Lab 9

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1 Classes of linear second-order PDEs

We usually come across three PDEs in mechanics: elliptic, hyperbolic, and parabolic, synonymous with the Poisson, Heat, and Wave equations.

Poisson: describes the value of the electric potential in space as a function of the charge density.

$$\nabla^2 U(x) = -4\pi\phi(x) \tag{1}$$

Heat: describes the temperature of a medium over time and space as a function of the heat sources and initial heat distribution. It has the same form as the diffusion equation.

$$\nabla^2 U(\vec{x}, t) = \alpha \frac{\partial U}{\partial t} \tag{2}$$

Wave: describes the displacement from equilibrium of the elements of the medium in time and space as a function of their initial conditions, boundary conditions, and properties of the medium.

$$\nabla^2 U(\vec{x}, t) = c^{-2} \frac{\partial^2 U}{\partial t^2} \tag{3}$$

Knowledge of the type of PDE gives us insight into how smooth the solution will be, how fast information propagates, and the effect of initial and boundary conditions.

2 Types of Boundary Conditions

Dirichlet These are values that the solution is to take on the boundary of the domain.
Neumann: These are values that the **derivative** of a solution is to take on the boundary of the domain.
Cauchy: These are values of both the **solution** and its **derivative** on the boundary.

3 Parabolic PDEs

This is synonymous with the heat equation. Let α be your diffusion coefficient. Solutions for these problems occur over all space at time zero, as well as the boundary conditions $U(x_{min}, t) = a(t)$ and $U(x_{max}, t) = b(t)$.

4 Forward Difference Method

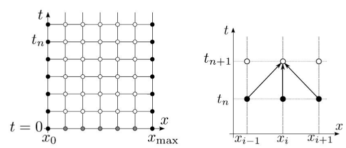

Figure 1. Note the blue nodes defining the boundary of this mesh. The empty nodes are the unknown values yet to be determined.

Express your second order differential as:

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} + \mathcal{O}(\Delta x^4) \tag{4}$$

and express a first order differential equation as:

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{(\Delta x)} + \mathcal{O}(\Delta x^2) \tag{5}$$

The Heat equation aka diffusion equation can be expressed as:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{6}$$

This is an explicit method because there is a way to determine new values (with respect to time) directly from the previously known values. It is helpful to rearrange the expression from (6) wherein any terms at time n are separate from those at time n + 1:

$$u_i^{n+1} = \phi u_{i-1}^n + (1 - 2\phi) u_i^n + \phi u_{i+1}^n \tag{7}$$

Let $\phi = \frac{\alpha \Delta t}{(\Delta x)^2}$

Stability criterion for the forward difference method is: $\frac{2\alpha\Delta t}{\Delta x^2} \leq 1$

We have to sacrifice spatial resolution or take very large series of steps to keep the integration stable. It is better to sacrifice spatial resolution because of the Δx^2 dependency. We can express this system as $\mathbf{Au}^n + \phi\, \mathbf{b}^n = \mathbf{u}^{n+1}$

5 Backward Difference Method

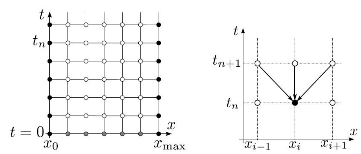

Figure 2. This is a method that manages error better than the previous method using an implicit process.

We can express the system similar to the Forward Difference Method in the following manner:

$$u_i^n = -\phi u_{i-1}^{n+1} + (1+2\phi)\, u_i^{n+1} - \phi u_{i+1}^{n+1} \tag{8}$$

Let $\phi = \frac{\alpha \Delta t}{(\Delta x)^2}$

This is an implicit method because we express known values as a function of unknown values.

Stability criterion for the backward difference method is unconditionally stable meaning it is stable for all choices of space and time. We can express this system as $\mathbf{A}^{-1}[\mathbf{u}^n + \phi\, \mathbf{b}^{n+1}] = \mathbf{u}^{n+1}$.

6 Crank-Nicolson Method

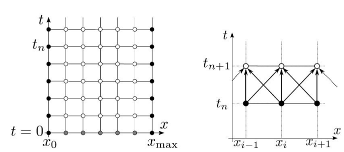

Figure 3. Note how this method solves for the unknowns.

This is an average of the forward and backward difference methods, wherein:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + (u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{(\Delta x)^2}\right] \tag{9}$$

We can express the system as:

$$-\phi u_{i-1}^{n+1} + (2+2\phi)\, u_i^{n+1} - \phi u_{i+1}^{n+1} = \phi u_{i-1}^n + (2-2\phi)\, u_i^n + \phi u_{i+1}^n$$

It is unconditionally stable for all choices of time step sizes and is the recommended choice for simple diffusion problems.The general case is expressed as: $\mathrm{A}^{-1}[Bu^n + \phi(b^n + b^{n+1})]$.