Revisão de Probabilidade

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Random Test

- It can be repeated under the same condition.
- There may be more than one result of each test, and all possible results of the test can be specified in advance.
- Before a test, we cannot determine which result will appear.

Examples:

- Toss two coins and check the outcome (front or back).
- Throw a dice and check the number of points that may appear.

Samples and Events

- Sample point: each possible result of a random test
- Sample space: a collection of all possible results of a random test
- Event: any subset of the sample space. If a sample point of event A occurs, event A occurs.
- Mutually Exclusive Events: Events that can not happen at the same time. Represented as $A \cdot B = \varnothing$ for two events A and B.
- Independent Events: The occurrence of one event does not affects the probability of the other event. $P(A \cap B) = P(A)P(B)$

Frequency and Probability

Frequency: Under the same conditions, perform tests for n times.

- $f(A) = n_A / n$

Probability: real function P(A)

- $0 \leq P(A) \leq 1$.
- inevitable event S, $P(S) = 1$.
- If $A \cdot B = \varnothing \Rightarrow P(A \cup B) = P(A) + P(B)$

Random Variable

- The random variable indicates a real-valued function that represents a random test with various possible results.
- It maps the sample space to real numbers.

Example:

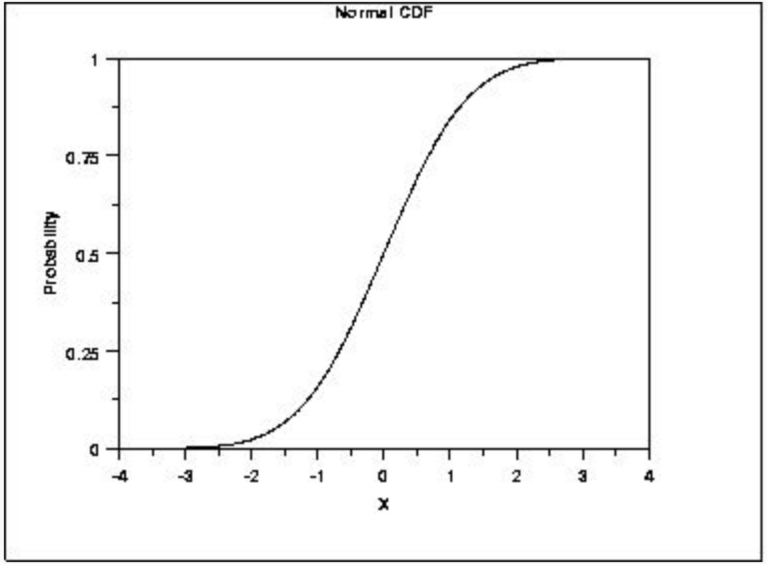
Coin toss: S = {head, tails}

X(head) = 0

X(tails) = 1

Distribution Function

- $F(x) = P\{X \leq x\}, -\infty < x < \infty$
- $F(\infty) = 1$
- $F(-\infty) = 0$
- $P(a \leq X \leq b) = F(b) - F(a)$
- If $a \leq b \Rightarrow F(a) \leq F(b)$

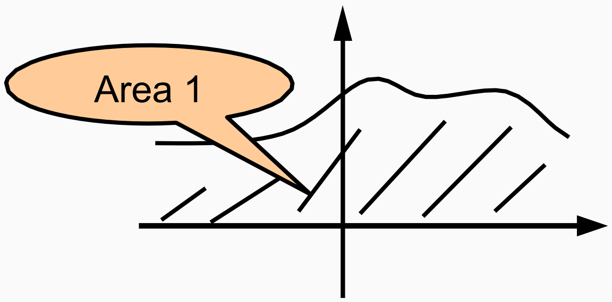


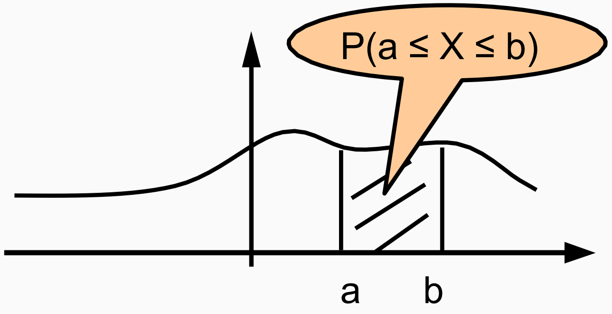
- the value of distribution function F(x) at x indicates the probability that X falls in the interval $(-\infty, x)$.

Probability Density Function

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

- $f(x) \geq 0$
- $\int_{-\infty}^{+\infty} f(x)dx = 1$
- $P\{x_1 < X \leq x_2\} = F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(x)dx$





Expectation

- Mathematical expectation or mean
- Most basic characteristics
- For discrete random variable:

$$E(X) = \sum_{k=1}^{\infty} x_k p_k$$

- For continuous random variable:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Variance

$$Var(X) = E\{[X - E(X)]^2\}$$

- Degree of dispersion
- Deviation between the random variable and its expectation.
- Standard deviation:

$$\sigma(X) = \sqrt{D(X)}$$

Covariance

Covariance: strength of linear correlation of two variables and the scale of these variables.

$$Cov(X,Y) = E(X - E(X))E(Y - E(Y))$$

Covariance matrix:

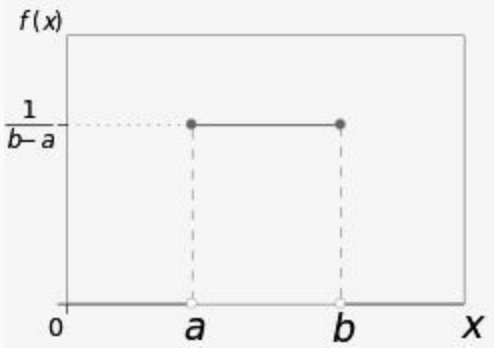
$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

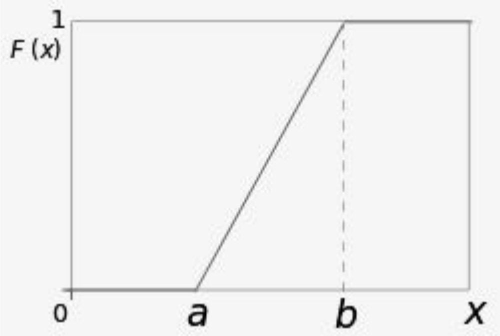
$$c_{ij} = Cov(X_i, X_j) = E\{[X_i - E(X_i)][X_j - E(X_j)]\}$$

Uniform Distribution

- Constant probability

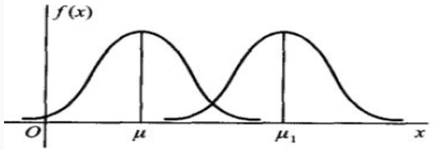
$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

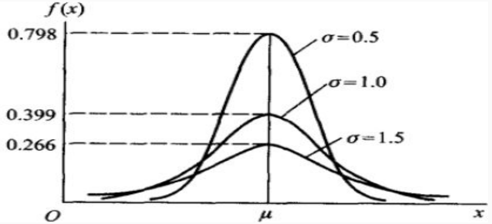




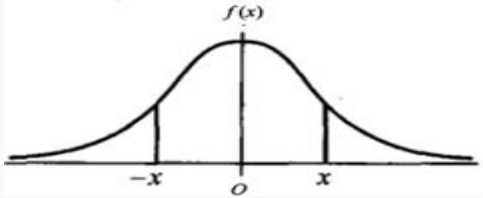
Normal Distribution

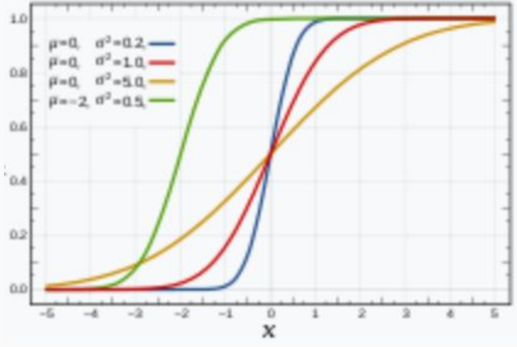
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$





- normal or Gaussian
- $\sigma > 0$
- Models many natural phenomena
- $E(X) = \mu$
- $Var(X) = \sigma^2$

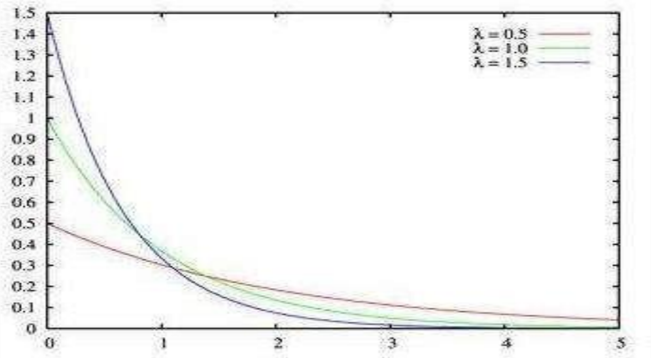




Exponential Distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, x > 0 \\ 0 \quad\quad ,otherwise \end{cases}$$

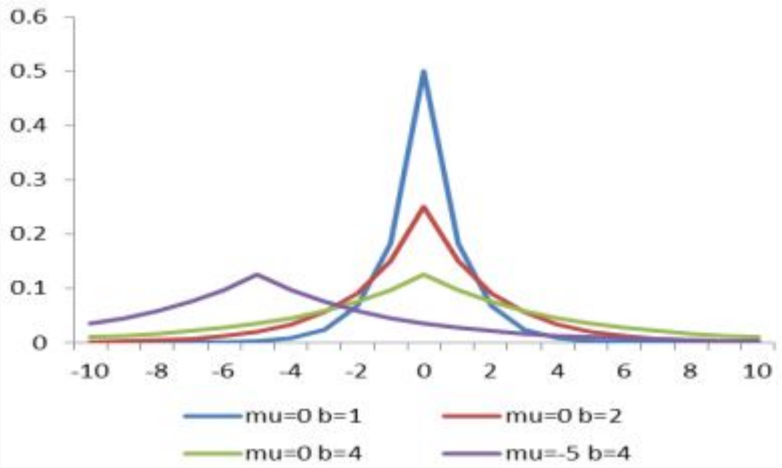
- $\lambda > 0$
- $E(x) = 1/\lambda$
- $Var(X) = 1/\lambda^2$



Laplace Distribution

$$Laplace(x; \mu, b) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}$$

- $E(X) = \mu$
- $Var(X) = 2b^2$



Discrete Random Variables

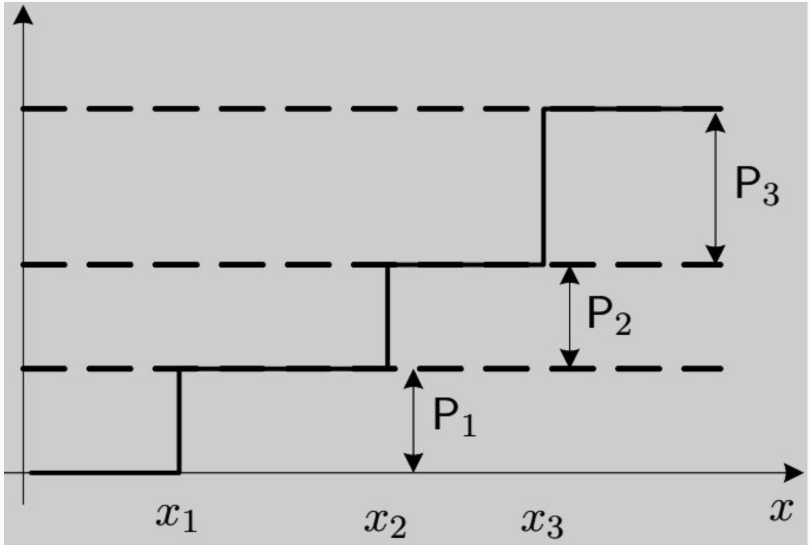
- Only assume discrete values

$$p_k \geq 0, k = 1,\ 2,\ \cdots.$$

$$\sum_{k=1}^{\infty} p_k = 1.$$

- Distribution law:

X	x_1	x_2	$\dots$	x_n	$\dots$
p_k	p_1	p_2	$\dots$	p_n	$\dots$



Bernoulli Distribution

Bernoulli distribution (0-1 distribution, two-point distribution, a-b distribution):

$$P\{X=k\}=p^k(1-p)^{1-k}, k=0,1 \quad (0<p<1),$$

$$E(X)=p, \ Var(X)=p(1-p).$$

X	0	1
p_k	$1-p$	p

Binomial Distribution

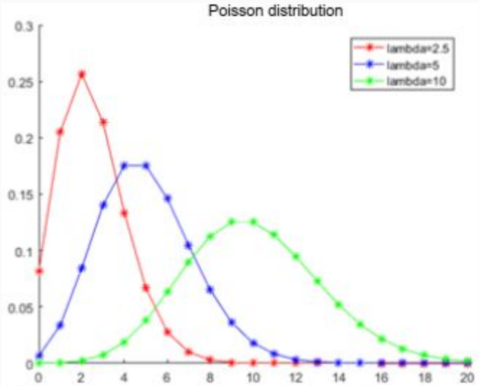
- Bernoulli experiments
- There are only two possible results per experiment
- The results of each experiment are independent of each other.
- X obeys binomial distribution with n and p parameters.
- Gives the chance of k successes in n tries.

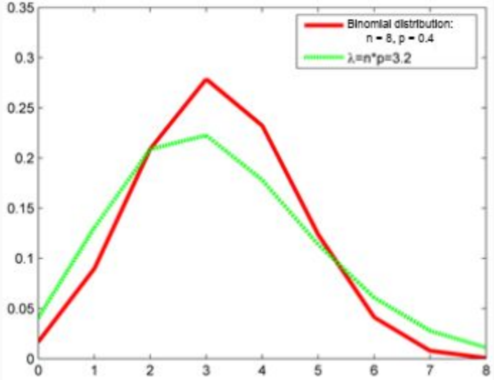
$$P(X = k) = C_n^k p^k (1 - p)^{n-k}, k = 0, 1, 2, \cdots n,$$

Poisson Distribution

$$P\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2,$$

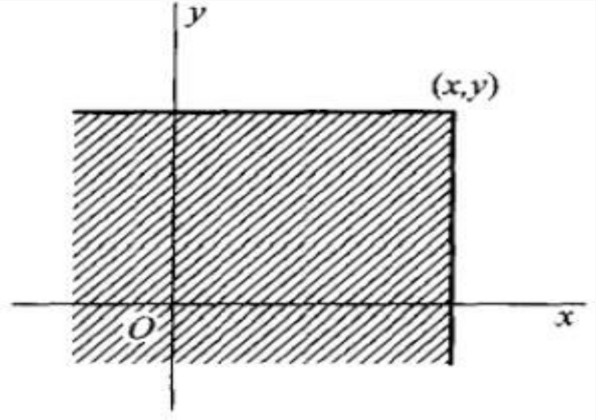
The mathematical models of Poisson distribution and Binomial distribution are both Bernoulli-type. Poisson distribution has the appropriately equal calculation as binomial distribution when n is very large and p very small.





Two-Dimensional Random Variable

- vector (X,Y) where X and Y are random variables in the same sample space
- $F(x,y) = P\{X \leq x, Y \leq y\}$
- F (x,y) is the probability of random point (X,Y) falling in the infinite rectangular field at the point (x,y) vertex and at the lower left of the point.



Joint Distribution Law

- Two-dimensional discrete random variable
- Joint distribution law of X and Y

X \ Y	x_1	x_2	. . .	x_i	. . .
y_1	e_{11}	e_{12}	. . .	e_{1i}	. . .
y_2	e_{21}	e_{22}	. . .	e_{2i}	. . .
$\vdots$	$\vdots$	$\vdots$		$\vdots$	
y_j	e_{j1}	e_{j2}	. . .	e_{ji}	. . .
$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Joint Probability Density

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v)\,du\,dv$$

- Marginal distribution
- F(x,∞) = F(x)
- F(∞,y) = F(y)

Conditional probability

- a measure of the probability of an event occurring given that another event has occurred.
- Commonly referred as the conditional probability of A given B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- Requires *P(B) > 0*

Bayes Formula

Bayes formula:

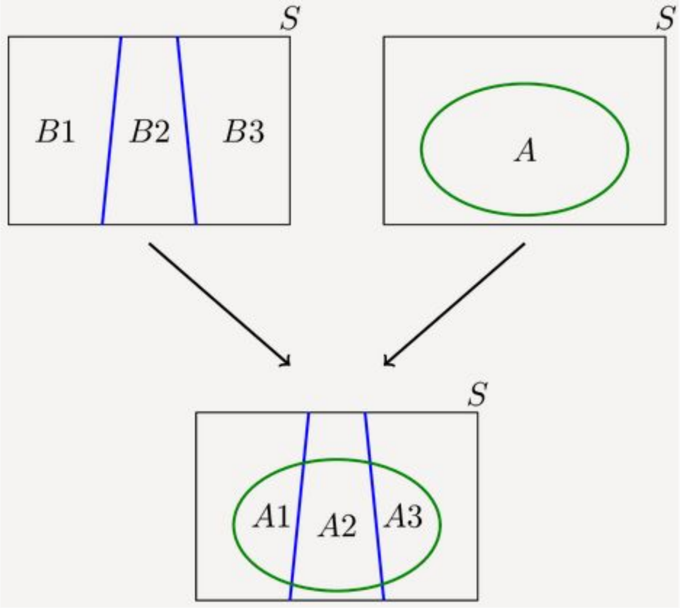
$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}$$

total probability:

$$P(A) = \sum_n P(A \mid B_n) P(B_n)$$

sample space divided in n pairwise disjoint

events B_n



Information Theory

- Studies how to measure information contained in a signal.
- Publication of "A Mathematical Theory of Communication" in 1948.
- Scientific and quantitative description of information.
- Proposed the concept of information entropy.

Information Quantity and Entropy

When an unlikely event happens, it provides more information than a very likely event.

Information quantity:

$$I(x) = -\log_2 p$$

Information entropy: The information contained in the source is the average uncertainty of all possible messages transmitted by the source.

$$Info(D) = -\sum_{i=1}^{m} p_i \log_2(p_i)$$