

$$ex \left( \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right) := \max_c \left\{ \phi \left( \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right) : \phi \left( \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right) = 0 \right\}.$$

$$\text{ex}\left(\begin{array}{c} \text{orange path of length 2} \\ \text{green triangle} \end{array}\right) \geq \frac{1}{2},$$



$(g)$

$\cdot$



$(g)$

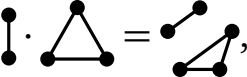
$=$

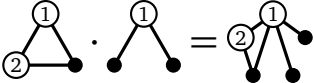


$(g)$

$\cdot$







$$\left(1\cancel{0}-2\begin{array}{c}\textcircled{1}\\\hline\bullet\end{array}\right)^2=\cancel{0}-4\begin{array}{c}\textcircled{1}\\\hline\bullet\end{array}+4\begin{array}{c}\textcircled{1}\\\hline\bullet\quad\bullet\end{array}\geq 0.$$

$$\left[ \left( 1\emptyset - 2 \begin{array}{c} \textcircled{1} \\ | \\ \bullet \end{array} \right)^2 \right] = \left[ \emptyset - 4 \begin{array}{c} \textcircled{1} \\ | \\ \bullet \end{array} + 4 \begin{array}{c} \textcircled{1} \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right] = \emptyset - 4 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + 4 \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \geq 0.$$

$$\frac{1}{2} - \text{diagram} = \left[ \frac{1}{2} \left( 1 \text{diagram} - 2 \text{diagram} \right)^2 \right] + \left[ \left( \text{diagram} - \text{diagram} - \text{diagram} + \text{diagram} \right)^2 \right] - \text{diagram} \geq 0.$$

LEVEL 21  
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$$H \text{ --- } \begin{array}{c} \bullet \\ | \\ \bullet \end{array} |E(H)| \geq 0.$$



$$\lim_{v \rightarrow 0} \frac{f(x + hv) - f(x)}{h} = f'(x)$$

$$\sigma_v^H := \lim_{i \rightarrow \infty} \frac{(H \text{ density in } (G_i - v)) - (H \text{ density in } G_i)}{i-1}$$

$$\partial_1 \text{ (edge)} = 2 \text{ (edge)} - 2 \text{ (edge with top node circled 1)}$$

$$\partial_1 \text{ (triangle)} = 3 \text{ (triangle)} - 2 \text{ (triangle with bottom-left node circled 1)} - \text{ (triangle with top node circled 1)}$$

$$d_H^e := \lim_{i \rightarrow \infty} \frac{(H \text{ density in } (G_i - e)) - (H \text{ density in } G_i)}{i-2}$$





$$\partial_e \text{ (edge) } = -2 \text{ (edge with nodes 1 and 2)}$$

$$\partial_e \text{ (V-shape) } = -2 \text{ (V-shape with nodes 1 and 2 at top)} - 2 \text{ (V-shape with nodes 1 and 2 at bottom)}$$

$$\partial_e \text{ (triangle) } = -6 \text{ (triangle with nodes 1 and 2 at top and bottom-left vertices)}$$

$$\frac{1}{4} \mathbb{I} \left[ \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{3} \end{array} a_{\text{swap}}(P_3 - e^3) \right] = \text{Diagram 1} + \text{Diagram 2} - 2 \text{Diagram 3} \geq 0$$



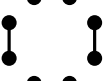
The equation shows a linear combination of three graph diagrams. The first diagram is a path of three nodes. The second and third diagrams are more complex, involving multiple paths and nodes. The final result is greater than or equal to zero.

But   $-$    $= \text{SOS} \geq 0$  and   $-$    $= \text{SOS} \geq 0.$

Thus





This proves   $=$    $\geq$   by induction.

$$\begin{array}{c} \textcircled{1} \\ \diagup \quad \diagdown \\ \bullet \qquad \bullet \end{array} (\mathcal{G}) := \lim_{i \rightarrow \infty} \mathbb{P} \left[ \sigma_i \left( \begin{array}{c} \textcircled{1} \\ \diagup \quad \diagdown \\ \bullet \qquad \bullet \end{array} \right) \text{ is a subgraph of } G_i \mid \sigma_i(1) = 1 \right]$$

$$\begin{aligned}
\partial_{\text{swap}}(P_3 - e^3) = & 2 \begin{array}{c} \textcircled{4} \\ | \\ \textcircled{3} \\ | \\ \textcircled{2} \end{array} + 2 \begin{array}{c} \textcircled{4} \\ | \\ \textcircled{3} \\ | \\ \textcircled{1} \end{array} + 2 \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{2} \\ | \\ \textcircled{3} \end{array} + 2 \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{2} \\ | \\ \textcircled{4} \end{array} + 2 \begin{array}{c} \textcircled{4} - \textcircled{3} \\ | \quad | \\ \textcircled{1} \quad \textcircled{2} \end{array} + 2 \begin{array}{c} \textcircled{2} - \textcircled{1} \\ | \quad | \\ \textcircled{3} \quad \textcircled{4} \end{array} \\
& - 2 \begin{array}{c} \textcircled{2} - \textcircled{1} \\ | \\ \bullet - \bullet \end{array} - 2 \begin{array}{c} \textcircled{1} - \textcircled{2} \\ | \\ \bullet - \bullet \end{array} - 2 \begin{array}{c} \textcircled{4} - \textcircled{3} \\ | \\ \bullet - \bullet \end{array} - 2 \begin{array}{c} \textcircled{3} - \textcircled{4} \\ | \\ \bullet - \bullet \end{array} - 2 \begin{array}{c} \textcircled{1} - \bullet \\ | \\ \textcircled{2} - \bullet \end{array} - 2 \begin{array}{c} \textcircled{3} - \bullet \\ | \\ \textcircled{4} - \bullet \end{array}
\end{aligned}$$