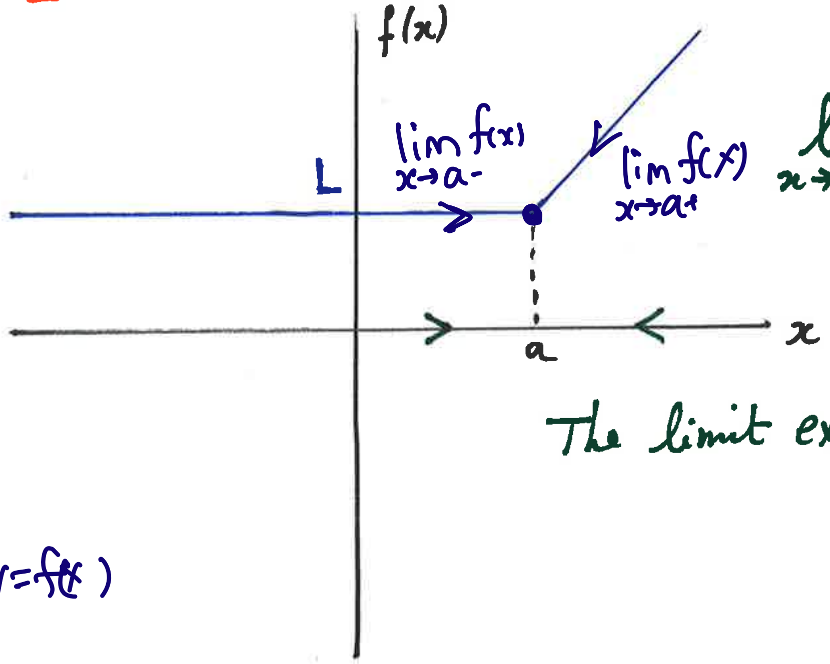
Lecture 3 §14.2

Limits

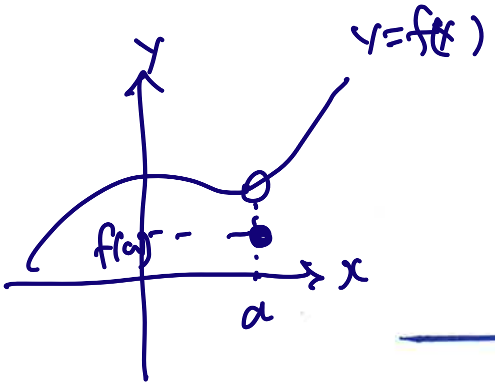
Recall that for a function of 1 variable $f(x)$, the limit of $f(x)$ as $x \to a$ exists whenever:

$$\left[\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \right]$$

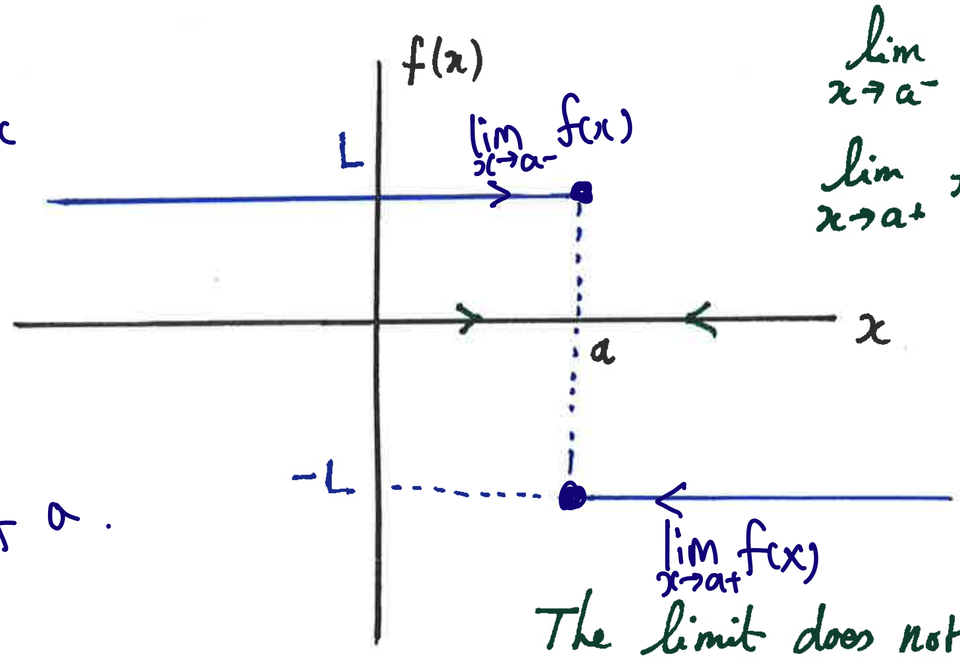


$\lim_{x \to a} f(x) = L$

The limit exists



$\lim_{x \to a} f(x)$ exists but $f(x)$ is not conti at a.



$\lim_{x \to a^-} f(x) = L$

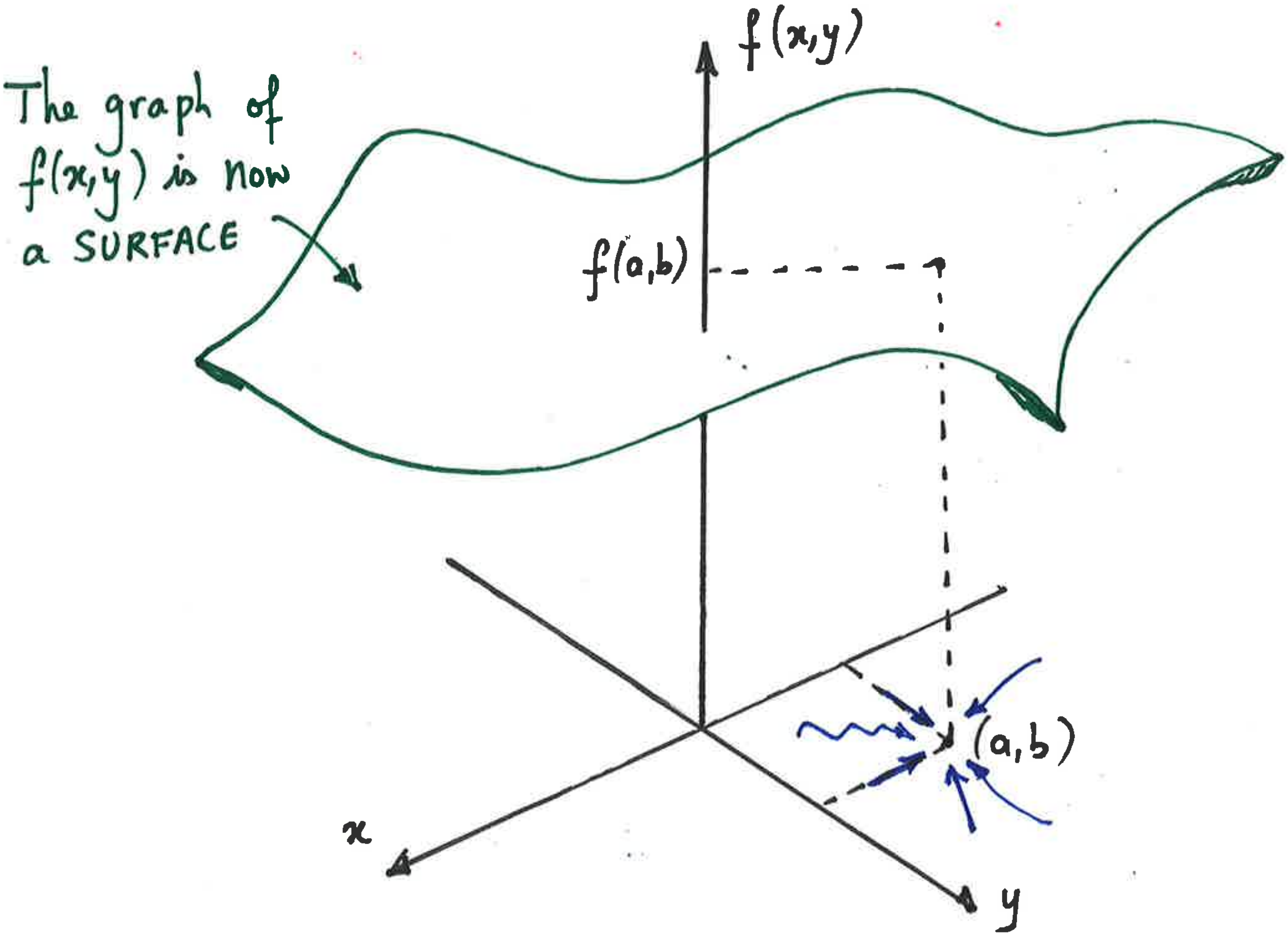
$\lim_{x \to a^+} f(x) = -L$

The limit does not exist (DNE)

We say $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists and equal to $f(a)$

Thus for a function of 1 variable, two directions are enough (from the left and from the right) suffice to decide whether a limit exists.

But for a function of 2 variables, things become much more complicated.



There are infinitely many directions and ways to approach the point (a,b) !!

$\lim_{(x,y)\to(a,b)} f(x,y)$ exists if and only if

$$\lim_{(x,y)\xrightarrow[\text{any curve}]{}(a,b)} f(x,y) \text{ are the same.}$$

Sometimes, a limit is found merely by direct substitution:

$$\lim_{(x,y)\to(4,3)} \frac{x^2-1}{3x+y} = \frac{4^2-1}{3\times4+3} = 1.$$

QUESTION: Find

$$\lim_{(x,y)\to(1,\pi/2)} \frac{x\sin y}{y}.$$

Answer:

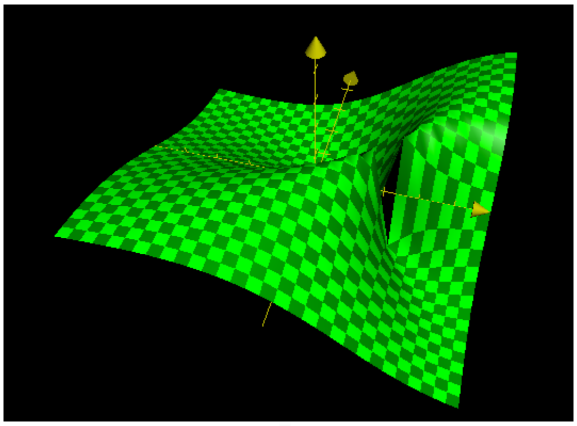
$$\lim_{(x,y)\to(1,\frac{\pi}{2})} \frac{x\sin y}{y} = \frac{1\cdot\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}$$

Of course, this is not always possible:

$$\lim_{(x,y)\to(1,0)} \frac{2(x-1)y}{(x-1)^2+y^2} = \frac{0}{0} = ??$$

Further investigation is necessary.



Let us see what happens when we approach the point $(1,0)$ along some specific paths, say along the straight lines

$$y = m(x-1) \quad , \text{ for some constant } m.$$

↑ line with slope m and passes through $(1,0)$

Then:

$$\frac{2(x-1)y}{(x-1)^2+y^2} = \frac{2m(x-1)^2}{(x-1)^2+m^2(x-1)^2} = \frac{2m}{1+m^2}.$$

The result depends on m, i.e. on the chosen path, and it follows that

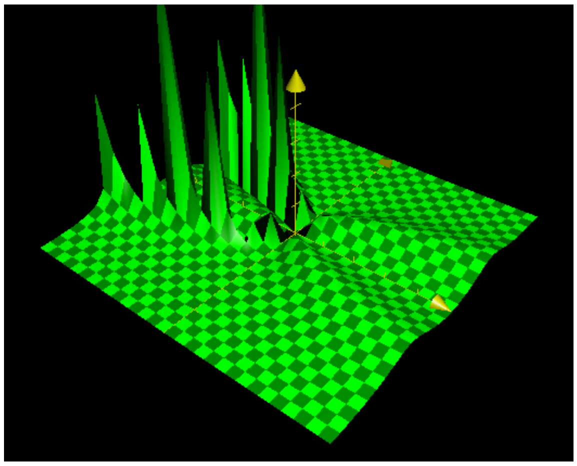
$$\lim_{(x,y)\to(1,0)} \frac{2(x-1)y}{(x-1)^2+y^2} \quad \text{does } \underline{\text{NOT}} \text{ exist.}$$

Therefore:

To show a limit does not exist, it suffices to exhibit two paths along which the limits differ.

Question: Investigate

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^6}.$$



Answer:

Hidden Step

See the exponents of x and y

Since y- exponent $= 2\times$ x-exponent,

we choose $y = m\sqrt{x}$

Step 1 Choose the curves $y = m\sqrt{x}$

Step 2

$$\frac{x^2y^2}{x^3+y^6} = \frac{x^2 \cdot m^2 \cdot x}{x^3 + m^6 \cdot x^3} = \frac{m^2}{1+m^6}$$

Step 3 Let $m=1$ and $m=2$.

Then

$\lim \frac{x^2y^2}{x^3+y^6}$ is $\frac{1}{2}$ and $\frac{2^2}{1+2^6}$, respectively.

Therefore, limit does not exist.

Thus we have seen that:

To show that a limit does NOT exist, it suffices to exhibit two paths along which the limits differ.

But how do we show that a limit DOES exist?

The first thing one needs is a guess on the limit, say you suspect $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

One useful tool to prove this is:

THE SQUEEZE THEOREM.

Suppose that for (x,y) near (a,b) it holds

$$|f(x,y) - L| \leq g(x,y)$$

"squeezer"

where $g(x,y)$ satisfies

$$\lim_{(x,y)\to(a,b)} g(x,y) = 0$$

Then indeed

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

$$0 \leq \underbrace{|f(x,y) - L|}_{\to 0} \leq g(x,y) \xrightarrow{(x,y)\to(a,b)} 0$$

How to choose $g(x,y)$?

1. sin and cos are bounded by 1

2. $|x| \leq \sqrt{x^2+y^2}$

   $|y| \leq \sqrt{x^2+y^2}$

3. $$\left.\begin{aligned} e^x &\geq 1+x \\ e^x &\geq 1+x+\frac{x^2}{2} \\ e^x &\geq 1+x+\frac{x^2}{2}+\frac{x^3}{6} \\ &\vdots \end{aligned}\right\} \text{Taylor expansion}$$

Example: We will use the squeeze theorem to prove that

$$\lim_{(x,y)\to(0,1)} \frac{(y-1)^3}{x^2+(y-1)^2} = 0.$$

We need to find $g(x,y)$ with

(i) $$\left|\frac{(y-1)^3}{x^2+(y-1)^2} - 0\right| \leq g(x,y)$$

(ii) $$\lim_{(x,y)\to(0,1)} g(x,y) = 0.$$

To get (i), note that

$$\left|\frac{(y-1)^3}{x^2+(y-1)^2}\right| \leq \left|\frac{(y-1)^3}{(y-1)^2}\right|$$

(dividing by a bigger number makes things smaller!)

$$= |y-1|$$

Which leads us to choose $g(x,y) = |y-1|$.

Remains to verify (ii):

$$\lim_{(x,y)\to(0,1)} g(x,y) = 0 \quad \text{indeed.}$$

By the squeeze theorem, we conclude that

$$\lim_{(x,y)\to(0,1)} \frac{(y-1)^3}{x^2+(y-1)^2} = 0$$, as announced.

Another way

$$|y-1| \leq \sqrt{x^2+|y-1|^2}$$

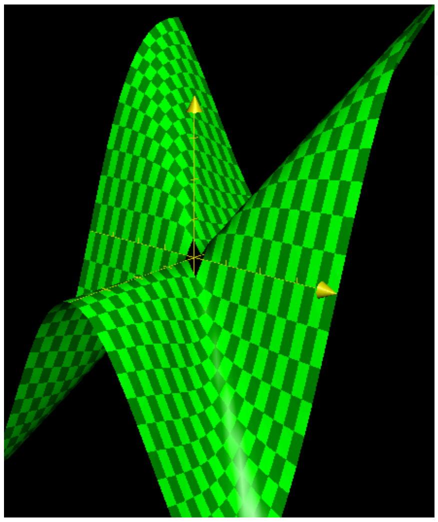
$$|x| \leq \sqrt{x^2+|y-1|^2}$$

$$\left|\frac{(y-1)^3}{x^2+(y-1)^2}\right| = \frac{|y-1|^3}{x^2+|y-1|^2} \leq \frac{(x^2+|y-1|^2)^{\frac{3}{2}}}{x^2+|y-1|^2} = (x^2+|y-1|^2)^{\frac{1}{2}} \to 0$$

as $(x,y)\to(0,1)$ $\square$

Question : Investigate

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}.$$



Answer :

Limit exists.

pf) Let $g(x,y) = 3(x^2+y^2)^{\frac{1}{2}}$

Then, since $|x| \le \sqrt{x^2+y^2}$, $|y| \le \sqrt{x^2+y^2}$

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| = \frac{3\,|x|^2\,|y|}{x^2+y^2} \le \frac{3(x^2+y^2)^{\frac{3}{2}}}{(x^2+y^2)}$$

$$= 3(x^2+y^2)^{\frac{1}{2}} \to 0$$

!! $g(x,y)$

$g(x,y)$ is defined by $3(x^2+y^2)^{\frac{1}{2}}$

By the squeeze theorem,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

Another important theorem about limits is:

Suppose that $\begin{cases} \lim_{(x,y)\to(a,b)} f(x,y) = L \\ \lim_{(x,y)\to(a,b)} g(x,y) = M \end{cases}$

Then:

(i) $\lim_{(x,y)\to(a,b)} \left(f(x,y) \pm g(x,y) \right) = L \pm M$

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)\, g(x,y) = LM$

(iii) $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = L/M$ provided $M \neq 0$ (!!)

(iv) If $F(s)$ is a function of one variable satisfying

$$\lim_{s\to L} F(s) \quad \text{exists}$$

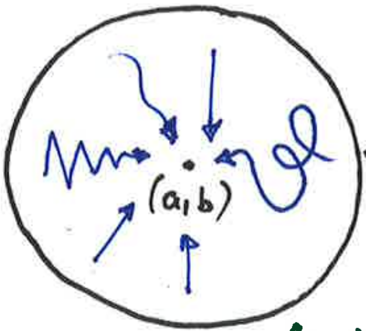
then:

$$\lim_{(x,y)\to(a,b)} F(f(x,y)) = F(L)$$

This is in particular the case for a continuous function $F(s)$.

Finally, the formal definition of a limit.

To cover all possible paths into the point (a,b), we surround it by a ball (a disk) of some radius $\delta > 0$:
"delta"



$$B_\delta((a,b)) = \{(x,y) \in \mathbb{R}^2 \,;\, (x-a)^2 + (y-b)^2 < \delta^2\}$$

"epsilon"

If, for every $\varepsilon > 0$ as small as you please, you can find a δ (depending on ε) such that:

$$(x,y) \in B_\delta((a,b)) \quad \text{implies} \quad |f(x,y) - L| < \varepsilon$$

then we say:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Example: Use the definition to show that $\displaystyle\lim_{(x,y)\to(0,0)} xy = 0$.

First pick a $\delta > 0$ and any point $(x,y) \in B_\delta((0,0))$. That means: $\quad x^2 + y^2 < \delta^2$.

But since $\quad xy \leq \frac{1}{2}(x^2+y^2)$ always (why?)

We get: $\quad |f(x,y) - 0| \leq \frac{1}{2}(x^2+y^2) < \delta^2/2 < \varepsilon$ we want this

Therefore, for any $\varepsilon > 0$ given, it suffices to choose $\delta^2/2 < \varepsilon$

i.e. $\quad \delta < \sqrt{2\varepsilon}$ to conclude that $\displaystyle\lim_{(x,y)\to(0,0)} xy = 0$ ■

CONTINUITY

We say a function $f(x,y)$ is <u>continuous</u> at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

<u>Example</u>: The function $f(x,y) := \dfrac{x^2-y^2}{x^2+y^2}$ is a rational function and is therefore continuous everywhere except perhaps when the denominator vanishes, that is at $(x,y)=(0,0)$.
We need to investigate: $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

Put $y = mx$. Then:

$$f(x, mx) = \frac{x^2 - m^2 x^2}{x^2 + m^2 x^2} = \frac{1-m^2}{1+m^2}$$

, which clearly depends on m, so the limit of $f(x,y)$ at $(0,0)$ does not exist. From this we conclude that $f(x,y)$ is continuous everywhere except at $(0,0)$. We symbolically write this set as:

$$\mathbb{R}^2 \setminus \{(0,0)\}$$

or

$$\{(x,y) \in \mathbb{R}^2 \,;\, (x,y) \neq (0,0)\}.$$

Question : Find the domain of continuity of

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$$

Answer :

From previous question, we know

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

Therefore f is continuous at $(0,0)$

when $(a,b) \neq (0,0)$, $f(x,y)$ is continuous at (a,b) because it is a rational function of continuous functions

Therefore, the domain of continuity is $\mathbb{R}^2$ (everywhere)

Examples

① $$\lim_{(x,y)\to(0,0)} \frac{1}{1-\underbrace{\log(x^2+y^2)}_{\downarrow \text{ as } (x,y)\to(0,0)\ -\infty}} = 0$$

② $$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^2+y^2}$$

$$e^r \geq 1+r+\frac{1}{2}r^2 \geq \frac{1}{2}r^2 \quad \text{if } r\geq 0$$

Let $r = \frac{1}{x^2+y^2}$

$$e^{\frac{1}{x^2+y^2}} \geq \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)^2$$

$$e^{-\frac{1}{x^2+y^2}} \leq 2(x^2+y^2)^2$$

$$\left|\frac{e^{-\frac{1}{x^2+y^2}}}{x^2+y^2} - 0\right| \leq \frac{2(x^2+y^2)^2}{x^2+y^2} = \overbrace{2(x^2+y^2)}^{\text{squeezer.}} \to 0$$

as $(x,y)\to(0,0)$.

Ans 0.

③ Check the continuity

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Ans : Not continuous.

Let $y = m\sqrt{x}$. Then

$$f(x, m\sqrt{x}) = \frac{x \cdot m^2 \cdot x}{x^2 + m^4 \cdot x^2} = \frac{m^2}{1+m^4}$$

Therefore

$$\lim_{\substack{(x,y)\to(0,0) \\ y=m\sqrt{x}}} f(x,y) = \frac{m^2}{1+m^4}$$ and this

depends on m. $\Rightarrow$ f is not continuou at $(0,0)$.

④ Check the continuity of

$$f(x,y) = \begin{cases} \dfrac{x^2y - xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

Sol Let $g(x,y) = 2\sqrt{x^2+y^2}$ Need to write at the last step.

Then, since $|x| \leq \sqrt{x^2+y^2}$, $|y| \leq \sqrt{x^2+y^2}$,

we have

$$|f(x,y) - 0| \leq \left| \frac{x^2y - xy^2}{x^2+y^2} \right|$$

$$\leq \frac{|x^2y| + |xy^2|}{x^2+y^2}$$

$$\leq \frac{|x||y|(|x| + |y|)}{x^2+y^2}$$

$$\leq \frac{\overset{|x|}{\sqrt{x^2+y^2}}\ \overset{|y|}{\sqrt{x^2+y^2}}\left(\overset{|x|}{\sqrt{x^2+y^2}} + \overset{|y|}{\sqrt{x^2+y^2}}\right)}{x^2+y^2}$$

$$\leq \frac{(x^2+y^2)\cdot 2\sqrt{x^2+y^2}}{x^2+y^2} = \overset{g(x,y)}{\overset{\shortparallel}{2\sqrt{x^2+y^2}}} \longrightarrow 0$$

as $(x,y) \to (0,0)$

Therefore, by the Squeeze Theorem, we conclude $f(x,y)$ is continuous at $(0,0)$.