3. Math primer and preface to Deep Learning

3.1 Overview on Linear algebra

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Index

- Linear Transformation and Composition
- Linear Operations
- System of Linear Equations
- Eigenvalues and Eigenvectors

Linear Transformation and Composition

Linear Transformation and Composition

Matrices represent linear transformations in vector space.

Linear transformations are functions with vectors as inputs and outputs.

Eg. $v' = \mathbf{M}v$

The vector v is transformed by matrix **M** resulting in vector v'.

Linear Transformation and Composition

A matrix composition describes the effect of multiple transformations.

Eg. $v' = \mathbf{AB}v$

The operation (multiplication) between matrices **A,B** is a matrix composition.

Function notation: Right to left.

Matrix multiplication is not commutative: eg. $v' = \mathbf{AB}v \neq \mathbf{BA}v$

Matrix multiplication is associative: eg. $v' = \mathbf{A}(\mathbf{B}v) = (\mathbf{AB})v$

Linear Operations

Linear Operations

Multiplying the vector v_{3x1} by the matrix $\mathbf{M}_{3x2}$ performs a linear transformation from the 2D space to the 3D space.

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} m_{11} \cdot v_1 + m_{12} \cdot v_2 \\ m_{21} \cdot v_1 + m_{22} \cdot v_2 \\ m_{31} \cdot v_1 + m_{32} \cdot v_2 \end{bmatrix}$$

General case: $\mathbf{M}_{\mathbf{mxn}} v_{nxc} = v'_{mxc}$

Linear Operations

The determinant is the scalar factor by which a vector space is changed by a transformation.

- det($\mathbf{M}$) > 1 Area increases;
- det($\mathbf{M}$) < 1 Area decreases;
- det($\mathbf{M}$) = 0 The transformation squishes the area to a lesser dimension.

Linear Operations

The dot product is the sum of the multiplication of each pair of numbers from two same length vectors.

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = A^t B$$

Linear Operations

The dot product can also be seen as the length of the projected a times the length of b.

$$dot(a, b) = |a||b| \cos \theta$$

- a□b > 0 Both vectors are pointing to the same directions;
- a□b < 0 Both vectors are pointing towards different directions;
- a□b = 0 Both vectors are orthogonal to each other.
- a□b = b□a

Linear Operations

The cross product in **2D** it gives the oriented area of a parallelogram formed by two vectors.

In **3D** it gives a vector which is orthogonal to two linearly independent vectors.

- cross(a,b) = cross(-b,a)

$$a \otimes b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 \cdot b_3 - b_2 \cdot a_3 \\ a_3 \cdot b_1 - b_3 \cdot a_1 \\ a_1 \cdot b_2 - b_1 \cdot a_2 \end{bmatrix}$$

Linear Operations

The transpose operator flips a matrix **A** into **A**^{t} taking the columns of **A** and turning into the rows of **A**^{t}.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}^t$$

Linear Operations

The inverse matrix **A^{-1}** is the unique matrix that does the inverse transformation described by **A**.

$$Av = v'$$

$$A^{-1}v' = v$$

$$AA^{-1} = A^{-1}A = I$$

System of Linear Equations

System of Linear Equations

It is collection of linear equations with shared variables.

The solution to a linear system satisfies all linear equations.

$$\begin{cases} x + 2y - 3z & = 1 \\ 2x - 2y + 4z & = 2 \\ 2x + y + 2z & = 3 \end{cases}$$

Solution: x = 0, y = 1, z = 1

System of Linear Equations

Systems of linear equations can be rewritten in matricial form.

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & -2 & 4 \\ 2 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

This allows us to use black-box methods such as Gaussian elimination, Gauss-Jordan, LU, Cholesky decomposition, Jacobi method, etc. To solve these equations.

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

The eigenvector v and eigenvalue λ pairs are a set of distinct pairs on a linear transformation **A** such that when the transformation **A**v is applied to to the non-zero vector v it doesn't change its direction and may end up only being scaled by a scalar λ.

$Av = \lambda v$