y_0,z_0)} \neq (0,0,0)$, then $\nabla F(x_0,y_0,z_0)$ is a vector (in $\mathbb{R}^3$) orthogonal to the level surface passing through (x_0,y_0,z_0) $F(x,y,z)=0$

A special case is when

$$F(x,y,z) = z - f(x,y).$$

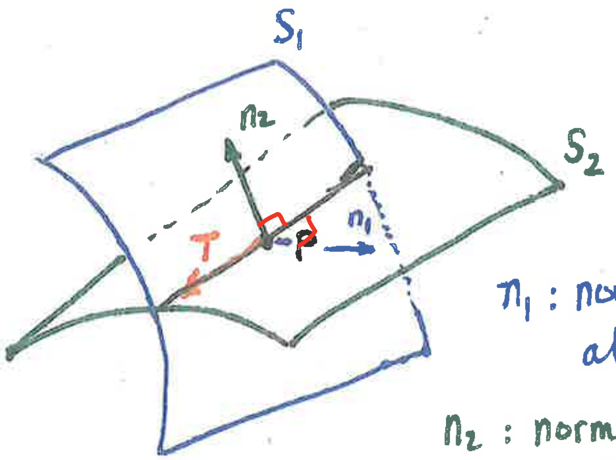
$F=0 \Leftrightarrow z=f(x,y)$

The corresponding 0-level surface is the graph $z=f(x,y)$. It passes through $(x_0,y_0,z_0) = (x_0,y_0,f(x_0,y_0))$. The vector

$$\nabla F(x_0,y_0,z_0) = \left(-\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1\right)$$

is therefore orthogonal to the graph of $z=f(x,y)$ at the point $(x_0,y_0,f(x_0,y_0))$. This is not a surprise, as we have already encountered this fact in Lecture 6.

Example: Find a vector tangent to the curve of intersection of the two cylinders $x^2+y^2=2$ and $y^2+z^2=2$ at the point $(1,-1,1)$.



n_1 : normal to S_1 at P

n_2 : normal to S_2 at P

$T = n_1 \times n_2$: tangent vector to intersection curve at P

Answer:

Set:

$$\begin{cases} F_1(x,y,z) = x^2+y^2-2 \\ F_2(x,y,z) = y^2+z^2-2 \end{cases}$$

By the previous observation:

$$\begin{cases} n_1 = \nabla F_1(1,-1,1) = (2x, 2y, 0)\big|_{(1,-1,1)} = (2,-2,0) \\ n_2 = \nabla F_2(1,-1,1) = (0, 2y, 2z)\big|_{(1,-1,1)} = (0,-2,2). \end{cases}$$

Hence:

$$T = n_1 \times n_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ -4 \end{pmatrix}$$

■

Q)

Question: Let $f(x,y) = x - y^2$ and consider the level curve

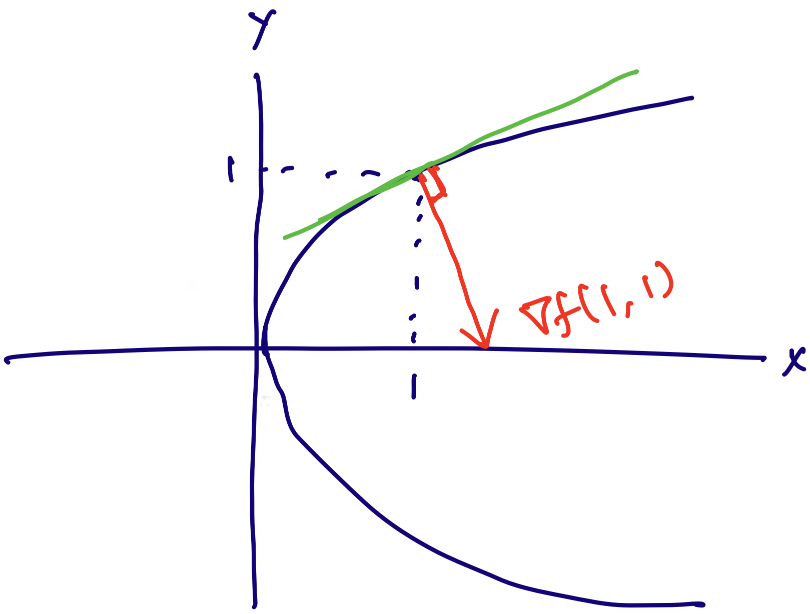
$$f(x,y) = 0.$$

What is the geometric significance of the vector $\nabla f(1,1)$?

Answer: "∇" : nabla

$\nabla f(1,1)$ is orthogonal to $\{(x,y) : f(x,y) = 0\}$

This means, $\nabla f(1,1)$ is orthogonal to the curve $x = y^2$ at $(1,1)$.



$$\nabla f(1,1) = (1, -2y)\big|_{(1,1)} = (1,-2)$$