ANA3 - Vaje 1

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1.

$$\int \frac{dt}{t \ln(t) \ln(\ln(t))}$$

$$u = \ln(t),\ \mathrm{d}u = \frac{1}{t}\,\mathrm{d}t$$

$$\int \frac{\mathrm{d}u}{u \ln(u)}$$

$$x = \ln(u),\ \mathrm{d}x = \frac{1}{u}\,\mathrm{d}u$$

$$\begin{aligned}\int \frac{\mathrm{d}x}{x} &= \ln(|x|) + C \\ &= \ln(|\ln(u)|) + C \\ &= \underline{\ln(|\ln(\ln(t))|) + C}\end{aligned}$$

2.

$$\int t^2 \sin(t)\,\mathrm{d}t$$

$$\begin{gathered}u = t^2,\ \mathrm{d}u = 2t\,\mathrm{d}t \\ \mathrm{d}v = \sin(t)\,\mathrm{d}t,\ v = -\cos(t)\end{gathered}$$

$$-t^2\cos(t) + 2t\int \cos(t)\,\mathrm{d}t$$

$$u = t,\ \mathrm{d}u = 1\,\mathrm{d}t$$
$$\mathrm{d}v = \cos(t)\,\mathrm{d}t,\ v = \sin(t)$$

$$-t^2\cos(t) + 2(t\sin(t) - \int \sin(t)\,\mathrm{d}t) = \underline{-t^2\cos(t) + 2t\sin(t) + 2\cos(t) + C}$$

3.
$$\int \frac{t^2+1}{(t-1)(t^2-1)}\,\mathrm{d}t$$

$$\int \frac{A+Bt}{(t-1)^2} + \int \frac{C}{(t+1)}\,\mathrm{d}t\,,\ At + A + Bt^2 + Bt + Ct^2 - 2tC + C = t^2 + 1$$

$$\begin{aligned} t^2 &: B + C = 1 & A &= \frac{1}{2} \\ t &: A + B - 2C = 0 & B &= \frac{1}{2} \\ 1 &: A + C = 1 & C &= \frac{1}{2} \end{aligned}$$

$$\frac{1}{2}\Big(\int \frac{1}{(t-1)^2}\,\mathrm{d}t + \int \frac{t}{(t-1)^2}\,\mathrm{d}t + \int \frac{1}{(t+1)}\,\mathrm{d}t\Big)$$

Vsak integral posebej rešim s substitucijo po vrsti $u_1 = t - 1$, $u_2 = t - 1$, $u_3 = t + 1$,.

4.
$$\int \frac{\mathrm{d}t}{1+\sin(t)}$$

Uporabim Weierstrassovo substitucijo, ki prevede funkcijo, ki vsebuje trigonometrične funkcije v racionalno funkcijo (`https://www.youtube.com/watch?v=vtsKCwmvaVY&t=131s`)

$$u = \tan\left(\frac{t}{2}\right),\ \mathrm{d}u = \frac{1+u^2}{2}\,\mathrm{d}t$$
$$\sin(t) = \frac{2u}{1+u^2}$$

$$\int \frac{1}{1+\frac{2u}{1+u^2}}\frac{2}{1+u^2}\,\mathrm{d}u = 2\int \frac{\mathrm{d}u}{(1+u)^2}$$

Integral rešim s substitucijo $x = 1 + u$

5.

$$\int_0^{\frac{\pi}{2}} \sin^2(t)\,\mathrm{d}t$$

Z identiteto $\sin^2(t) = \frac{1-\cos(2t)}{2}$ prevedem integral na stopnjo nižje.

$$\int_0^{\frac{\pi}{2}} \frac{1-\cos(2t)}{2}\,\mathrm{d}t = \frac{1}{2}\Big(\int_0^{\frac{\pi}{2}}\mathrm{d}t - \int_0^{\frac{\pi}{2}}\cos(2t)\,\mathrm{d}t\Big)$$

V drugem integralu uvedem spremenljivko $u = 2t$.

6.

$$\int_0^1 \frac{\mathrm{d}t}{t^2+a^2},\ a>0$$

Poznamo rešitev integrala $\int \frac{\mathrm{d}x}{x^2+1} = \arctan(x) + C$ in želimo to uporabiti.

$$\frac{1}{a^2}\int_0^1 \frac{\mathrm{d}t}{(\frac{t}{a})^2+1} = \frac{1}{a^2}\arctan\left(\frac{t}{a}\right)a\Big|_0^1 = \underline{\frac{1}{a}\arctan\left(\frac{1}{a}\right)}$$

7.

$$\int_0^1 \frac{\mathrm{d}t}{t^2-a^2}$$

$$\int_0^1 \frac{A}{t-a} + \frac{B}{t+a},\ At + Aa + Bt - Ba = 1$$

$$\begin{aligned} t &: A + B = 0 & A &= \frac{1}{a} \\ 1 &: Aa - Ba = 1 & B &= -\frac{1}{a} \end{aligned}$$

$$\frac{1}{a}\int_0^1 \frac{\mathrm{d}t}{t-a} - \frac{1}{a}\int_0^1 \frac{\mathrm{d}t}{t+a}$$

Integrala rešim s substitucijo po vrsti $u_1 = t - a$, $u_2 = t + a$

8.

$$\int_0^1 \frac{\mathrm{d}t}{t^2+x},\ x\in\mathbb{R}$$

Ločiti moramo primere za $x>0,\ x<0$ in $x=0$.

(a) $x > 0$: lahko uporabimo isti postopek kot v nalogi (6).

(b) $x < 0$: Zapišemo $\int_0^1 \frac{\mathrm{d}t}{t^2-(-x)}$ in uporabimo isti postopek kot v nalogi (7).

(c) $x = 0$: Očitno

9.
$$\int_0^{\infty} t^n e^{-t}\,\mathrm{d}t\,,\ n \in \mathbb{N}$$

Opazimo Eulerjevo Gama funkcijo $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,\mathrm{d}t$. Za naravno števila (naša naloga) ima Gama funkcija lepo formulo in sicer $\Gamma(n) = (n-1)!$

V našem primeru je rešitev $\Gamma(n+1) = n!$

10.
$$\int_0^1 t^2(1-t)^{2023}$$

Podobno lahko opazimo Eulerjevo Beta funkcijo $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}$. Za naravni števili m, n lahko Beta funkcijo izrazimo z Gama funkcijo in sicer:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

V našem primeru je rešitev $B(3, 2024) = \frac{2!\ 2023!}{2026!}$