Simulation and modeling of natural processes

Week 3: Dynamical systems and numerical integration

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General introduction to dynamical systems

What is a dynamical system?

- A system that varies through time.
- The system is described by its state $s(t)$.
- The state $\boldsymbol{s}$ can be multivariate (a vector)

$$\boldsymbol{s}(t) = (s_1(t), ..., s_n(t))^{\mathrm{T}}.$$

- Examples: pendulum, population evolution, ...

Different classes of descriptions

- **Discrete** dynamical systems are described by *recurrence relations*

$$s(t+\Delta t) = f(s(t)),$$

  where f is a function and Δt the time increment.

- **Continuous** dynamical systems are described by *differential equations*

$$\dot{s} \equiv \frac{\mathrm{d}s}{\mathrm{d}t} = f(s(t)).$$

Generality of the description: continuous case

What happens if we have second derivatives?

$$\dot{s} + \ddot{s} = f(s)$$

Let us then define

$$\begin{aligned} \dot{s} &= y, \\ \dot{y} &= f(s) - y. \end{aligned}$$

By defining $\boldsymbol{u}(t) \equiv (s(t), y(t))$ we have a system of the form

$$\dot{\boldsymbol{u}} = g(\boldsymbol{u}).$$

Generality of the description: discrete case

"Second derivatives" equivalent to presence of a $t - \Delta t$ term

$$s(t + \Delta t) = f(s(t)) + s(t - \Delta t).$$

Similarly to continuous case

$$\begin{aligned} y(t + \Delta t) &= s(t), \\ s(t + \Delta t) &= f(s(t)) + y(t), \end{aligned}$$

and by defining $\boldsymbol{u}(t) \equiv (s(t), y(t))$ we have a system of the form

$$\boldsymbol{u}(t + \Delta t) = g(\boldsymbol{u}(t)).$$

Generality of the description: continuous case

What happens if f explicitly depends on t?

$$\dot{s} = f(s, t)$$

Let us then define

$$\begin{aligned} \dot{s} &= f(s, y) \\ \dot{y} &= 1, \end{aligned}$$

with $y(0) = 0$. By defining $\boldsymbol{u}(t) \equiv (s(t), y(t))$ we have a system of the form

$$\dot{\boldsymbol{u}} = g(\boldsymbol{u}).$$

End of module

General introduction to dynamical systems

Coming next

The random walk

The random walk

The random walk

Time evolution of particles, drunk people, ...

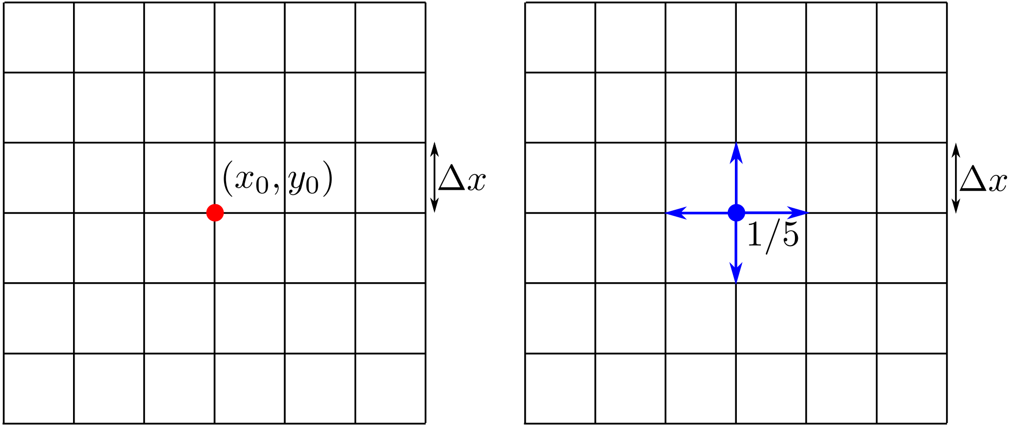
$$x(t+\Delta t) = x(t) + \Delta t \cdot v_x(t),$$
$$y(t+\Delta t) = y(t) + \Delta t \cdot v_y(t),$$

where (v_x, v_y) is chosen randomly in the ensemble

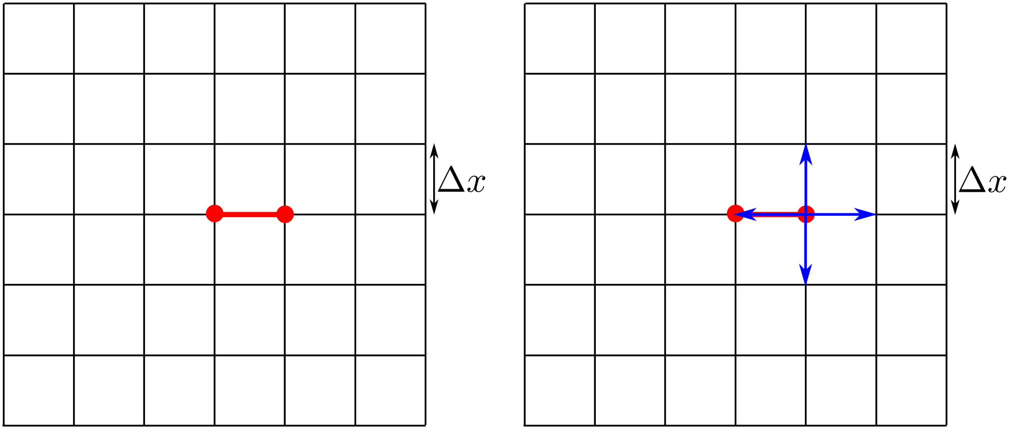
$$\{(\Delta x/\Delta t, 0), (-\Delta x/\Delta t, 0), (0,0), (0, \Delta y/\Delta t), (0, -\Delta y/\Delta t)\},$$

where Δx and Δy are the length of the displacements in direction x and y.

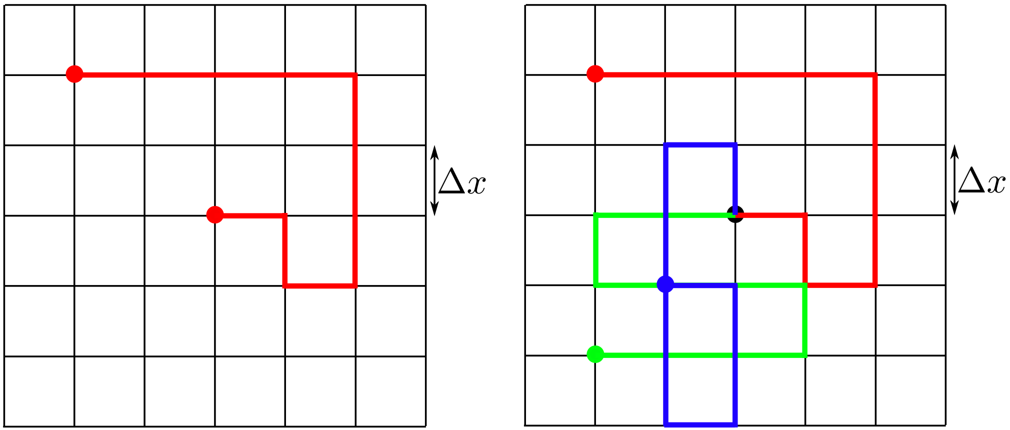
The random walk: illustration



The random walk: illustration



The random walk: illustration



The random walk: mathematical analysis

What is the “average” behavior of many random walkers?

- Initial position of the walker is $(x(0), y(0)) = (x_0, y_0)$.
- After a time $t = N \cdot \Delta t$ the position of a random walker is

$$
\begin{aligned}
x(t) &= x(t - \Delta t) + \Delta t \cdot v_x(t - \Delta t) \\
&= x(t - 2\Delta t) + \Delta t \cdot \left(v_x(t - 2\Delta t) + v_x(t - \Delta t)\right), \\
&= x(0) + \Delta t \sum_{n=1}^{N} v_x(t - n \cdot \Delta t).
\end{aligned}
$$

The random walk: mathematical analysis

What is the average position?

$$\langle x(t)\rangle = x_0 + \Delta t \sum_{n=1}^{N} \langle v_x(t - n \cdot \Delta t)\rangle,$$

where for any time, τ, one has

$$\langle v_x(\tau)\rangle = \frac{1}{5}\left(-\frac{\Delta x}{\Delta t}\right) + \frac{3}{5} \cdot 0 + \frac{1}{5}\frac{\Delta x}{\Delta t} = 0,$$

and therefore

$$\langle x(t)\rangle = x_0.$$

The random walk: mathematical analysis

What is the average displacement?

$$d(t) = \sqrt{\langle (x(t) - x_0)^2 \rangle + \langle (y(t) - y_0)^2 \rangle}$$

Therefore with $d_x^2 = \langle (x(t) - x_0)^2 \rangle$

$$\begin{aligned} d_x^2 &= \Delta t^2 \left\langle \sum_{m=1}^{N} v_x(t - m\Delta t) \sum_{n=1}^{N} v_x(t - n\Delta t) \right\rangle , \\ &= \Delta t^2 \sum_{m,n=1}^{N} \langle v_x(t - m\Delta t) v_x(t - n\Delta t) \rangle . \end{aligned}$$

The random walk: mathematical analysis

Statistical independence

$$\Rightarrow \langle v_x(t - m\Delta t) v_x(t - n\Delta t)\rangle = 0, \qquad m \neq n.$$

And for $m = n$

$$\langle v_x^2 \rangle = \frac{1}{5}\left(\frac{\Delta x^2}{\Delta t^2}\right) + \frac{3}{5} \times 0 + \frac{1}{5}\left(\frac{\Delta x^2}{\Delta t^2}\right) = \frac{2\Delta x^2}{5\Delta t^2}.$$

The random walk: mathematical analysis

The sum can be separated in two terms

$$d_x^2 = \Delta t^2 \left(\sum_{\substack{m,n=1 \\ m \neq n}}^{N} \langle v_x(t - m\Delta t) v_x(t - n\Delta t) \rangle + \sum_{m=1}^{N} \langle v_x(t - m\Delta t)^2 \rangle \right),$$

$$= \sum_{m=1}^{N} \frac{2\Delta x^2}{5} = N\frac{2\Delta x^2}{5} = t\frac{2\Delta x^2}{5\Delta t}.$$

The random walk: mathematical analysis

Doing similarly for d_y^2 on gets

$$d = \sqrt{d_x^2 + d_y^2} = \sqrt{t\frac{4\Delta x^2}{5\Delta t}}.$$

Finally by defining the diffusion coefficient $D \equiv \frac{\Delta x^2}{5\Delta t}$

$$d = \sqrt{4Dt}.$$

Equivalent to

$$\partial_t \rho(x, y, t) = D(\partial_x^2 + \partial_y^2)\rho(x, y, t).$$

The random walk: 2D diffusion

Parameters of the simulation

$$t = 100,\ \Delta x = \Delta t = 1,\ (x_0, y_0) = (0, 0).$$

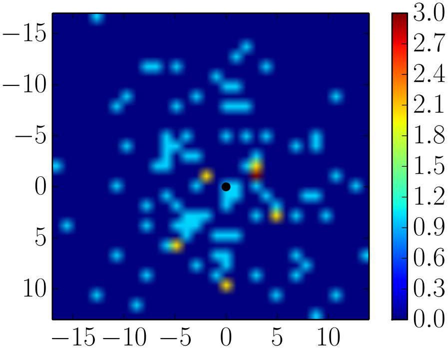
Transport parameters

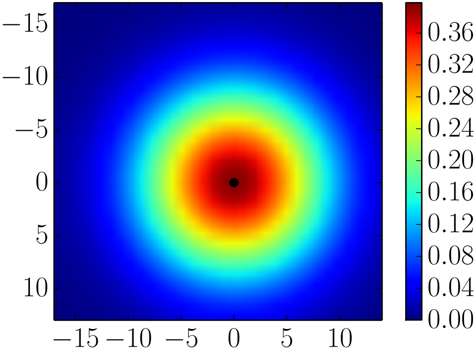
$$D = \frac{1}{5}, \quad d = \sqrt{80} \cong 8.9443.$$

Let us measure the average position and the average distance at time t for a different number of walkers.

The random walk: $N = 100$

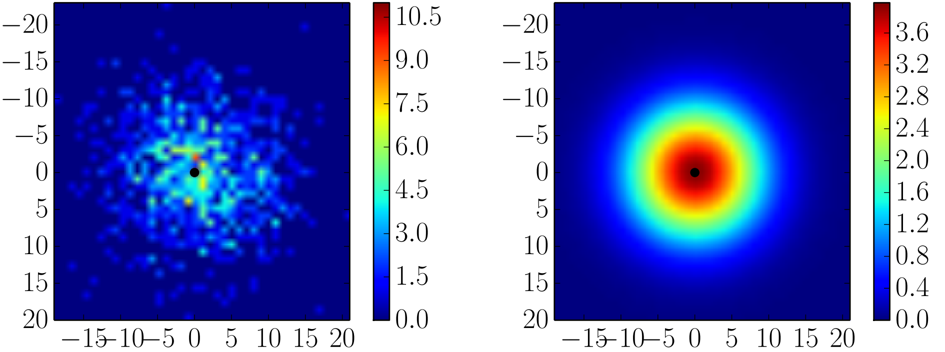
$$\langle (x, y) \rangle = (-0.63, -0.01), \quad d = 9.262$$





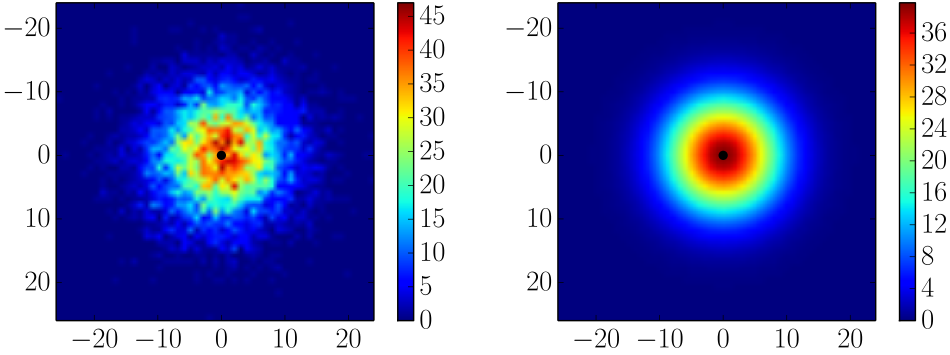
The random walk: $N = 1000$

$$\langle (x, y) \rangle = (0.382, -0.094), \quad d = 9.037$$



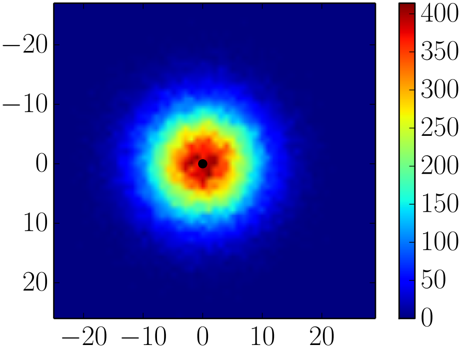
The random walk: $N = 10000$

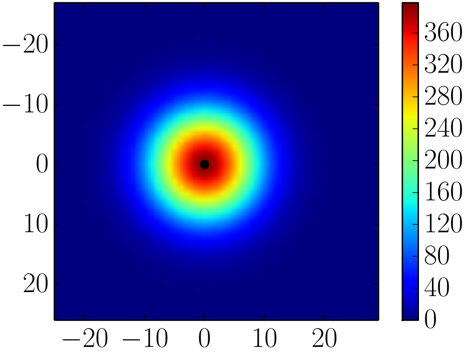
$$\langle (x, y) \rangle = (0.0034, 0.0335), \quad d = 8.972$$



The random walk: $N = 100000$

$$\langle (x, y) \rangle = (-0.00214, -0.0013), \quad d = 8.938$$





$\boxed{\textit{End of module}}$

The random walk

$\boxed{\textit{Coming next}}$

Growth of a population

Growth of a population

Growth of a population

- Let us note the number of individuals of a population $P(t)$, and $P(0) = P_0$.
- Only two processes are affecting the number of individuals: the number of births $B(t)$ and the number of deaths $D(t)$.
- The variation of $P(t)$ can be expressed as

$$\dot{P}(t) = B(t) - D(t).$$

- A possible model for births and deaths is given by

$$B(t) = r_b P(t), \quad D(t) = r_d P(t).$$

Growth of a population

Total variation of the population

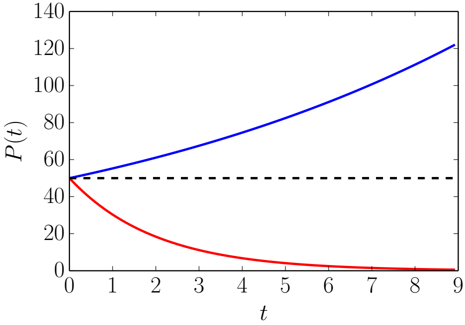
$$\dot{P}(t) = rP(t), \quad r = r_b - r_d.$$

The solution of this equation

$$P(t) = P_0 e^{rt}.$$

Growth of a population

- $r > 0$ exponential growth.
- $r < 0$ exponential decay.
- $r = 0$ constant.



Growth of a population

- Of course an infinite growth of population is not realistic.
- One can add a target "capacity" C to the population by $r \to 0$ as $P \to C$

$$r = v\left(1 - \frac{P}{C}\right).$$

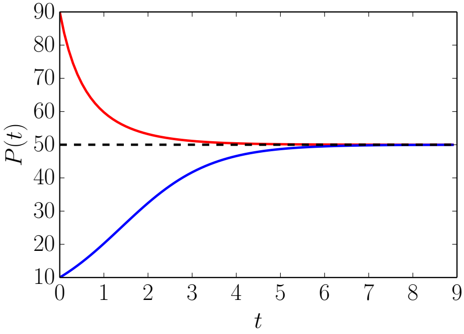
- The model becomes

$$\dot{P} = v\left(1 - \frac{P}{C}\right)P$$

Growth of a population

The solution is a logistic function

$$P(t) = \frac{C}{1 + \frac{C - P_0}{P_0} e^{-vt}}$$



$$C = 50, \quad v = 1, \quad P_0 = 10 \text{ (blue)},\ 90 \text{ (red)}.$$

Growth of a population

- $P_0 < C$ growth of population and $P \to C$.
- $P_0 > C$ decay of population and $P \to C$.
- Except if $P_0 = 0$ then $P(t) = 0$.

This system has two fixed points: C (attractive) and 0 (repulsive).

$\boxed{\textit{End of module}}$

Growth of a population

$\boxed{\textit{Coming next}}$

Balance equations

Balance equations I

Balance equations

As shown with the growth of a population example the basic idea is to write a **balance equation**

$$\underbrace{\text{variation of a quantity}}_{\text{ex: } \dot{P}} = \underbrace{\text{creation rate}}_{\text{ex: } B(t)} - \underbrace{\text{destruction rate}}_{\text{ex: } D(t)}.$$

Or to reuse the generic formalism of continuous dynamical systems

$$\dot{s} = f(s) = \text{creation rate} - \text{destruction rate}.$$

Balance equations

For discrete dynamical systems this equation becomes

$$s(t+\Delta t) - s(t) = \Delta t\,(\text{creation rate} - \text{destruction rate})\,.$$

Both formulations are equivalent in the limit $\Delta t \to 0$ since

$$\lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} = \dot{s}.$$

Discrete balance equations can be directly implemented on a computer.

Lotka–Volterra model (“Prey-Predator”)

Let a be the antelope population and c the cheetah population.

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \underbrace{k_a a(t)}_{(1)} - \underbrace{k_{c,a} c(t) a(t)}_{(2)},$$

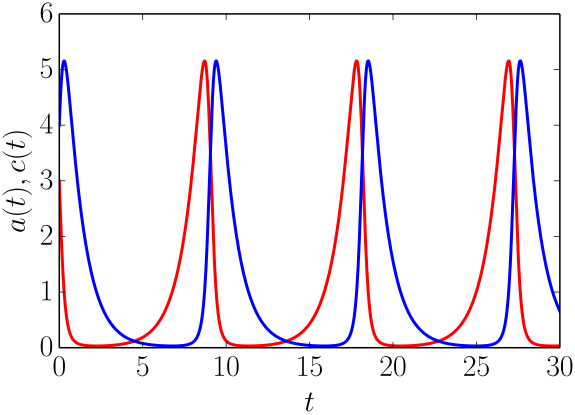
$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\underbrace{k_c c(t)}_{(3)} + \underbrace{k_{a,c} a(t) c(t)}_{(4)}$$

- The antelopes reproduce exponentially fast (infinite amount of food).
- The antelopes are being eaten by the cheetah proportionally to their chance to meet.
- The cheetahs die exponentially fast of starvation.
- The cheetahs are reproducing proportionally to their chance to eat antelopes.

Lotka–Volterra model ("Prey-Predator")

Time evolution

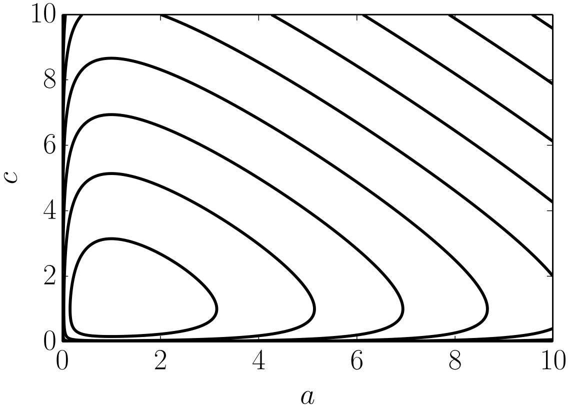
Periodicity



Lotka–Volterra model (“Prey-Predator”)

Parametric plot

Initial condition sensitivity



$\boxed{\textit{End of module}}$

Balance equations I

$\boxed{\textit{Coming next}}$

Balance equations II

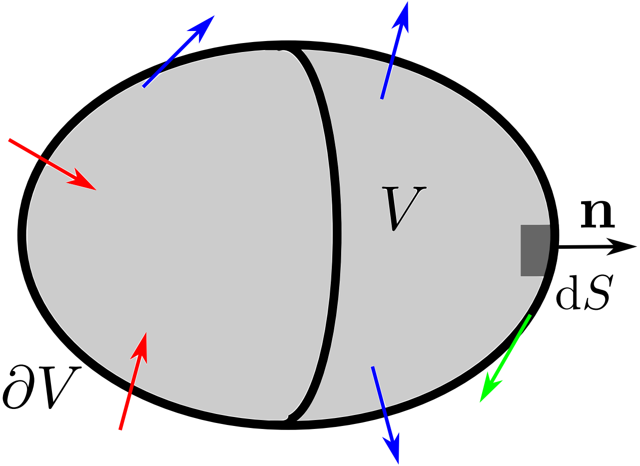
Balance equations II

Space–time dependent equations

- For certain situations time dependence is not sufficient (e. g. weather forecast).
- Necessity to add spatial dependence.
- The balance is now done in a volume $V \in \mathbb{R}^d$ and must not only take into account the destruction/creation processes but also the fluxes through the surface of V, noted ∂V.

The flux through of a quantity is the amount of s crossing ∂V per unit time.

Control volume



The continuity equation

- Let us define $\rho(t, \boldsymbol{x})$ the volume density of the property one wants to study (mass, electric charge, ...)

$$s(t) = \int_V \rho(t, \boldsymbol{x}) \mathrm{d}V.$$

- Balance equation for $s(t)$ in V reads

$$\dot{s} = -\underbrace{\text{flux}}_{=\boldsymbol{j}} \text{ through surface} + \underbrace{\text{volumic destruction/creation rate}}_{=\Sigma}.$$

The continuity equation

By defining $\Sigma = \int_V \sigma \mathrm{d}V$

$$\boxed{\dot{s} + \oint_{\partial V} \boldsymbol{j} \cdot \boldsymbol{n} \mathrm{d}S = \Sigma}$$

By using the Gauss–Ostrogradski theorem

$$\int_V \frac{\partial \rho}{\partial t} \mathrm{d}V + \int_V \boldsymbol{\nabla} \cdot \boldsymbol{j} \mathrm{d}V = \int_V \sigma \mathrm{d}V,$$

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = \sigma}$$

Example: The diffusion equation, $\rho \to C$

Fick's law $\boldsymbol{j} = -D\boldsymbol{\nabla} C$, and $\sigma = 0$

$$\frac{\partial C}{\partial t} = D\boldsymbol{\nabla}^2 C,$$

where D is the diffusion coefficient.

Example: The Navier–Stokes equations, $\rho \to \rho\boldsymbol{u}$

Momentum flux $\boldsymbol{j} = \rho\boldsymbol{u}\boldsymbol{u} + p\boldsymbol{I} - \boldsymbol{\tau}$, and $\sigma = 0$

$$\frac{\partial \rho\boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{\nabla}p + \boldsymbol{\nabla}\cdot\boldsymbol{\tau}.$$

For a Newtonian incompressible fluid

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\boldsymbol{\nabla}^2\boldsymbol{u},$$

with ν the kinematic viscosity.

$\boxed{\textit{End of module}}$

Balance equations II

$\boxed{\textit{Coming next}}$

Integration of ordinary differential equations

Integration of ordinary differential equations

Integration of ordinary differential equations

Let us define a one-dimensional differential equation

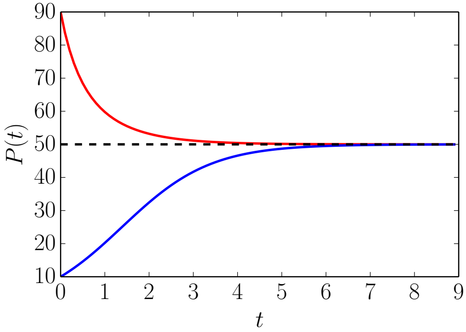
$$\dot{s} = f(s,t), \quad t_0 \leq t \leq t_f, \text{ and } s(t = t_0) = s_0.$$

where f is a function of s and t, and s_0 is a prescribed initial condition. Depending on the form of f, $s(t)$ may be difficult to obtain analytically.

Example: Growth of a population

$$\dot{s} = f(s,t), \quad f(s,t) = \nu\left(1 - \frac{s}{C}\right)s.$$

$$\boxed{s(t) = \frac{C}{1 + \left(\frac{C - s_0}{s_0}\right)e^{-\nu t}}}$$



Explicit Euler scheme, $\dot{s} = f(s,t)$

Find approximate solution at $t_1 = t_0 + \Delta t$ (Taylor exp.)

$$s(t_0 + \Delta t) = s(t_0) + \Delta t \cdot \dot{s}(t_0) + \underbrace{\mathcal{O}(\Delta t^2)}_{\text{error}}$$

With the definition of the differential equation we find

$$s(t_1) = s(t_0) + \Delta t \cdot f(s, t_0).$$

Explicit Euler scheme (continued)

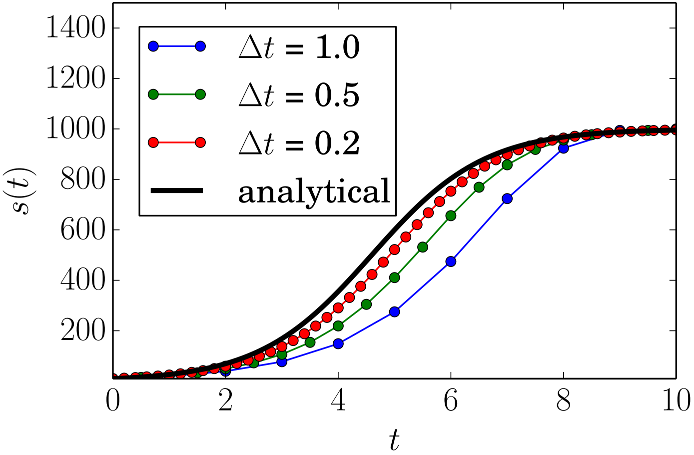
Iteratively one finds the explicit or forward Euler scheme

$$s(t_{n+1}) = s(t_n) + \Delta t \cdot f(s, t_n), \quad t_{n+1} = t_n + \Delta t.$$

Remarks

- It looks a lot like a discrete dynamical system!
- $s(t_{n+1})$ depends **explicitly** on $s(t_n)$.
- Straightforward to implement on a computer.

Example: growth of a population



$\boxed{\textit{End of module}}$

Integration of ordinary differential equations

$\boxed{\textit{Coming next}}$

Error of the approximation

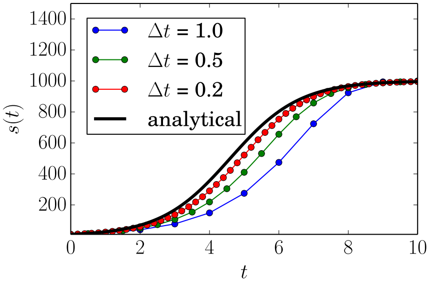
Error of the approximation

Error of the approximation

Evaluation of the error

$$E(\Delta t) = \sqrt{\frac{1}{N+1}\sum_{i=0}^{N}(s(t_i) - s_{\Delta t}(t_i))^2},$$

with $N = (t_f - t_0)/\Delta t$



Remark

- If no analytical solution exists one can replace $s(t_i) \to s_{\min(\Delta t)}(t_i)$.

Evaluation of the error

Evaluation of the error, $s_i = s(t_i)$

$$
\begin{aligned}
s_1 &= s_0 + \Delta t \cdot f(s_0, t_0) + \mathcal{O}(\Delta t^2), \\
s_2 &= s_1 + \Delta t \cdot f(s_1, t_1) + \mathcal{O}(\Delta t^2), \\
\Rightarrow s_2 &\cong s_0 + \Delta t \cdot \{f(s_0, t_0) + f\left[s_0 + \Delta t \cdot f(s_0, t_0), t_0 + \Delta t\right]\} + 2\mathcal{O}(\Delta t^2).
\end{aligned}
$$

The error accumulates: after $2\Delta t$

$$s(t_0 + 2\Delta t) - s_{\Delta t}(t_0 + 2\Delta t) \sim 2 \times \mathcal{O}(\Delta t^2).$$

Evaluation of the error

Defining $E_i = s(t_i) - s_{\Delta t}(t_i)$, we have

$$E_0 = s(t_0) - s_{\Delta t}(t_0) = 0, \cdots,$$
$$E_N = s(t_N) - s_{\Delta t}(t_N) = N\mathcal{O}(\Delta t^2).$$

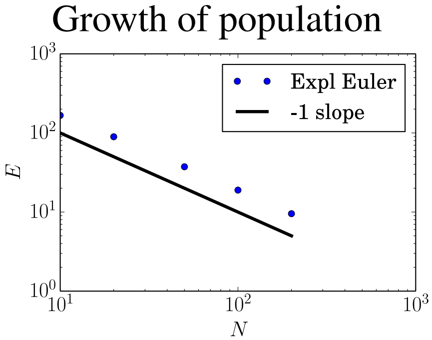
Therefore

$$E = \sqrt{\frac{1}{N+1}\sum_{i=0}^{N} E_i^2} \sim \sqrt{\frac{\mathcal{O}(\Delta t^4)}{N+1}\sum_{i=0}^{N} i^2} \sim \sqrt{\Delta t^4 N^2} \sim \mathcal{O}(\Delta t),$$

where we used that $N = (t_f - t_i)/\Delta t \sim \mathcal{O}(\Delta t^{-1})$.

Order of the scheme

- The forward Euler scheme is of **order** 1, since $E \sim \mathcal{O}(\Delta t^1)$.
- Δt is divided by a factor $M \Rightarrow$, E is decreased by a factor M.
- A scheme is of order k if $E \sim \mathcal{O}(\Delta t^k)$.



$\boxed{\textit{End of module}}$

Error of the approximation

$\boxed{\textit{Coming next}}$

Implicit Euler scheme

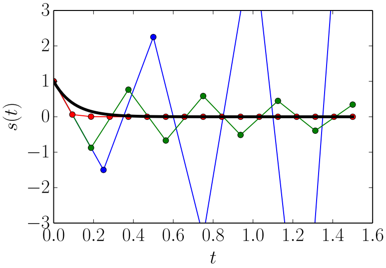
Implicit Euler scheme

Stability of explicit schemes

Example: $\dot{s} = -10s(t), \quad t_0 = 0, \quad t_f = 1.5, \quad s_0 = 1.$

Solution: $s = s_0 \exp(-10t)$. Euler: $s(t + \Delta t) = s(t) - 10\Delta t \cdot s(t)$.

Explicit Euler scheme



- Difficult to solve **stiff** problems with large Δt.
- Since only information about t_{n-1} is know when going to t_n "too big" jumps are possible.

The implicit Euler scheme

Instead of s_{j+1} we approximate s_{j-1}

$$s_{j-1} = s_j - \Delta t \cdot f(s, t_j) + \mathcal{O}(\Delta t^2)$$

Our differential equation is now approximated by the *backward Euler scheme*

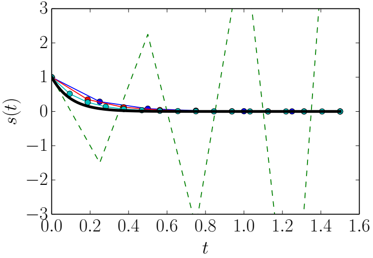
$$s_j = s_{j-1} + \Delta t \cdot f(s_j, t_j) \Leftrightarrow \boxed{s_{j+1} = s_j + \Delta t \cdot f(s_{j+1}, t_{j+1})}.$$

This class of approximation is called *implicit*.

Implicit Euler scheme (continued)

Example: $\dot{s} = -10s(t), \quad t_0 = 0, \quad t_f = 1.5, \quad s_0 = 1.$
Solution: $s = s_0 \exp\left(-10t\right).$

Implicit Euler scheme



- Implicit scheme

$$s_{j+1} = s_j - 10\Delta t s_{j+1},$$
$$s_{j+1} = s_j/(1 + 10\Delta t).$$

- Explicit scheme

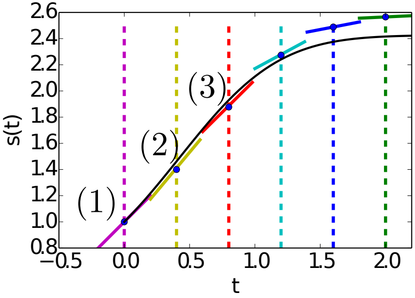
$$s_{j+1} = s_j(1-10\Delta t).$$

- Instability solved.

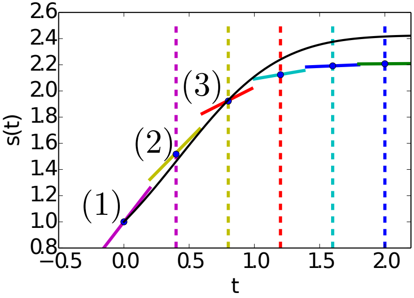
Explicit vs Implicit (continued)

Graphical interpetation

Explicit



Implicit



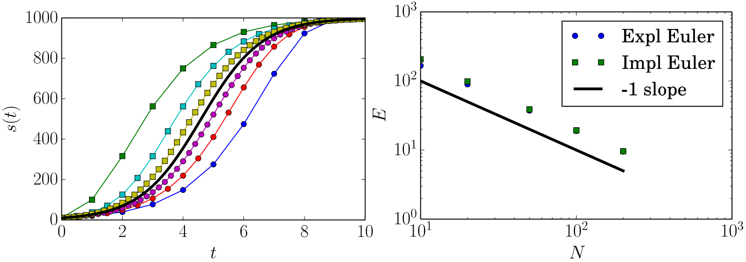
Explicit vs Implicit

Explicit

- Explicit equation to solve
- Conditionally stable (on stiff problems)
- Simple implementation

Implicit

- Implicit equation to solve
- Unconditionnaly stable
- Difficult implementation



Explicit/Implicit solution (left) and error (right) on the population growth problem.

$\boxed{\textit{End of module}}$

Implicit Euler scheme

$\boxed{\textit{Coming next}}$

Numerical integration of partial differential equations

Numerical integration of partial differential equations

The diffusion equation

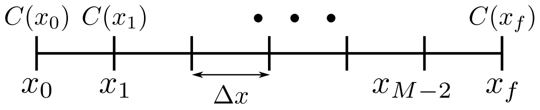
In 1D the diffusion equation reads

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2}, \quad t \in [t_0, t_f], \quad x \in [x_0, x_f],$$

with the following spatial boundary conditions $C(x = x_0, t) = C_0$ et $C(x = x_f, t) = C_1$ and initial condition $C(x, t_0) = C_x$.

Not only time must be discretized but also space.

The diffusion equation: space discretization



- The physical space is divided in $M+1$ equidistant points with $M = (x_f - x_0)/\Delta x$.
- The set of points is defined such as $x_{i+1} = x_i + \Delta x$ and $x_f = x_M$.
- The concentration is the defined on each point with $C_i \equiv C(x_i)$.
- The time dependence is understood here.

The diffusion equation: space discretization

The Taylor expansion of C_{i+1} and C_{i-1} to second order reads

$$C_{i+1} = C_i + \Delta x \frac{\partial C_i}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 C_i}{\partial x^2},$$
$$C_{i-1} = C_i - \Delta x \frac{\partial C_i}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 C_i}{\partial x^2}.$$

By summing these equations one gets ($x_i \in [x_1, ..., x_{M-1}]$)

$$\frac{\partial^2 C_i}{\partial x^2} = \frac{C_{i+1} - 2C_i + C_{i-1}}{\Delta x^2}$$

The diffusion equation: space discretization

The discretization of the Laplacian leads to

$$\frac{\partial C_i(t)}{\partial t} = D\frac{C_{i+1}(t) - 2C_i(t) + C_{i-1}(t)}{\Delta x^2}.$$

By defining $\boldsymbol{C}(t) = (C(x_0,t), C(x_1,t), ..., C(x_M,t))^{\mathrm{T}}$ and $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 1 & -2 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Remark

- The first and last lines of $\boldsymbol{A}$ correspond to boundary conditions.

The diffusion equation: time discretization

We can now rewrite the diffusion equation under a matrix form

$$\frac{\partial \boldsymbol{C}(t)}{\partial t} = \frac{D}{\Delta x^2}\boldsymbol{A}\boldsymbol{C}(t).$$

Explicit time discretization

$$\frac{\boldsymbol{C}(t_{i+1}) - \boldsymbol{C}(t_i)}{\Delta t} = \frac{D}{\Delta x^2}\boldsymbol{A}\boldsymbol{C}(t_i).$$

Implicit time discretization

$$\frac{\boldsymbol{C}(t_{i+1}) - \boldsymbol{C}(t_i)}{\Delta t} = \frac{D}{\Delta x^2}\boldsymbol{A}\boldsymbol{C}(t_{i+1}).$$

The diffusion equation: time discretization

Explicit time discretization

$$\boldsymbol{C}(t_{i+1}) = \left(\frac{\Delta t D}{\Delta x^2}\boldsymbol{A} + \boldsymbol{I}\right)\boldsymbol{C}(t_i).$$

Implicit time discretization

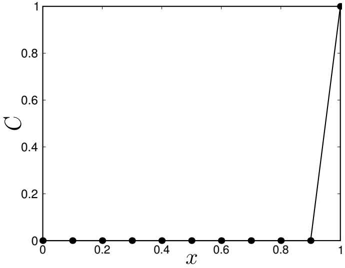
$$\boldsymbol{C}(t_{i+1}) = \left(-\frac{\Delta t D}{\Delta x^2}\boldsymbol{A} + \boldsymbol{I}\right)^{-1}\boldsymbol{C}(t_i).$$

Remarks

- Implicit scheme needs matrix inversion (it is an art!) but stable.
- Explicit scheme easy to implement but unstable (CFL).

Numerical example

Initial condition

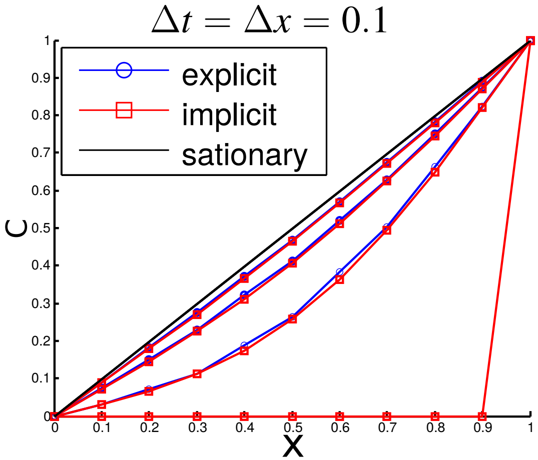


Boundary conditions

$$C(x_0, t_i) = 0, \quad C(x_f, t_i) = 1, \forall t_i.$$

For $t \to \infty$ admits a stationary solution.

Numerical example



$\boxed{\textit{End of module}}$

Numerical integration of partial differential equations

$\boxed{\textit{End of Week 3}}$

Thank you for your attention!