Ex. 1

$$y = x^T x, \qquad x \in R^N$$

$$\frac{\partial y}{\partial x} = ?$$

Solution

$$a = \frac{\partial y}{\partial x} \in R^N$$

$$y = \sum_i x_i^2$$

$$a_i = \frac{\partial y}{\partial x_i} = 2 \sum_j \delta_{ij} x_j = 2x_i$$

$$a = 2x$$

Ex. 2

$$y = tr(AB), \qquad A, B \in R^{N \times N}$$

$$\frac{\partial y}{\partial A} = ?$$

Solution

$$a = \frac{\partial y}{\partial A} \in R^{N \times N}$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

$$tr(AB) = \sum_i \sum_k A_{ik} B_{ki}$$

$$a_{mn} = \frac{\partial tr(AB)}{\partial A_{mn}} = \sum_i \sum_k \delta_{im} \delta_{kn} B_{ki} = B_{nm}$$

$$a = B^T$$

Ex. 3

$$y = x^T Ac, \quad A \in R^{N\times N}; \quad x \in R^N; \quad c \in R^N$$

$$\frac{\partial y}{\partial A} = ?$$

Solution

$$a = \frac{\partial y}{\partial A} \in R^{N\times N}$$

$$(Ac)_i = \sum_k A_{ik} c_k$$

$$x^T Ac = \sum_i x_i \sum_k A_{ik} c_k$$

$$a_{mn} = \frac{\partial x^T Ac}{\partial A_{mn}} = \sum_i x_i \sum_k \delta_{im}\delta_{kn} c_k = x_m c_n$$

Ex. 4

$$J = \|X - AS\|^2, \quad A \in R^{N\times R}; \quad S \in R^{R\times M}$$

$$\frac{\partial J}{\partial S} = ?$$

Solution

$$(X - AS)_{ij} = X_{ij} - \sum_{k=1}^{R} A_{ik} S_{kj} = M_{ij}$$

$$\|M\|^2 = tr(MM^T) = \sum_{i,j=1}^{N,M} M_{ij}^2$$

$$\frac{\partial J}{\partial S_{mn}} = \sum_{i,j=1}^{N,M} \frac{\partial M_{ij}^2}{\partial S_{mn}} = 2 \sum_{i,j=1}^{N,M} M_{ij} \frac{\partial M_{ij}}{\partial S_{mn}} =$$

$$= 2 \sum_{i,j=1}^{N,M} M_{ij} \frac{\partial}{\partial S_{mn}} \sum_{k=1}^{R} A_{ik} S_{kj} = 2 \sum_{i,j=1}^{N,M} M_{ij} \sum_{k=1}^{R} A_{ik} \delta_{km} \delta_{jn} =$$

$$= 2 \sum_{i,j=1}^{N,M} M_{ij} A_{im} \delta_{jn} = 2 \sum_{i}^{N} M_{in} A_{im}$$