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The faculty name

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Table of Contents

List of Figures

List of Tables

1

Advanced Financial Econometrics Exam s22o

1.1 Time-varying parameter models

Consider the Stochastic Poisson Autoregressive (SPA) model:

$$Y_t|\delta_t \sim Pois(\delta_t),$$

such that $p(Y_t = y_t|\delta_t) = \frac{\delta_t^{y_t}}{y_t!}e^{-\delta_t}$, where $\delta_t = \omega exp(\mu_t)$, with $0 < \omega < \infty$ and:

$$\mu_{t+1} = \phi\mu_t + \eta\epsilon_t,$$

with $0 \leq \eta < \infty$ and $\epsilon_t \overset{iid}{\sim} N(0,1)$, and $\epsilon_t \perp Y_t$ (to read: ϵ_t is independent from Y_t).

1.1.1 1. Derive the condition that ensures the weak stationarity of $(Y_t)_{t\in Z}$

.

1.1.2 2. Compute $E[Y_{t+h}|\delta_t]$

.

1.1.3 3. Derive an approximated likelihood function for the SPA model using a structural Hidden Markow model.

1.1.4 4. Write a code to simulate from the SPA model.

1.1.5 5. Write a code to estimate the SPA model by maximizing the approximated likelihood function you derived in point 3). Impose constraints that ensure weak stationarity of $(Y_t)_{t\in Z}$

1.1.6 6. Simulate T = 500 observations from the SPA model with $\omega = 1$,$\phi = 0.95$, $\eta = 0.15$. Estimate the SPA model on the simulated series using the true parameters as the starting values for the maximization.

Consider now the Stochastic Poisson Mixtue Autoregressive (SPMA) model, which is defined as the SPA but with:

$$\mu_{t+1} = \begin{cases} \phi_1\mu_t + \eta_1\epsilon_{1,t}, & \text{if } Z_t = 0 \\ \phi_2\mu_t + \eta_2\epsilon_{2,t}, & \text{if } Z_t = 1 \end{cases}$$

with $0 \leq \eta_i < \infty$ and $\eta_{i,t} \overset{iid}{\in} N(0,1)$ for $i = 1,2$ with $\eta_{1,t} \perp \epsilon_{2,s}$ for all t,s. The random variable Z_t is assumed independently Bernoulli distributed with success rate $\alpha \in (0,1)$, $Z_t \overset{iid}{\in} Ber(\alpha)$, and $Y_t \perp Z_t$, $Z_t \perp \epsilon_{i,s}$ fo all i,s.

1.1.7 7. Derive Condittions that ensure the weak stationarity of $(\mu_t)_{t\in Z}$.

1.1.8 8. Compute $E[\mu_{t+k}|\mu_t]$.

1.1.9 9. Derive an approximated likelihood function for the SPMA model using a structured Hidden Markov Model.

1.1.10 10. Write a code to simulate from the SPMA model.

1.1.11 11. Write a code to estimate the SPMA model by maximizing the approximated likelihood function you derived in point 9), Impose constraints that ensure weak stationarity of $(Y_t)_{t\in Z}$.

1.1.12 12. Simulate T=500 observations from the SPMA model with $\omega = 1$, $\phi_1 = 0.95$, $\phi_2 = 0.99$, $\eta_1 = 0.15$, $\eta_2 = 0.2$. Estimate the SPMA model on the simulated series using the true parameters as the starting values for the maximization.

Let Y_t be the number of trades at time t, for $i = 1, 2$. The file NumberTrades.txt contains the number of trades at the 30-second frequency for American Express (AXP, first column) and Bank of America (BA, second column) for 21 January 2005.

1.1.13 13. Compute descriptive statistics of the two series, and comment on your results.

1.1.14 14. For each series, estimate the SPA and SPMA models. Which specification is selected according to the BIC criteria (use the value of the approximated likelihood at its optimum to calculate the BIC)?

1.1.15 15. For each series, in a 2 x 1 figure plot the smoothed intensity $\hat{\lambda}_{t|T} = E[\lambda_t|Y_{1:T}]$ for $t = 1, ..., T$, estimated using the SPA and SPMA models. Comment on your results.

1.2 Dynamic models of the term structure

Consider the article by Christensen, Posch and van der Wel (Journal of Econometrics, 2016).
The Euler equation is

$$\frac{du_C}{u_C} = (\rho-(r_t-\delta))dt - \frac{u_{CC}(C_t, A_t)}{u_C(C_t, A_t)}C_K\sigma^2K_tdt + \frac{u_{CC}(C_t, A_t)}{u_C(C_t, A_t)}C_{A\eta}(A_t)dB_t + \frac{u_{CA}(C_t, A_t)}{u_C(C_t, A_t)}\eta(A_t)dB_t + \frac{u_{CC}(C_t, A_t)}{u_C(C_t, A_t)}C_K\sigma K_tdZ_t.$$

1.2.1 (a) Explain the meaning of the parameters, variables, and functions in the Euler equation.

1.2.2 (b) Use the Euler equation (1) to derive the relation $\underbrace{\rho - \frac{1}{dt}E_t[\frac{du_C}{u_C}]}_{\text{cost of forgone consumption}} = \underbrace{r_t - \delta + \frac{u_{CC}(C_t, A_t)}{u_C(C_t, A_t)}C_K\sigma^2K_t}_{\text{certainty equivalent rate of return}}$. Explain and interpret the equation.

1.2.3 (c) What is the relation between (2) and the risk-free rate r_t^f? Explain.

1.2.4 (d) Use the Euler equation (1) and Itô lemma to derive the consumption path

$$dC_t = \frac{u'(C_t)}{u''(C_t)}(\rho - (r_t - \delta))dt - \sigma^2C_KK_tdt - \frac{1}{2}(C_A^2\eta(A_t)^2 + C_K^2\sigma^2K_t^2)\frac{u'''(C_t)}{u''(C_t)}d_t + C_{A\eta}(A_t)dB_t + C_K\sigma K_tdZ_t$$

. Explain the steps involved.

1.2.5 (e) Now consider the special case of the AK-Vasicek model with loggarithmic preferences. The equilibrium dynamics are given by

$$dlnC_t = (r_t - \rho - \delta - \frac{1}{2}\sigma^2)dt + \sigma dZ_t \tag{4a}$$

$$dlnY_t = (\frac{\kappa\gamma}{r_t} - \frac{1}{2}\eta^2/r_t^2 + r_t - \kappa - \rho - \delta - \frac{1}{2}\delta^2)dt + \frac{\eta}{r_t dB_t} + \sigma dZ_t \tag{4b}$$

$$dr_t = \kappa(\gamma - r_t)dt + \eta dB_t. \tag{4c}$$

Use (4) with $s - t = \Delta$ derive

$$ln(\frac{C_t}{C_{t-\Delta}}) - \int_{t-\Delta}^{t} r_\nu^f d\nu = -(\rho - \frac{1}{2}\sigma^2)\Delta + \epsilon_{C,t}, \tag{5.a}$$

$$ln(\frac{Y_t}{Y_{t-\Delta}}) - \int_{t-\Delta}^{t} r_\nu^f d\nu = \kappa\gamma \int_{t-\Delta}^{t} \frac{1}{(r_\nu^f + \delta + \sigma^2)dv} - \frac{1}{2}\eta^2 \int_{t-\Delta}^{t} \frac{1}{(r_\nu^f + \delta + \sigma^2)^2 dv} - (\kappa + \rho - \frac{1}{2}\sigma^2)\Delta + \epsilon_{Y,t} \tag{5b}$$

Explain the steps.

1.2.6 (f) How are the integrals in (5) implemented in the estimation procedures?

1.2.7 (g) Explain the GMM and MEF estimation procedures. Do a simulation study to compare the efficiency of the two procedures. Show and explain your work.

1.3 High Frequency econometrics

1.3.1 (a) In the first two parts, we study the covariance. Suppose that the log prices of two assets are given by

$$X_{1,t} = x_1 + \mu_1 t + \sigma_{11} W_{t,1} + \sigma_{12} W_{t,2},$$

$$X_{2,t} = x_2 + \mu_2 t + \sigma_{21} W_{t,1} + \sigma_{22} W_{t,2}$$

for $t \in [0, 1]$. Here x_i, μ_i, σ_{ij}, $i, j = 1, 2$, are constants, and $(W_{t,1})_{t\geq 0}$ and $(W_{t,2})_{t\geq 0}$ are mutually independent standard Brownian motions. Let's define

$$C_n = \alpha_n \sigma_{i=1}^{n} (X_{1,\frac{i}{n}} - X_{1,\frac{i-1}{n}})(X_{2,\frac{i}{n}} - X_{2,\frac{i-1}{n}}).$$

Try to find a suitable normalizing factor α_n (that depend on n) and finite non-zero real number C (that may depend on above constants, such that

$$\lim_{n\to\infty} E[C_n] = C$$

Is this possible or impossible? Justify your answer.

1.3.2 (b) Continuing from part (a), define $S_n = \sqrt{n}(C_n - C)$. Find out (with justification)

$$\lim_{n\to\infty} E[S_n^2]$$

[If this limit is S^2, then a possible conjecture is that the asymptotic limit is $N(0, S^2)$.] If your answer from part (a) is "impossible", perform this analysis with some reasonable choices C and α_n.

1.3.3 (c) In this part, we are interested in asset prices with irregular observation times. Suppose that the log asset price follows:

$$X_t = x + \mu t + \sigma W_t \text{ for t} \in [0,1],$$

where x, μ and σ are constants, and $(W_t)_{t\geq 0}$ is a standard Brownian motion. At each stage n, we have access to observations $X_{t^n_i}, i = 0, 1, ..., n$ such that $t^n_i = f(\frac{i}{n})$. Here, $f : [0,1] \longrightarrow$ [0,1] is a strictly increasing function that satisfies $f(t) - f(s) \leq A(t-s)$ for each $0 \leq s < t \leq 1$, where A is a constant. Let's define

$$IO_n = \beta_n \sum_{i=1}^{n} (X_{t^n_i} - X_{t^n_{i-1}})^2.$$

Try to find a suitable normalizing factor β_n (that may depend on n) such that

$$\lim_{n\to\infty} E[IO_n] = \sigma^2.$$

Is this possible or impossible? Justify your answer.

1.3.4 Simulate 1000 independent replications of the model and the estimator in part (c) with parameters $\mu = 0.05$, $\sigma = 0.2$, $f(x) = x^2$, and $n = 1000$. Report the bias and the mean squared error IO_n (with respect to σ^2), and plot the vales of IQ_n and σ^2. If your answer from part (c) is "impossible", perform this analysis with some reasonable choice of β_n.

1.3.5 (e) Now, your task is to use an empirical data. AAPL2013 and MSFT2013 data contain 5-second sampled stock prices of Apple and Microsoft in the first 100 days (with 6.5 trading hours) of 2013, i.e. the dimensions are 4681x100. Using the estimator in part (a) and individual realized volatilities, compute realized correlation between two stocks for each of 100 days and by using three sampling schemes, namely 1,5, and 15 minutes. Produces three time series, plots, and report means and variances of these time series. Provide brief comments on results. If your answer from part (a) is "impossible", perform this analysis with some reasonable choices of C and α_n.

2

Advanced Financial Econometrics Exam s21o

2.1 Time-varying parameter models

Consider the following time varying volatility model:

$$Y_t|(S_t = j, F_{t-1}) \sim N(0, \sigma^2_{j,t}), \tag{1}$$

where S_t is a Categorical random variable with state space $(1, ..., J)$, and $P(S_t = j) = \omega_j$, with $\omega_j > 0$ and $\sum_{j=1}^{J} \omega_j = 1$. Assume a score driven recursion for $\sigma^2_{j,t}$ of the following form:

$$\sigma^2_{j,t+1} = \omega_j + \alpha_j u_{j,t} + \beta_j \sigma^2_{j,t}, j = 1, ..., J, \tag{2}$$

where $u_{j,t} = I^{-1}_{j,t} \nabla_{j,t}$ and

$$\nabla_{jj,t} = \frac{\partial lnp(y_t|F_{t-1}}{\partial \sigma^2_{j,t}}, I_{j,t} = E\left[\frac{\partial lnp(y_t|S_t = j, F_{t-1}}{\partial \sigma^2_{j,t}}|S_t = j, F_{t-1}\right],$$

are the score and conditional Fisher information quantity, respectively. Note that the score is taken with respect to the density of the random variable $Y_t|F_{t-1}$, while the information quantity is computed with respect to the density and distribution of the random variable $Y_t|(S_t = j, F_{t-1})$.

2.1.1 a) Derive $\mu_{j,t}$ and provide the following representation of the conditional variance recursion:

$\sigma^2_{j,t+1} = \omega_j + A_{j,t} y^2_t + B_{j,t} \sigma^2_{t,j}, j = 1, ..., J.$

i) Define $A_{j,t}$ and $B_{j,t}$ are.

ii) Derive sufficient conditions for $\omega_j, \alpha_j, \beta_j$ such that $\sigma^2_{j,t} > 0$ for all j and t.

iii) Discuss in words what you believe are the pros and cons of this specification.

iv) Compute $VAR(Y_{t+1}|F_t)$.

v) Define $\theta = (\omega', \theta'_j, j = 1, ..., J)'$, where $\omega = (\omega_1, ..., \omega_J)'$ and $\theta'_j = (\omega_j, \alpha_j, \beta_j)'$ for $j = 1, ..., J$. Write down the log likelihood of the model for a sequence of T observations.

2.1.2 b) Consider the Independent Mixture GARCH model of Haas et al. (2004) where Y_t is defined as in equation (1), but $\sigma^2_{j,t}$ is defined as:

$$\sigma^2_{j,t+1} = \omega_j + A_j y^2_t + B_j \sigma^2_{j,t}.$$

What do you believe is the main difference between the score driven model (2) and the one in (3)?

2.1.3 c) Write a function that computes the Maximum Likelihood Estimator (MLE) of the score driven model defined by the system of equations (1) and (2). Impose the following constraints in the optimization: $\omega_j \in (0,2)$, $\beta_j \in (0,1)$, $\alpha_j \in (0,1)$, $\omega_j \in (0,1)$ for $j = 1, ..., J$. Furthermore, impose that $\sum_{j=1}^{J} \omega_j = 1$, and the sufficient conditions on α_j and β_j that you derived in point a)ii) to ensure $\sigma_{j,t}^2 > 0$ for all j,t. Set $\sigma_{j,1} = \hat{\sigma}^2$, where $\hat{\sigma}^2$ is the sample variance of the time series you use to estimate the model. You can employ any optimizer. If you do not manage to write the code for the general case with J states, set J=2.

2.1.4 d) Write a function that computes the Maximum Likelihood Estimator (MLE) of the Independent Mixture GARCH model defined by the system of equations (1) and (3). Impose the following constraints in the optimization: $\omega_j \in (0,2)$, $\beta_j \in (0,1)$, $\alpha_j \in (0,1)$, $\omega_j \in (0,1)$ for $j = 1, ..., J$. Furthermore, impose that $\sum_{j=1}^{J} \omega_j = 1$, and $\alpha_j + \beta_j < 1$ for all j. You can employ any optimizer. If you do not manage to write the code for the general case with J states, set J=2.

2.1.5 e) Consider the dji30ret dataset in the rugarch package which reports the Dow Jones 30 Constituents closing value log returns from 1987-03-16 to 2009-02-03 from Yahoo Finance. Select one time series of your choice, call this Y_t^*, and multiply it by 100. Define the demeaned series $Y_t = Y_t^* - \hat{\mu}$, where $\hat{\mu}$ is the empirical mean of the series. The series $(Y_1, ..., Y_T)$ is the series you will use in the following points.

2.1.6 f) Report descriptive statistics of your choice and discuss the stylized facts of the series.

2.1.7 Estimate:

i) the score driven model defined by the system of equations (1) and (2) with J=2,

ii) the Independent Mixture GARCH model defined by the system of equations (1) and (3) with J=2

iii) a GARCH(1,1) model with Gaussian shocks: $y_t = \sigma_t \epsilon_t, \epsilon \overset{iid}{\sim} N(0,1)$, where $\sigma_{t+1}^2 = \omega + A y_t^2 + B \sigma_t^2$. You can use the rugarch package for this point.

iv) For each model, discuss the estimated parameters.

2.1.8 h) Compute the sequence of filtered variances $\sigma_t^2 = VAR(Y_t|F_{t-1})$ for $t = 1, ..., T$ (note that $\sigma_1^2 = \hat{\sigma}^2$, see points d) and e)) for the score driven and Independent Mixture GARCH. Compare in a figure the two filtered variances with that of the GARCH(1,1). Discuss your results.

References:

Haas, M., Mittnik, S., and Paolella, M.S. (2004). Mixed normal conditional heteroskedasticity. Journal of financial Econometrics, 2(2):211-250.

2.2 Dynamic models of the term structure

Consider the article by Christensen and Varneskov (2021). Investor uses foreign currencies to hedge a stock-bond portfolio in discrete time while observing interest rates, stock and bond prices, and foreign currency exchange rates at higher frequency than the rebalancing interval. Let $w_{c,t}$ be the position in the country c portfolio from t to t+1, $c = 0, 1, ..., C$, $t = 0, 1, ..., T$, with c=0 the home country (the US), $R_{c,t+1}$ the return in county c, and $S_{c,t+1}$ the spot exchange rate, measured in USD/FCU.

2.2.1 Question 1

Explain that the unhedged return on the investment in country c is

$$R^u_{c,t+1} = R_{c,t+1}\frac{S_{c,t+1}}{S_{c,t}}.$$

Let $F_{c,t}$ be the one-period forward exchange rate, and $I_{c,t}$ the interest rate in county c. Equation (1) in the article presents the hedged portfolio return as

$$R^h_{t+1} = \sum_{c=0}^{C}\omega_{c,t}R^u_{c,t+1} + \sum_{c=0}^{C}\theta_{c,t}\frac{F_{c,t}}{S_{c,t}} - \sum_{c=0}^{C}\theta_{c,t}\frac{S_{c,t+1}}{S_{c,t}}. \tag{4}$$

2.2.2 Question 2 - explain the meaning of $\theta_{c,t}$.

2.2.3 Question 3

Give an arbitrage argument that

$$\frac{F_{c,t}}{S_{c,t}} = \frac{1+I_{0,t}}{1+I_{c,t}} \tag{5}$$

.

2.2.4 Question 4 - Use logarithms to express (5) as an interest rate differential.

2.2.5 Question 5 - What are the net exposures to foreign currencies?

2.2.6 Question 6 - Give a defintion of the quadratic variation (QV) of $r^h_{t+1} = log(R_{t+1^h})$

2.2.7 Question 7 - Give a definition of the quadratic covariation (QC) of two vector-valued processes.

2.2.8 Question 8 - What are the optimal net exposures to foreign currencies? Explain how you use QV and QC in your answer.

2.2.9 Question 9 - Explain how to estimate QC in Question 8 using realized kernels. Give an equation.

2.2.10 Questions 10 - Explain how to estimate the optimal net exposures to foreign currencies in Question 8 using multivariate GARCH. Give an equation.

2.2.11 Question 11 - Explain how to optain an alternative estimate of QC by using a pre-averaging estimator instead of realized kernels. Give an equation.

2.2.12 Question 12 - How is your answer to Question 10 on estimation of optimal net exposures to foreing currencies using multivariate GARCH affected by the use of the alternative estimate of QC from Question 11 instead of the one from Question 9?

2.2.13 Question 13 - Discuss pros and cons of the alternative estimators of QC from question 9 and 11

2.2.14 Question 14 - What are the values of C and T in the empirical analysis in the article?

2.3 High frequency econometrics

Suppose that the log asset price follows:

$$X_t = x + \mu t + \int_0^t \sigma_s dW_s \text{ for } t \in [0,1] \tag{6}$$

where x, μ are constants, $(\sigma_s)_{s\geq 0}$ is a volatility term, and $(W_s)_{s\geq 0}$ is a standard Brownian motion.

2.3.1 (a) In this part, we are interested in estimating the object

$$S = \int_0^1 \sigma_t^3 dt$$

Given access to observations $X_{\frac{i}{n}}$, $0 \leq i \leq n$, let's try to estimate S by

$$S_n = \sqrt{n} \sum_{i=1}^{n} (X_{\frac{i}{n}} - X_{\frac{i-1}{n}})^3.$$

Under the model in (6) with constant volatility, i.e. $\sigma_t = \sigma > 0$ for all $t \in [0,1]$, it is natural to ask:

$$\text{Is } \lim_{n\to\infty} E[S_n] = S?$$

Justify your answer.

2.3.2 (b) Given access to observations $X_{\frac{i}{n}}$, $0 \leq i \leq n$, and supposing that n is an even positive integer, we consider

$$V_n = \alpha_n \frac{1}{2} \left[\sum_{i=1}^{n/2} (X_{\frac{2i}{n}} - X_{\frac{2i-2}{n}})^4 + \sum_{i=1}^{(n-2)/2} (X_{\frac{2i+1}{n}} - X_{\frac{2i-1}{n}})^4 \right]$$

The idea of V_n is to use even and odd observations separately, and to see if the average of two realized quarticities produces any improvements. Under the model in (6) with constant volatility, i.e. $\sigma_t = \sigma > 0$ for all $t \in [0,1]$, try to find a suitable normalizing factor α_n (that may depend on n) such that

$$\lim_{n\to\infty} E[RQ_n] = \sigma^4.$$

Is this possible or impossible? Justify your answer.

2.3.3 (c) In this part, we are interested in asset prices with irregular observation times. In particular, at each stage n, during time $[0,1]$ under model in (6), we have access o observations $X_{t_i^n}$, $i = 1, ..., n$ such that $t_i^n = (i/n)^2$. Lets try to understand realized quarticity in this setting and define

$$RQ_n = \beta_n \sum_{i=1}^{n} (X_{t_i^n} - X_{t_{i-1}^n})^4$$

Is this possible or impossible? Justify your answer.

2.3.4 (d) In this part, we are back to the setting of the regular observations $X_{i/n}$, $0 \leq i \leq n$.

However the asset price is based on the following dynamics:

$$X_t = x + \sigma W_t + c \times 1_{(t \geq t_0)} \text{ for t} \in [0,1],$$

where $x, \sigma, c \neq 0$ and $t_0 \in (0,1)$ are constants, $(W_t)_{t\geq 0}$ is a standard Brownian motion. Note that the model contains a jump of size c. Based on this X, we again consider a similiar estimator:

$$Q_n = \gamma_n \sum_{i=1}^{n} (X_{\frac{i}{n}} - X_{\frac{i-1}{n}})^4.$$

Try to find a suitable normalizing factor γ_n (that may depend on n) and finite non-zero real number Q (that may depend on above constants) such that

$$\lim_{n\to\infty} E[Q_n] = Q.$$

Is this possible or impossible? Justify your answer.

2.3.5 (e) Simulate 1000 independent replications of the model in (6) and the estimator in part (c) with parameters $\mu = 0.05$, constant volatility $\sigma = 0.2$, and $n = 1000$. Report the bias and the mean squared error of RQ_n (with respect to σ^4) and plot the values of RQ_n and σ^4. If your answer in part (c) is "impossible", perform this analysis with some choice of β_n.