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On the Existence and the Coefficients of the Implicit Equation of Rational Surfaces

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The existence of the implicit equation of rational surfaces can be proved by three techniques: elimination theory, undetermined coefficients, and the theory of field extensions. The methods of elimination theory and undetermined coefficients also reveal that the implicit equation can be written with coefficients from the coefficient field of the parametric polynomials. All three techniques can be implemented as implicitization algorithms. For each method, the theoretical limitations of the proof and the practical advantages and disadvantages of the algorithm are discussed. Our results are important for two reasons. First, we caution that elimination theory cannot be generalized in a straightforward manner from rational plane curves to rational surfaces to show the existence of the implicit equation; thus other rigorous methods are necessary to bypass the vanishing of the resultant in the presence of base points. Second, as an immediate consequence of the coefficient relationship, we see that the implicit representation involves only rational (or real) coefficients if a parametric representation involves only rational (or real) coefficients. The existence of the implicit equation means every rational surface is a subset of an irreducible algebraic surface. The subset relation can be proper and this may cause problems in certain applications in computer aided geometric design. This anomaly is illustrated by an example.

1. INTRODUCTION

In computer aided geometric design (CAGD), surfaces are typically represented either parametrically or implicitly. A surface represented parametrically by rational functions (ratios of polynomials) is known as a *rational* surface. The parametrization of a rational surface can be written as

$$X = \frac{\xi(s,t)}{\omega(s,t)},\quad Y = \frac{\eta(s,t)}{\omega(s,t)},\quad Z = \frac{\zeta(s,t)}{\omega(s,t)}, \tag{1}$$

where $\omega(s, t)$, $\xi(s, t)$, $\eta(s, t)$, $\zeta(s, t)$ are polynomials in the indeterminates s, t. A surface represented implicitly by a polynomial $f(X, Y, Z) = 0$ is known as an *algebraic surface*. The purpose of this paper is to clarify the relationship between rational and algebraic surfaces.

An algebraic surface given by $f(X, Y, Z) = 0$ is said to be irreducible if f is irreducible; that is, if f has no nonconstant factors of lower degrees. The irreducibility of a polynomial depends on its coefficient field. For example, the polynomial $x_1^2 - 2x_2^2$ is irreducible over the field of rational numbers, but reducible over the field of real numbers. Let $\mathbf{C}$ be the field of complex numbers and let $K[x_1, \cdots, x_k]$ denote the ring of polynomials in the indeterminates $x_1, \cdots, x_k$ with coefficients in the field $K \subseteq \mathbf{C}$. Unless otherwise stated, a polynomial of $K[x_1, \cdots, x_n]$ is said to be irreducible if it is irreducible in $\mathbf{C}[x_1, \cdots, x_n]$.

We show that a parametrization of a rational surface always satisfies an *irreducible* polynomial equation identically. That is, there always exists an irreducible polynomial equation $f(X, Y, Z) = 0$ such that

$$f\left(\frac{\xi(s,t)}{\omega(s,t)}, \frac{\eta(s,t)}{\omega(s,t)}, \frac{\zeta(s,t)}{\omega(s,t)}\right) \equiv 0.$$

The equation $f = 0$ is called an *implicit equation* of the parametrization and the polynomial f will be called an *implicit polynomial*.

To show the existence of an implicit equation it suffices to find a polynomial $f(X, Y, Z) \in \mathbf{C}[X, Y, Z]$, not necessarily irreducible, such that $f(\xi/\omega, \eta/\omega, \zeta/\omega) \equiv 0$. For if f is reducible, we have $f \equiv f_1^{n_1} \cdots f_k^{n_k}$ where the f_i's are distinct and irreducible since $\mathbf{C}[X, Y, Z]$ is a unique factorization domain. But the $f_i(\xi/\omega, \eta/\omega, \zeta/\omega)$'s are elements of the field $\mathbf{C}(s, t)$ of rational functions in the indeterminates s, t and a field has no zero divisors, hence $f(\xi/\omega, \eta/\omega,$

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$\zeta/\omega) \equiv 0$ means there must be a factor f_j such that $f_j(\xi/\omega, \eta/\omega, \zeta/\omega) \equiv 0$. Consequently, to prove that a parametrization satisfies an irreducible polynomial identically, we need only prove that it satisfies a polynomial (irreducible or not) identically. This fact is used in our existence proofs.

We shall also show that if the coefficients of the *parametric polynomials* $\omega(s, t)$, $\xi(s, t)$, $\eta(s, t)$, $\zeta(s, t)$ are elements of a field $K \subseteq \mathbf{C}$, then the coefficients of the corresponding implicit polynomial can also be taken to be elements of K.

The implicit polynomial of a rational surface has to be unique up to a constant factor. For if a parametrization satisfies two distinct irreducible polynomial equations $f = 0$ and $g = 0$ identically, then the rational surface represented by the parametrization is part of each of the irreducible algebraic surfaces given by $f = 0$ and $g = 0$. That is, the rational surface is part of the common intersection of two distinct irreducible algebraic surfaces, which is a space curve. Hence, up to constant factors, there is only one implicit equation for a rational surface.

2. EXISTENCE OF THE IMPLICIT EQUATION OF RATIONAL SURFACES

The existence of the implicit equation of rational surfaces is an important fact [1]. Very often this is asserted by appealing to elimination theory and claiming that the implicit equation exists because it can be obtained by eliminating the parameters (for example, Salmon [15]). In CAGD, this technique has been resurrected for rational plane curves by Sederberg *et al.* [17]. Goldman *et al.* [9], and De Montaudouin and Tiller [7]. But unlike the case of plane curves, we illustrate that the straightforward application of elimination theory does not cover all possibilities for rational surfaces unless we are more careful (Chionh [2], Manocha and Canny [13]). Fortunately two alternative rigorous proofs are available: one using the method of undetermined coefficients and the other applying the theory of field extensions.

2.1. *Elimination Theory*

Using elimination theory (van der Waerden [19]), the existence of the implicit equation for rational surfaces is established by actually producing the implicit equation. Given a system of homogeneous polynomial equations having the same number of equations and variables, the resultant of the equations exists by elimination theory. We call the resultant a *bivariate resultant* or a *multivariate resultant* if the number of variables of the homogeneous system is two or more than two respectively. Bivariate resultants can be found by the Sylvester determinant (van der Waerden [20]). Multivariate resultants can be found by the Macaulay quotient (Macaulay [11, 12], of which bivariate resultants are a special case. For the special case of three equations of the same degree, the methods of Sylvester (Salmon [16]) and Dixon [8] can also be used to construct the resultant.

2.1.1. *Multivariate Resultants.*

The parametrization of a rational surface can be written with projective parameters,

$$X = \frac{\xi(r, s, t)}{\omega(r, s, t)}, \quad Y = \frac{\eta(r, s, t)}{\omega(r, s, t)}, \quad Z = \frac{\zeta(r, s, t)}{\omega(r, s, t)},$$

where $\omega(r, s, t)$, $\xi(r, s, t)$, $\eta(r, s, t)$, $\zeta(r, s, t)$ are now homogeneous polynomials of the same degree obtained by homogenizing the respective parametric polynomials with a homogenizing variable r. Three *auxiliary equations* can be derived from the parametrization:

$$\begin{aligned} \xi(r, s, t) - X\,\omega(r, s, t) &= 0 \\ \eta(r, s, t) - Y\,\omega(r, s, t) &= 0 \\ \zeta(r, s, t) - Z\,\omega(r, s, t) &= 0. \end{aligned} \tag{2}$$

Since this is a system of homogeneous polynomial equations (in the variables r, s, t) having the same number of equations and variables, the resultant exists by elimination theory. If the resultant of the auxiliary equations is $\operatorname{res}(X, Y, Z) \not\equiv 0$, then the rational surface is part of the algebraic surface given by $\operatorname{res}(X, Y, Z) = 0$. This follows because the vanishing of the resultant is a necessary condition for the corresponding system of equations to have a solution. This means $\operatorname{res}(\xi/\omega, \eta/\omega, \zeta/\omega) \equiv 0$. Hence the parametrization satisfies an irreducible polynomial relation identically (see Section 1).

For example, for the plane parametrization

$$X = \frac{s^2}{r^2}, \quad Y = \frac{t^2}{r^2}, \quad Z = \frac{r^2 - s^2 - t^2}{r^2},$$

the resultant of the auxiliary equations

$$\begin{aligned} -X \quad r^2 + s^2 \qquad &= 0 \\ -Y \quad r^2 \qquad + t^2 &= 0 \\ (1 - Z)\, r^2 - s^2 - t^2 &= 0 \end{aligned}$$

is found to be $(X + Y + Z - 1)^4$ by using the Macaulay quotient (Chionh [2]). Clearly $X + Y + Z - 1 = 0$ is the implicit equation of the given plane. In this example we used r^2, s^2, t^2 instead of r, s, t to illustrate that the resultant is a power $\mu \geq 1$ of the implicit polynomial (Chionh and Goldman [3]).

But the resultant can be identically zero. For example, the following is a parametrization of the sphere $X^2 + Y^2 + Z^2 - 1 = 0$:

$$X = \frac{2rs}{r^2 + s^2 + t^2}, \quad Y = \frac{2rt}{r^2 + s^2 + t^2}, \quad Z = \frac{r^2 - s^2 - t^2}{r^2 + s^2 + t^2}.$$

The resultant of the auxiliary equations

$$\begin{array}{rlrlrl} -X & r^s + 2\,rs & - & X & s^2 - & X \quad t^2 = 0 \\ -Y & r^2 + 2\,rt & - & Y & s^2 - & Y \quad t^2 = 0 \\ (1 - Z) & r^2 & - & (1 + Z) & s^2 - & (1 + Z)\ t^2 = 0 \end{array}$$

is identically zero.

This outcome is expected because the vanishing of the resultant is a necessary condition for the system of equations to have a common solution, and the auxiliary equations of this sphere parametrization always have the solutions $(r : s : t) = (0 : 1 : i), (0 : 1 : -i)$ for any X, Y, Z.

From this discussion, it can be seen that whenever the equations

$$\omega(r, s, t) = 0, \quad \xi(r, s, t) = 0, \quad \eta(r, s, t) = 0, \quad \zeta(r, s, t) = 0, \tag{3}$$

have a common solution, the resultant of the auxiliary equations will vanish identically. A common solution $(r' : s' : t')$ of Equations (3) is known as a *base point* of the parametrization. Consequently, when a parametrization has base points, straightforward elimination fails to produce the implicit polynomial.[1] Though not obvious, it is still possible to show the existence of the implicit equation using multivariate resultants by applying perturbations; this method is discussed in detail in Chionh and Goldman [5].

2.1.2. *Bivariate Resultants.* There are other ways of constructing the implicit equation using resultants. For example, using affine parameters (s, t) instead of projective parameters $(r : s : t)$, the auxiliary equations (2) become nonhomogeneous polynomial equations in two variables s, t. Using a bivariate resultant, the variables s and t can be eliminated successively to obtain a polynomial relation involving only X, Y, Z. But the problem of producing an identically zero expression persists, which again makes an existence proof nontrivial, though still possible (Chionh [2]).

2.2. *Undetermined Coefficients*

To use the method of undetermined coefficients to prove the existence of the implicit equation, we consider the parametrization of the rational surface in projective coordinates $(w : x : y : z)$:

$$w : x : y : z = \omega(r, s, t) : \xi(r, s, t) : \eta(r, s, t) : \zeta(r, s, t). \tag{4}$$

Note that the degrees of the homogeneous parametric polynomials are the same; let it be n.

Consider a homogeneous polynomial $F(w, x, y, z)$ of degree m with indeterminate coefficients $\alpha_1, \ldots, \alpha_M$, where $M = \binom{m+3}{3}$ is the number of terms of F. Let

$$F(\omega(r, s, t), \xi(r, s, t), \eta(r,s,t), \zeta(r, s, t)) \equiv G(r, s, t).$$

Clearly G is a homogeneous polynomial in r, s, t of degree mn with $N \le \binom{mn+2}{2}$ terms. We wish to determine the α_i's such that $G \equiv 0$. This means that the coefficients of G, which are homogeneous linear polynomials in the α_i's, must all vanish. Thus to find the α_i's, we simply solve a homogeneous linear system of N equations in M variables. But $M > N$ for large enough m since M is of order $O(m^3)$ but N is of order $O(m^2)$. By the theory of linear algebra, there is always a non-zero solution for the α_i's since this is an underdetermined system. Hence we have proved the existence of a polynomial F which is satisfied identically by the parametrization. Therefore by the argument in Section 1 the parametrization satisfies an irreducible polynomial relation identically.

2.3. *The Theory of Field Extensions*

By exploiting the uniqueness of the degree of transcendency of field extensions (van der Waerden [20]), we obtain a very concise alternative proof. The proof is also constructive because it leads to an algorithm for finding the implicit polynomial (Chionh [2]). We present some relevant facts from the theory of field extensions before giving the proof.

2.3.1. *Algebraic and Transcendental Field Extensions.* If K, E are fields and $K \subseteq E$, then E is called an *extension* of K. If $S \subseteq E$, then $K(S)$ denotes the smallest subfield of E containing the set $K \cup S$.

Let E be an extension of the field K. If $\alpha \in E$ and $q(\alpha) = 0$ for some nonzero polynomial $q(x) \in K[x]$, then the element α is *algebraic over* K. If every element of E is algebraic over K, then E is an *algebraic extension* of K.

Again let E be an extension of the field K. If $\alpha_1, \cdots, \alpha_n \in E$ satisfy a nonzero polynomial relation with coefficients in K, then $\alpha_1, \cdots, \alpha_n$ are *algebraically dependent* over K; otherwise, they are *algebraically independent* over K.

Now let E be an algebraic extension of $K(\alpha_1, \cdots, \alpha_n)$, where $\alpha_1, \cdots, \alpha_n$ are algebraically independent over K. The set of elements $\alpha_1, \cdots, \alpha_n$ is called a *transcendency basis* of E over K. The number n is called the *degree of transcendency* of E over K. Since the degree of transcendency is unique (van der Waerden [20]), any $n + 1$ or more elements of E are algebraically dependent over K.

[1] In the case of plane curves, base points do not cause any problem because they can always be removed by discarding the GCD of the parametric polynomials.

The uniqueness of the degree of transcendency of extensions is the main result we apply to prove the existence of the implicit equation for rational surfaces.

2.3.2. *The Proof.* Consider the parameters s, t as two indeterminates. Clearly the extension $\mathbf{C}(s, t)$ of $\mathbf{C}$ has degree of transcendency two over $\mathbf{C}$. Given the parametrization (1), X, Y, Z are three elements of $\mathbf{C}(s, t)$. Thus X, Y, Z must satisfy a polynomial relation over $\mathbf{C}$. That is, there is a polynomial $f \in \mathbf{C}[x_1, x_2, x_3]$ such that f is satisfied when $x_1 = X$, $x_2 = Y$, $x_3 = Z$. Hence $f(X, Y, Z) \equiv 0$. Therefore by the argument in Section 1 the parametrization satisfies an irreducible polynomial identically.

3. COEFFICIENTS OF THE IMPLICIT EQUATION

We now show that the implicit equation can be written with coefficients from the coefficient field of the parametric polynomials.

Let $\omega(s, t)$, $\xi(s, t)$, $\eta(s, t)$, $\zeta(s, t) \in K[s, t]$, where $K \subseteq \mathbf{C}$ is a field. Replace the coefficients of the irreducible implicit polynomial $f(X, Y, Z) \in \mathbf{C}[X, Y, Z]$ by the indeterminates $\alpha_1, \cdots, \alpha_M$ where M is the number of terms in f. Let

$$f\left(\frac{\xi(s, t)}{\omega(s, t)}, \frac{\eta(s, t)}{\omega(s, t)}, \frac{\zeta(s, t)}{\omega(s, t)}\right) \equiv \frac{\phi(s, t)}{\omega(s, t)^m},$$

where m is the degree of f. Clearly the coefficients of $\phi(s, t)$ are homogeneous linear polynomials in the α_i's with coefficients in K. If $\phi \equiv 0$, then the coefficient of each term of $\phi(s, t)$ must be zero. This leads to a system of homogeneous linear equations in the α_i's with coefficients in K. The existence of the implicit polynomial f guarantees the existence of a solution to this system of linear equations. By linear algebra, the values of the α_i's, which are a solution, are in K. Hence f, or a constant multiple of f, is in $K[X, Y, Z]$.

The coefficient relationship can also be proved by elimination theory when the resultant $\mathrm{res}(X, Y, Z)$ of the auxiliary equations (2) does not vanish identically. In this case it can be proved that $\mathrm{res} = f^\mu$ where $\mu \geq 1$ and $f(X, Y, Z)$ is the implicit polynomial (Chionh and Goldman [3]). By construction, $f^\mu = \mathrm{res} \in K[X, Y, Z]$ where K is the coefficient field of the parametrization. So f^μ and at least one of its partial derivatives with respect to X, Y, Z will have a GCD $f^{\mu-1}$. This means $f^{\mu-1} \in K[X, Y, Z]$. Hence $f \equiv f^\mu / f^{\mu-1}$ is in $K[X, Y, Z]$.

This observation has a simple but important implication: if the coefficient field of the parametric polynomials is $K \subseteq \mathbf{C}$, it suffices to look for a polynomial $f \in K[X, Y, Z]$ and irreducible over K, which is satisfied identically by the parametrization. Such a polynomial will be irreducible in $\mathbf{C}[X, Y, Z]$ and hence is the implicit polynomial.

To see this, note that there is an irreducible $g \in K[X, Y, Z]$ which is the implicit polynomial. Thus if a polynomial $f \in K[X, Y, Z]$ can be found which is irreducible in $K[X, Y, Z]$ and is satisfied identically by the parametrization, then f and g must have a nonconstant common factor since the algebraic surfaces they represent have the rational surface as a common component. For if f and g has no common factors, then the intersection of the algebraic surfaces, represented by $f = 0$ and $g = 0$ is just some space curves. But g is irreducible; hence f is a multiple of g. But $f/g \in K[X, Y, Z]$ since both f, $g \in K[X, Y, Z]$. This means f/g is a constant since f is irreducible in $K[X, Y, Z]$.

This is an important observation because for some fields $K \subseteq \mathbf{C}$, the problem of determining if a polynomial f is reducible in $K[x_1, \cdots, x_k]$ is simpler than the more general problem of determining if f is reducible in $\mathbf{C}[x_1, \cdots, x_k]$. For example, there are algorithms to factor a polynomial in $\mathbf{Q}[x_1, \cdots, x_k]$, where $\mathbf{Q}$ is the field of rational numbers (Davenport *et al.* [6]).

4. A COMPARISON OF THE THREE METHODS

Each of the three existence proofs has advantages and disadvantages in theoretical considerations and practical implementations. In the following comparison of the three methods, n is the degree of parametrization, that is, the degree of the homogeneous parametric polynomials in (4).

The existence proof by elimination theory is straightforward unless the resultant vanishes identically. But when the resultant does not vanish, the method gives a concise expression for the implicit polynomial and reveals that the implicit equation can be written with coefficients from the coefficient field of the parametric polynomials. This way of finding the implicit polynomial has two advantages. First, the resultant can be expressed expeditiously in determinant form by using the Macaulay quotient. Moreover, the order of the matrix in this algorithm is only $4n^2 - n$ and the entries are linear polynomials in X, Y, Z. Symbolic expansion of the determinant can be costly (Hoffmann [10]), but other more practical techniques are available for computing the determinant (Manocha and Canny [14]). Moreover, some operations do not require such an expansion. For example, we can easily use the determinant form to determine if a point (X, Y, Z) lies on the surface. Second, as explained in Section 3, the resultant $\mathrm{res}(X, Y, Z)$ is a power $\mu \geq 1$ of the implicit polynomial; thus if $\mu > 1$, then res and one of the partial derivatives of res with respect to X, Y, Z will have $f(X, Y, Z)$ as a common factor. Hence a GCD rather than a factorization algorithm can be used to find the implicit polynomial. This is an important advantage because factorization is definitely more difficult than the computation of a GCD.

The method of undetermined coefficients is simple to understand and fully rigorous. More important, it reveals the coefficient relationship between the parametric and implicit representations. The significance of this relationship has been elaborated in Section 3. If n is the degree of the homogeneous parametric polynomial, the degree of the implicit polynomial can be as high as n^2. Thus potentially the linear system to be solved has $\binom{n^3+2}{2}$ equations with $\binom{n^2+3}{3}$ variables. Factoring is then required to find the implicit polynomial, unless the degree m of the polynomial with undetermined coefficients happens to be equal to the degree of the implicit polynomial. Theoretically, factoring can be avoided by searching for the right m, which is always between 1 and n^2 (Chionh and Goldman [4]). The right m is simply the smallest degree for which the system of homogeneous linear equations in the undetermined coefficients has a solution.

The proof by the theory of field extensions is most elegant and concise. It can be implemented as an implicitization algorithm, but many details need to be filled in. In our implementation (Chionh [2]), a matrix of order as large as n^4 with nonlinear polynomials in X, Y, Z must be expanded and factored to find the implicit polynomial.

In summary, we have Table 1.

5. INCOMPLETE RATIONAL SURFACES

We have shown that every rational surface is a subset of an irreducible algebraic surface. To highlight that the subset relationship can be proper we shall call a rational surface *incomplete* if it is a proper subset of the corresponding algebraic surface. It is known that if a surface parametrization has base points, they blow up to rational curves on the algebraic surface (Semple and Roth [18, Chap. VI, VII]). Consequently, a rational surface can be incomplete because of such rational curves.

To illustrate incomplete rational surfaces, we consider the plane given by the parametrization

$$X = \frac{st}{rs + rt + st}, \quad Y = \frac{rt}{rs + rt + st}, \quad Z = \frac{rs}{rs + rt + st}, \tag{5}$$

which has three base points $(r:s:t) = (1:0:0)$, $(0:1:0)$, and $(0:0:1)$. The implicit equation of this plane is clearly

$$X + Y + Z - 1 = 0. \tag{6}$$

TABLE 1

	Elimination theory	Undetermined coefficients	Field extensions
Existence proof	Yes	Yes	Yes
Concise expression	Yes (no base points)	No	No
Coefficient relationship	Yes (no base points)	Yes	No
Size of matrix	$4n^2 - n$ linear entries	$\binom{n^2+3}{3}$ by $\binom{n^3+2}{2}$ constant entries	n^4 nonlinear entries
Factoring	No (no base points)	Avoidable	Yes

Note that the line l_x which is the intersection of the implicit plane (6) and the plane $X = 0$, except for the two points $(0, 0, 1)$ and $(0, 1, 0)$, is missing from the parametric plane (5), since to have $X = 0$, s or t must be zero, so Y or Z must be zero as well. Similarly, all points of the line l_y which is the intersection of the implicit plane and the plane $Y = 0$ and all points of the line l_z which is the intersection of the implicit plane and the plane $Z = 0$, except the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, are missing from the parametric plane. That is, to account for all the points of the implicit plane, we have to complete the parametric plane with the lines l_x, l_y, l_z. Note that in this case the base points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ blow up to the lines l_x, l_y, l_z, respectively (Chionh [2]).

6. CONCLUSION

We have discussed three methods for showing the existence of the implicit equation for rational surfaces: elimination theory, undetermined coefficients, and the theory of field extensions. All these proofs can be implemented as algorithms to implicitize rational surfaces. A comparison of the proofs in terms of theoretical and practical considerations was also given.

Unfortunately, none of these methods give the degree of the implicit polynomial. At best they only provide some bounds on the degree. For a precise formula relating the implicit and parametric degrees, see Chionh and Goldman [4].

We have also shown that the implicit equation can be written with coefficients from the coefficient field of the parametric polynomials. This is very pleasant because a parametrization with rational (or real) coefficients will again have an implicit equation with rational (or real) coefficients.

The fact that a rational surface can be a proper subset of an irreducible algebraic surface was illustrated by an example. The implication of this anomaly to some intersection algorithms used in CAGD is discussed in Chionh [2].

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