Dissertation
zur Erlangung des Naturwissenschaftlichen Doktorgrades

Building the ULTIMATE Dark Matter Detector

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“To our knowledge this is the coolest known galaxy outside of the Local Group.“

Pieter van Dokkum et al.

“Sometimes science is more art than science.“

Rick Sanchez

“I’ll do it myself - because it needs to work!“

Darryl Masson

“Wow!“

Marc Schumann, 6th May 2019, 10:07h

Contents

Chapter 1

TexStuff

This chapter is supposed to deliver beautifully formated LaTex output that I can save as images and insert into talks, papers, exercise sheets and so on...

$$
\begin{aligned}
& R_\mathrm{e} \\
\frac{\text{number of stars}}{\text{resolution element}} \quad \frac{\text{luminosity of the stars}}{\text{resolution element}} & \\
R_\mathrm{e} &= 1.6\,\mathrm{kpc} \\
D_\mathrm{sbf} &= (19.9 \pm 2.8)\ \mathrm{Mpc} \\
\Rightarrow\ L_{V,606} &= (7.7 \pm 0.8) \cdot 10^7\, L_\odot \\
\Rightarrow\ M_\mathrm{stars} &= (1.5 \pm 0.4) \cdot 10^8\, \mathrm{M}_\odot \\
M_\mathrm{stars} \approx 1.5 \cdot 10^8\, \mathrm{M}_\odot & \\
M_\mathrm{total} = 0.4^{+1.2}_{-0.3} \cdot 10^8\, \mathrm{M}_\odot & \\
\mu < 24\, \frac{\mathrm{mag}}{\mathrm{arcsec}} & \\
R_\mathrm{e} < 1.5\,\mathrm{kpc} & \\
m_\mathrm{stars} \approx 10\,\%\, m_\mathrm{vis} & \\
m_\mathrm{gas} \approx 90\,\%\, m_\mathrm{vis} &
\end{aligned} \tag{1.1}
$$

RTG Fall Workshop 2019

$40\,\mathrm{t}$

$\mathbf{40\,t}$

$\sigma_{\mathrm{SI}} = 2.5 \cdot 10^{-49}\,\mathrm{cm}^2$

$m_\chi = 40\,\frac{\mathrm{GeV}}{\mathrm{c}^2}$

$\boldsymbol{\sigma_{\mathrm{SI}} = 2.5 \cdot 10^{-49}\,\mathrm{cm}^2}$

$\boldsymbol{m_\chi = 40\,\frac{\mathrm{GeV}}{\mathrm{c}^2}}$

$98\,\%$

$\tilde{t}_{\mathrm{drift}} \leq 10\,\mathrm{s}\ (98\,\%)$

$\varepsilon_{\mathrm{tot}} = \varepsilon_{\mathrm{coll}} \cdot \varepsilon_{\mathrm{loss}} \cdot \varepsilon_{\mathrm{conv}} \cdot 0.5$

$\varepsilon^{214}_{\mathrm{coll}} > \varepsilon^{218}_{\mathrm{coll}}$

U_{coll}

$$\left(\frac{N_{^{212}\mathrm{Bi}}}{N_{^{212}\mathrm{Po}}}\right)_{\mathrm{meas}} = (0.57 \pm 0.01)\,, \quad \left(\frac{N_{^{212}\mathrm{Bi}}}{N_{^{212}\mathrm{Po}}}\right)_{\mathrm{lit}} = 0.5625$$

Astroparticle School 2019

$$\varepsilon_{\text{tot}} = \varepsilon_{\text{coll}} \cdot \varepsilon_{\text{loss}} \cdot \varepsilon_{\text{angle}} \cdot \varepsilon_{\text{conv}} = \frac{A^{\text{Po}}_{\text{MonXe}}}{A^{222\text{Rn}}_{\text{RAD7}}} \tag{1.2}$$

$$t_{\text{drift}} \leq \tilde{t}_{\text{drift}} = \frac{\tilde{d}}{\tilde{v}_{\text{drift}}} = \frac{h(\vec{r}_0) + r(\vec{r}_0)}{\mu\left(E_{\text{eff}}, p, T\right) \cdot E_{\text{eff}}(\vec{r}_0)} \tag{1.3}$$

$$t_{\text{drift}} \leq \tilde{t}_{\text{drift}} = \frac{\tilde{d}}{\tilde{v}_{\text{drift}}} = \frac{{\color{blue}h(\vec{r}_0)} + {\color{blue}r(\vec{r}_0)}}{{\color{green}\mu\left(E_{\text{eff}}, p, T\right)} \cdot {\color{red}E_{\text{eff}}(\vec{r}_0)}} \tag{1.4}$$

$$\varepsilon^{\text{geo}}_{\text{coll}} = 2 - \frac{A_{^{214}\text{Po}}}{A_{^{218}\text{Po}}} \qquad \mathbf{^{212}Bi} \quad \mathbf{^{212}Po} \quad \mathbf{^{216}Po} \tag{1.5}$$

t_{drift}	actual drift time
$\tilde{t}_{\text{drift}}$	drift time estimation
$\tilde{d}$	drift distance estimation
$\tilde{v}_{\text{drift}}$	drift velocity estimation
μ	$^{222}\text{Rn}^{+}$ ion mobility in helium
E_{eff}	electrical drift field (vector sum)
p	pressure of the helium gas
T	temperature of the helium gas
$\vec{r}_0$	initial position of the $^{218}\text{Po}^{+}$ ion drift

Chapter 2

Construction of a Radon Emanation Chamber

2.1 Signal Formation in a Radon Emanation Chamber

2.1.1 Radon Emanation

2.1.2 Ion Drift in Electrical Fields

2.1.3 Alpha Decay Detection with a PIN Diode

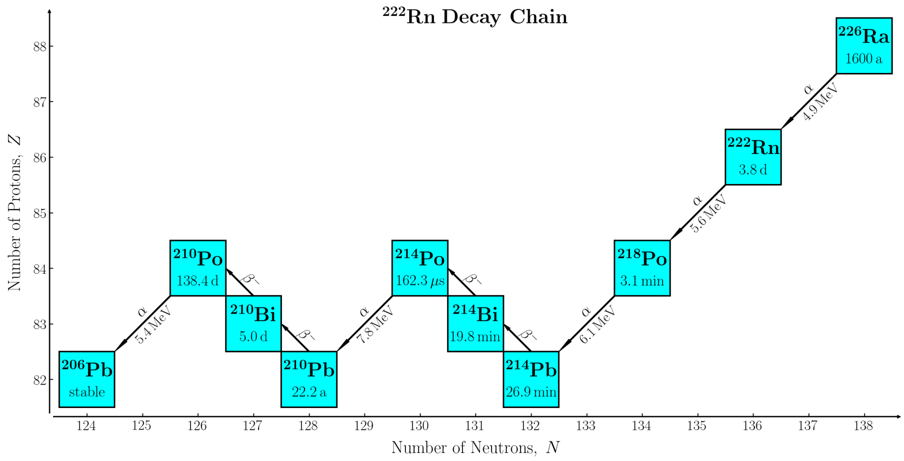

Figure 2.1: ^{222}Rn Decay Chain

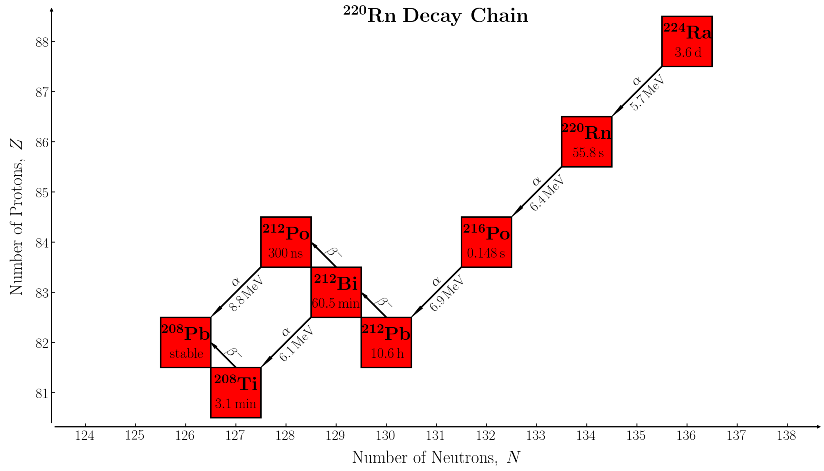

Figure 2.2: ^{220}Rn Decay Chain

2.2 Analysis

- The sebsequent decay of Radon daughters is modelled by the Bateman equations[**bateman**]:

$$\begin{aligned}\frac{\mathrm{d}N_{^{222}\mathrm{Rn}}}{\mathrm{d}t} &= -\lambda_{^{222}\mathrm{Rn}} \\ \frac{\mathrm{d}N_{^{218}\mathrm{Po}}}{\mathrm{d}t} &= -\lambda_{^{218}\mathrm{Po}} + \lambda_{^{222}\mathrm{Rn}} \\ \frac{\mathrm{d}N_{^{214}\mathrm{Pb}}}{\mathrm{d}t} &= -\lambda_{^{214}\mathrm{Pb}} + \lambda_{^{218}\mathrm{Po}} \\ \frac{\mathrm{d}N_{^{214}\mathrm{Bi}}}{\mathrm{d}t} &= -\lambda_{^{214}\mathrm{Bi}} + \lambda_{^{214}\mathrm{Pb}} \\ \frac{\mathrm{d}N_{^{214}\mathrm{Po}}}{\mathrm{d}t} &= -\lambda_{^{214}\mathrm{Po}} + \lambda_{^{214}\mathrm{Bi}}\end{aligned} \tag{2.1}$$

- Hereby a general explicit solution is given by:

$$N_n(t) = \sum_{i=1}^{n} \left\{ N_i(0) \cdot \left[\prod_{j=i}^{n-1} \lambda_j \right] \cdot \left[\sum_{j=i}^{n} \left(\frac{e^{-\lambda_j t}}{\prod_{k=i,k\neq j}^{n} (\lambda_k - \lambda_j)} \right) \right] \right\} \tag{2.2}$$

- From (2.2) the solutions for the Bateman model (2.1) are explictly derived as follows:

$$N_1(t) = e^{-\lambda_1 t}\, N_1(0) \tag{2.3}$$

$$\begin{aligned} N_2(t) = {} & e^{-\lambda_1 t} \left(+N_1(0)\frac{\lambda_1}{\lambda_2-\lambda_1} \right) \\ & + e^{-\lambda_2 t} \left(-N_1(0)\frac{\lambda_1}{\lambda_2-\lambda_1} + N_2(0) \right) \end{aligned} \tag{2.4}$$

$$\begin{aligned} N_3(t) = {} & e^{-\lambda_1 t} \left(+N_1(0)\frac{\lambda_2}{\lambda_3-\lambda_1}\frac{\lambda_1}{\lambda_2-\lambda_1} \right) \\ & + e^{-\lambda_2 t} \left(-N_1(0)\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_2-\lambda_1} + N_2(0)\frac{\lambda_2}{\lambda_3-\lambda_2} \right) \\ & + e^{-\lambda_3 t} \left(+N_1(0)\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_3-\lambda_1} - N_2(0)\frac{\lambda_2}{\lambda_3-\lambda_2} + N_3(0) \right) \end{aligned} \tag{2.5}$$

$$\begin{aligned} N_4(t) = {} & e^{-\lambda_1 t} \left(+N_1(0)\frac{\lambda_3}{\lambda_4-\lambda_1}\frac{\lambda_2}{\lambda_3-\lambda_1}\frac{\lambda_1}{\lambda_2-\lambda_1} \right) \\ & + e^{-\lambda_2 t} \left(-N_1(0)\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_2-\lambda_1} + N_2(0)\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2} \right) \\ & + e^{-\lambda_3 t} \left(+N_1(0)\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_3-\lambda_1} - N_2(0)\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2} + N_3(0)\frac{\lambda_3}{\lambda_4-\lambda_3} \right) \\ & + e^{-\lambda_4 t} \left(-N_1(0)\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda_1}{\lambda_4-\lambda_1} + N_2(0)\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2} - N_3(0)\frac{\lambda_3}{\lambda_4-\lambda_3} + N_4(0) \right) \end{aligned} \tag{2.6}$$

$$\begin{aligned} N_5(t) = {} & e^{-\lambda_1 t} \left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_1}\frac{\lambda_3}{\lambda_4-\lambda_1}\frac{\lambda_2}{\lambda_3-\lambda_1}\frac{\lambda_1}{\lambda_2-\lambda_1} \right) \\ & + e^{-\lambda_2 t} \left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_2-\lambda_1} + N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2} \right) \\ & + e^{-\lambda_3 t} \left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_3-\lambda_1} - N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2} + N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3} \right) \\ & + e^{-\lambda_4 t} \left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda_1}{\lambda_4-\lambda_1} + N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2} - N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3} + N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4} \right) \\ & + e^{-\lambda_5 t} \left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\frac{\lambda_1}{\lambda_5-\lambda_1} - N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2} + N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3} - N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4} + N_5(0) \right) \end{aligned} \tag{2.7}$$

- Accordingly from (2.3), (2.4), (2.5), (2.6) and (2.7) one can derive the corresponding activities:

$$A_1(t) = \lambda_1 \, N_1(t) \tag{2.8}$$

$$A_2(t) = \lambda_2 \, N_2(t) \tag{2.9}$$

$$A_3(t) = \lambda_3 \, N_3(t) \tag{2.10}$$

$$A_4(t) = \lambda_4 \, N_4(t) \tag{2.11}$$

$$A_5(t) = \lambda_5 \, N_5(t) \tag{2.12}$$

- One can calculate:

$$
\begin{aligned}
N_2^{\mathrm{dec}} &= \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} A_2(t)\,\mathrm{d}t \\
&= \lambda_2 \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} N_2(t)\,\mathrm{d}t \\
&= \lambda_2 \left[\left(+N_1(0)\frac{\lambda_1}{\lambda_2-\lambda_1} \right) \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} e^{-\lambda_1 t}\,\mathrm{d}t \right. \\
&\qquad \left. + \left(-N_1(0)\frac{\lambda_1}{\lambda_2-\lambda_1} + N_2(0) \right) \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} e^{-\lambda_2 t}\,\mathrm{d}t \right] \\
&= N_1(0) \left[\frac{\lambda_2}{\lambda_2-\lambda_1} \left(e^{-\lambda_1 t^i_{\mathrm{meas}}} - e^{-\lambda_1 t^f_{\mathrm{meas}}} \right) \right. \\
&\qquad \left. - \frac{\lambda_1}{\lambda_2-\lambda_1} \left(e^{-\lambda_2 t^i_{\mathrm{meas}}} - e^{-\lambda_2 t^f_{\mathrm{meas}}} \right) \right] \\
&\qquad + N_2(0) \left(e^{-\lambda_2 t^i_{\mathrm{meas}}} - e^{-\lambda_2 t^f_{\mathrm{meas}}} \right)
\end{aligned} \tag{2.13}
$$

$$
\Rightarrow \quad N_1(0) = \frac{N_2^{\mathrm{dec}} - N_2(0)\left(e^{-\lambda_2 t^i_{\mathrm{meas}}} - e^{-\lambda_2 t^f_{\mathrm{meas}}} \right)}{\frac{\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t^i_{\mathrm{meas}}} - e^{-\lambda_1 t^f_{\mathrm{meas}}} \right) - \frac{\lambda_1}{\lambda_2-\lambda_1}\left(e^{-\lambda_2 t^i_{\mathrm{meas}}} - e^{-\lambda_2 t^f_{\mathrm{meas}}} \right)} \tag{2.14}
$$

$$
\begin{aligned}
N_5^{\mathrm{dec}} &= \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} A_5(t)\,\mathrm{d}t \\
&= \lambda_5 \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} N_5(t)\,\mathrm{d}t \\
&= \lambda_5 \left[\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_1}\frac{\lambda_3}{\lambda_4-\lambda_1}\frac{\lambda_2}{\lambda_3-\lambda_1}\frac{\lambda_1}{\lambda_2-\lambda_1} \right) \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} e^{-\lambda_1 t}\,\mathrm{d}t \right. \\
&\qquad + \left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_2-\lambda_1} + N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2} \right) \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} e^{-\lambda_2 t}\,\mathrm{d}t \\
&\qquad + \left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_3-\lambda_1} - N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2} + N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3} \right) \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} e^{-\lambda_3 t}\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda_1}{\lambda_4-\lambda_1}+N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}-N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}+N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\right)\int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} e^{-\lambda_4 t}\,\mathrm{d}t \\
&+\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\frac{\lambda_1}{\lambda_5-\lambda_1}-N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}+N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}-N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}+N_5(0)\right)\int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} e^{-\lambda_5 t}\,\mathrm{d}t\Bigg] \\
=\lambda_5\Bigg[&\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_1}\frac{\lambda_3}{\lambda_4-\lambda_1}\frac{\lambda_2}{\lambda_3-\lambda_1}\frac{\lambda_1}{\lambda_2-\lambda_1}\right)\frac{1}{\lambda_1}\left(e^{-\lambda_1 t^i_{\mathrm{meas}}}-e^{-\lambda_1 t^f_{\mathrm{meas}}}\right) \\
&+\left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_2-\lambda_1}+N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\right)\frac{1}{\lambda_2}\left(e^{-\lambda_2 t^i_{\mathrm{meas}}}-e^{-\lambda_2 t^f_{\mathrm{meas}}}\right) \\
&+\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_3-\lambda_1}-N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}+N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\right)\frac{1}{\lambda_3}\left(e^{-\lambda_3 t^i_{\mathrm{meas}}}-e^{-\lambda_3 t^f_{\mathrm{meas}}}\right) \\
&+\left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda_1}{\lambda_4-\lambda_1}+N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}-N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}+N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\right)\frac{1}{\lambda_4}\left(e^{-\lambda_4 t^i_{\mathrm{meas}}}-e^{-\lambda_4 t^f_{\mathrm{meas}}}\right) \\
&+\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\frac{\lambda_1}{\lambda_5-\lambda_1}-N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}+N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}-N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}+N_5(0)\right)\frac{1}{\lambda_5}\left(e^{-\lambda_5 t^i_{\mathrm{meas}}}-e^{-\lambda_5 t^f_{\mathrm{meas}}}\right)\Bigg] \\
=&\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_1}\frac{\lambda_3}{\lambda_4-\lambda_1}\frac{\lambda_2}{\lambda_3-\lambda_1}\frac{\lambda_1}{\lambda_2-\lambda_1}\right)\frac{\lambda_5}{\lambda_1}\left(e^{-\lambda_1 t^i_{\mathrm{meas}}}-e^{-\lambda_1 t^f_{\mathrm{meas}}}\right) \\
&+\left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_2-\lambda_1}+N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\right)\frac{\lambda_5}{\lambda_2}\left(e^{-\lambda_2 t^i_{\mathrm{meas}}}-e^{-\lambda_2 t^f_{\mathrm{meas}}}\right) \\
&+\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_3-\lambda_1}-N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}+N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\mathrm{meas}}}-e^{-\lambda_3 t^f_{\mathrm{meas}}}\right) \\
&+\left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda_1}{\lambda_4-\lambda_1}+N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}-N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}+N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\mathrm{meas}}}-e^{-\lambda_4 t^f_{\mathrm{meas}}}\right) \\
&+\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\frac{\lambda_1}{\lambda_5-\lambda_1}-N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}+N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}-N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}+N_5(0)\right)\left(e^{-\lambda_5 t^i_{\mathrm{meas}}}-e^{-\lambda_5 t^f_{\mathrm{meas}}}\right) \\
=&\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_1}\frac{\lambda_3}{\lambda_4-\lambda_1}\frac{\lambda_2}{\lambda_3-\lambda_1}\frac{\lambda_1}{\lambda_2-\lambda_1}\right)\frac{\lambda_5}{\lambda_1}\left(e^{-\lambda_1 t^i_{\mathrm{meas}}}-e^{-\lambda_1 t^f_{\mathrm{meas}}}\right) \\
&+\left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_2-\lambda_1}\right)\frac{\lambda_5}{\lambda_2}\left(e^{-\lambda_2 t^i_{\mathrm{meas}}}-e^{-\lambda_2 t^f_{\mathrm{meas}}}\right) \\
&+\left(+N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\right)\frac{\lambda_5}{\lambda_2}\left(e^{-\lambda_2 t^i_{\mathrm{meas}}}-e^{-\lambda_2 t^f_{\mathrm{meas}}}\right) \\
&+\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_3-\lambda_1}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\mathrm{meas}}}-e^{-\lambda_3 t^f_{\mathrm{meas}}}\right) \\
&+\left(-N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\mathrm{meas}}}-e^{-\lambda_3 t^f_{\mathrm{meas}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(+N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\mathrm{meas}}}-e^{-\lambda_3 t^f_{\mathrm{meas}}}\right)\\
&+\left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda_1}{\lambda_4-\lambda_1}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\mathrm{meas}}}-e^{-\lambda_4 t^f_{\mathrm{meas}}}\right)\\
&+\left(+N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\mathrm{meas}}}-e^{-\lambda_4 t^f_{\mathrm{meas}}}\right)\\
&+\left(-N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\mathrm{meas}}}-e^{-\lambda_4 t^f_{\mathrm{meas}}}\right)\\
&+\left(+N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\mathrm{meas}}}-e^{-\lambda_4 t^f_{\mathrm{meas}}}\right)\\
&+\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\frac{\lambda_1}{\lambda_5-\lambda_1}\right)\left(e^{-\lambda_5 t^i_{\mathrm{meas}}}-e^{-\lambda_5 t^f_{\mathrm{meas}}}\right)\\
&+\left(-N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\right)\left(e^{-\lambda_5 t^i_{\mathrm{meas}}}-e^{-\lambda_5 t^f_{\mathrm{meas}}}\right)\\
&+\left(+N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\right)\left(e^{-\lambda_5 t^i_{\mathrm{meas}}}-e^{-\lambda_5 t^f_{\mathrm{meas}}}\right)\\
&+\left(-N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\right)\left(e^{-\lambda_5 t^i_{\mathrm{meas}}}-e^{-\lambda_5 t^f_{\mathrm{meas}}}\right)\\
&+\left(+N_5(0)\right)\left(e^{-\lambda_5 t^i_{\mathrm{meas}}}-e^{-\lambda_5 t^f_{\mathrm{meas}}}\right)\\
=&\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_1}\frac{\lambda_3}{\lambda_4-\lambda_1}\frac{\lambda_2}{\lambda_3-\lambda_1}\frac{\lambda_1}{\lambda_2-\lambda_1}\right)\frac{\lambda_5}{\lambda_1}\left(e^{-\lambda_1 t^i_{\mathrm{meas}}}-e^{-\lambda_1 t^f_{\mathrm{meas}}}\right)\\
&+\left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_2-\lambda_1}\right)\frac{\lambda_5}{\lambda_2}\left(e^{-\lambda_2 t^i_{\mathrm{meas}}}-e^{-\lambda_2 t^f_{\mathrm{meas}}}\right)\\
&+\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_3-\lambda_1}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\mathrm{meas}}}-e^{-\lambda_3 t^f_{\mathrm{meas}}}\right)\\
&+\left(-N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda_1}{\lambda_4-\lambda_1}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\mathrm{meas}}}-e^{-\lambda_4 t^f_{\mathrm{meas}}}\right)\\
&+\left(+N_1(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\frac{\lambda_1}{\lambda_5-\lambda_1}\right)\left(e^{-\lambda_5 t^i_{\mathrm{meas}}}-e^{-\lambda_5 t^f_{\mathrm{meas}}}\right)\\
&+\left(+N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\right)\frac{\lambda_5}{\lambda_2}\left(e^{-\lambda_2 t^i_{\mathrm{meas}}}-e^{-\lambda_2 t^f_{\mathrm{meas}}}\right)\\
&+\left(-N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\mathrm{meas}}}-e^{-\lambda_3 t^f_{\mathrm{meas}}}\right)\\
&+\left(+N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\mathrm{meas}}}-e^{-\lambda_4 t^f_{\mathrm{meas}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-N_2(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\right)\left(e^{-\lambda_5 t^i_{\text{meas}}}-e^{-\lambda_5 t^f_{\text{meas}}}\right)\\
&+\left(+N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\text{meas}}}-e^{-\lambda_3 t^f_{\text{meas}}}\right)\\
&+\left(-N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\text{meas}}}-e^{-\lambda_4 t^f_{\text{meas}}}\right)\\
&+\left(+N_3(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\right)\left(e^{-\lambda_5 t^i_{\text{meas}}}-e^{-\lambda_5 t^f_{\text{meas}}}\right)\\
&+\left(+N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\text{meas}}}-e^{-\lambda_4 t^f_{\text{meas}}}\right)\\
&+\left(-N_4(0)\frac{\lambda_4}{\lambda_5-\lambda_4}\right)\left(e^{-\lambda_5 t^i_{\text{meas}}}-e^{-\lambda_5 t^f_{\text{meas}}}\right)\\
&+\left(+N_5(0)\right)\left(e^{-\lambda_5 t^i_{\text{meas}}}-e^{-\lambda_5 t^f_{\text{meas}}}\right)\\
=\ & N_1(0)\Bigg[\left(+\frac{\lambda_4}{\lambda_5-\lambda_1}\frac{\lambda_3}{\lambda_4-\lambda_1}\frac{\lambda_2}{\lambda_3-\lambda_1}\frac{\lambda_1}{\lambda_2-\lambda_1}\right)\frac{\lambda_5}{\lambda_1}\left(e^{-\lambda_1 t^i_{\text{meas}}}-e^{-\lambda_1 t^f_{\text{meas}}}\right)\\
&+\left(-\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_2-\lambda_1}\right)\frac{\lambda_5}{\lambda_2}\left(e^{-\lambda_2 t^i_{\text{meas}}}-e^{-\lambda_2 t^f_{\text{meas}}}\right)\\
&+\left(+\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\frac{\lambda_1}{\lambda_3-\lambda_1}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\text{meas}}}-e^{-\lambda_3 t^f_{\text{meas}}}\right)\\
&+\left(-\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda_1}{\lambda_4-\lambda_1}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\text{meas}}}-e^{-\lambda_4 t^f_{\text{meas}}}\right)\\
&+\left(+\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\frac{\lambda_1}{\lambda_5-\lambda_1}\right)\left(e^{-\lambda_5 t^i_{\text{meas}}}-e^{-\lambda_5 t^f_{\text{meas}}}\right)\Bigg]\\
&+N_2(0)\Bigg[\left(+\frac{\lambda_4}{\lambda_5-\lambda_2}\frac{\lambda_3}{\lambda_4-\lambda_2}\frac{\lambda_2}{\lambda_3-\lambda_2}\right)\frac{\lambda_5}{\lambda_2}\left(e^{-\lambda_2 t^i_{\text{meas}}}-e^{-\lambda_2 t^f_{\text{meas}}}\right)\\
&+\left(-\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_3-\lambda_2}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\text{meas}}}-e^{-\lambda_3 t^f_{\text{meas}}}\right)\\
&+\left(+\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\frac{\lambda_2}{\lambda_4-\lambda_2}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\text{meas}}}-e^{-\lambda_4 t^f_{\text{meas}}}\right)\\
&+\left(-\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\frac{\lambda_2}{\lambda_5-\lambda_2}\right)\left(e^{-\lambda_5 t^i_{\text{meas}}}-e^{-\lambda_5 t^f_{\text{meas}}}\right)\Bigg]\\
&+N_3(0)\Bigg[\left(+\frac{\lambda_4}{\lambda_5-\lambda_3}\frac{\lambda_3}{\lambda_4-\lambda_3}\right)\frac{\lambda_5}{\lambda_3}\left(e^{-\lambda_3 t^i_{\text{meas}}}-e^{-\lambda_3 t^f_{\text{meas}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_4-\lambda_3}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\text{meas}}}-e^{-\lambda_4 t^f_{\text{meas}}}\right)\\
&+\left(+\frac{\lambda_4}{\lambda_5-\lambda_4}\frac{\lambda_3}{\lambda_5-\lambda_3}\right)\left(e^{-\lambda_5 t^i_{\text{meas}}}-e^{-\lambda_5 t^f_{\text{meas}}}\right)\Bigg]\\
+N_4(0)&\left[\left(+\frac{\lambda_4}{\lambda_5-\lambda_4}\right)\frac{\lambda_5}{\lambda_4}\left(e^{-\lambda_4 t^i_{\text{meas}}}-e^{-\lambda_4 t^f_{\text{meas}}}\right)+\left(-\frac{\lambda_4}{\lambda_5-\lambda_4}\right)\left(e^{-\lambda_5 t^i_{\text{meas}}}-e^{-\lambda_5 t^f_{\text{meas}}}\right)\right]\\
+N_5(0)&\left(e^{-\lambda_5 t^i_{\text{meas}}}-e^{-\lambda_5 t^f_{\text{meas}}}\right)
\end{aligned}
\tag{2.15}
$$

$$
\Rightarrow \quad N_1(0)=\frac{N_5^{\text{dec}}-\sum_{i=2}^{5}\left[N_i(0)\sum_{j=i}^{5}\left[\left(e^{-\lambda_j t^{\text{i}}_{\text{meas}}}-e^{-\lambda_j t^{\text{f}}_{\text{meas}}}\right)(-1)^j\,\frac{\lambda_5}{\lambda_j}\prod_{k=i+1}^{5}\left[\frac{\lambda_{k-1}}{\lambda_{k+\theta(i-k-1)\cdot(i-k)}-\lambda_{(k-1)-\theta(k-i-2)(k-i-1)}}\right]\right]\right]}{\sum_{j=1}^{5}\left[\left(e^{-\lambda_j t^{\text{i}}_{\text{meas}}}-e^{-\lambda_i t^{\text{f}}_{\text{meas}}}\right)(-1)^{j+1}\,\frac{\lambda_5}{\lambda_j}\prod_{k=2}^{5}\left[\frac{\lambda_{k-1}}{\lambda_{k+\theta(j-k-1)\cdot(j-k)}-\lambda_{(k-1)-\theta(k-j-2)(k-j-1)}}\right]\right]}
\tag{2.16}
$$

$$
\tag{2.17}
$$

with

$$
\theta:\mathbb{Z}\to\{0,1\},\quad z\mapsto\begin{cases}0 & \text{, for } z<0\\ 1 & \text{for } z\geq 0\end{cases}
\tag{2.18}
$$

the discrete Heaviside step function.

-

$$
\begin{aligned}
N_2^{\mathrm{dec}} + N_2^{\mathrm{dec}} &= \int_{t_{\mathrm{meas}}^{\mathrm{i}}}^{t_{\mathrm{meas}}^{\mathrm{f}}} A_2(t) + A_5(t)\,\mathrm{d}t \\
&= \int_{t_{\mathrm{meas}}^{\mathrm{i}}}^{t_{\mathrm{meas}}^{\mathrm{f}}} A_2(t)\,\mathrm{d}t + \int_{t_{\mathrm{meas}}^{\mathrm{i}}}^{t_{\mathrm{meas}}^{\mathrm{f}}} A_5(t)\,\mathrm{d}t
\end{aligned}
$$

- From the N_{meas} recorded counts during the measurement interval $\left[t_{\mathrm{meas}}^i,\, t_{\mathrm{meas}}^f\right]$ the activity $A_{t_{\mathrm{meas}}^i}$ at time t_{meas}^i can be computed:

$$
A_{t_{\mathrm{meas}}^i} = \frac{1}{\varepsilon_{\mathrm{fill}} \cdot \varepsilon_{\mathrm{yield}}} \left(N - N_{\mathrm{bg}} \cdot t_{\mathrm{data}}\right) \cdot \lambda_{^{222}\mathrm{Rn}} \tag{2.19}
$$

$A_{t_{\mathrm{meas}}^i}$	activity at time t_{meas}^i
N_{meas}	number of recorded events
R_{bkg}	background rate
t_{meas}	$= t_{\mathrm{meas}}^f - t_{\mathrm{meas}}^i$, lifetime of the measurement run
$\lambda_{^{222}\mathrm{Rn}}$	$= 3.8232(8)\,\mathrm{d}$, decay rate of $^{222}\mathrm{Rn}$
$\varepsilon_{\mathrm{coll}}$	$= X.X\,\%$, collection efficiency

- Consider a long-lived radioactive Isotope of type A of approximately constant activity A_A with a comparatively short-lived daughter of type B. In secular equilibrium the number of type B isotopes is constant in time as they decay with the same rate as they are produced via the decay of the type A progenitors. Hence the equilibrium activity A_B^{eq} can be calculated from the initial activity of the mother isotope A_A:

$$
\begin{aligned}
\frac{\mathrm{d}N_B}{\mathrm{d}t} &= \lambda_A\, N_A - \lambda_B\, N_B \overset{!}{=} 0 \\
\Rightarrow \quad A_B^{\mathrm{eq}} &= \lambda_B\, N_B^{\mathrm{eq}} = \lambda_A\, N_A = A_A
\end{aligned} \tag{2.20}
$$

Now consider a $^{226}\mathrm{Ra}$ contaminated sample emanating radon with an emanation rate of R_{em}. Assuming again a secular equilibrium one can calculate the composition of the measured radon activity:

$$
\frac{\mathrm{d}N_{^{222}\mathrm{Rn}}}{\mathrm{d}t} = R_{\mathrm{em}} + R_{\mathrm{em}}^{\mathrm{bkg}} - \lambda_{^{222}\mathrm{Rn}}\, N_{^{222}\mathrm{Rn}} = 0 \tag{2.21}
$$

$$
\Rightarrow \quad A_{^{222}\mathrm{Rn}}^{\mathrm{eq}} = R_{\mathrm{em}} + R_{\mathrm{em}}^{\mathrm{bkg}} \tag{2.22}
$$

Hereby the background emanation rate $R_{\mathrm{em}}^{\mathrm{bkg}}$ can be infered directly from a background activity measurement utilizing (2.20).

- The number of decays N_{cts} that are recorded during the measurement interval $\left[t^i_{\mathrm{meas}},\, t^f_{\mathrm{meas}}\right]$ can be calculated by computing the following integral:

$$
\begin{aligned}
N_{\mathrm{cts}} &= \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} A(t)\,\mathrm{d}t \\
&= \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} \lambda_{^{222}\mathrm{Rn}}\, N(t)\,\mathrm{d}t \\
&= \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} \lambda_{^{222}\mathrm{Rn}}\, N(t^i_{\mathrm{meas}})\, e^{-\lambda_{^{222}\mathrm{Rn}} t}\,\mathrm{d}t \\
&= \lambda_{^{222}\mathrm{Rn}}\, N(t^i_{\mathrm{meas}}) \int_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} e^{-\lambda_{^{222}\mathrm{Rn}} t}\,\mathrm{d}t \\
&= \lambda_{^{222}\mathrm{Rn}}\, N(t^i_{\mathrm{meas}}) \left[-\frac{1}{\lambda_{^{222}\mathrm{Rn}}}\, e^{-\lambda_{^{222}\mathrm{Rn}} t}\right]_{t^i_{\mathrm{meas}}}^{t^f_{\mathrm{meas}}} \\
&= -N(t^i_{\mathrm{meas}}) \left(e^{-\lambda_{^{222}\mathrm{Rn}} t^f_{\mathrm{meas}}} - e^{-\lambda_{^{222}\mathrm{Rn}} t^i_{\mathrm{meas}}}\right) \qquad (2.23)
\end{aligned}
$$

$$
\Rightarrow \quad N(t^i_{\mathrm{meas}}) = \frac{N_{\mathrm{cts}}}{e^{-\lambda_{^{222}\mathrm{Rn}} t^i_{\mathrm{meas}}} - e^{-\lambda_{^{222}\mathrm{Rn}} t^f_{\mathrm{meas}}}} \qquad (2.24)
$$

With (2.24) one can then express the initial activity $A(t^i_{\mathrm{meas}})$ at time t^i_{meas} as a function of the measured counts N_{cts}:

$$
A(t^i_{\mathrm{meas}}) = \lambda_{^{222}\mathrm{Rn}}\, N(t^i_{\mathrm{meas}}) = \lambda_{^{222}\mathrm{Rn}}\, \frac{N_{\mathrm{cts}}}{e^{-\lambda_{^{222}\mathrm{Rn}} t^i_{\mathrm{meas}}} - e^{-\lambda_{^{222}\mathrm{Rn}} t^f_{\mathrm{meas}}}} \qquad (2.25)
$$

- By extrapolating $A(t^i_{\mathrm{meas}})$ back to time t_{proc} and correcting it for additional emanation contributions of the measurement setup Δ one can compute the activity $A(t_{\mathrm{proc}})$ of the gas sample at time t_{proc} when the radon enriched gas sample from the emanation vessel is processed further on is given by

$$
\begin{aligned}
A(t_{\mathrm{proc}}) &= \tilde{A}(t_{\mathrm{proc}}) - \Delta \\
&= A(t_{\mathrm{meas}}) \cdot e^{+\lambda_{^{222}\mathrm{Rn}} \cdot (t_{\mathrm{meas}} - t_{\mathrm{proc}})} \qquad (2.26)
\end{aligned}
$$

whereas

$$
\begin{aligned}
\Delta &= \Delta_1 k_1 + \Delta_2 k_1 k_2 + \Delta_3 k_1 k_2 k_3 + \Delta_4 k_1 k_2 k_3 k_4 + ... \\
&= \sum_{i=1}^{n} \left(\Delta_i \cdot \prod_{j=1}^{i} k_j\right)
\end{aligned}
$$

$$= \sum_{i=1}^{n} \left(\Delta_i \cdot \prod_{j=1}^{i} e^{\left(\lambda_{222\mathrm{Rn}} \cdot \left(t'_j - t_j\right)\right)} \right) \tag{2.27}$$

corresponds to the accumulated contribution of n additional radon emanation sources. Given the saturation emanation activities of the emanation containers B_i, one can further calculate their contribution

$$\Delta_i = B_i \cdot \left(1 - e^{-\lambda_{222\mathrm{Rn}} \cdot \left(t'_i - t_i\right)}\right) \ . \tag{2.28}$$

- Once computed, $A(t_{\mathrm{proc}})$ can be extrapolated

$$A_{\mathrm{Rn}} = A(t_{\mathrm{proc}}) \cdot e^{(\cdot (t_{\mathrm{proc}} - t_0))} \tag{2.29}$$

$$A_{\mathrm{em}} = A(t_{\mathrm{proc}}) \cdot (1 -) \tag{2.30}$$

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