The Lebesgue Integral

From σ-algebras to simple functions;
building up to defining the Lebesgue Integral

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Abstract

In this paper we work our way up to defining the *Lebesgue Intergral* by introducing relevent results from Set Theory and Measure Theory. We make comparisions between the Lebesgue Intergral and the Reimann Intergral, noting the relative stregths and weaknesses of each.

Over the course of the paper, we will touch on other uses of Measures Theory, including it's use in Axiomatic Set Theory in the definition of a random variable. Finally, we will look very briefly at $\mathcal{L}^p$ spaces; we'll simply state what they are and make vauge but interesting statements about what it means for the $\mathcal{L}^p$s to be normed vector spaces, and that $\mathcal{L}^2$ in particular is an inner product space.

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1 Introduction

1.1 The Definition

The aim of this paper is to eventually define the Lebesgue Intergral - so lets just do exactly that:

Definition 1.1 (*Lebesgue Integration*)**.** Given

- a Measure Space; $(X, \mathcal{A}, \mu)$,
- an $\mathcal{A}/\bar{\mathcal{B}}$ Measuarable Function $u : X \to \bar{\mathbb{R}}$,
- positive functions u^+ and u^- s.t. $u = u^+ - u^-$,
- the sets $S^\pm$ of *all* Simple Functions ($S \subset \mathcal{E}^+_\mu$) on $(X, \mathcal{A}, \mu)$ s.t. $s \in S^\pm \Rightarrow s \leq u^\pm$,

we define the μ-integral of u - which we write as $\int u \, d(\mu)$ - in terms of the integrals of the positive functions u^+ and u^-. The intergral of a general positive function - so including our $u^\pm$ - is defined as

$$\int u^+ \, d(\mu) := \sup\{I_\mu(s) : \quad s \in \mathcal{E}^+ \quad \text{and} \quad s \leq u\}, \tag{1.1$'$}$$

and going back to our more general u,

$$\int u \, d(\mu) := \int u^+ \, d(\mu) - \int u^- \, d(\mu). \tag{1.1}$$

If the domain of our function is $\mathbb{R}^n$, with the 'standard' n-dimensional Borel σ-algebra and Lebesgue Measure on it - i.e. $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}, \lambda^n)$ - then this becomes the λ^n-integral (read as Lebesgue integral instead of lambda integral - or we just call it 'the integral') and we write

$$\int u \, d(\lambda^n) = \int u(x) \, d(\lambda^n) := \int u(x) \, d(x) \tag{1.1$''$}$$

That's quite a long definition, one which raises more questions than it answers. The first (and biggest) issue here is that, even assuming someone has taken a course in Analysis, there are many undefined terms;

- What are Measures and σ-algebras? And what is a Measure Space?
- What is the Borel σ-algerbra
- What is a Measuarable Function? What is a Simple Function?
- What is I_μ and why can it seemingly only be used on positive Simple Functions?

That last bullet point touches on the second key issue with this definition; why. Why any of this. What about this definition of what we call the integral makes it any better than the Reimann Integral? Do they even describe similar things? This is not at all immedately obvious, since the Reimann intergral is defined on functions $f : Y \to \mathbb{R}$ where Y is some compact[1] subest of R and this integral is defined on...well, any set at all.

Even if we can somehow convince ourselves they are compatible, it still seems like a convoluted defintion - what about Simple Functions is so special that the intergrals of all other functions are defined in terms of them (or to be more precise, why are general functions are defined in terms of positive functions, and then positive functions defined in terms of positive simple ones)?

[1]Define a compact set

Interestingly (to me, at least) the Reinmann intergral is defined in a similarish way - we find the upper Reimann intergral of a function by finding all the Step Functions which bound the function from above, define/find the intergrals of these using their 'steppyness', then take the infinium - we do the same thing from below, and then if these limits coiencied then we call that the Reinmann integral. In order to understand the motivations behind Lebegsue Intgration better, let's take a slightly closer look at the similarites and differences between these two.

1.2 The Motivation

Our goal when we integrate a function is to find the 'space' or area between the graph and the x-axis (or the Abscissa - but I'll just say x-axis). It is diffficult to say how much space an aribitray shape takes up, but we can work out the area of rectangles very easily - by calculating the (base $\times$ height).

Picture of approximating a graph with rectangles

In the case where the domain of the function (the 'x-axis'), is $\mathbb{R}^2$, we are actually calculating a volume by finding the 'space' between the graph - still, the 'area of a rectangle' (in this case, the volume of a cuboid) is calculated in much the same way; the size of the base (which is now the area of the base) $\times$ the height.

Picture of the appromimatimg a 3D graph with cuboids

With this is mind, lets clear up some of the terms we are going to use:

Lebesgue integration is the method of integration defined above. Just like Rieman Interation, it is used to find the space between the graph of a function and the domain - however, the domain can be any set at all. Thinking back to rectangles and cuboids (size of base $\times$ height), if we wanted make sense of what it means for there to be space between the graph and the x-axis, we should be able to make sense of the 'size' of parts of the domain.

The Lebesgue Measure seems like it'd then be the technique for assigning sizes to, or 'measuring' aribtray sets; but it is not! It's specifically for when we give sizes to the subsets of $\mathbb{R}^n$, and ones which coicied with our usual idea of the size of a set in $\mathbb{R}^n$ - i.e. it is used when we are considering functions $\mathbb{R}^n \supset Y \to \mathbb{R}$.

The Lebesgue Integral is what we get when we do Lebesgue Intergration on a set with the Lebesgue Measure on it. i.e. it's the intergral of functions $\mathbb{R}^n \supset Y \to \mathbb{R}$. This is the (basically) the same domain as the Riemann integral and so we'd want to check that these two things match up - and then see if the Lebesgue integral is better in some way.

To see the motivation behind the Lebesgue integral over Riemann, let's look at the canonical example of a function for which the Reimann integral fails; $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \mathbb{1}_{\mathbb{Q}}(x)$.

Picture of this function

First, let's define the Riemann integral. The definition of the Riemann integral involes many steps, and so it's quite long. First, lets give the definition of a step function;

Definition 1.2. (*Riemann Intergration*) Given;

- A function, $f : \mathbb{R} \to \mathbb{R}$
- A