$$\require{color}$$

Symmetry reduced Flag-hierarchies

-

Daniel Brosch

March 28, 2022

Contents

- Semidefinite programming
- Symmetry reduction basics
- Representation theory (of S_n)
- Flag-Algebras
- Lasserre style hierarchy
- Razborov style hierarchy
- Advanced ideas

$$ \inf\{\langle C,X\rangle : \langle A_i, X\rangle = b_i \forall i, X\succcurlyeq 0\} $$

$$V = m_1S_1\oplus m_2S_2\oplus \ldots \oplus m_kS_k$$

Semidefinite programming

$$ \inf_X\{\langle C,X\rangle : \langle A_i, X\rangle = b_i \forall i,

X\succcurlyeq 0\} $$

\fcolorbox{orange}{black}{$X\succcurlyeq 0$}\} $$

The anatomy of an SDP

$$ \inf_X\{{\color{orange}\langle} C{\color{orange},}X{\color{orange}\rangle} : {\color{orange}\langle} A_i{\color{orange},}X{\color{orange}\rangle} = b_i \forall i, X\succcurlyeq 0\} $$

$$ \inf_X\{\langle C,X\rangle : \langle A_i,X\rangle = b_i \forall i, X\succcurlyeq 0\} $$

$\color{orange} \langle\cdot,\cdot\rangle$ denotes the matrix inner product: $$\langle A,B\rangle = \sum_{i,j=1}^n A_{ij} B_{ij}$$

$\color{orange} C, A_i$ are symmetric data matrices

${\color{orange} X\succcurlyeq 0}$ means that X is **positive semidefinite**: $$v^TXv \geq 0 \text{ for all vectors } v.$$

SDPs are **linear optimization problems** over a **convex cone**. We can solve them in **polynomial time!**

SDPs for polynomial optimization

We often want to minimize a polynomial under some polynomial constraints:

$$\min_x \{{\color{green}f}(x): {\color{red}g_i}(x)\geq 0 \quad\forall i\}.$$

We can rewrite this problem as

$$ \max_{\lambda, s} \{\lambda : {\color{green}f}-\lambda = {\color{orange}s_0} + \sum_i {\color{orange}s_i} {\color{red}g_i} \geq 0\} $$

where the ${\color{orange}s_j}$ are **nonnegative polynomials**.

Sums-of-Squares

We can relax the nonnegative polynomials ${\color{orange}s_j}$ to be **sums of squares** of polynomials:

\begin{align}
{\color{orange}s_j}(x) &= \sum_t p_t(x)^2 = \left({\color{red}[x]}^Tv_t\right)^2\\
&= {\color{orange}s_j}(x) \geq 0.
\end{align}

Given a polynomial basis $${\color{red}[x]}=\{1, x_1, x_2, x_1^2, x_1 x_2, \ldots\}$$ we can write the

$${\color{orange}s_j}(x) ={\color{red}[x]}^T\left(\sum_t v_t v_t^T\right) {\color{red}[x]} = {\color{red}[x]}^T X {\color{red}[x]},$$

where X is **positive semidefinite**.

Comparing coefficients leads to a standard SDP!

SDP Symmetry reduction basics

What is a symmetry?

Let ${\color{orange}\sigma}$ be a **permutation** of $\{1,\ldots, n\}$. We let ${\color{orange}\sigma}$ act on the *indices of X simultanously*:
$${\color{orange}\sigma}(X) = \left(X_{{\color{orange}\sigma}(i){\color{orange}\sigma}(j)}\right)_{i,j=1}^n.$$

If X is positive semidefinite, then $\sigma(X)$ is as well!

When does an SDP have a symmetry?

An SDP has symmetry ${\color{orange}\sigma}$ if for each constraint $\langle A_i,X\rangle=b_i$ there is a j such that $${\color{orange}\sigma}(A_i)=A_j, \quad b_i=b_j$$ and the objective is symmetric: $${\color{orange}\sigma}(C)=C.$$

This holds if symmetry ${\color{orange}\sigma}$ maps feasible solutions to feasible solutions with the same objective value.

How do we exploit symmetries?

The set of all symmetries forms a **group ${\color{orange}G}$** of permutations. As the feasible set of an SDP is **convex**, we can **average** feasible solutions:
$$\mathcal{R}(X) = \frac{1}{|{\color{orange}G}|}
\sum_{{\color{orange}\sigma}\in {\color{orange}G}}
{\color{orange}\sigma}(X)$$
$\mathcal{R}(X)$ is again feasible, with the same objective value as X!

Symmetric optimal solutions

In the case of ${\color{orange}G}=D_{10}=C_5\times Z_2$ we can restrict X to have the **pattern**

$$\begin{pmatrix} {\color{red}A} & {\color{orange}B} & {\color{green}C} & {\color{green}C} & {\color{orange}B}\\ {\color{orange}B} & {\color{red}A} & {\color{orange}B} & {\color{green}C} & {\color{green}C}\\ {\color{green}C} & {\color{orange}B} & {\color{red}A} & {\color{orange}B} & {\color{green}C}\\ {\color{green}C} & {\color{green}C} & {\color{orange}B} & {\color{red}A} & {\color{orange}B} \\ {\color{orange}B} & {\color{green}C} & {\color{green}C} & {\color{orange}B} & {\color{red}A}\\ \end{pmatrix}$$

If X^* is an optimal solution, then so is $\mathcal{R}(X^*)$. This solution is invariant under actions of ${\color{orange}G}$:

$${\color{orange}\sigma}(\mathcal{R}(X^*)) = \mathcal{R}(X^*).$$

We can restrict the SDP to optimize only over invariant matrices:

$$\begin{align}\max\quad & \langle C, X\rangle \\ \text{s.t.}\quad & \langle A_i, X\rangle = b_i\quad \text{for all } i\\ & X\succeq 0,\ X = \mathcal{R}(X). \\ \end{align}$$

Block-diagonalization

$$\begin{pmatrix} {\color{red}A} & {\color{orange}B} &
{\color{green}C} & {\color{green}C} & {\color{orange}B}\\
{\color{orange}B} & {\color{red}A} & {\color{orange}B} &
{\color{green}C} & {\color{green}C}\\ {\color{green}C} &
{\color{orange}B} & {\color{red}A} & {\color{orange}B} &
reen}C} &
ange}B} \\
{green}C} &
trix} $$

In general we may still get multiple bigger blocks

- ${\color{red}A} + {\color{orange}B} + {\color{green}C} \geq 0$,
- ${\color{red}A} +\frac{\sqrt{5}-1}{4}{\color{orange}B} - \frac{\sqrt{5}+1}{4}{\color{green}C} \geq 0$,
- ${\color{red}A} -\frac{\sqrt{5}+1}{4}{\color{orange}B}+\frac{\sqrt{5}-1}{4}{\color{green}C} \geq 0$.

But the sum of the block sizes is often significantly lower than n!

How do we find the block-diagonalization?

Representation theory

$$V = m_1W_1\oplus m_2W_2\oplus \ldots \oplus m_kW_k$$

A ${\color{orange}G}$-module is

- a **vectorspace** V,
- a **group** ${\color{orange}G}$, and
- a **group homomorphism** $\rho:{\color{orange}G}\to \mathrm{GL}(V)$.

Together, they let ${\color{orange}G}$ act on vectors:
$$\color{orange}\boxed{gv := \rho(g)v}$$

Example: ${\color{orange}S_n}$-module

The vectorspace of polynomials up to degree d in n variables:
$$\mathbb{R}[x_1,\ldots,x_n]_{\leq d}$$ is an ${\color{orange}S_n}$-module by defining
$${\color{orange}\sigma}(x_i) := x_{{\color{orange}\sigma}(i)}$$
and extending the operation to polynomials.

Irreducible modules

We call a ${\color{orange}G}$-module V **irreducible**, if V and $\{0\}$ are the only submodules of V.

The $\color{orange}S_2$-module
$$V = \mathrm{span}\{1,x_1,x_2\}$$
is not irreducible:
\begin{align} V =& \enspace{\color{orange}\mathrm{span}\{1\}} \oplus {\color{orange}\mathrm{span}\{x_1+x_2\}}\\&\oplus {\color{green}\mathrm{span}\{x_1 - x_2\}}\\ \end{align}

Maschke's Theorem

We can decompose $\color{orange}G$-modules into irreducible submodules:

$$ V = W_1\oplus \ldots \oplus W_k.$$

$$\begin{align} V =& \enspace \mathrm{span}\{1,x_1,x_2\}\\ =&\enspace {\color{orange}\mathrm{span}\{1\}} \oplus {\color{orange}\mathrm{span}\{x_1 + x_2\}}\\ &\oplus {\color{green}\mathrm{span}\{x_1 - x_2\}}\\ \simeq&\enspace {\color{orange}2S^{(2)}} \oplus {\color{green}S^{(1,1)}}\\ \end{align}$$

Sorting by equivalent irreducible modules we see that

$$V\simeq c_1 S_1\oplus \ldots\oplus c_n S_n,$$

where the c_i are nonnegative integers, and the S_i form a set of **all irreducible G-modules** up to isomorphism.

The irreducible modules S^λ of S_n are called **Specht modules**.

Schur's Lemma

The main ingredient of block-diagonalization

Let $\color{orange}M$, $\color{orange}N$ be two **irreducible** G-modules over a ring R. Let ${\color{limegreen}\varphi} : {\color{orange}M}\to {\color{orange}N}$ be a homomorphism.

- If $\color{orange}M$ and $\color{orange}N$ are not isomorphic, then ${\color{limegreen}\varphi} \equiv 0$.
- If ${\color{orange}M}\simeq {\color{orange}N}$ and R is an algebraically closed field, then ${\color{limegreen}\varphi} = c\mathrm{I}$ for a $c\in R$.

Representation theory of ${\color{orange}S_n}$