A Note on the Bartlett Decomposition of a Wishart Matrix

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SUMMARY

By suitably factorizing the generalized variance, a concise derivation of the Bartlett decomposition of a central Wishart matrix is given. The method is extended to give a decomposition of a p-dimensional non-central Wishart matrix of rank t ($\leqslant p$), in terms of a t-dimensional non-central Wishart matrix of rank t, $(p-t)$ independent χ^2 variates and $\frac{1}{2}(p-t)(p+t-1)$ independent unit normal variates.

1. INTRODUCTION

BY using random orthogonal transformations, Wijsman (1957) and Kshirsagar (1959) have derived the Bartlett decomposition of a central Wishart matrix. These derivations appear to be easier than the earlier ones. Recently, Kshirsagar (1963) has used the same method to obtain the Bartlett decomposition of a p-dimensional non-central Wishart matrix in linear, planar and cubic cases. Our purpose here is to give a simpler derivation of the Bartlett decomposition by suitably factorizing the generalized variance.

2. BARTLETT DECOMPOSITION: CENTRAL WISHART MATRIX

Let $\mathbf{B}$ be a $p \times p$ positive definite symmetric matrix having the Wishart density

$$f(\mathbf{B}) = C\exp(-\tfrac{1}{2}\operatorname{tr}\mathbf{B})|\mathbf{B}|^{\frac{1}{2}(N-p-1)}, \tag{2.1}$$

where

$$C^{-1} = 2^{\frac{1}{2}pN}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma\{\tfrac{1}{2}(N-p+i)\}. \tag{2.2}$$

Obviously the above density may be written as

$$f(\mathbf{B}) = C\exp\left(-\tfrac{1}{2}\sum_{i=1}^{p} b_{ii}\right)\prod_{i=1}^{p-1}(b_{ii}-\mathbf{b}'_{(i)}\mathbf{B}_{ii}^{-1}\mathbf{b}_{(i)})^{\frac{1}{2}(N-p-1)}\, b_{pp}^{\frac{1}{2}(N-p-1)}, \tag{2.3}$$

where the matrix $\mathbf{B}_{ii}$ is obtained from $\mathbf{B}$ by omitting its first i columns and rows, i.e.

$$\mathbf{B}_{ii} = \begin{bmatrix} b_{i+1\,i+1} & b_{i+1\,i+2} & \cdots & b_{i+1\,p} \\ b_{i+2\,i+1} & b_{i+2\,i+2} & \cdots & b_{i+2\,p} \\ \cdot & \cdot & \cdots & \cdot \\ b_{p\,i+1} & b_{p\,i+2} & & b_{pp} \end{bmatrix}, \tag{2.4}$$

and

$$\mathbf{b}'_{(i)} = (b_{i\,i+1}, b_{i\,i+2}, \ldots, b_{ip}) \quad (i = 1, 2, \ldots, p-1). \tag{2.5}$$

Now we change the quadratic forms $\mathbf{b}'_{(i)}\mathbf{B}_{ii}^{-1}\mathbf{b}_{(i)}$ to their respective sums of squares $\mathbf{z}'_{(i)}\mathbf{z}_{(i)}$, say, where

$$\mathbf{z}'_{(i)} = (z_{i\,i+1}, z_{i\,i+2}, \ldots, z_{ip}) \quad (i = 1, 2, \ldots, p-1). \tag{2.6}$$

It follows that the density for the b_{ii} and z_{ij} is

$$f(b_{11}, b_{22}, \ldots, b_{pp};\, z_{12}, z_{13}, \ldots, z_{1p};\, z_{23}, z_{24}, \ldots, z_{2p};\, \ldots, z_{p-1\,p})$$
$$= C\exp\left\{-\tfrac{1}{2}\sum_{i=1}^{p} b_{ii}\right\}\prod_{i=1}^{p-1}(b_{ii}-\mathbf{z}'_{(i)}\mathbf{z}_{(i)})^{\frac{1}{2}(N-p+i-2)}\, b_{pp}^{\frac{1}{2}(N-2)}. \tag{2.7}$$

A further transformation

$$b_{ii} = \chi_i^2 + \mathbf{z}'_{(i)}\mathbf{z}_{(i)} \quad (i = 1, 2, \ldots, p-1), \quad b_{pp} = \chi_p^2, \tag{2.8}$$

reduces the density (2.7) to the density

$$f(\chi_1^2, \chi_2^2, \ldots, \chi_p^2;\, z_{12}, z_{13}, \ldots, z_{1p};\, z_{23}, z_{24}, \ldots, z_{2p};\, \ldots, z_{p-1\,p})$$
$$= C\exp\left(-\tfrac{1}{2}\sum_{i=1}^{p}\chi_i^2 - \tfrac{1}{2}\sum_{i=1}^{p-1}\mathbf{z}'_{(i)}\mathbf{z}_{(i)}\right)\prod_{i=1}^{p}(\chi_i^2)^{\frac{1}{2}(N-p+i-2)}, \tag{2.9}$$

which represents the joint density of p χ^2 variates and $\frac{1}{2}p(p-1)$ independent unit normal variates. We note that χ_i^2 $(i = 1, 2, \ldots, p)$ has a χ^2 distribution with $N-p+i$ degrees of freedom. We further note that $p+\frac{1}{2}p(p-1)$ variates, i.e. p χ^2 and $\frac{1}{2}p(p-1)$ unit normal variates, are all mutually independent.

It is easily seen that the above decomposition also follows by setting

$$\begin{aligned}
&\chi_i^2 = z_{ii}^2 \quad (i = 1, 2, \ldots, p-1),\\
&b_{ii} = z_{ii}^2 + \mathbf{z}'_{(i)}\mathbf{z}_{(i)} \quad (i = 1, 2, \ldots, p-1),\\
&b_{ij} = z_{ii}z_{ij} + \mathbf{z}'_{(i)}\mathbf{z}_{(j)} \quad (j > i, i = 1, 2, \ldots, p-1),\\
&b_{pp} = z_{pp}^2 = \chi_p^2.
\end{aligned} \tag{2.10}$$

In this case we may write

$$\mathbf{B} = \mathbf{T}\mathbf{T}', \tag{2.11}$$

where $\mathbf{T}$ is the upper triangular matrix

$$\mathbf{T} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & \cdots & z_{1p} \\ 0 & z_{22} & z_{23} & \cdots & z_{2p} \\ 0 & 0 & z_{33} & \cdots & z_{3p} \\ \cdot & \cdot & \cdot & \cdots & \vdots \\ 0 & 0 & 0 & & z_{pp} \end{bmatrix}. \tag{2.12}$$

3. Bartlett Decomposition: Non-central Wishart Matrix

Let $\mathbf{B}$ be a $p\times p$ positive definite symmetric matrix having the p-dimensional non-central Wishart density of rank t $(\leqslant p)$. Then following Anderson (1946, p. 419) the density of $\mathbf{B}$ may be written as

$$f(\mathbf{B}\,|\,t) = C_1\exp(-\tfrac{1}{2}\operatorname{tr}\mathbf{B})\,|\mathbf{B}|^{\frac{1}{2}(N-p-1)}\,\Phi(\mathbf{X}), \tag{3.1}$$

where

$$\Phi(\mathbf{X}) = \int |\mathbf{I}-\mathbf{X}\mathbf{X}'|^{\frac{1}{2}(N-2t-1)} \exp(\omega_{p-t+1} x_{p-t+1\, p-t+1} + \omega_{p-t+2} x_{p-t+2\, p-t+2} + \ldots + \omega_p x_{pp})\, d\mathbf{X}, \tag{3.2}$$

the $t \times t$ matrix $\mathbf{X}$ is

$$\mathbf{X} = \begin{bmatrix} x_{p-t+1\, p-t+1} & x_{p-t+1\, p-t+2} & \ldots & x_{p-t+1\, p} \\ x_{p-t+2\, p-t+1} & x_{p-t+2\, p-t+2} & \ldots & x_{p-t+2\, p} \\ . & . & \ldots & . \\ x_{p\, p-t+1} & x_{p\, p-t+2} & \ldots & x_{pp} \end{bmatrix}, \tag{3.3}$$

and the range of integration with respect to $\mathbf{X}$ is determined by the condition that the matrix $(\mathbf{I}-\mathbf{X}\mathbf{X}')$ is positive semidefinite. In the equation (3.2) $\omega^2_{p-t+1}, \omega^2_{p-t+2}, \ldots, \omega^2_p$ are the roots of the equation

$$|\mathbf{K}^2 \mathbf{B}_t - \omega^2 \mathbf{I}| = 0, \tag{3.4}$$

where the matrix $\mathbf{B}_t$ is obtained from the matrix $\mathbf{B}$ by omitting its first $(p-t)$ rows and columns. Here the notation $\mathbf{B}_t$ is used in preference to the previous notation $\mathbf{B}_{p-t\, p-t}$ of Section 2. The matrix $\mathbf{K}$ of the equation (3.4) is

$$\mathbf{K} = \begin{bmatrix} k_{p-t+1} & 0 & \ldots & 0 \\ 0 & k_{p-t+2} & \ldots & 0 \\ . & . & \ldots & 0 \\ 0 & 0 & \ldots & k_p \end{bmatrix}, \tag{3.5}$$

and the constant C_1 of the equation (3.1) is

$$C_1^{-1} = 2^{\frac{1}{2}pN} \pi^{\frac{1}{4}(p^2+2t^2-p)} \exp(-\tfrac{1}{2} \operatorname{tr} \mathbf{K}^2) \prod_{i=t+1}^{p} \Gamma\{\tfrac{1}{2}(N-p-t+i)\} \prod_{i=1}^{t} \Gamma\{\tfrac{1}{2}(N-2t+i)\}. \tag{3.6}$$

Now we write the density (3.1) as

$$f(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_t | t) = C_1 \exp(-\tfrac{1}{2} \operatorname{tr} \mathbf{B}_1 - \tfrac{1}{2} \operatorname{tr} \mathbf{B}_t) |\mathbf{B}_1 - \mathbf{B}_2 \mathbf{B}_t^{-1} \mathbf{B}_2'|^{\frac{1}{2}(N-p-1)} |\mathbf{B}_t|^{\frac{1}{2}(N-p-1)} \Phi(\mathbf{X}), \tag{3.7}$$

where the matrices $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_t$ are given by

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_2' & \mathbf{B}_t \end{bmatrix}. \tag{3.8}$$

Further, setting

$$\mathbf{B}_1 = \mathbf{B}_2 \mathbf{B}_t^{-1} \mathbf{B}_2' + \mathbf{D}, \quad \mathbf{B}_2 = \mathbf{Z}\mathbf{B}_t^{\frac{1}{2}}, \tag{3.9}$$

we find that the density of $\mathbf{D}$, $\mathbf{Z}$ and $\mathbf{B}_t$ is

$$f(\mathbf{D}, \mathbf{Z}, \mathbf{B}_t | t) = C_1 \exp(-\tfrac{1}{2} \operatorname{tr} \mathbf{D} - \tfrac{1}{2} \operatorname{tr} \mathbf{Z}\mathbf{Z}') |\mathbf{D}|^{\frac{1}{2}(N-p-1)} \exp(-\tfrac{1}{2} \operatorname{tr} \mathbf{B}_t) |\mathbf{B}_t|^{\frac{1}{2}(N-t-1)} \Phi(\mathbf{X}). \tag{3.10}$$

Obviously the densities of the matrices $\mathbf{D}$, $\mathbf{Z}$ and $\mathbf{B}_t$ are independent. $\mathbf{D}$ has a central Wishart density, $\mathbf{Z}$ has a $(p-t) \times t$-variate normal density and $\mathbf{B}_t$ has a

t-dimensional non-central Wishart density of rank t. The $(p-t)\times(p-t)$ matrix $\mathbf{D}$ may now be expressed, as in Section 2, in terms of $(p-t)$ χ^2 variates and $\frac{1}{2}(p-t)(p-t-1)$ independent unit normal variates. Thus the p-dimensional non-central Wishart matrix of rank t may be decomposed into $(p-t)$ χ^2 variates, $\frac{1}{2}(p-t)(p-t-1)+(p-t)t = \frac{1}{2}(p-t)(p+t-1)$ independent unit normal variates, and a t-dimensional non-central Wishart matrix of rank t. By setting $t = 1$, $t = 2$ and $t = 3$, we can deduce the results given by Kshirsagar (1963).

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