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# Using multivariate resultants to find the implicit equation of a rational surface

Eng-Wee Chionh[1] and Ronald N. Goldman[2]

[1] Department of Information Systems and Computer Science, National University of Singapore, Kent Ridge, Sinapore 0511, Republic of Singapore
e-mail: isccew@nusvm.bitnet
[2] Department of Computer Science, Rice University, P.O. Box 1892, Houston, TX 77251, USA

Given a parametrization of a rational surface, the absence of base points is shown to be a necessary and sufficient condition for the auxiliary resultant to be a power of the implicit polynomial. The method of resultants also reveals other important properties of rational surface representations, including the coefficients of the implicit equation, the relationship between the implicit and parametric degrees, the degree of each coordinate variable of the implicit equation, and the number of correspondence of the parametrization.

**Key words:** Implicitization – Implicit degree – Parametric degree – Multivariate resultants – Number of correspondence – Rational surfaces – Unfaithful parametrizations

*Offprint requests to:* E.-W. Chionh

## 1 Introduction

In computer-aided geometric design (CAGD), both parametric and implicit representations are used to describe shapes (Böehm et al. 1984; Bajaj 1989). Each representation has its unique strenghts. For example, the implicit form quickly determines whether a point lies on a shape; the parametric form easily generates points along the shape. More importantly, hybrid representations can lead to better algorithms. For example, consider finding the intersection of a parametric curve $P(t)=(X(t), Y(t), Z(t))$ and an implicit surface $f(X, Y, Z)=0$: the intersection points can conveniently be computed by solving the univariate polynomial equation $f(X(t), Y(t), Z(t))=0$ (Sederberg and Parry 1986).

Because in CAGD shapes are usually approximated by rational surfaces, it is useful to develop rigorous and efficient techniques for finding the implicit equation of rational surfaces. In this paper, we use multivariate resultants (Lefschetz 1953) to explore the relationships between the implicit and parametric representations of a rational surface. We first establish a necessary and sufficient condition under which this method is applicable. When the method applies, it yields many important results: the implicit equation can be obtained speedily in determinant form; the coefficients of the implicit equation can be taken to be in the coefficient field of the parametric equations; several degree relationships can be formulated, one of which leads to an algorithm for deciding if a parametrization is faithful; and several useful simplifying observations can be made.

The main contribution of this paper is to develop techniques based on multivariate resultants for deriving many of the fundamental properties of rational surfaces. Some of these properties and techniques can be found scattered in literature of algebraic geometry; but instead of gleaning them separately, we unite these results through the focus of a single constructive tool – the multivariate resultant.

The rest of this paper consists of six sections. Section 2 contains preliminaries on rational surfaces and resultants. Section 3 discusses implicitization. Section 4 concerns the coefficients of the implicit representation. Section 5 establishes several degree relationships. Section 6 makes some simplifying observations. Section 7 summarizes our results.

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## 2 Preliminaries on rational surfaces and resultants

Some basic properties of rational surfaces are presented below. Base points and the auxiliary resultant, two important concepts associated with a rational parametrization, are then defined. A fundamental relationship between base points and the auxiliary resultant is that base points annihilate the auxiliary resultant. Thus, the auxiliary resultant is a non-zero expression if and only if there are no base points.

### 2.1 Rational and irreducible algebraic surfaces

A rational surface is a rational transformation of a plane in a 3-space. Using projective coordinates $(r{:}s{:}t)$ for the plane and affine coordinates $(X, Y, Z)$ for the 3-space, we can express a rational surface as

$$X=\frac{\xi(r,s,t)}{\omega(r,s,t)}, \quad Y=\frac{\eta(r,s,t)}{\omega(r,s,t)}, \quad Z=\frac{\zeta(r,s,t)}{\omega(r,s,t)}, \tag{1}$$

wherever $\omega(r, s, t) \neq 0$. The *parametric polynomials* $\omega, \xi, \eta, \zeta$ must be homogeneous of the same degree. This is because projective coordinates are ratios: $(r{:}s{:}t)$ and $(\alpha r{:}\alpha s{:}\alpha t)$ mean the same parameter point for any $\alpha \neq 0$; consequently, both ratios must correspond to the same surface point $(X, Y, Z)$. We call this common degree the *parametric degree* of the parametrization. The coefficients of the parametric polynomials usually belong to a subfield $K$ of the field of complex numbers $\mathbf{C}$; for example, $K$ can be $\mathbf{Q}$, the field of rational numbers. We assume that the greatest common divisor (GCD) of the parametric polynomials is a constant. If this is true, the parametrization is called a *proper parametrization.*

In CAGD, usually affine parameters $(s, t)$ are used instead of projective parameters $(r{:}s{:}t)$. However, the conversion from affine to projective parameters is straightforward: we simply introduce a homogenizing parameter $r$ to homogenize the parametric polynomials so that they all have the same degree. For example, the plane $X+Y+Z-1=0$ can be parametrized as

$$X=st+s+t+1, \quad Y=-st, \quad Z=-s-t. \tag{2}$$

The parametrization can be re-written in the form of Eq. (1) by setting

$$X=\frac{st+rs+rt+r^2}{r^2}, \quad Y=\frac{-st}{r^2}, \quad Z=\frac{-rs-rt}{r^2}. \tag{3}$$

By applying the theory of field extensions (van der Waerden 1950), the method of undetermined coefficients (Chionh and Goldman 1990), or various elimination techniques (Chionh 1990), we know there always exists a non-zero polynomial $f(X, Y, Z)$ irreducible over $\mathbf{C}$ such that

$$f\left(\frac{\xi(r,s,t)}{\omega(r,s,t)}, \frac{\eta(r,s,t)}{\omega(r,s,t)}, \frac{\zeta(r,s,t)}{\omega(r,s,t)}\right) \equiv 0. \tag{4}$$

A polynomial is irreducible over a field $K$ if it cannot be written as the product of two nonconstant polynomials whose coefficients are in $K$. For example, $X^2+Y^2$ is irreducible over $\mathbf{Q}$, but reducible over $\mathbf{C}$. The polynomial $f$ is known as the *implicit polynomial* of the parametrization. Implicit polynomials are unique up to constant factors if and only if the parametrization indeed represents a surface rather than curves or points (Semple and Roth 1985). For this reason, we can refer to *the* implicit polynomial of a rational surface. A surface represented implicitly by a polynomial equation $g(X, Y, Z)=0$ is known as an *algebraic surface.* An algebraic surface is irreducible if its defining polynomial is irreducible over $\mathbf{C}$. The above observations can be summarized as:

**Theorem 1.** *Every rational parametrization satisfies some implicit equations. In particular, every rational surface is a subset of a unique irreducible algebraic surface.*

This theorem assures us that the problem of implicitizing rational surfaces is well-defined.

### 2.2 Base points of a parametrization

The *base points* of a rational surface parametrization are the common intersections of the four plane curves

$$\omega(r,s,t)=0, \quad \xi(r,s,t)=0, \quad \eta(r,s,t)=0, \quad \zeta(r,s,t)=0 \tag{5}$$

defined by the parametric polynomials.

For example, Eq. (3) is a parametrization with two base points $(r{:}s{:}t)=(0{:}1{:}0), (0{:}0{:}1)$. They are found by solving the system of homogeneous equations:

$$\begin{aligned} r^2 &= 0 \\ st+rs+rt+r^2 &= 0 \\ -st &= 0 \\ -rs-rt &= 0. \end{aligned}$$

Algorithms to detect the presence of base points and to identify them are described in Chionh (1990).

### 2.3 Auxiliary resultant of a parametrization

Given a rational-surface parametrization, we can form three *auxiliary* equations of the parametrization:

$$\begin{aligned} &\xi(r,s,t) - X\,\omega(r,s,t) = 0 \\ &\eta(r,s,t) - Y\omega(r,s,t) = 0 \\ &\zeta(r,s,t) - Z\,\omega(r,s,t) = 0. \end{aligned} \tag{6}$$

Clearly the auxiliary equations are a system of three homogeneous equations in three variables $r$, $s$, $t$, whose coefficients are linear polynomials in $X$, $Y$, $Z$.

A resultant is a polynomial (with integer coefficients) in the coefficients (considered as indeterminates) of a system of homogeneous equations that vanishes if and only if the homogeneous system has a common non-trivial solution (Macaulay 1903). By elimination theory, we know that the resultant of $k$ homogeneous polynomial equations in $k$ variables exists (van der Waerden 1950). When $k=2$, the resultant can be found by many methods; Sylvester's dialytic method (Uspensky 1948) is perhaps the best known. If the $k$ equations are all linear, the resultant is the determinant of the coefficient matrix, a fact known as the *fundamental theorem of elimination.* When $k \geq 2$, the resultant can be found by the methods of Macaulay (1903, 1923) as a quotient of two determinants. Appendix I provides an algorithm adapted from one of Macaulay's methods for systems of equal degree equations (Macaulay 1923). For a survey of resultant expressions, see Chionh and Goldman (1989).

Consequently, the resultant of the auxiliary equations, called here the *auxiliary resultant*, exists and can be computed. The auxiliary resultant, denoted as $res(X, Y, Z)$, is either identical to zero or a non-zero polynomial in $X$, $Y$, $Z$; it cannot be a non-zero constant, because $\omega(r,s,t) \not\equiv 0$ and the auxiliary equations are satisfied by any $(r:s:t)$ with the corresponding surface point $(X, Y, Z)$.

As an example, the auxiliary equations of the parametrization

$$X = s+t, \qquad Y = s, \qquad Z = t$$

are

$$s+t-rX=0, \qquad s-rY=0, \qquad t-rZ=0,$$

and the auxiliary resultant is

$$\begin{vmatrix} -X & 1 & 1 \\ -Y & 1 & 0 \\ -Z & 0 & 1 \end{vmatrix} = -X+Y+Z$$

by the fundamental theorem of elimination.

### 2.4 Base points annihilate the auxiliary resultant

We can now derive the following fundamental relationship between base points and the auxiliary resultant.

**Theorem 2.** *The auxiliary resultant is identical to zero if and only if the parametrization has base points.*

*Proof.* It there is a base point, then it is a solution of the auxiliary Eq. (6) for any $X$, $Y$, $Z$. Because the vanishing of the resultant is a necessary condition for common solutions, we have $res(X, Y, Z)=0$ for any $X$, $Y$, $Z$. That is, $res(X, Y, Z) \equiv 0$.

Conversely, suppose $res(X, Y, Z) \equiv 0$, but there are no base points. Because the vanishing of the resultant is a sufficient condition for common solutions for any $(X, Y, Z)$, there is a $(a:b:c)$ satisfying the auxiliary equations. However, $\omega(a,b,c) \neq 0$, otherwise $(a:b:c)$ is a base point. Hence, the entire affine 3-space $(X, Y, Z)$ is represented by the parametrization. Yet this is impossible due to the existence of implicit equations (see Theorem 1).

## 3 Implicitization

In this section, we show that the auxiliary resultant of a rational parametrization is a power of the implicit polynomial when the parametrization has no base points. This relationship, and the ability to express resultants as quotients of determinants, means there is a quick implicitization method when it is acceptable to express the implicit polynomial in terms of a quotient of determinants.

### 3.1 Power of the implicit polynomial

First we prove:

**Theorem 3.** *If the auxiliary resultant $res(X, Y, Z)$ of a rational surface parametrization does not vanish identically, then the algebraic surface $res(X, Y, Z)=0$ and the rational suface represents the same set of points.*

*Proof.* Consider the auxiliary equations. Inasmuch as the vanishing of the resultant is a necessary condition for common solutions, the rational surface is a subset of the algebraic surface $res(X, Y, Z)=0$. On the other hand, while the vanishing of the resultant is a sufficient condition for a common solution, for each point on the algebraic surface $res(X, Y, Z)=0$ there corresponds a parameter point $(a:b:c)$ satisfying the auxiliary equations. However, $\omega(a:b:c)\neq 0$ for otherwise $(a:b:c)$ is a base point and $res(X, Y, Z)\equiv 0$ by Theorem 2. Hence, the algebraic surface is a subset of the rational surface.

Now we can prove that:

**Theorem 4.** *If the auxiliary resultant $res(X, Y, Z)$ of a rational surface parametrization does not vanish identically, then*

$$res(X, Y, Z)=f^{l}(X, Y, Z),$$

*where $f$ is the implicit polynomial of the parametrization and $l \geqq 1$ is an integer.*

*Proof.* Because $\mathbf{C}[X, Y, Z]$, the ring of polynomials in the indeterminates $X$, $Y$, $Z$ over the field of complex numbers, is a unique factorization domain (UFD) (Fraleigh 1989), we have

$$res(X, Y, Z)\equiv f_1^{l_1}(X, Y, Z) \ldots f_k^{l_k}(X, Y, Z),$$

where the $f_i$s are distinct and irreducible. Let $f$ be the unique irreducible implicit polynomial whose existence is guaranteed by Theorem 1. By Theorem 3, the algebraic surface $f_1^{l_1} \ldots f_k^{l_k}=0$ is a subset of the algebraic surface $f=0$. This is possible only if $k=1$ and $f=f_1$, because $f$ is irreducible.

### 3.2 Finding the implicit polynomial

By Theorem 4, if the auxiliary resultant $res \not\equiv 0$, then $res=f^{l}$ and $res'=l f^{l-1} f'$, where $f$ is the implicit polynomial and the superscript "′" denotes the partial derivative with respect to one of the variables $X$, $Y$, $Z$ that appear in $res$. Because $f$ is irreducible and the degree of $f'$ is less than the degree of $f$, $f$ and $f'$ have no common factors. Hence, the GCD of $res$ and $res'$ is $f^{l-1}$. Consequently, the implicit polynomial $f$ can be obtained from the auxiliary resultant as

$$f=\frac{res}{\mathrm{GCD}(res, res')}. \tag{7}$$

This means the implicit polynomial can be found without factorization, which is much more difficult than GCD computations.

Theorem 2 indicates that the auxiliary resultant of a rational surface parametrization fails to give the implicit polynomial when the parametrization has base points. For implicitation methods using resultants in the presence of base points, see Chionh and Goldman (1991) and Manocha and Canny (1991).

## 4 Coefficients of the implicit equation

Let $K \subseteq \mathbf{C}$ be the coefficient field of a rational surface parametrization whose auxiliary resultant $res \not\equiv 0$. We shall show that the implicit polynomial $f$ can be taken to be in $K[X, Y, Z]$ the ring of polynomials in the indeterminates $X$, $Y$, $Z$ with coefficients in $K$.

Because resultants are polynomials with integer coefficients in the coefficients of the associated polynomial equations, we have $res \in K[X, Y, Z]$. It follows that $res' \in K[X, Y, Z]$. Therefore, by Eq. (7), we have $f \in K[X, Y, Z]$, because the GCD can be obtained by using Euclid's algorithm (Knuth 1969), which involves only addition, subtraction, multiplication, and division – operations that are closed in the field $K$.

It is pleasant to know that the coefficient field of the implicit representation is no larger than the coefficient field of the parametric representation. More significantly, this result implies that if a parametrization with coefficients in a field $K \subseteq \mathbf{C}$ satisfies $g \in K[X, Y, Z]$ identically and if $g$ is irreducible over $K$, then $g$ must also be irreducible over $\mathbf{C}$.

# 5 Implicit and parametric degrees

It is known that for a rational-surface parametrization there exists an integer $\mu \geqq 1$ such that a general surface point corresponds to $\mu$ parameter points (Zariski 1971). The number $\mu$ will be referred to as *the number of correspondence* of the parametrization. When $\mu = 1$, the parametrization is said to be *faithful*, otherwise it is *unfaithful*.

By Bézout's theorem (van der Waerden 1950), the degree of the implicit polynomial $f$ is equal to the number of intersections of the algebraic surface $f=0$ and two planes. But the intersection of two planes is a line. By using two different representations of the line, we shall derive a formula relating the degree of the implicit polynomial $f$, the degree of the auxiliary resultant *res*, the parametric degree, and the number of correspondence $\mu$.

In the following discussion, instead of the affine space $(X, Y, Z)$, we consider the projective space $(w:x:y:z)$, because it facilitates the discussion of points at infinity. In projective space, the algebraic surface $f(X, Y, Z)=0$ becomes $F(w, x, y, z)=0$, where

$$F(w, x, y, z) \equiv w^m f\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$$

and $m$ is the degree of $f$. Note that $F$ is a homogeneous polynomial degree $m$ in $w$, $x$, $y$, $z$ and is irreducible if $f$ is irreducible. Similarly, the rational surface can be written as:

$$w:x:y:z = \omega(r, s, t) : \xi(r, s, t) : \eta(r, s, t) : \xi(r, s, t). \tag{8}$$

## 5.1 Implicit and parametric line representations

A projective line $(w:x:y:z)$ can be represented implicitly as the intersection of two projective planes:

$$\begin{aligned} &aw+bx+cy+dz=0 \\ &a'w+b'x+c'y+d'z=0. \end{aligned} \tag{9}$$

Alternatively, the line can also be represented parametrically as

$$w:x:y:z = \alpha_0 u+\beta_0 v : \alpha_1 u+\beta_1 v : \alpha_2 u+\beta_2 v : \alpha_3 u + \beta_3 v,$$

where $(\alpha_0:\alpha_1:\alpha_2:\alpha_3)$, $(\beta_0:\beta_1:\beta_2:\beta_3)$ are two arbitrary distinct points on the line

## 5.2 Degree relationship

Using the implicit representation of a line, we can find the intersection of the rational surface [Eq. (8)] and the line [Eq. (9)] by solving

$$\begin{aligned} &a\,\omega(r, s, t) + b\,\xi(r, s, t) + c\,\eta(r, s, t) + d\,\zeta(r, s, t) = 0 \\ &a'\omega(r, s, t) + b'\xi(r, s, t) + c'\eta(r, s, t) + d'\zeta(r, s, t) = 0. \end{aligned}$$

By Bézout's theorem, there are in general $n^2$ solutions $(r:s:t)$, where $n$ is the parametric degree. Note that when there are no base points, each solution $(r:s:t)$ represents a point of the rational surface.

Using the parametric representation of the line, we can find the intersection of the rational surface and the line by solving

$$\begin{aligned} &\alpha_0 u+\beta_0 v : \alpha_1 u+\beta_1 v : \alpha_2 u+\beta_2 : \alpha_3 u+\beta_3 v \\ &\quad = \omega(r, s, t) : \xi(r, s, t) : \eta(r, s, t) : \zeta(r, s, t). \end{aligned}$$

If $\alpha_0 u+\beta_0 v \neq 0$, this is equivalent to solving

$$\begin{aligned} &\xi(r, s, t) - \frac{\alpha_1 u+\beta_1 v}{\alpha_0 u+\beta_0 v}\,\omega(r, s, t) = 0 \\ &\eta(r, s, t) - \frac{\alpha_2 u+\beta_2 v}{\alpha_0 u+\beta_0 v}\,\omega(r, s, t) = 0 \\ &\eta(r, s, t) - \frac{\alpha_3 u+\beta_3 v}{\alpha_0 u+\beta_0 v}\,\omega(r, s, t) = 0. \end{aligned} \tag{10}$$

The common solutions are then given by the resultant of Eq. (10),

$$res\left(\frac{\alpha_1 u+\beta_1 v}{\alpha_0 u+\beta_0 v}, \frac{\alpha_2 u+\beta_2 v}{\alpha_0 u+\beta_0 v}, \frac{\alpha_3 u+\beta_3 v}{\alpha_0 u+\beta_0 v}\right) = 0,$$

where *res* is the auxiliary resultant. However, by Theorem 4, we have $res = f^l$, *where* $f$ is the implicit polynomial. Hence, we must solve

$$f^l\left(\frac{\alpha_1 u+\beta_1 v}{\alpha_0 u+\beta_0 v}, \frac{\alpha_2 u+\beta_2 v}{\alpha_0 u+\beta_0 v}, \frac{\alpha_3 u+\beta_3 v}{\alpha_0 u+\beta_0 v}\right) = 0$$

or equivalently

$$\frac{F^l(\alpha_0 u+\beta_0 v, \alpha_1 u+\beta_1 v, \alpha_2 u+\beta_2 v, \alpha_3 u+\beta_3 v)}{(\alpha_0 u+\beta_0 v)^{ml}} = 0,$$

where $F$ is the homogeneous implicit polynomial of degree $m$.
For indeterminates $\alpha$, $\beta$, $u$, $v$, $F(\alpha_0 u+\beta_0 v, \alpha_1 u+\beta_1 v, \alpha_2 u+\beta_2 v, \alpha_3 u+\beta_3 v)$ does not have the factor $(\alpha_0 u+\beta_0 v)$, because $F(w, x, y, z)$ is irreducible. Thus, by selecting a line that does not meet the rational surface at infinity, and noting that in general a line intersects an irreducible algebraic surface at distinct points (Chionh 1990), we obtain $m$ distinct solutions, each occurring with multiplicity $l$. That is, $l$ is the number of correspondence $\mu$.
Consequently, by equating the number of solutions obtained from the two representations of the same line, we obtain

$$ml=n^2. \tag{11}$$

From this we can conclude that

- The number of correspondence $\mu=l$.
- The implicit degree is $m=n^2/l$.
- The degree of the auxiliary resultant is $ml=n^2$.

We summarize these results as separate theorems:

**Theorem 5.** *If the auxiliary resultant of a parametrization is $f^l(X, Y, Z)$ and $f$ is irreducible, then the parametrization is a surface parametrization whose implicit polynomial is $f$ and whose number of correspondence is $l$.*

**Theorem 6.** *If a rational surface parametrization of degree $n$ has no base points, then $\mu m=n^2$, where $m$ is the implicit degree and $\mu$ is the number of correspondence. If the parametrization is faithful, then $m=n^2$*

**Theorem 7.** *If the auxiliary resultant of a parametrization of degree $n$ does not vanish identically, then its degree is $n^2$ in $X$, $Y$, $Z$.*

To illustrate Theorem 5, consider the parametrization

$$\begin{aligned} w:x:y:z = {} & r^2+s^2+t^2 \\ & :r^2 \\ & :s^2 \\ & :t^2. \end{aligned}$$

The implicit polynomial is clearly $X+Y+Z-1$. Using Macaulay's method, we compute the auxiliary resultant to be $(X+Y+Z-1)^4$. This result is expected, because in general each point of the rational surface corresponds to the four parameter points $(r:s:t)$, $(r:s:-t)$, $(r:-s:t)$, and $(r:-s:-t)$.

## 5.3 Detecting unfaithful parametrizations

By Theorem 5, we can find the number of correspondence $\mu$ without factorization by using Eqs. (7) and (11).
It is simpler if we just want to know whether a parametrization is unfaithful. From Section 3.2, *res* and *res'* have a non-constant factor $f^{l-1}$ if and only if $l>1$. This means the bivariate resultant of *res* and *res'*, that is, the discriminant of *res* (with respect to one of $X$, $Y$, $Z$) vanishes if and only if the parametrization is unfaithful.

## 5.4 Degrees in $w$, $x$, $y$, $z$

Our method for finding the degree of $F$, the homogeneous implicit polynomial, can be extended to find the (highest) degrees of $w$, $x$, $y$, $z$ in $F$.
For example, to find the degree of $w$, we consider the intersections of the line

$$\begin{aligned} bx+cy+dz &= 0 \\ b'x+c'y+d'z &= 0 \end{aligned} \tag{12}$$

and the rational surface [Eq. (8)]. The intersections are given by solving

$$\begin{aligned} b\xi(r,s,t)+c\eta(r,s,t)+d\zeta(r,s,t) &= 0 \\ b'\xi(r,s,t)+c'\eta(r,s,t)+d'\zeta(r,s,t) &= 0. \end{aligned} \tag{13}$$

If there are $m_w$ solutions independent of $a$, $b$, $c$, $a'$, $b'$, $c'$, then a generic line [Eq. (12)] intersects the rational surface at $(w:x:y:z)=(1:0:0:0)$ at least $m_w$ times, provided there are no base points. Note that the $m_w$ solutions are due to the intersection of the three plane curves

$$\xi(r,s,t)=0, \qquad \eta(r,s,t)=0, \qquad \zeta(r,s,t)=0.$$

Consequently $m_w$ is the multiplicity of the point $(1:0:0:0)$ on the rational surface. Therefore, the lowest degree term in $F^\mu(1, x, y, z)$ has to be $m_w$. This means the degree of $w$ in $F$ scaled by $\mu$ is $n^2-m_w$.
One way to find the value of $m_w$ is to use the $u$-resultant heuristically (Chionh 1990). That is, we find two $u$-resultants of Eq. (13) for two sets of random values $b$, $c$, $d$, $b'$, $c'$, $d'$. The degree of the GCD of the two $u$-resultants almost always gives $m_w$.
The degrees of $x$, $y$, $z$ in $F$ can be found similarly.

## 6 Reduced auxiliary resultant

The degrees of the auxiliary equations of a proper parametrization may be reduced, because the pairwise GCD's of $(\xi, \omega)$, $(\eta, \omega)$, and $(\zeta, \omega)$ may not be constant. For example, none of the pairwise GCD's of the parametrization

$$
\begin{aligned}
w{:}x{:}y{:}z = {} & rst \\
& {:}r(r+s+t)^2 \\
& {:}s(r+s+t)^2 \\
& {:}t(r+s+t)^2
\end{aligned}
$$

for the surface $(X+Y+Z)^2 - XYZ = 0$ is constant. If we discard these pairwise GCD's from the auxiliary equations, we arrive at auxiliary equations of lower degrees. This degree reduction is certainly computationally advantageous. In this section, we shall investigate the effect of such degree optimization.

### 6.1 Factor of the auxiliary resultant

Let the pairwise GCD's be

$$\mathrm{GCD}(\omega, \xi) = g_x \qquad \mathrm{GCD}(\omega, \eta) = g_y$$
$$\mathrm{GCD}(\omega, \zeta) = g_z.$$

The *reduced auxiliary equations* of a parametrization are defined as

$$X_w^* \equiv \frac{\xi(r,s,t) - X\,\omega(r,s,t)}{g_x(r,s,t)} \equiv \xi_w(r,s,t) - X\,\omega_x(r,s,t)$$
$$Y_w^* \equiv \frac{\eta(r,s,t) - Y\omega(r,s,t)}{g_y(r,s,t)} \equiv \eta_w(r,s,t) - Y\omega_y(r,s,t)$$
$$Z_w^* \equiv \frac{\zeta(r,s,t) - Z\,\omega(r,s,t)}{g_z(r,s,t)} \equiv \zeta_w(r,s,t) - Z\,\omega_z(r,s,t).$$

The resultant of the reduced auxiliary equations will be referred to as the *reduced auxiliary resultant*. In the subsequent discussion, the auxiliary and reduced auxiliary resultants are denoted by *res* and *res**, respectively.

For $k+1$ homogeneous polynomials $g, h, f_2, \ldots, f_k$ in $k$ variables, we have

$$[gh, f_2, \ldots, f_k] = [g, f_2, \ldots, f_k]\,[h, f_2, \ldots, f_k],$$

where $[\ldots]$ denotes the resultant of the equations "…". This property is known as the Product Theorem (Macaulay 1964). Consequently, we can write

$$
\begin{aligned}
res = res^*\,&[g_x, g_y, g_z][g_x, g_y, Z_w^*][g_x, Y_w^*, g_z] \\
&[X_w^*, g_y, g_z] \\
&\cdot[g_x, Y_w^*, Z_w^*][X_w^*, g_y, Z_w^*][X_w^*, Y_w^*, g_z]. \qquad (14)
\end{aligned}
$$

Thus, the reduced auxiliary resultant is always a factor of the auxiliary resultant. More precisely, we have the following theorem describing the relationship between the two auxiliary resultants.

**Theorem 8.** *When there are no base points, the auxiliary and reduced auxiliary resultants differ only by a consant factor; when there are base points, the reduced auxiliary resultant either is identical to zero or is a multiple of the implicit polynomial.*

The proof and examples for this theorem are given in the following sections.

### 6.2 Power of the implicit polynomial

When $res \not\equiv 0$, the parametrization has no base points by Theorem 2. Hence, the curves given by

$$g_x(r,s,t) = 0$$
$$g_y(r,s,t) = 0$$
$$g_z(r,s,t) = 0$$

have no common intersections; this means $[g_x, g_y, g_z]$ is a non-zero constant. More importantly, because by Bézout's theorem two plane curves always intersect in the complex projective plane, we see that $\mathrm{GCD}(g_x, g_y)$, $\mathrm{GCD}(g_x, g_z)$, $\mathrm{GCD}(g_y, g_z)$ are all constants.

Consider the equations

$$
\begin{aligned}
g_x(r,s,t) &= 0 \\
g_y(r,s,t) &= 0 \\
Z_w^* \equiv \zeta_w(r,s,t) - Z\,\omega_z(r,s,t) &= 0.
\end{aligned}
$$

For any common intersection of $g_x = 0$ and $g_y = 0$, we have $\omega_z = 0$, because $\omega_z$ has the factors $g_x$ and $g_y$ – but $\zeta_w \neq 0$; otherwise there is a base point. Hence, $[g_x, g_y, Z_w^*]$ is a non-zero constant. Similarly, $[g_x, Y_w^*, g_z)$ and $[X_w^*, g_y, g_z]$ are non-zero constants.

Now consider the equations

$$
\begin{aligned}
g_x(r,s,t) &= 0 \\
Y_w^* \equiv \eta_w(r,s,t) - Y\omega_y(r,s,t) &= 0 \\
Z_w^* \equiv \zeta_w(r,s,t) - Z\,\omega_z(r,s,t) &= 0.
\end{aligned}
$$

For any point on the curve $g_x = 0$, we have $\omega_y = 0$ and $\omega_z = 0$, because $\omega_y$ and $\omega_z$ both have $g_x$ as

a factor. However, $\eta_w$ and $\zeta_w$ cannot both be zero; otherwise there is a base point. Hence, $[g_x, Y_w^*, Z_w^*]$ is a non-zero constant. Similarly, $[X_w^*, g_y, Z_w^*]$ and $[X_w^*, Y_w^*, g_z]$ are non-zero constants.
Consequently, by Eq. (14), if the auxiliary resultant does not vanish identically, the reduced auxiliary resultant and the auxiliary resultant differ by at most a constant factor; hence, both are the same power of the implicit polynomial.
For example, the parametrization

$$\begin{aligned} w:x:y:z = rst \\ &:r^2(2r+s+t) \\ &:s^2(r+2s+t) \\ &:t^2(r+s+2t) \end{aligned}$$

has the same auxiliary and reduced auxiliary resultants.
Both resultants are:

$$\begin{aligned} &108-9(X+Y+Z)-24(XY+XZ+YZ) \\ &\quad -112XYZ+3X^2(Y+Z)+3Y^2(X+Z) \\ &\quad +3Z^2(X+Y)+4XYZ(X+Y+Z) \\ &\quad -XYZ(X^2+Y^2+Z^2) \\ &\quad +13XYZ(XY+XZ+YZ) \\ &\quad +32(XYZ)^2+2(XYZ)^2(X+Y+Z) \\ &\quad -(XYZ)^3. \end{aligned}$$

## 6.3 Identically zero or a multiple of the implicit polynomial

We shall show that if $res \equiv 0$, then either $res^* \equiv 0$ or the implicit polynomial $f$ is a factor of $res^*$.

### 6.3.1 Vanishing reduced auxiliary resultant

The parametrization

$$\begin{aligned} w:x:y:z = (r-s)(r-t)(s-t) \\ &:(r-s)(2r-s-t)\,t \\ &:(r-t)(r-2s+t)\,s \\ &:(s-t)(r+s-2t)\,r \end{aligned}$$

illustrates the case in which both $res$ and $res^*$ are identical to zero.

This can easily be seen, because the constituent polynomials of the auxiliary equations (reduced or otherwise) all vanish at the parametric point (1:1:1).

### 6.3.2 Reduced auxiliary resultant and extraneous factors

If $res^* \not\equiv 0$, then it is a multiple of the implicit polynomial $f$. To see this, note that $res^*$ and $f$ must both vanish on those $(X, Y, Z)$ representing points on the rational surface. This means the surfaces represented by $res^*=0$ and $f=0$ both contain the rational surface as a subset. However, a rational surface can only be a subset of a unique irreducible algebraic surface; consequently, $res^*$ must have $f$ as a factor.
It is possible that $res^*$ has other factors besides $f$; that is, $res^*$ may have extraneous factors. To show this, we need only produce an example.
Consider the proper parametrization

$$\begin{aligned} w:x:y:z = rst \\ &:r(r+s+t)^2 \\ &:s(r+s+t)^2 \\ &:t(r+s+t)^2. \end{aligned}$$

The auxiliary resultant vanishes identically, because of the existence of base points $(0:1:-1)$, $(1:0:-1)$, and $(1:-1:0)$. The reduced auxiliary resultant, computed by the Macaulay quotient, is

$$X^2Y^2Z^2((X+Y+Z)^2-XYZ).$$

Clearly, $X^2$, $Y^2$, $Z^2$ are extraneous factors.
Note that the implicit polynomial $(X+Y+Z)^2-XYZ$ is found by factoring $res^*$ over the field of rational numbers and is identified by back-substitution. It is not necessary to factor over the field of complex numbers because of the observations in Sect. 4.
The appearance of extraneous factors in the reduced auxiliary resultant $res^*$ when $res \equiv 0$ suggests that the recovery of the implicit polynomial from a zero auxiliary resultant using perturbations will introduce extraneous factors. Although we shall not pursue this topic here, this is indeed the case (Chionh 1990). Thus, the reduced auxiliary resultant provides us with some insight into the general case.

## 7 Conclusion

We have shown that in contrast to the case of plane rational curves (Sederberg et al. 1984), the method of resultants does not always succeed in implicitizing rational surfaces. A necessary and sufficient condition for the direct applicability of resultants to implicitization is that the parametrization has no base points.

When a parametrization has no base points, we showed that many important properties of the implicit representation can be derived by using multivariate resultants. We demonstrated that the auxiliary resultant is a power of the implicit polynomial; the coefficients of the implicit representation can be taken from the coefficient field of the parametric representation; the implicit degree $m$ scaled by the number of correspondence $\mu$ is the square of the parametric degree $n(n^2=ml)$; the degrees of the variables $w$, $x$, $y$, $z$ of the homogeneous implicit polynomial can be computed by examining the intersections of two generic curves, each of which is a linear combination of three of the four homogeneous parametric polynomials; the number of correspondence is easily computed; there is a quick method for detecting unfaithful parametrizations; and it is better to use reduced auxiliary resultants. Because multivariate resultants can be computed by the Macaulay quotient, all our results are constructive.

The results and method in this paper are valid provided that the auxiliary resultant does not vanish, that is, when base points are absent. However, it is known that all these results and methods, except the computation of $\mu$ and the detection of unfaithful parametrizations, either remain valid or can be adjusted to achieve similar outcomes when there are base points (Chionh 1990). We conjecture that the computation of $\mu$ and the detection of unfaithful parametrizations can also be done in a similar fashion even when there are base points. Thus, the study of this special case of no base points not only produces useful results, but also provides helpful clues for studying the general case.

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## Appendix I: A Macaulay Algorithm

The following algorithm is adapted from Macaulay (1923) to construct the resultant for same-degree polynomial equations.

**Input** $k$ homogeneous polynomials $f_1, \ldots, f_k$ of the same degree $n$ in $x_1, \ldots, x_k$

$d \leftarrow (k-1)(n-1)$

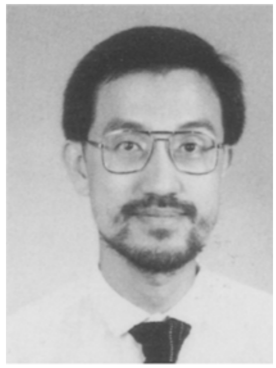
$\Omega_0^* \leftarrow \{x_1^{i_1} \ldots x_k^{i_k} \mid i_1 + \ldots + i_k = d,\ i_1 + \ldots + i_j \leq j(n-1) \text{ for } 1 \leq j \leq k-1\}$

**Comment** $\Omega_0^*$ has $\frac{n}{k-1}\binom{kn-1}{k-2}$ monomials

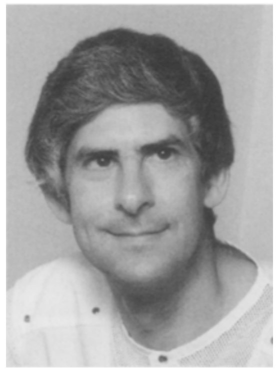
$\Omega_1^* \leftarrow \Omega_0^*$
for $i$ from 2 to $k-1$ do
  $\Omega_i^* \leftarrow \{x_1^{j_1} \ldots x_k^{j_k} \in \Omega_{i-1}^* \mid j_i < n\}$

**Comment** $x_1^n\, \Omega_0^*, \ldots, x_k^n\, \Omega_{k-1}^*$ are disjoint sets
$\Omega^* \leftarrow x_1^n\, \Omega_0^* \cup \ldots \cup x_k^n\, \Omega_{k-1}^*$
for $i$ from 1 to $k$ do
  for each monomial $\omega^{(d)}$ in $\Omega_{i-1}^*$, compute the polynomial $\omega^{(d)} f_i$
$D^* \leftarrow$ coefficient matrix of these polynomials with $\Omega^*$ as column indices
$F_1 \leftarrow \emptyset$
for $i$ from 1 to $k-2$ do
  $F_i \leftarrow$ set of powers $\in \Omega_i^*$ divisible by at least one of $x_{i+1}^n, \ldots, x_{k-1}^n$
$F_0 \leftarrow F_1$
$F \leftarrow x_1^n\, F_0 \cup \ldots \cup x_{k-1}^n\, F_{k-2}$
if $F \neq \emptyset$ then
  $M^* \leftarrow$ the minor of $D^*$ with
    column indices $F$ and
    rows from polynomials $f_1\, F_0, \ldots, f_{k-1}\, F_{k-2}$
else $M^* \leftarrow 1$

**Output** $|D^*|/|M^*|$



ENG-WEE CHIONH is a lecturer at the National University of Singapore. His principal research interest is computer geometric modeling. He received a BSc with 1st Class Honours in mathematics from Nanyang University, Singapore, in 1974, and M Math and PhD in computer science from the University of Waterloo, Canada, in 1977 and 1990, respectively. He is a member of ACM and of the Singapore Computer Society.



RON GOLDMAN attended the Massachusetts Institute of Technology, where he graduated with a BS in Mathematics in 1968. He continued his education at Johns Hopkins University, receiving his MA in 1972 and his PhD in 1973, both in Pure Mathematics. Dr. Goldman's current research interests lie in the mathematical representation, manipulation, and analysis of shape using computers. He is particularly interested in algorithms for polynomial and piecewise polynomial curves and surfaces, and he has investigated both parametrically and implicitly represented geometry. His work includes research in computer-aided geometric design, solid modeling, computer graphics, and splines. He has published over fifty articles in journals, books, and conference proceeding on these and related topics. Before returning to academia, Dr. Goldman worked for ten years in industry solving problems in computer graphics, geometric modeling, and computer-aided design. He served as a Mathematician at Manufacturing Data Systems Inc., where he helped to implement one of the first industrial solid modeling systems. Later he worked as a Senior Design Engineer at Ford Motor Company, enhancing the capabilities of their corporate graphics and computer-aided design software. From Ford he moved on to Control Data Corporation, where he was a Principal Consultant for the development group devoted to computer-aided design and manufacture. His responsibilities included data base design, algorithms, education, acquisitions, and research. Dr. Goldman left Control Data Corporation in 1987 to become an Associate Professor of Computer Science at the University of Waterloo in Ontario, Canada. He joined the faculty at Rice University on Houston, Texas as a Professor of Computer Science in July 1990.