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Quantum–Kinetic Dark Energy (QKDE):

An effective dark energy framework with the Einstein–Hilbert metric sector intact and a time–dependent scalar kinetic normalization

($c_T^2 = 1$, Planck mass constant; $\alpha_B = \alpha_M = \alpha_T = \alpha_H = 0$, $\alpha_K > 0$ in EFT–DE)

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Note. Earlier drafts circulated under the working title *Scalar–Cost Dark Energy (SCDE)*.

Abstract

A minimal, GR–preserving dark energy framework—*Quantum–Kinetic Dark Energy* (QKDE)—is formulated in which only the scalar kinetic normalization carries a background time dependence $K(t) > 0$ while the Einstein–Hilbert sector remains unmodified. The unitary–gauge action (time diffeomorphisms broken) $S = \int d^4x \sqrt{-g}\,[\frac{1}{2}M_{\rm pl}^2 R + K(t)X - V(\phi)]$ admits an exact mapping to the EFT–DE language with $\alpha_K = K\dot{\phi}^2/(H^2 M_{\rm pl}^2) > 0$ and $\alpha_B = \alpha_M = \alpha_T = \alpha_H = 0$ (luminal tensors, constant Planck mass). From this action a closed first–order background system in $N = \ln a$ is obtained; scalar perturbations propagate with $c_s^2 = 1$ and satisfy $\Phi = \Psi$, and linear growth obeys the GR equation. Consequently, late–time signatures enter solely through the expansion history $H(a)$ and the induced growth $D(a)$. Two concrete specifications of K are analyzed: (i) a curvature–motivated form $K = 1 + \alpha R/M^2$ equipped with a fully algebraic, iteration–free identity for K'/K; and (ii) a phenomenological running $K = 1 + K_0(1+z)^p$. A reproducible numerical pipeline (state vector, tolerances, and identity checks) is detailed together with a Fisher setup based on exact sensitivity equations for distances, $H(z)$, and $f\sigma_8(z)$. Stability and admissibility reduce to $K(t) > 0$ and the nonvanishing of the algebraic denominator in the curvature case. The framework yields sharp, falsifiable linear–scale predictions: $\mu(a,k) = \Sigma(a,k) = 1$, $\varpi(a,k) = 0$, $c_T^2 = 1$; any robust deviation from these null tests lies outside the QKDE baseline.

Keywords: Baryon acoustic oscillations (BAO); cosmology; dark energy; effective field theory of dark energy (EFT–DE); growth of structure; kinetic normalization; linear perturbations; redshift–space distortions (RSD); scalar fields; weak gravitational lensing.

Notation and symbols

Overdot $\dot{( )}$ denotes d/dt; prime $( )'$ denotes d/dN with $N = \ln a$. Units $c = \hbar = 1$; metric signature $(-,+,+,+)$. The reduced Planck mass is $M_{\rm pl} = (8\pi G)^{-1/2}$. Spatial flatness (FRW) is assumed unless noted.

Table 0.1: Core symbols used throughout

Symbol	**Meaning**
$a(t)$	Scale factor
$N \equiv \ln a$	E-fold time
$H \equiv \dot{a}/a$	Hubble parameter
$E \equiv H'/H$	Logarithmic Hubble derivative
R	FRW Ricci scalar $= 6(2H^2 + \dot{H}) = 6H^2(2+E)$
$M_{\rm pl}$	Reduced Planck mass
$\phi(t)$	Homogeneous scalar field
$V(\phi)$	Scalar potential
$X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$	Kinetic invariant; FRW: $X = \frac{1}{2}\dot{\phi}^2$
$K(t) > 0$	Time-dependent kinetic normalization
ρ_i, p_i	Energy density/pressure, $i \in \{m, r, \phi\}$
$\Omega_i \equiv \rho_i/(3M_{\rm pl}^2 H^2)$	Density parameters; $\sum_i \Omega_i = 1$ (flat)
$w_\phi \equiv p_\phi/\rho_\phi$	Scalar equation of state
$q \equiv -1 - H'/H$	Deceleration parameter
Φ, Ψ	Bardeen potentials (Newtonian gauge); here $\Phi = \Psi$
$D(a)$	Linear matter growth factor
$f \equiv d\ln D/d\ln a$	Linear growth rate
$\chi(z)$	Comoving distance
D_A, D_L	Angular-diameter and luminosity distances
D_V, $F_{\rm AP}$	BAO summary distances
r_d	Sound horizon at drag epoch
η	Conformal time
$z = e^{-N} - 1$	Redshift–e-folds map

Table 0.2: EFT–DE and growth shorthand

Symbol	**Meaning**
α_K	Kinetic EFT coeff.; here $\alpha_K = K\,\dot{\phi}^2/(H^2 M_{\rm pl}^2) > 0$
α_B, α_M, α_T, α_H	Braiding, M_* running, tensor speed, beyond-Horndeski (all $= 0$)
$\mu(a,k)$	Poisson modifier (linear); here $\mu = 1$
$\Sigma(a,k)$	Lensing modifier (linear); here $\Sigma = 1$
$\eta(a,k) \equiv \Phi/\Psi - 1$	Gravitational slip; here $\eta = 0$
σ_8	RMS matter fluct. in $8\,h^{-1}$ Mpc at $z = 0$
$f\sigma_8$	RSD observable $f(z)\,\sigma_{8,0}\,D(z)/D(0)$
$s \equiv \phi'$	Scalar e-fold derivative (background state)
$y_H \equiv \ln H$	Log-Hubble variable (background state)

Disambiguation. This work reserves $E \equiv H'/H$ (no use of $E(z)$). When the normalized Hubble rate is needed, it is written explicitly as $H(z)/H_0$. The symbol η denotes conformal time; the gravitational slip uses $\eta(a,k)$ and is contextually distinct.

Data and code availability All equations and algorithms required to reproduce the results are stated explicitly in the text (notably Eqs. (3.12), (7.7), and (7.13)), with numerical tolerances, step control, and diagnostics summarized in Sec. 8 and Table 8.1. A compact reference implementation may be built directly on the state vector (ϕ, ϕ', H) using the identities in Secs. 3–4.

Competing interests The author declares no competing interests.

Author contribution Single-author manuscript. Conceptualization, formal analysis, derivations, numerical implementation, validation, and writing were performed by the author.

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Contents

List of Figures

List of Tables

1 Introduction

The observed late–time acceleration of the Universe admits two broad explanatory directions: modifications of gravity on cosmological scales or additional cosmic components with negative pressure. Bounds from the binary–neutron–star merger GW170817 and its electromagnetic counterpart enforce a luminal tensor speed and an effectively constant Planck mass at low redshift, disfavoring large sectors of modified–gravity space [1, 2]. In this context it is natural to isolate dark–energy constructions that preserve the Einstein–Hilbert metric sector and alter only the matter content.

This work develops a minimal, covariant dark–energy framework—*Quantum–Kinetic Dark Energy* (QKDE)—in which only the normalization of the scalar kinetic term varies slowly in time while the metric sector remains that of General Relativity (GR). The defining action is

$$S = \int \mathrm{d}^4x \sqrt{-g}\Big[\frac{1}{2}M_{\rm pl}^2 R + K(t)\,X - V(\phi)\Big], \qquad X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi, \tag{1.1}$$

with minimally coupled matter and radiation. The function $K(t)$ is a *background* (unitary–gauge) coefficient: it depends on cosmic time only and introduces no new perturbative operators in the scalar sector beyond a time–dependent normalization. In the Effective Field Theory of Dark Energy (EFT–DE) language [3–5], the Bellini–Sawicki set satisfies

$$\alpha_B = \alpha_M = \alpha_T = 0, \qquad \alpha_K = \frac{K\,\dot{\phi}^2}{H^2 M_{\rm pl}^2} > 0, \tag{1.2}$$

so the tensor speed is luminal and the Planck mass is constant, consistent with GW170817 constraints [1, 2]. Because $P(X,\phi,t) = K(t)X - V(\phi)$ has $P_{,XX} = 0$, the scalar sound speed is $c_s^2 = 1$ (and, with $K(t) > 0$, no ghosts), excluding gradient instabilities and scale–dependent growth on linear/subhorizon scales [6].

What is—and is not—modified.
Modified: a single background function $K(t) > 0$ multiplying the scalar kinetic operator.
Not modified: the metric sector (pure Einstein–Hilbert), the tensor speed ($c_T^2 = 1$), the Planck mass (constant), and linear GR relations ($\mu = \Sigma = 1,\ \eta = 0$).

Motivation. Quantum field theory in curved spacetime (QFTCS) predicts curvature–suppressed operators such as $R\,X$ and $R_{\mu\nu}\partial^\mu\phi\,\partial^\nu\phi$, generated by loops and/or integrating out heavy fields [7–14]. On a homogeneous background these operators are degenerate with a time–dependent rescaling of X. QKDE isolates precisely this *background imprint* by promoting the scalar normalization to $K(t)$ while deliberately avoiding nonminimal couplings or braiding terms in the metric sector. Canonical quintessence is recovered when $K \equiv 1$ [15, 16]; general k–essence extends to $P(X,\phi)$ with $P_{,XX} \neq 0$ and admits $c_s^2 \neq 1$ [6, 17]. QKDE occupies the conservative corner of EFT–DE with only α_K nonzero.

Two structural features make the framework predictive. First, because only the scalar normalization runs, the background expansion $H(a)$, the kinetic invariant X, and the scalar equation of state $w_\phi(a)$ follow from a closed first–order system in e–fold time $N = \ln a$, derived directly from (1.1) and the Einstein equations (Secs. 2–3). Second, at linear order the Bardeen potentials satisfy $\Phi = \Psi$ and the GR Poisson equation holds; the scalar mode is pressure supported with $c_s^2 = 1$ and does not cluster on subhorizon scales (Sec. 5). Consequently, all late–time signatures enter through the background: distances and growth respond solely to $H(a)$ and its induced growth history $D(a)$ [18–21].

Scope and contributions. The analysis is derived from first principles; every equation used later is either obtained from the covariant action, stress–energy tensor, and Einstein equations, or recalled with explicit cross–references. No phenomenological template is assumed without definition and derivation.

1. A covariant definition of QKDE is stated in (1.1) together with minimal coupling and the positivity prior $K(t) > 0$. Immediate implications are $\alpha_T = \alpha_M = \alpha_B = 0$ and a luminal tensor sector (Sec. 6).

2. A closed background system in e–fold variables is assembled: the scalar equation, the Raychaudhuri relation H'/H, and exact expressions for X, ρ_ϕ, and R/H^2 (Sec. 3). Well–posedness and admissibility are specified.

3. Two kinetic normalizations are analyzed that preserve the GR metric sector: a curvature–motivated form $K = 1 + \alpha R/M^2$ that captures QFTCS renormalization at background level, and a phenomenological running $K = 1 + K_0 \mathrm{e}^{-pN}$ ($\Leftrightarrow 1 + K_0(1+z)^p$) (Sec. 4). Exact identities for K'/K are provided; in the curvature case, an algebraic, iteration–free expression depending only on (ϕ, ϕ', H) and known sources is derived (Eq. (4.10)).

4. Linear perturbations are developed from the perturbed Klein–Gordon equation to the quadratic action and the Mukhanov–Sasaki system, yielding $c_s^2 = 1$, $\Phi = \Psi$, and the GR Poisson equation on subhorizon scales (Sec. 5).

5. The EFT–DE mapping is exhibited explicitly, with $\alpha_K > 0$ and $\alpha_B = \alpha_M = \alpha_T = 0$ (Eq. (1.2)), and QKDE is situated within existing theory space (Sec. 6.4).

6. Forecast–ready relations are recorded for $\chi(z)$, $D_A(z)$, $D_L(z)$, $D_V(z)$, $F_{\mathrm{AP}}(z)$, and the GR growth observables $D(a)$, $f(a)$, and $f\sigma_8(z)$, each expressed solely in terms of the background solution (Sec. 7).

7. A reproducible numerical pipeline is provided: state vector, tolerances, algebraic evaluation of K'/K, and diagnostic identities (Friedmann closure, Raychaudhuri, Klein–Gordon, Ricci) used as invariants of the integration (Sec. 8); parameter sensitivities and Fisher assembly are derived from exact variational equations (Sec. 9).

Admissibility and assumptions. Spatial flatness is assumed; the expanding branch satisfies $H(t) > 0$; and ghost freedom requires $K(t) > 0$. Smoothness $K \in C^1$, $V \in C^2$ ensures local existence/uniqueness of the e–fold system. When early–time anchors are used, a conservative prior $K(z) \to 1$ for $z \geq z_{\mathrm{drag}}$ is adopted; otherwise the sound horizon r_d is recomputed from its definition (Sec. 7.2). For the curvature–motivated case, the algebraic denominator in (4.10) must remain nonzero along the solution; this condition is monitored alongside $K > 0$.

Relation to prior work. Canonical quintessence modifies the background through $V(\phi)$ with fixed kinetic normalization [15, 16]; k–essence admits general $P(X, \phi)$ and can alter linear phenomenology via $c_s^2 \neq 1$ [6, 17]. QKDE retains canonical propagation ($c_s^2 = 1$) but incorporates a background kinetic running $K(t)$ that encapsulates curvature–induced wavefunction renormalization [7, 9, 12–14] while preserving the Einstein–Hilbert metric sector. In EFT–DE terms [3–5], it isolates a single nonzero parameter α_K consistent with GW170817 [1, 2].

Notation and organization. Units with $c = \hbar = 1$ are used; the metric signature is $(-,+,+,+)$; an overdot denotes d/dt, and a prime denotes d/dN with $N = \ln a$. The reduced Planck mass is

$M_{\rm pl} = (8\pi G)^{-1/2}$. Section 2 states the covariant equations and stability prerequisites. Section 3 derives the closed background system in e–fold variables. Section 4 specifies and analyzes the kinetic normalizations $K(t)$ and the algebraic closure for K'/K. Section 5 develops the linear perturbations and the Mukhanov–Sasaki system. Section 6 presents the EFT–DE mapping and linear viability and, in Sec. 6.4, situates QKDE within existing theory space. Section 7 records background and growth observables. Sections 8–11 cover implementation, parameter sensitivities and Fisher setup, limiting cases, and conclusions.

2 Covariant formulation and core background equations

Assumptions and conventions. (1) The metric sector is Einstein–Hilbert with constant reduced Planck mass $M_{\rm pl}$. (2) Matter and radiation are minimally coupled. (3) The kinetic normalization K is a *background* function of cosmic time only ($K = K(t) > 0$; unitary gauge in the EFT sense). (4) Spatially flat FRW background. (5) Units $c = \hbar = 1$; metric signature $(-,+,+,+)$. For later use, $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$.

This section collects and *derives* the background relations used throughout, starting from the covariant action introduced in (1.1):

$$S = \int d^4x\,\sqrt{-g}\Big[\tfrac{1}{2}M_{\rm pl}^2 R + K(t)\,X - V(\phi)\Big], \qquad X \equiv -\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi, \tag{2.1}$$

with $K(t) > 0$ and minimally coupled matter and radiation.

2.1 Stress–energy tensor and fluid variables

For a general scalar Lagrangian $P(X,\phi,t)$, the stress–energy tensor is $T_{\mu\nu} = P_{,X}\,\partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}P$ (standard result; see [6]). Applying this to $P = K(t)X - V(\phi)$ gives

$$T^{(\phi)}_{\mu\nu} = K(t)\,\partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}[K(t)\,X - V(\phi)]\,. \tag{2.2}$$

On a spatially flat FRW background $ds^2 = -dt^2 + a^2(t)d\vec{x}^2$, homogeneity implies $X = \frac{1}{2}\dot{\phi}^2$ and therefore

$$\rho_\phi = K\,\frac{\dot{\phi}^2}{2} + V(\phi), \qquad p_\phi = K\,\frac{\dot{\phi}^2}{2} - V(\phi), \qquad w_\phi = \frac{K\dot{\phi}^2/2 - V}{K\dot{\phi}^2/2 + V}. \tag{2.3}$$

2.2 Scalar equation of motion and modified friction

The Euler–Lagrange equation for $P(X,\phi,t)$ is $\nabla_\mu(P_{,X}\nabla^\mu\phi) - P_{,\phi} = 0$ [6]. For $P = K(t)X - V(\phi)$ one obtains

$$K\,\Box\phi + (\nabla_\mu K)\,\nabla^\mu\phi - V_{,\phi} = 0. \tag{2.4}$$

Since $K = K(t)$, $(\nabla_\mu K)\nabla^\mu\phi = g^{00}\dot{K}\,\dot{\phi} = -\dot{K}\,\dot{\phi}$. Using $\Box\phi = -(\ddot{\phi} + 3H\dot{\phi})$ on FRW gives the background equation

$$K(t)(\ddot{\phi} + 3H\dot{\phi}) + \dot{K}(t)\,\dot{\phi} + V_{,\phi} = 0, \tag{2.5}$$

i.e. the canonical Klein–Gordon relation with an additional, time–dependent friction term $\dot{K}\,\dot{\phi}$.

2.3 Einstein equations, Raychaudhuri form, and Ricci scalar

Because the metric sector is pure GR (no non-minimal coupling and constant $M_{\rm pl}$),

$$H^2 = \frac{1}{3M_{\rm pl}^2}(\rho_m + \rho_r + \rho_\phi), \qquad \dot{H} = -\frac{1}{2M_{\rm pl}^2}\Big(\rho_m + \tfrac{4}{3}\rho_r + \rho_\phi + p_\phi\Big). \tag{2.6}$$

Substituting $\rho_\phi + p_\phi = K\,\dot\phi^2$ into the second relation yields the Raychaudhuri form

$$\dot{H} = -\frac{1}{2M_{\rm pl}^2}\left(\rho_m + \tfrac{4}{3}\rho_r + K\,\dot\phi^2\right), \tag{2.7}$$

and spatial flatness implies the FRW Ricci scalar

$$R = 6\left(2H^2 + \dot{H}\right). \tag{2.8}$$

2.4 Continuity relations and covariant source

Minimal coupling implies the standard matter and radiation continuity equations,

$$\dot\rho_m + 3H\rho_m = 0, \qquad \dot\rho_r + 4H\rho_r = 0. \tag{2.9}$$

Because $P(X, \phi, t)$ depends explicitly on the background time through $K(t)$, the scalar stress tensor is not separately conserved. The associated Noether identity for explicit coordinate dependence gives

$$\nabla_\mu T^\mu{}_{\nu(\phi)} \;=\; -\,\partial_\nu P(X,\phi,t) \;=\; -\,X\,\partial_\nu K, \tag{2.10}$$

which on FRW reduces to the exact background relation

$$\dot\rho_\phi = \tfrac{1}{2}\dot{K}\,\dot\phi^2 + K\,\dot\phi\,\ddot\phi + V_{,\phi}\dot\phi, \tag{2.11}$$

$$\dot\rho_\phi + 3H(\rho_\phi + p_\phi) = \tfrac{1}{2}\dot{K}\,\dot\phi^2 + K\,\dot\phi\,(\ddot\phi + 3H\dot\phi) + V_{,\phi}\dot\phi = -\,\tfrac{1}{2}\,\dot{K}\,\dot\phi^2, \tag{2.12}$$

after using (2.5). Thus the scalar component exchanges energy with the preferred background clock at a rate set by $\dot{K}$. The *total* stress–energy remains conserved by the Bianchi identity, and (2.6)–(2.7) hold identically.[1]

2.5 Stability and propagation speed

For general $P(X, \phi, t)$, the scalar sound speed is $c_s^2 = P_{,X}/(P_{,X} + 2XP_{,XX})$ [6]. Here $P = K(t)X - V(\phi)$ gives $P_{,X} = K(t)$ and $P_{,XX} = 0$, hence

$$c_s^2 = 1, \qquad \text{no gradient instability.} \tag{2.13}$$

Ghost freedom requires $P_{,X} > 0$, i.e. $K(t) > 0$. The metric sector is Einstein–Hilbert, so the tensor speed is luminal and the Planck mass is constant, consistent with GW170817/GRB bounds [1, 2].

[1] For a Lagrangian density $\mathcal{L}(g_{\mu\nu}, \phi, \partial\phi; x)$ with explicit coordinate dependence, diffeomorphism invariance implies $\nabla_\mu T^\mu{}_\nu = -\partial_\nu\mathcal{L}$ evaluated on shell. For $P = K(t)X - V(\phi)$, $\partial_\nu P = X\,\partial_\nu K$, yielding (2.10). See e.g. [6, Sec. 2].

2.6 EFT–DE mapping (for later use)

Expansion of the quadratic action in the EFT–DE basis [3–5] yields

$$\alpha_K = \frac{K(t)\,\dot{\phi}^2}{H^2 M_{\rm pl}^2}, \qquad \alpha_B = \alpha_M = \alpha_T = 0, \tag{2.14}$$

i.e. a pure–kinetic background running without braiding, Planck–mass evolution, or tensor speed excess. This is the inflation/EFT pattern of time–dependent background coefficients in unitary gauge (cf. 22).

Consistency diagnostics (used later in numerics). The combination $3M_{\rm pl}^2 H^2 - \rho_m - \rho_r - \rho_\phi$ vanishes identically by (2.6); monitoring its fractional magnitude serves as an energy-closure check. Differentiating (2.6) and using (2.9) and (2.12) reproduces (2.7) exactly; this provides an independent Raychaudhuri-consistency diagnostic during integration.

Section 2 at a glance.

Action: $S = \int \mathrm{d}^4x\sqrt{-g}\left[\tfrac{1}{2}M_{\rm pl}^2 R + K(t)X - V(\phi)\right], \quad X \equiv -\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi.$

Stress tensor: $T^{(\phi)}_{\mu\nu} = K\,\partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}(KX - V)\,.$

Background: $\rho_\phi = K\dfrac{\dot\phi^2}{2} + V, \quad p_\phi = K\dfrac{\dot\phi^2}{2} - V, \quad w_\phi = \dfrac{K\dot\phi^2/2 - V}{K\dot\phi^2/2 + V}.$

KG: $K(\ddot\phi + 3H\dot\phi) + \dot K\,\dot\phi + V_{,\phi} = 0.$

Einstein: $H^2 = \dfrac{\rho_m + \rho_r + \rho_\phi}{3M_{\rm pl}^2}, \quad \dot H = -\dfrac{\rho_m + \frac{4}{3}\rho_r + K\dot\phi^2}{2M_{\rm pl}^2}, \quad R = 6(2H^2 + \dot H).$

Source: $\nabla_\mu T^\mu{}_{\nu(\phi)} = -\,\partial_\nu K\,X \;\Rightarrow\; \dot\rho_\phi + 3H(\rho_\phi + p_\phi) = -\tfrac{1}{2}\dot K\,\dot\phi^2.$

EFT–DE: $\alpha_K = \dfrac{K\dot\phi^2}{H^2 M_{\rm pl}^2}, \quad \alpha_B = \alpha_M = \alpha_T = 0, \quad c_s^2 = 1.$

Equations (2.3)–(2.14) constitute the covariant and background foundation used in the subsequent e–fold formulation, linear perturbations, and observables.

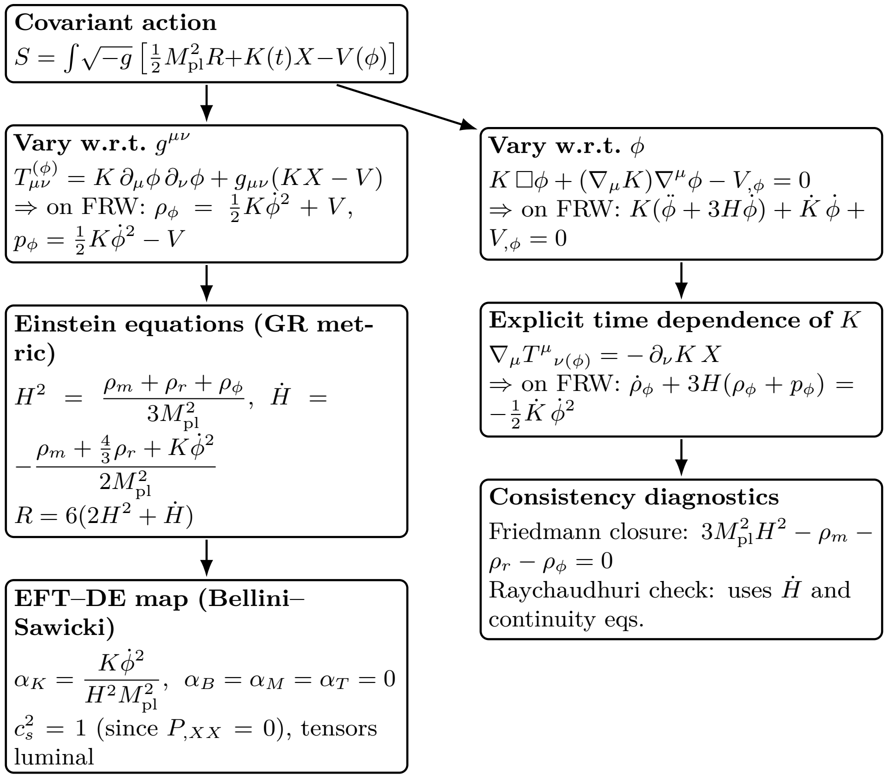

Figure 2.1: Dependency flow for the core background relations in Sec. 2. Every equation used later descends from the covariant action by variation and specialization to FRW, with $K(t)$ entering only as a background kinetic normalization.

3 E-fold reformulation and closed background system

It is convenient to reparametrize time by the number of e-folds,

$$N \equiv \ln a, \qquad (\,)' \equiv \frac{d}{dN} = \frac{1}{H}\frac{d}{dt}, \tag{3.1}$$

a choice standard in cosmology and inflationary dynamics [23, 24]. For any sufficiently smooth background function $f(t)$,

$$\dot f = Hf', \qquad \ddot f = H^2 f'' + HH'f', \tag{3.2}$$

which follows directly by the chain rule and $\dot N = H$. It is also convenient to denote $E \equiv H'/H$.

3.1 Klein–Gordon equation in e-fold time

Inserting (3.2) into the FRW Klein–Gordon equation (2.5), $K(\ddot\phi + 3H\dot\phi) + \dot K\,\dot\phi + V_{,\phi} = 0$, gives

$$K\Big(H^2\phi'' + HH'\phi' + 3H^2\phi'\Big) + \dot K\,(H\phi') + V_{,\phi} = 0.$$

Dividing by KH^2 and using $K'/K = \dot K/(HK)$ yields the exact e-fold form

$$\boxed{\phi'' + \left(3 + \frac{H'}{H} + \frac{K'}{K}\right)\phi' + \frac{V_{,\phi}}{H^2K} = 0}. \tag{3.3}$$

The kinetic invariant and scalar energy density become

$$\boxed{X = \tfrac{1}{2}H^2\phi'^2, \qquad \rho_\phi = K\,\frac{H^2}{2}\phi'^2 + V(\phi)}, \tag{3.4}$$

which follows from $X = \frac{1}{2}\dot\phi^2$ and (2.3).

3.2 Hubble derivative, Ricci scalar, and redshift

From the Raychaudhuri relation (2.7), $\dot H = -(\rho_m + \frac{4}{3}\rho_r + K\,\dot\phi^2)/(2M_{\rm pl}^2)$, and $\dot H = H^2(H'/H)$, one finds

$$\boxed{\frac{H'}{H} = -\frac{1}{2M_{\rm pl}^2 H^2}\left(\rho_m + \tfrac{4}{3}\rho_r + K\,H^2\phi'^2\right)}. \tag{3.5}$$

Expressed in density parameters $\Omega_i \equiv \rho_i/(3M_{\rm pl}^2 H^2)$ for $i \in \{m, r\}$,

$$\boxed{\frac{H'}{H} \;=\; -\frac{1}{2}\left(3\,\Omega_m + 4\,\Omega_r + \frac{K\,\phi'^2}{M_{\rm pl}^2}\right)}, \tag{3.6}$$

and spatial flatness gives the FRW Ricci scalar ratio

$$\boxed{\frac{R}{H^2} = 6\left(2 + \frac{H'}{H}\right)}. \tag{3.7}$$

The e-fold/redshift mapping and minimally coupled background scalings are

$$\boxed{z(N) = e^{-N} - 1}, \qquad \rho_m(N) = \rho_{m0}\,e^{-3N}, \quad \rho_r(N) = \rho_{r0}\,e^{-4N}, \tag{3.8}$$

which follow from $\nabla_\mu T^\mu{}_\nu$ conservation for dust and radiation on FRW.

Closure, equation of state, and deceleration. With $\Omega_\phi \equiv 1 - \Omega_m - \Omega_r$ and (3.4),

$$\boxed{\Omega_\phi(N) = \frac{K\,\phi'^2}{6\,M_{\rm pl}^2} + \frac{V(\phi)}{3M_{\rm pl}^2 H^2}, \qquad w_\phi(N) = \frac{\frac{K\,\phi'^2}{6M_{\rm pl}^2} - \frac{V}{3M_{\rm pl}^2 H^2}}{\frac{K\,\phi'^2}{6M_{\rm pl}^2} + \frac{V}{3M_{\rm pl}^2 H^2}}}, \tag{3.9}$$

and from (3.5),

$$\boxed{w_{\rm eff}(N) = -1 - \frac{2}{3}\frac{H'}{H}, \qquad q(N) = -1 - \frac{H'}{H}}. \tag{3.10}$$

3.3 Closed autonomous system

Equations (3.3), (3.5), and (3.4) form a closed, first-order autonomous system once $K(N)$ and $V(\phi)$ are specified (subject to $K > 0$). For stable numerics, the state vector

$$\mathbf{y} \equiv (\phi,\; s,\; y_H) \equiv (\phi,\; \phi',\; \ln H) \tag{3.11}$$

evolves according to

$$s' = -\left(3 + \frac{H'}{H} + \frac{K'}{K}\right) s - \frac{V_{,\phi}}{H^2 K}, \tag{3.12a}$$

$$y_H' = \frac{H'}{H} = -\frac{1}{2M_{\rm pl}^2 H^2}\left(\rho_m + \tfrac{4}{3}\rho_r + K\, H^2 s^2\right), \tag{3.12b}$$

$$X = \tfrac{1}{2}H^2 s^2, \qquad \rho_\phi = K\,\frac{H^2}{2}s^2 + V(\phi), \tag{3.12c}$$

with ρ_m, ρ_r from (3.8) and $H = e^{y_H}$. The evaluation of K'/K for the two specifications appears in Sec. 4. The denominators H^2 and H^2K are nonzero by construction ($H > 0$, $K > 0$).

3.4 Well-posedness and admissible initial data

Let $\mathbf{y}' = \mathbf{F}(N, \mathbf{y})$ on a domain where $K(N) > 0$ and $H(N) > 0$. For any smooth $K(N)$ and $V(\phi)$, the right-hand side $\mathbf{F}$ is continuous and locally Lipschitz in $\mathbf{y}$; the Picard–Lindelöf theorem then guarantees local existence and uniqueness through any $N_{\rm ini}$ with admissible initial data [25]. A matter-era initialization consistent with $\rho_m \gg \rho_r, \rho_\phi$ is

$$\phi'(N_{\rm ini}) = 0, \qquad H(N_{\rm ini}) = H_0\sqrt{\Omega_{m0}e^{-3N_{\rm ini}} + \Omega_{r0}e^{-4N_{\rm ini}}}\,, \tag{3.13}$$

while $\phi(N_{\rm ini})$ is fixed by enforcing $\Omega_\phi(0) = 1 - \Omega_{m0} - \Omega_{r0}$ using (3.9). Global continuation holds as long as $K > 0$ and $H > 0$.

Diagnostics (for numerics). For consistency checks during integration, the GR identities defined in Sec. 8 are monitored: the Friedmann closure $\mathcal{C}_F = 1 - \Omega_m - \Omega_r - \Omega_\phi$ and the Raychaudhuri residual $\mathcal{C}_R$ in e-fold form. Both vanish analytically; numerically they should remain $\ll 1$ at the chosen tolerances.

Limits. Setting $K \equiv 1$ reduces the system to canonical quintessence. With $V = \text{const}$ and $\phi' = 0$ the system reproduces ΛCDM.

Table 3.1: E-fold reformulation: core identities, evolution equations, and diagnostics used in Sec. 3. All relations are exact given the assumptions in Sec. 2.

Quantity	E-fold form / Definition	Ref.
Time variable	$N \equiv \ln a, \quad (\,)' \equiv \mathrm{d}/\mathrm{d}N = H^{-1}\mathrm{d}/\mathrm{d}t$	(3.1)
Chain rules	$\dot f = Hf', \quad \ddot f = H^2 f'' + HH'f'$	(3.2)
Klein–Gordon	$\phi'' + \left(3 + \frac{H'}{H} + \frac{K'}{K}\right)\phi' + \frac{V_{,\phi}}{H^2 K} = 0$	(3.3)
Kinetic & ρ_ϕ	$X = \frac{1}{2}H^2\phi'^2, \quad \rho_\phi = K\,\frac{H^2}{2}\phi'^2 + V(\phi)$	(3.4)
H'/H (fluids)	$\frac{H'}{H} = -\frac{1}{2M_{\rm pl}^2 H^2}\left(\rho_m + \frac{4}{3}\rho_r + K\,H^2\phi'^2\right)$	(3.5)
H'/H (Ω)	$\frac{H'}{H} = -\frac{1}{2}\left(3\Omega_m + 4\Omega_r + \frac{K\,\phi'^2}{M_{\rm pl}^2}\right)$	(3.6)
Ricci scalar	$\frac{R}{H^2} = 6\left(2 + \frac{H'}{H}\right)$	(3.7)
Redshift map	$z(N) = e^{-N} - 1, \quad \rho_m = \rho_{m0}e^{-3N}, \quad \rho_r = \rho_{r0}e^{-4N}$	(3.8)
Closure & w_ϕ	$\Omega_\phi = \frac{K\,\phi'^2}{6M_{\rm pl}^2} + \frac{V}{3M_{\rm pl}^2 H^2}, \quad w_\phi = \frac{\frac{K\phi'^2}{6M_{\rm pl}^2} - \frac{V}{3M_{\rm pl}^2H^2}}{\frac{K\phi'^2}{6M_{\rm pl}^2} + \frac{V}{3M_{\rm pl}^2H^2}}$	(3.9)
Effective EoS	$w_{\rm eff} = -1 - \frac{2}{3}\frac{H'}{H}, \quad q = -1 - \frac{H'}{H}$	(3.10)
State vector	$\mathbf{y} = (\phi,\, s,\, y_H) \equiv (\phi,\, \phi',\, \ln H)$	§3.3
Autonomous sys.	$s' = -\left(3 + \frac{H'}{H} + \frac{K'}{K}\right)s - \frac{V_{,\phi}}{H^2K}; \;\; y_H' = \frac{H'}{H}; \;\; X = \frac{1}{2}H^2 s^2$	(3.12)
K'/K (note)	From chosen $K(N)$: phenom. (4.14) or curvature (4.10)	Sec. 4
Diagnostics	Friedmann closure $\mathcal{C}_F = 1 - \Omega_m - \Omega_r - \Omega_\phi$; Raychaudhuri residual $\mathcal{C}_R$ (both $\to 0$ analytically)	Sec. 8

4 Kinetic normalization $K(N)$: specifications and derived identities

Only the *background* kinetic normalization is allowed to vary in time while the metric sector remains Einstein–Hilbert. In EFT language (unitary gauge), a time–dependent background coefficient multiplying $X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$ is consistent with the Bellini–Sawicki corner $\alpha_B = \alpha_M = \alpha_T = 0$ and produces a pure $\alpha_K > 0$ [3–5]. Throughout this section,

$$E \;\equiv\; \frac{H'}{H}, \qquad \frac{R}{H^2} = 6\,(2 + E) \quad (\text{cf. } (3.7)).$$

Dimensions: K is dimensionless; R has mass dimension 2; the ratio α/M^2 is dimensionless.

UV motivation (operator origin). In the covariant derivative expansion of a light scalar coupled to gravity, integrating out heavy fields or evaluating loop corrections in curved spacetime generates higher–dimension operators such as

$$\Delta\mathcal{L} \;\supset\; \frac{c_R}{M^2}\,R\,(\partial\phi)^2 \quad \text{and} \quad \frac{c_{\mathcal{G}}}{M^2}\,\mathcal{G}_{\mu\nu}\,\partial^\mu\phi\,\partial^\nu\phi, \tag{4.1}$$

suppressed by a UV scale M with Wilson coefficients $c_R, c_{\mathcal{G}} = \mathcal{O}(1)$ by power counting [7, 12, 26]. On an FRW background the second operator is degenerate with the first at background level, and both reduce to a time–dependent rescaling of X. This motivates a curvature–renormalized kinetic normalization $K(t) = 1 + \mathcal{O}(R/M^2)$ without introducing any metric–sector modifications or braiding. The EFT–DE mapping then yields $\alpha_K > 0$ with $\alpha_B = \alpha_M = \alpha_T = 0$ [5].

4.1 Curvature–motivated specification

A minimal curvature–suppressed form is

$$\boxed{K(N) = 1 + \alpha\,\frac{R(N)}{M^2}}, \tag{4.2}$$

with $R/H^2 = 6(2+E)$. Differentiation gives

$$\boxed{\frac{K'}{K} = \frac{\alpha}{M^2}\frac{R'}{1+\alpha R/M^2}}. \tag{4.3}$$

Writing $R = H^2 r$ with $r \equiv 6(2+E)$ and using $(H^2)' = 2EH^2$,

$$\frac{R'}{H^2} \;=\; r' + 2Er \;=\; 6\Big(E' + 4E + 2E^2\Big). \tag{4.4}$$

From (3.5),

$$E \;=\; -\frac{1}{2M_{\rm pl}^2 H^2}\Big(\rho_m + \tfrac{4}{3}\rho_r + KH^2\phi'^2\Big). \tag{4.5}$$

Differentiating (4.5) with respect to N, using $\rho_m' = -3\rho_m$, $\rho_r' = -4\rho_r$, and the background scalar equation (3.12a), yields

$$E' = -\frac{1}{2M_{\rm pl}^2 H^2}\Big[-3\rho_m - \tfrac{16}{3}\rho_r + (KH^2\phi'^2)'\Big] - 2E^2, \tag{4.6}$$

$$(KH^2\phi'^2)' = -\,H^2\Big(K'\phi'^2 + 6K\phi'^2\Big) - 2\,\phi' V_{,\phi}. \tag{4.7}$$

Substituting (4.6)–(4.7) into (4.4) shows that R'/H^2 is *affine* in K':

$$\frac{R'}{H^2} \;=\; A \;+\; B\,K', \qquad B = \frac{3\,\phi'^2}{M_{\rm pl}^2}, \qquad A = 24E + \frac{18K\,\phi'^2}{M_{\rm pl}^2} + \frac{9\rho_m + 16\rho_r + 6\phi' V_{,\phi}}{M_{\rm pl}^2 H^2}. \tag{4.8}$$

Defining

$$c \;\equiv\; \frac{\alpha H^2/M^2}{1+\alpha R/M^2}, \tag{4.9}$$

and inserting (4.8) into (4.3) gives an *algebraic*, iteration–free solution:

$$\boxed{\frac{K'}{K} \;=\; \frac{c\,A}{1 - c\,B\,K} \;=\; \frac{\dfrac{\alpha H^2}{M^2}\,\dfrac{A}{1+\alpha R/M^2}}{1 - \dfrac{\alpha H^2}{M^2}\,\dfrac{3K\,\phi'^2}{M_{\rm pl}^2\,(1+\alpha R/M^2)}}}. \tag{4.10}$$

All quantities on the right are functions of the state vector (ϕ, ϕ', H) and known sources via (3.12). The denominator in (4.10) must remain nonzero.

Background limits (consistency). During radiation domination $R = 0$ so $K \simeq 1$ and $K'/K \simeq 0$; during matter domination $E \simeq -3/2$ so $R/H^2 \simeq 3$ and $K \simeq 1 + 3\alpha H^2/M^2$ (slowly varying); in de Sitter $E = 0$ so $R/H^2 = 12$, giving a constant shift $K = 1 + 12\alpha H^2/M^2$. These limits follow directly from (3.7) and (4.2).

Evaluation protocol. At each step in N: (i) compute $E = H'/H$ from (3.12b); (ii) form $R/H^2 = 6(2+E)$; (iii) assemble A and c; (iv) evaluate K'/K via (4.10). No recursion in R' or K' is required.

Parameter priors and early–time safety. Positivity requires $K(N) = 1 + \alpha R/M^2 > 0$ over the integration range. To preserve standard pre–drag early physics, enforce

$$\left| \frac{\alpha}{M^2} R(N) \right| \ll 1 \quad \text{for all } z \geq z_{\text{drag}}, \tag{4.11}$$

ensuring $K \to 1$ before BAO/CMB anchoring [27, 28]. If $\alpha < 0$, a conservative bound is $\alpha > -M^2/\max_N R(N)$, monitored together with the non–vanishing denominator in (4.10).

4.2 IR slow running (RG–like form)

A minimal background flow may be posed as a *constant fractional running* of $K - 1$:

$$\frac{d \ln |K-1|}{dN} = -p \quad \Longrightarrow \quad \boxed{K(N) = 1 + K_p \, e^{-p(N-N_p)}}. \tag{4.12}$$

This is the unique two–parameter solution of a constant "beta function" $\beta_K \equiv d \ln |K-1|/d \ln a = -p$, analogous to standard RG flows [29, 30]. Choosing the pivot N_p yields the equivalent redshift form

$$\boxed{K(z) = 1 + K_0(1+z)^p} \qquad \Longleftrightarrow \qquad \boxed{K(N) = 1 + K_0 \, e^{-pN}}, \tag{4.13}$$

with $K_0 = K_p \, e^{pN_p}$. The exact identities used in the background system are

$$\boxed{K' = -p\,(K-1), \qquad \frac{K'}{K} = -p\,\frac{K-1}{K}}. \tag{4.14}$$

The form in (4.13) is the minimal analogue of the CPL expansion for $w(z)$ [31, 32]: it is controlled by two parameters, remains regular for large excursions, and reduces locally to a linearized perturbation about a pivot if $|p(N - N_p)| \ll 1$.

Positivity, early–time behavior, and interpretation. If $p > 0$, then $K \to 1$ toward the future and $|K - 1|$ grows toward the past; if $p < 0$, the running turns on toward the future. Enforcing

$$1 + K_0 \, e^{-pN} \; > \; 0 \quad \forall \, N \in [N_{\min}, 0] \tag{4.15}$$

guarantees ghost freedom ($K > 0$). The running modifies only the background normalization of X and introduces no new perturbative operators, so $\alpha_B = \alpha_M = \alpha_T = 0$ and $c_s^2 = 1$ remain exact (Secs. 2.5, 6).

4.3 Admissibility and stability

Ghost freedom requires

$$\boxed{K(N) > 0 \quad \text{for all relevant } N.} \tag{4.16}$$

No gradient instability arises because $P_{,XX} = 0 \Rightarrow c_s^2 = 1$, and the metric sector remains GR with luminal tensors [1, 2]. For the curvature–motivated case, the algebraic denominator in (4.10) must remain nonzero; for sufficiently small $|\alpha R/M^2|$ this is automatic and can be monitored alongside $K > 0$ during numerical integration.

Table 4.1: Background-only kinetic normalizations $K(N)$ used in Sec. 4. Each card lists the definition, the exact identity for K'/K (by reference to the main text), admissibility notes for numerics, and the EFT–DE corner.

(a) Curvature-motivated

Item	**Expression / Notes**
Definition	$K = 1 + \alpha\, R/M^2$ (Eq. (4.2)), with $R/H^2 = 6(2 + E)$ and $E = H'/H$.
K'/K identity	Closed, iteration-free algebraic form (Eq. (4.10)); uses affine split $R'/H^2 = A + B\,K'$ with $B = 3\phi'^2/M_{\rm pl}^2$ and A from Eq. (4.8).
Admissibility	Require $K > 0$ on $N \in [N_{\rm min}, 0]$ and non-vanishing denominator in Eq. (4.10). Early-time safety prior $\lvert\alpha R/M^2\rvert \ll 1$ before $z_{\rm drag}$ (Eq. (4.11)).
Implementation	Evaluate E, then R/H^2, form (A, B), compute c and return K'/K from Eq. (4.10); no iteration.
EFT–DE corner	$(\alpha_B, \alpha_M, \alpha_T) = (0, 0, 0)$; $\alpha_K > 0$ (Eq. (2.14)).

(b) IR slow running (RG–like)

Item	**Expression / Notes**
Definition	$K(N) = 1 + K_0\, e^{-pN}$ (Eq. (4.13)).
K'/K identity	Exact: $K' = -p\,(K - 1)$, $\dfrac{K'}{K} = -p\,\dfrac{K-1}{K}$ (Eq. (4.14)).
Admissibility	Enforce $K(N) > 0$ on $[N_{\rm min}, 0]$ via Eq. (4.15). Monotonicity set by p.
Implementation	Direct evaluation of K and K'/K at each step; trivial numerically.
EFT–DE corner	$(\alpha_B, \alpha_M, \alpha_T) = (0, 0, 0)$; $\alpha_K > 0$ (Eq. (2.14)).

5 Linear perturbations at first order

This section develops scalar–metric perturbations around the spatially flat FRW background defined earlier. All results follow from the covariant action (1.1), the stress–energy tensor (2.2), and the background relations in Secs. 2–3, working strictly to linear order in perturbations.

Conventions. Overdots denote d/dt. Unless explicitly noted, a prime in this section denotes $d/d\eta$ with η the conformal time; in the growth subsection (Sec. 5.5) a prime denotes $d/d\ln a$. The Fourier convention is $f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, f(\mathbf{k})$, so $\nabla^2 \to -k^2$.

5.1 Newtonian gauge and primary variables

Work in Newtonian gauge,

$$ds^2 = -(1+2\Psi)\,dt^2 + a^2(t)\,(1-2\Phi)\,d\vec{x}^2, \qquad \phi(t,\vec{x}) = \phi_0(t) + \delta\phi(t,\vec{x}). \tag{5.1}$$

Because the metric sector is Einstein–Hilbert (no non-minimal coupling $F(\phi)R$ and no braiding $G(\phi,X)\Box\phi$), the scalar sector contributes no anisotropic stress at linear order for $P(X,\phi,t)$ models [18, 33]. Neglecting late-time photon/neutrino shear on subhorizon scales therefore gives

$$\Phi = \Psi. \tag{5.2}$$

5.2 Perturbed Klein–Gordon equation (derived)

Starting from the covariant equation (2.4), $K\,\Box\phi + (\nabla_\mu K)\nabla^\mu\phi - V_{,\phi} = 0$, and expanding to first order, note that $K = K(t) \Rightarrow \delta K = 0$ and $(\nabla_\mu K)\nabla^\mu\phi = -\dot{K}\,\dot{\phi}$ at all orders. Using the standard expansion of $\Box\phi$ in Newtonian gauge (e.g., [18]) and subtracting the background equation (2.5) yields, in *cosmic time*,

$$K\left(\delta\ddot{\phi} + 3H\,\delta\dot{\phi} + \frac{k^2}{a^2}\delta\phi\right) + \dot{K}\,\delta\dot{\phi} + V_{,\phi\phi}\,\delta\phi \;=\; 4K\,\dot{\phi}_0\,\dot{\Phi} - 2\,V_{,\phi}\,\Phi. \tag{5.3}$$

No additional perturbative operators arise relative to canonical quintessence because $\delta K = 0$. The *conformal-time* form follows by $d/dt = a^{-1}d/d\eta$ and $\mathcal{H} \equiv aH$:

$$K\Big(\delta\phi'' + 2\mathcal{H}\,\delta\phi' + k^2\delta\phi\Big) + a^2 V_{,\phi\phi}\,\delta\phi \;=\; 4K\,\phi_0'\,\Phi' - 2a^2 V_{,\phi}\,\Phi. \tag{5.4}$$

For completeness, the scalar-fluid perturbations read (cf. [18])

$$\delta\rho_\phi = K\left(\dot{\phi}_0\,\delta\dot{\phi} - \dot{\phi}_0^2\,\Phi\right) + V_{,\phi}\,\delta\phi, \tag{5.5}$$

$$\delta p_\phi = K\left(\dot{\phi}_0\,\delta\dot{\phi} - \dot{\phi}_0^2\,\Phi\right) - V_{,\phi}\,\delta\phi, \tag{5.6}$$

$$(\rho_\phi + p_\phi)\,\theta_\phi = K\,\dot{\phi}_0\,\frac{k^2}{a^2}\,\delta\phi, \tag{5.7}$$

and, because $P_{,XX} = 0$ for $P = K(t)X - V(\phi)$, the rest-frame sound speed is $c_s^2 = 1$ with vanishing intrinsic non-adiabatic pressure [6].

5.3 Quadratic action and Mukhanov–Sasaki equation

Define the gauge-invariant Mukhanov–Sasaki (MS) variable and pump field

$$v \;\equiv\; a\left(\delta\phi + \frac{\dot{\phi}_0}{H}\Phi\right), \qquad z^2 \;\equiv\; a^2\frac{K\,\dot{\phi}_0^2}{H^2}. \tag{5.8}$$

Expanding (1.1) to second order in scalar perturbations and integrating out nondynamical constraints yields the canonical MS form

$$S^{(2)} = \frac{1}{2}\int d\eta\, d^3x \left[(v')^2 - (\nabla v)^2 + \frac{z''}{z}\, v^2\right], \tag{5.9}$$

whose variation gives

$$v'' + \left(c_s^2 k^2 - \frac{z''}{z}\right) v = 0, \qquad c_s^2 = \frac{P_{,X}}{P_{,X} + 2X P_{,XX}} = 1, \tag{5.10}$$

with the last equality exact for $P = K(t)X - V(\phi)$ [6, 23, 34].

Super- and subhorizon behavior. On superhorizon scales ($k \to 0$), the comoving curvature $\mathcal{R} \equiv v/z$ is conserved at leading order,

$$\mathcal{R}' \;=\; \mathcal{O}\left(k^2\right), \tag{5.11}$$

which in Newtonian gauge implies the adiabatic relation

$$\delta\phi \;\simeq\; -\frac{\dot{\phi}_0}{H}\,\Phi \qquad (k \ll aH), \tag{5.12}$$

useful for setting initial conditions [23, 34]. On subhorizon scales ($k \gg aH$), $c_s^2 = 1$ ensures pressure support and $\delta\phi$ oscillates on the sound horizon; the scalar does not cluster appreciably.

5.4 Einstein constraints and the Poisson equation

The linearized GR constraints in Newtonian gauge (e.g., [18]) are

$$00: \quad -3H(\dot{\Phi} + H\Phi) - \frac{k^2}{a^2}\Phi = 4\pi G\left(\delta\rho_m + \delta\rho_r + \delta\rho_\phi\right), \tag{5.13}$$

$$0i: \quad \dot{\Phi} + H\Phi = 4\pi G\left[(\rho_m + p_m)\theta_m + (\rho_r + p_r)\theta_r + (\rho_\phi + p_\phi)\theta_\phi\right]. \tag{5.14}$$

With $\Phi = \Psi$ and negligible radiation shear at late times, the subhorizon limit ($k \gg aH$) reduces to the GR Poisson equation

$$-\frac{k^2}{a^2}\,\Phi \;=\; 4\pi G\,\rho_m\,\delta_m, \tag{5.15}$$

i.e., no modification of the gravitational coupling and no slip. In the EFT–DE language this corresponds to

$$\mu(a,k) = 1, \qquad \Sigma(a,k) = 1, \qquad \eta(a,k) \equiv \frac{\Phi}{\Psi} - 1 = 0 \quad \text{(linear order, all } k\text{)}. \tag{5.16}$$

5.5 Growth of matter perturbations (GR form)

Let $D(a)$ be the linear growth factor of matter overdensities in the subhorizon GR limit. Combining (5.15) with mass conservation and the Euler equation for pressureless matter gives

$$D'' + \left(2 + \frac{H'}{H}\right) D' - \frac{3}{2}\,\Omega_m(a)\,D = 0, \tag{5.17}$$

where here a prime denotes $d/d\ln a$, and $\Omega_m(a) \equiv \rho_m/(3M_{\rm pl}^2 H^2)$. In QKDE, all linear-level signatures enter (5.17) only through the background $H(a)$ determined by Sec. 3; no scale dependence is induced.

Key takeaway. The scalar mode is luminal and ghost-free ($c_s^2 = 1$, $K > 0$), the metric potentials satisfy $\Phi = \Psi$, and the GR Poisson equation holds on linear/subhorizon scales. Consequently, late-time linear observables (RSD, weak lensing, ISW) are affected only through the background expansion $H(a)$ and the induced growth history $D(a)$.

6 EFT–DE mapping, linear viability, and relations to existing theories

This section records the Effective Field Theory of Dark Energy (EFT–DE) map of the covariant action (1.1), states linear-viability conditions, and situates the construction within the broader landscape of dark-energy and modified-gravity frameworks. All statements refer to the background and first-order perturbative level defined in Secs. 2–5.

6.1 Bellini–Sawicki functions (derivation and specialization)

The EFT–DE description encodes linear dynamics in the Bellini–Sawicki functions $\{\alpha_K, \alpha_B, \alpha_M, \alpha_T\}$ and an effective Planck mass M_*^2 [3–5]. For minimally coupled $P(X, \phi, t)$ models (no $F(\phi)R$, no braiding), the general result reads

$$\alpha_K = \frac{2X(P_{,X} + 2XP_{,XX})}{H^2 M_*^2}, \qquad \alpha_B = 0, \qquad \alpha_M \equiv \frac{d\ln M_*^2}{d\ln a} = 0, \qquad \alpha_T = 0, \tag{6.1}$$

with $M_*^2 = M_{\rm pl}^2$ in the Einstein–Hilbert metric sector. Specializing to

$$P(X, \phi, t) = K(t)\,X - V(\phi),$$

gives $P_{,X} = K(t)$ and $P_{,XX} = 0$, hence

$$\boxed{\alpha_K = \frac{2X\,K}{H^2 M_{\rm pl}^2} = \frac{K\,\dot\phi^2}{H^2 M_{\rm pl}^2} \geq 0, \qquad \alpha_B = \alpha_M = \alpha_T = 0} \tag{6.2}$$

and $\alpha_H = 0$ (no beyond-Horndeski operator). Using $X = \frac{1}{2}H^2\phi'^2$ (Sec. 3) also yields $\alpha_K = K\,\phi'^2/M_{\rm pl}^2$ in $N \equiv \ln a$. This corner saturates the GW170817 constraint by construction through $\alpha_T = 0$ and $M_*^2 = \text{const}$ [1, 2].

Table 6.1: EFT–DE and phenomenological functions for the QKDE construction (linear order).

	α_K	α_B	α_M	α_T	μ	Σ
QKDE	$K\dot{\phi}^2/(H^2M_{\rm pl}^2)\ \geq 0$	0	0	0	1	1

6.2 Phenomenological response functions μ, Σ, and slip

In the quasi-static, subhorizon regime the linear response to matter is often summarized by [35]

$$-\frac{k^2}{a^2}\Psi \;=\; 4\pi G\,\mu(a,k)\,\rho_m\,\Delta_m, \tag{6.3}$$

$$-\frac{k^2}{a^2}(\Phi+\Psi) \;=\; 8\pi G\,\Sigma(a,k)\,\rho_m\,\Delta_m, \tag{6.4}$$

with Δ_m the comoving-gauge matter perturbation. For (6.2) together with $c_s^2 = 1$ (Sec. 5), the linearized Einstein constraints reduce exactly to GR at all k:

$$\boxed{\mu(a,k)=1, \qquad \Sigma(a,k)=1, \qquad \eta(a,k)\equiv\frac{\Phi}{\Psi}-1=0}\,, \tag{6.5}$$

up to negligible late-time shear from standard relativistic species. As a result, RSD, WL, and ISW respond solely through the background $H(a)$ entering the GR growth equation (5.17).

6.3 Linear stability and admissibility (scalar and tensor)

Ghost freedom of the scalar mode requires positivity of the kinetic coefficient:

$$\boxed{K(t)>0 \quad\Rightarrow\quad \alpha_K\geq 0\ \ \text{(equality when } \phi'=0)}\,. \tag{6.6}$$

Because $P_{,XX}=0$ for $P=K(t)X-V(\phi)$, the sound speed is luminal,

$$c_s^2=\frac{P_{,X}}{P_{,X}+2XP_{,XX}}=1, \tag{6.7}$$

so no gradient (Laplace) instability arises [6]. The tensor sector remains Einstein–Hilbert with $\alpha_T=0$ and $M_*^2=\text{const}$ ($\alpha_M=0$), satisfying GW170817/GRB limits [1, 2]. Background well-posedness additionally requires $H>0$ and, for the curvature-motivated K in Sec. 4, a non-vanishing denominator in (4.10).

6.4 Relations to existing dark-energy and modified-gravity frameworks

The location in theory space is clarified by comparing covariant operators, EFT maps, and linear phenomenology.

6.4.1 Canonical quintessence

Setting $K\equiv 1$ in (1.1) reproduces canonical quintessence [15, 16]. Then $\alpha_K=\dot{\phi}^2/(H^2M_{\rm pl}^2)$ and $\alpha_B=\alpha_M=\alpha_T=0$. The present construction differs only by a *background* time dependence in the kinetic normalization, appearing as the friction $\dot{K}\,\dot{\phi}$ in (2.5) and as $z^2=a^2K\dot{\phi}^2/H^2$ in (5.8). Linear gravitational relations remain GR (Sec. 5).

6.4.2 General k-essence

k-essence permits $P(X, \phi)$ with $P_{,XX} \neq 0$ [6, 17]. The present theory corresponds to the *linear-X* subclass $P(X, \phi, t) = K(t)X - V(\phi)$, for which

$$P_{,X} = K(t), \qquad P_{,XX} = 0 \implies c_s^2 = 1. \tag{6.8}$$

Consequently no low-sound-speed clustering or scale-dependent growth is generated at linear order. Since K depends only on time (unitary-gauge background), the EFT–DE language [3–5] is the natural description.

6.4.3 Horndeski and beyond-Horndeski

In the Bellini–Sawicki parametrization [5], the construction sits at the conservative corner

$$\alpha_K > 0, \qquad \alpha_B = \alpha_M = \alpha_T = 0 \quad (\alpha_H = 0), \tag{6.9}$$

i.e., no braiding, no Planck-mass running, and luminal tensors—consistent with GW170817 [1, 2]. Many Horndeski/Galileon models feature $\alpha_B \neq 0$ and/or $\alpha_M \neq 0$ and are tightly constrained post-GW170817 [35, 36]. GLPV/DHOST adds $\alpha_H \neq 0$ [37]; this is absent here.

6.4.4 $f(R)$ gravity (Jordan-frame scalar–tensor)

Metric $f(R)$ gravity is equivalent to a scalar–tensor theory with $F(\phi)R$, implying $\alpha_M \neq 0$ and typically $\alpha_B \neq 0$ [33, 38]. Linear phenomenology departs from GR via $\mu \neq 1$ and $\Sigma \neq 1$. The present model keeps the metric sector strictly Einstein–Hilbert, hence $\alpha_M = \alpha_B = \alpha_T = 0$ and $\mu = \Sigma = 1$.

6.4.5 Interacting dark energy / fifth-force models

Dark-matter–dark-energy couplings modify the matter continuity/Euler equations and produce effective fifth forces [35, 39]. Here matter and radiation are minimally coupled [Eqs. (2.9)], and the Poisson equation is purely GR [Eq. (5.15)]. Any detection of scale-dependent $\mu(a, k) \neq 1$ or gravitational slip $\eta \neq 0$ at linear order would falsify this baseline.

6.4.6 Phantom regimes and energy conditions

With $K > 0$, the scalar obeys $\rho_\phi + p_\phi = K\dot{\phi}^2 \geq 0$, enforcing $w_\phi \geq -1$ (from (2.3)). Phantom behavior would require $K < 0$ and thus a ghost, which is excluded by (6.6).

6.4.7 Inflationary EFT analogy

Time-dependent coefficients in the EFT of Inflation provide a useful analogy [22]: a single running background coefficient multiplies the kinetic operator while the metric sector remains GR. Here this reduces to $z^2 = a^2 K\dot{\phi}^2/H^2$ with $c_s^2 = 1$ [Eqs. (5.8)–(5.10)].

6.4.8 Observational degeneracies and falsifiability

Because linear phenomenology is GR-like, late-time background histories $H(a)$ are degenerate with those obtained from smooth $w(z)$ quintessence. The combination of *(i)* $\mu = \Sigma = 1$, $\eta = 0$, *(ii)* luminal tensors with constant $M_{\rm pl}$, and *(iii)* a single running parameter $\alpha_K \geq 0$ is distinctive and falsifiable: evidence for $\mu \neq 1$, $\Sigma \neq 1$, or $c_T^2 \neq 1$ at late times would rule out this framework.

Key takeaway. At linear order the construction occupies the most conservative EFT–DE corner: $\Phi = \Psi$, $\mu = \Sigma = 1$, $c_s^2 = c_T^2 = 1$, $M_{\rm pl} = \text{const}$, with the only running parameter $\alpha_K \geq 0$ set by the background kinetic normalization $K(t)$.

Table 6.2: EFT–DE (Bellini–Sawicki) parameters and linear phenomenology (quasi-static, subhorizon). Entries marked "$\neq 0$ (generic)" are typically nonzero but model dependent; "$= 0$" indicates identically zero within the class.

Model / Class	$\boldsymbol{\alpha_K}$	$\boldsymbol{\alpha_B}$	$\boldsymbol{\alpha_M}$	$\boldsymbol{\alpha_T}$	c_s^2	(μ, Σ)
QKDE (this work)	$\frac{K\dot{\phi}^2}{H^2 M_{\rm pl}^2} > 0$	0	0	0	1	$(1, 1)$
Canonical quintessence	$\frac{\dot{\phi}^2}{H^2 M_{\rm pl}^2} > 0$	0	0	0	1	$(1, 1)$
Minimally coupled k-essence ($P(X, \phi)$)	> 0	0	0	0	$\neq 1$ (general)	$(1, 1)$
$f(R)$ gravity (metric)	model dep.	$\neq 0$ (generic)	$\neq 0$ (generic)	$0^\dagger$	model dep.	$(\neq 1, \neq 1)$
Horndeski (generic)	> 0	$\neq 0$ (generic)	$\neq 0$ (generic)	$0^\dagger$	model dep.	model dep.
Beyond-Horndeski (DHOST)	> 0	$\neq 0$ (generic)	$\neq 0$ (generic)	$0^\dagger$	model dep.	model dep.

† Viable late-time models consistent with GW170817 enforce $\alpha_T \simeq 0$.

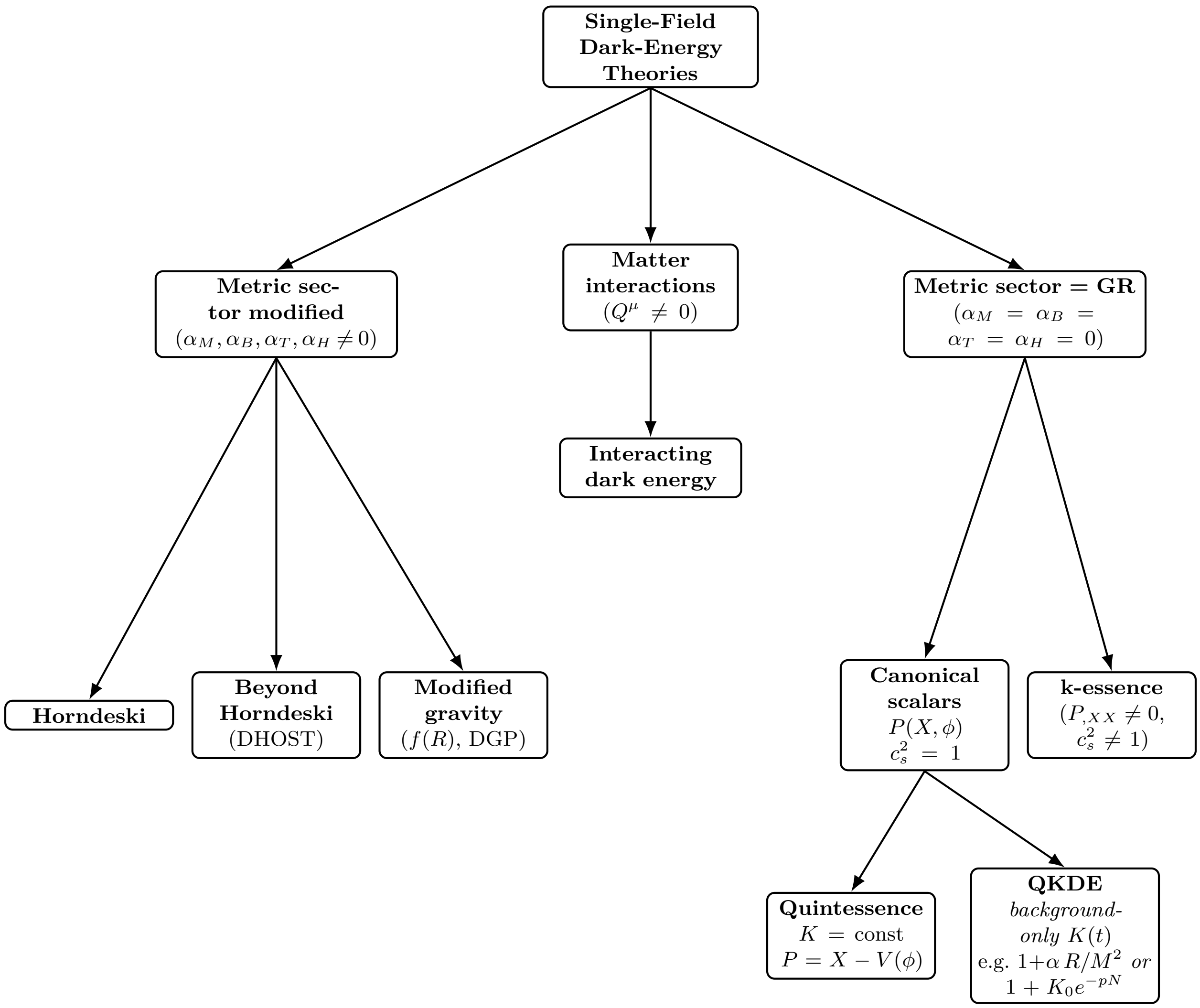

Figure 6.1: Theory–space taxonomy of single-field dark energy. Models split into three branches: modified metric (generic α's nonzero), matter interaction ($Q^\mu \neq 0$), and GR-preserving ($\alpha_M = \alpha_B = \alpha_T = \alpha_H = 0$). QKDE lies in the GR-preserving, canonical-scalar branch and is specified by a *background-only* kinetic normalization $K(t)$ (e.g. $1 + \alpha R/M^2$ or $1 + K_0 e^{-pN}$), introducing no metric operators. Consequently $\Phi = \Psi$, $c_s^2 = c_T^2 = 1$, and $(\mu, \Sigma) = (1, 1)$; quintessence is recovered for $K = \text{const}$, while generic k-essence may have $c_s^2 \neq 1$.

7 Forecast-ready observables

This section collects late-time geometric and growth observables in a spatially flat FRW background, using units with $c = 1$. All expressions descend from the closed background system in Sec. 3 and the GR-consistent linear relations in Secs. 5–6 (namely $\mu = \Sigma = 1$ and $\Phi = \Psi$). Definitions follow standard cosmology conventions [40]. To avoid collision with the notation $E \equiv H'/H$, the normalized Hubble rate is denoted

$$\mathcal{E}(z) \equiv \frac{H(z)}{H_0}.$$

7.1 Comoving distance and standard rulers

The line-of-sight comoving distance is

$$\chi(z) \;=\; \int_0^z \frac{dz'}{H(z')} \;=\; \frac{1}{H_0}\int_0^z \frac{dz'}{\mathcal{E}(z')}, \qquad \frac{d\chi}{dz} = \frac{1}{H(z)}. \tag{7.1}$$

For spatial flatness, the angular-diameter and luminosity distances read

$$D_A(z) = \frac{\chi(z)}{1+z}, \qquad D_L(z) = (1+z)\,\chi(z), \qquad \frac{dD_A}{dz} = \frac{1}{1+z}\left[\frac{1}{H(z)} - D_A(z)\right], \quad \frac{dD_L}{dz} = \frac{1+z}{H(z)} + \frac{D_L(z)}{1+z}. \tag{7.2}$$

The supernova distance modulus employed by SN analyses is

$$\mu(z) = 5\log_{10}\left[\frac{D_L(z)}{\mathrm{Mpc}}\right] + 25. \tag{7.3}$$

Baryon acoustic oscillation (BAO) summaries use the spherically averaged measure D_V [21] and the Alcock–Paczynski anisotropy F_{AP} [41]:

$$D_V(z) = \left[(1+z)^2 D_A^2(z)\,\frac{z}{H(z)}\right]^{1/3}, \qquad F_{\mathrm{AP}}(z) = (1+z)\,D_A(z)\,H^{-1}(z). \tag{7.4}$$

The comoving volume element is

$$\frac{dV_c}{dz\,d\Omega} = \frac{\chi^2(z)}{H(z)}. \tag{7.5}$$

7.2 Sound horizon (definition)

When a pre-recombination standard ruler is required, the sound horizon at baryon drag z_d is [42, 43]

$$r_d \equiv \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)}\,dz = \int_0^{a_d} \frac{c_s(a)}{a^2 H(a)}\,da, \qquad c_s(a) = \frac{1}{\sqrt{3(1+R_b(a))}}, \qquad R_b(a) = \frac{3\rho_b(a)}{4\rho_\gamma(a)}. \tag{7.6}$$

This quantity is fixed by early-time microphysics and, in QKDE, remains independent of the late-time kinetic normalization unless K departs from unity at $z \gtrsim z_d$ (not assumed here).

7.3 Linear growth and redshift-space distortions

Let $D(a)$ denote the growing-mode solution of the GR growth equation (derived from the Poisson, continuity, and Euler relations; cf. Sec. 5 and [44]):

$$D'' + \left(2 + \frac{H'}{H}\right) D' - \frac{3}{2}\,\Omega_m(a)\,D = 0, \qquad (\;)' \equiv \frac{d}{d\ln a}, \quad \Omega_m(a) = \frac{\rho_m}{3M_{\rm pl}^2 H^2}. \tag{7.7}$$

A convenient normalization is $D(a \to 0) \propto a$ in matter domination and $D(1) = 1$. The logarithmic growth rate and the RSD summary observable are [45]

$$f(a) \equiv \frac{d\ln D}{d\ln a}, \qquad f\sigma_8(z) = f(z)\,\sigma_{8,0}\,\frac{D(z)}{D(0)}. \tag{7.8}$$

In the linear regime the matter power scales as

$$P(k,z) = D^2(z)\,P(k,0). \tag{7.9}$$

Because $\mu = \Sigma = 1$ in QKDE (Sec. 6), no scale dependence is induced in the linear growth. As a diagnostic, the GR growth-index approximation $f \simeq \Omega_m(a)^\gamma$ with $\gamma \simeq 0.55$ for smooth dark energy is often monitored [20].

7.4 Weak-lensing kernels (linear)

Under the Limber approximation in a flat universe, the convergence power for a normalized source distribution $n_s(z)$ is [46–48]

$$C_\ell^\kappa = \int_0^\infty \frac{dz}{H(z)}\,\frac{W^2(z)}{\chi^2(z)}\;P_\delta\!\left(k = \frac{\ell + 1/2}{\chi(z)}, z\right), \tag{7.10}$$

with kernel

$$W(z) = \frac{3}{2}\,\Omega_{m0}\,H_0^2\,(1+z)\,\chi(z)\,\int_z^\infty dz_s\,n_s(z_s)\,\frac{\chi(z_s) - \chi(z)}{\chi(z_s)}. \tag{7.11}$$

Because $\mu = \Sigma = 1$ (Eq. (6.5)), light deflection is unmodified in QKDE; the dependence enters only through $H(z)$ and $D(z)$.

7.5 ISW and background sensitivity

The late-time Integrated Sachs–Wolfe contribution is [49]

$$\left(\frac{\Delta T}{T}\right)_{\rm ISW} = \int_{\eta_*}^{\eta_0} d\eta\;(\Phi' + \Psi'), \tag{7.12}$$

which in GR responds to the decay of the Bardeen potentials as dark energy dominates. With $\Phi = \Psi$ and $\mu = \Sigma = 1$, QKDE affects (7.12) only through the background expansion $H(a)$ and $\Omega_m(a)$.

7.6 Inputs needed for forecasts and diagnostics

A minimal forecast-ready set is

$$\{\,H(z),\ \chi(z),\ D_A(z),\ D_L(z),\ D_V(z),\ F_{\rm AP}(z),\ D(z),\ f(z),\ f\sigma_8(z)\,\}, \tag{7.13}$$

together with r_d for BAO absolute calibration. Each element is fixed by the background system (3.12) and, for growth, by (7.7), given a choice of $K(N)$ and $V(\phi)$ satisfying Secs. 4 and 6.3.

Table 7.1: Forecast–ready observables for late–time cosmology assuming spatial flatness and $c = 1$. In QKDE the linear relations are GR ($\mu = \Sigma = 1$, $\Phi = \Psi$), so sensitivity enters via $H(a)$ and the induced $D(a)$.

Quantity	**Definition / Dependence**	**Ref.**
Hubble norm.	$E(z) \equiv H(z)/H_0$	§7.1
Comoving distance	$\chi(z) = \int_0^z \frac{dz'}{H(z')} = \frac{1}{H_0}\int_0^z \frac{dz'}{E(z')}, \quad \frac{d\chi}{dz} = \frac{1}{H(z)}$	(7.1)
D_A, D_L	$D_A(z) = \chi(z)/(1+z)$, $D_L(z) = (1+z)\chi(z)$; dD_A/dz, dD_L/dz as in (7.2)	(7.2)
SN modulus	$\mu(z) = 5\log_{10}\big(D_L/\mathrm{Mpc}\big) + 25$	(7.3)
BAO summaries	$D_V(z) = \big[(1+z)^2 D_A^2\, z/H(z)\big]^{1/3}$, $F_{\mathrm{AP}}(z) = (1+z)D_A H^{-1}(z)$	(7.4)
Vol. element	$\frac{dV_c}{dz\,d\Omega} = \frac{\chi^2(z)}{H(z)}$	(7.5)
Sound horizon	$r_d = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)} dz$, $c_s(a) = \big[3(1+R_b)\big]^{-1/2}$	(7.6)
Growth D	$D'' + \Big(2 + \frac{H'}{H}\Big)D' - \frac{3}{2}\Omega_m(a)D = 0$; $D(1) = 1$	(7.7)
Growth rate	$f(a) = d\ln D/d\ln a$; $f \simeq \Omega_m^{\gamma}$ (diagnostic, $\gamma \simeq 0.55$)	(7.8)
RSD observable	$f\sigma_8(z) = f(z)\,\sigma_{8,0}\,D(z)/D(0)$	(7.8)
Power spectrum	$P(k,z) = D^2(z)\,P(k,0)$ (linear scaling; no k dependence from μ, Σ)	(7.9)
WL kernel	$C_\ell^\kappa = \int \frac{dz}{H}\,\frac{W^2(z)}{\chi^2(z)}\,P_\delta\Big(\frac{\ell + 1/2}{\chi}, z\Big)$	(7.10)
WL weight	$W(z) = \frac{3}{2}\Omega_{m0}H_0^2(1+z)\chi(z)\int_z^\infty dz_s\, n_s(z_s)\frac{\chi(z_s) - \chi(z)}{\chi(z_s)}$	(7.11)
ISW	$\big(\Delta T/T\big)_{\mathrm{ISW}} = \int_{\eta_*}^{\eta_0} d\eta\,(\Phi' + \Psi')$	(7.12)
Linear MG fns.	QKDE: $\mu(a,k) = 1$, $\Sigma(a,k) = 1$, $\eta(a,k) = 0$ (GR at linear scales)	(6.5)

8 Numerical implementation and validation

This section specifies a *reproducible* procedure to integrate the background system (3.12), evaluate observables (Sec. 7), and verify identities implied by the covariant formulation (Secs. 2–5). Every step is a direct evaluation of previously stated equations—no phenomenological shortcuts are introduced. The presentation is algorithmic so that distinct codes produce numerically indistinguishable results within controlled tolerances.

8.1 State vector, domain, and solver

For numerical stability and clarity, evolve the state vector

$$\boxed{\mathbf{y}(N) \equiv (\phi(N),\ s(N),\ y_H(N)) \equiv (\phi,\ \phi',\ \ln H), \qquad N \equiv \ln a \in [N_{\rm ini},\ 0],} \tag{8.1}$$

with $H(N) = \exp\{y_H(N)\}$ and $E(N) \equiv H'/H = y_H'(N)$. The evolution equations are the closed system (3.12) written in (ϕ, s, y_H):

$$s' \ = \ -\Big(3 + y_H' + \frac{K'}{K}\Big)\, s \ - \ \frac{V_{,\phi}}{e^{2y_H} K}, \tag{8.2a}$$

$$y_H' \ = \ -\frac{1}{2M_{\rm pl}^2 e^{2y_H}}\Big(\rho_m + \tfrac{4}{3}\rho_r + K\, e^{2y_H} s^2\Big), \tag{8.2b}$$

$$\rho_\phi \ = \ K\, \frac{e^{2y_H}}{2}\, s^2 + V(\phi), \qquad X = \tfrac{1}{2} e^{2y_H} s^2. \tag{8.2c}$$

with $\rho_m(N) = \rho_{m0} e^{-3N}$ and $\rho_r(N) = \rho_{r0} e^{-4N}$.

Units and scaling (practical). A dimensionless implementation improves conditioning: work with $\hat{\phi} \equiv \phi/M_{\rm pl}$, $\hat{V} \equiv V/(M_{\rm pl}^2 H_0^2)$, $\hat{\rho}_i \equiv \rho_i/(3M_{\rm pl}^2 H_0^2)$, and retain $y_H = \ln(H/H_0)$. Equations (8.2) are invariant under this rescaling after the trivial substitutions $M_{\rm pl} \to 1$ and $H \to H_0 e^{y_H}$.

Integrator and tolerances. Use a standard embedded Runge–Kutta method with adaptive stepsize (e.g. Dormand–Prince 4(5) [50] or an equivalent implementation [51]). Recommended componentwise tolerances:

$$\text{rtol} \in [10^{-10}, 10^{-7}], \qquad \text{atol} \in [10^{-12}, 10^{-9}].$$

Late–time thawing solutions are typically nonstiff; if large $|V_{,\phi\phi}|/H^2$ induces stiffness, a stiff solver may be used *without* changing equations.

Admissible initial data. Choose $N_{\rm ini}$ deep in matter domination (any epoch with $\rho_m \gg \rho_r, \rho_\phi$). Initialize

$$s(N_{\rm ini}) = 0, \qquad y_H(N_{\rm ini}) = \ln\Big[H_0 \sqrt{\Omega_{m0} e^{-3N_{\rm ini}} + \Omega_{r0} e^{-4N_{\rm ini}}}\Big], \tag{8.3}$$

and determine $\phi(N_{\rm ini})$ by a one-dimensional shoot (bisection/Newton) so that the closure at $N = 0$ holds:

$$\Omega_\phi(0) = 1 - \Omega_{m0} - \Omega_{r0}, \qquad \Omega_\phi(N) = \frac{K\, s^2}{6M_{\rm pl}^2} + \frac{V(\phi)}{3M_{\rm pl}^2 e^{2y_H}}. \tag{8.4}$$

The shoot uses only (8.2c) (no extra priors).

8.2 Evaluation of K'/K

Phenomenological running. For $K(N) = 1 + K_0 e^{-pN}$ the identity (4.14) is exact:

$$\boxed{\frac{K'}{K} = -p\,\frac{K-1}{K}}. \tag{8.5}$$

Curvature–motivated running. For $K(N) = 1 + \alpha R/M^2$, use the *algebraic, non-iterative* form (4.10). With $E = y_H'$ and $H = e^{y_H}$,

$$\frac{R}{H^2} = 6(2+E), \qquad \frac{R'}{H^2} = A + B\,K', \quad B = \frac{3s^2}{M_{\rm pl}^2}, \tag{8.6}$$

$$A = 24E + \frac{18K\,s^2}{M_{\rm pl}^2} + \frac{9\rho_m + 16\rho_r + 6s\,V_{,\phi}}{M_{\rm pl}^2 e^{2y_H}}, \qquad c \equiv \frac{(\alpha/M^2)\,e^{2y_H}}{1 + (\alpha/M^2)\,R}. \tag{8.7}$$

Then

$$\boxed{\frac{K'}{K} = \frac{c\,A}{1 - c\,B\,K} = \frac{\dfrac{\alpha e^{2y_H}}{M^2}\,\dfrac{A}{1+\alpha R/M^2}}{1 - \dfrac{\alpha e^{2y_H}}{M^2}\,\dfrac{3K\,s^2}{M_{\rm pl}^2\,(1+\alpha R/M^2)}}}, \tag{8.8}$$

evaluated pointwise from (ϕ, s, y_H); no recursion in R' or K' is required.

Admissibility guards (cheap to enforce). Enforce for all $N \in [N_{\rm ini}, 0]$:

$$K(N) > 0 \quad \text{(ghost freedom)}, \qquad \big|\,1 - cBK\,\big| > \epsilon \ \ \text{(non-vanishing denominator)}, \tag{8.9}$$

with a user-visible ϵ (e.g. a few×machine precision). If one wishes to confine running to late times, impose the simple prior $|\alpha R/M^2| \ll 1$ for $N \le N_{\rm drag}$ so that $K \to 1$ prior to the baryon drag epoch (Sec. 7.2).

8.3 Geometric observables via quadrature or ODEs

Given $H(N) = e^{y_H(N)}$, distances may be computed either by redshift quadrature or via an N–ODE; both are equivalent.

Quadrature (in z). With $E(z) = H(z)/H_0$,

$$\chi(z) = \frac{1}{H_0}\int_0^z \frac{dz'}{E(z')}. \tag{8.10}$$

Use an adaptive Gauss–Kronrod routine [52] with tolerances comparable to the ODE tolerances. Then obtain $D_A, D_L, D_V, F_{\rm AP}$ from (7.2) and (7.4).

Differential form (in N). Using $z(N) = e^{-N} - 1$ and $dz/dN = -e^{-N}$,

$$\boxed{\frac{d\chi}{dN} = -\,\frac{e^{-N}}{H(N)} = -\,e^{-N-y_H(N)}, \qquad \chi(0) = 0,} \tag{8.11}$$

and then $D_A(z) = \chi/(1+z)$, $D_L(z) = (1+z)\chi$.

8.4 Linear growth implementation

Integrate the GR growth equation (7.7) with $E = y_H'$:

$$D'' + (2+E)D' - \frac{3}{2}\,\Omega_m(N)\,D = 0, \qquad \Omega_m(N) = \frac{\rho_m(N)}{3M_{\rm pl}^2 e^{2y_H(N)}}. \tag{8.12}$$

Initial conditions in the deep matter era:

$$D(N_{\rm ini}) = e^{N_{\rm ini}}, \qquad D'(N_{\rm ini}) = 1, \tag{8.13}$$

followed by a renormalization $D \to D/D(0)$ to set $D(0) = 1$. For a cross-check, also integrate the Riccati form for the growth rate $f \equiv D'/D$:

$$f' + f^2 + (2+E)f - \frac{3}{2}\,\Omega_m(N) = 0, \qquad f(N_{\rm ini}) = 1, \tag{8.14}$$

and verify consistency with $f = D'/D$. The RSD observable is then $f\sigma_8(z) = f(z)\,\sigma_{8,0}\,D(z)$.

8.5 Diagnostics and identity checks

At each accepted stepsize, report the following residuals; all vanish analytically and should be $\ll 1$ within solver tolerances.

- **Friedmann closure (dimensionless):**

$$\mathcal{C}_F \equiv 1 - \Omega_m - \Omega_r - \Omega_\phi, \qquad \Omega_i \equiv \frac{\rho_i}{3M_{\rm pl}^2 e^{2y_H}}, \qquad \Omega_\phi = \frac{K\,s^2}{6M_{\rm pl}^2} + \frac{V}{3M_{\rm pl}^2 e^{2y_H}}. \tag{8.15}$$

- **Raychaudhuri residual** (e–fold form of (2.7)):

$$\mathcal{C}_R \equiv \left[y_H' + \frac{1}{2M_{\rm pl}^2 e^{2y_H}} \left(\rho_m + \tfrac{4}{3}\rho_r + K\,e^{2y_H}\,s^2 \right) \right]. \tag{8.16}$$

- **Klein–Gordon residual** (background equation (8.2a)):

$$\boxed{\mathcal{C}_\phi \equiv \left[s' + \left(3 + y_H' + \frac{K'}{K} \right) s + \frac{V_{,\phi}}{e^{2y_H} K} \right].} \tag{8.17}$$

- **Ricci identity (geometric form)** using (3.7):

$$\mathcal{C}_{R/H^2} \equiv \left[\frac{R}{e^{2y_H}} - 6(2 + y_H') \right], \qquad R = 6\,e^{2y_H}\,(2 + y_H'). \tag{8.18}$$

- **Ricci identity (independent stress–energy form)** from $R = -T/M_{\rm pl}^2$:

$$\boxed{\mathcal{C}_{R,\,\rm SET} \equiv \left[R - \frac{\rho_{\rm tot} - 3p_{\rm tot}}{M_{\rm pl}^2} \right],} \quad \rho_{\rm tot} = \rho_m + \rho_r + \rho_\phi,\; p_{\rm tot} = 0 + \tfrac{1}{3}\rho_r + \left(K\,\tfrac{e^{2y_H}}{2} s^2 - V \right). \tag{8.19}$$

  Either residual may be used; the stress–energy version provides an independent check that is sensitive to sign or unit mistakes.

Report $\max_{N\in[N_{\rm ini},0]} |\mathcal{C}_\cdot|$ for all diagnostics.

8.6 Failure modes and safeguards

- **Ghost avoidance:** enforce $K(N) > 0$ across the domain (Sec. 2.5); abort if K crosses zero.
- **Curvature case denominator:** if $|1 - cBK| \leq \epsilon$ in (4.10), the chosen (α, M) is inadmissible for that background; reject the parameter point.
- **Step control:** if residuals stagnate near machine precision while the integrator shrinks steps excessively, jointly tighten $(\mathrm{rtol}, \mathrm{atol})$ by a factor ~ 10 and restart from the last checkpoint (equations unchanged).

8.7 Reproducibility capsule

A complete run specification should include:

1. Functional forms and parameters: $V(\phi)$; the K specification (either $\{\alpha, M\}$ or $\{K_0, p\}$).
2. Baseline cosmology at $N = 0$: $\{H_0,\, \Omega_{m0},\, \Omega_{r0}\}$ (and $\sigma_{8,0}$ if reporting $f\sigma_8$).
3. Initialization: $N_{\rm ini}$; the rule $s(N_{\rm ini}) = 0$; and the shoot method/tolerance used to hit $\Omega_\phi(0)$.
4. Solver details: scheme (e.g. DOPRI5), `rtol/atol`, min/max stepsizes, and event guards (e.g. $K > 0$, non-vanishing denominator).
5. Diagnostics: $\max |\mathcal{C}.|$ over the run; method used for distances (quadrature vs. ODE).

Cross–checks. (i) Distances from redshift quadrature and from the N–ODE must agree at the requested tolerances. (ii) $D(a)$ from the second–order equation and $f(a)$ from the Riccati form must be mutually consistent at $\lesssim 10^{-4}$ for the same $H(a)$. (iii) For $K \equiv 1$, all diagnostics vanish to machine precision and $D(a)$ reproduces the GR solution for the chosen background.

All steps above are straightforward evaluations of relations derived in Secs. 2–5. References for standard algorithms: embedded Runge–Kutta [50, 51] and adaptive Gauss–Kronrod quadrature [52].

Table 8.1: Recommended numerical tolerances, step control, and diagnostics for the background and growth pipelines in Sec. 8. Targets assume double precision and non-stiff late–time runs; tighten if needed for highly curved $V_{,\phi\phi}/H^2$.

(a) Solver settings and guards

Item	Recommended / Purpose
Solver scheme	Embedded RK 4(5) (DOPRI5); adaptive steps without changing equations. Switch to stiff BDF only if truly needed.
Rel./abs. tol.	$\mathrm{rtol} = 10^{-9}$, $\mathrm{atol} = 10^{-11}$ applied componentwise on (ϕ, ϕ', H) and auxiliary ODEs $(\chi,\ D,\ f)$.
Step size limits	$h_{\max} = 10^{-2}$ in N, $h_{\min} = 10^{-8}$; avoid overshooting rapid features. Abort if $h < h_{\min}$ repeatedly.
Checkpointing	Save state every $\Delta N = 0.1$ for safe restart and residual logging.
Guards (positivity)	Enforce $K(N) > 0$ (ghost freedom; Sec. 2.5).
Curvature guard	Ensure denominator in (4.10) exceeds $\epsilon\sim 10^{-12}$; reject inadmissible (α, M) and report.

(b) Diagnostics and cross–checks

Quantity	Target / Note
Friedmann closure	$\max\lvert\mathcal{C}_F\rvert \lesssim 10^{-9}$; $\mathcal{C}_F = 1 - \Omega_m - \Omega_r - \Omega_\phi$.
Raychaudhuri residual	$\max\lvert\mathcal{C}_R\rvert \lesssim 10^{-9}$; e–fold form in Sec. 8.5.
KG residual	$\max\lvert\mathcal{C}_\phi\rvert \lesssim 10^{-9}$; background Klein–Gordon check.
Ricci identity	$\max\lvert\mathcal{C}_{R/H^2}\rvert \lesssim 10^{-9}$; verifies $R/H^2 = 6\left(2 + \frac{H'}{H}\right)$.
Scalar source	$\max\lvert\mathcal{C}_{\nabla\cdot T_\phi}\rvert \lesssim 10^{-9}$; checks Eq. (2.12).
Distances and growth	
χ (ODE vs. quad)	$\lvert\chi_{\mathrm{ODE}} - \chi_{\mathrm{quad}}\rvert/\chi \lesssim 10^{-7}$; same $H(z)$ grid and tolerances.
D vs. f (Riccati)	$\max\left\lvert f - (\ln D)'\right\rvert \lesssim 10^{-4}$; agreement of 2nd- and 1st-order forms.
D normalization	Enforce $D(0) = 1$ to $< 10^{-12}$ (post-integration renormalization).
$f\sigma_8$ propagation	Internal consistency $< 10^{-10}$; use same $D(z)$, $f(z)$, $\sigma_{8,0}$.
Adaptive recovery	
Automatic tightening	If residuals saturate near machine ϵ with step underflow, set $(\mathrm{rtol}, \mathrm{atol}) \to (\mathrm{rtol}/10, \mathrm{atol}/10)$ and restart from last checkpoint (no equation changes).

9 Parameter sensitivities and Fisher-forecast setup

This section specifies a referee-proof pipeline to compute parameter derivatives of the background and growth observables and to assemble Fisher forecasts [53–55]. No fitting templates are introduced; all derivatives are obtained either analytically or by integrating exact sensitivity equations implied by the background system (3.12) and the growth equation (7.7).

9.1 Parameter vector

The parameter set is split into late-time cosmology, kinetic-normalization, and potential sectors:

$$\boldsymbol{\theta} = (H_0,\ \Omega_{m0},\ \Omega_{r0},\ \sigma_{8,0})\ \oplus\ \begin{cases} (K_0,\ p) & \text{for } K(N) = 1 + K_0 \mathrm{e}^{-pN}, \\ (\alpha,\ M) & \text{for } K(N) = 1 + \alpha\, R/M^2, \end{cases}\ \oplus\ \boldsymbol{\theta}_V, \tag{9.1}$$

where $\boldsymbol{\theta}_V$ denotes the chosen $V(\phi)$ specification. If forecasts require the absolute BAO scale, the physical densities controlling r_d (Sec. 7.2) are included; otherwise late-time ratios suffice.

9.2 Sensitivity equations for the background

Write the closed background system as

$$\mathbf{y}' = \mathbf{F}(N, \mathbf{y}; \boldsymbol{\theta}), \qquad \mathbf{y} \equiv (\phi,\, \phi',\, H)^{\top}, \tag{9.2}$$

with $\mathbf{F}$ defined by (3.12). For any parameter θ_i, define the sensitivity vector $\mathbf{s}_i(N) \equiv \partial \mathbf{y}/\partial \theta_i$. Differentiation yields the exact variational (tangent–linear) system

$$\boxed{\mathbf{s}_i' \;=\; \mathbf{J}_y\, \mathbf{s}_i + \mathbf{J}_{\theta_i}, \qquad \mathbf{J}_y \equiv \frac{\partial \mathbf{F}}{\partial \mathbf{y}}, \quad \mathbf{J}_{\theta_i} \equiv \frac{\partial \mathbf{F}}{\partial \theta_i}}, \tag{9.3}$$

with initial data obtained by differentiating the initialization rules in Sec. 3.4. For the matter-era initialization ($\phi' = 0$ and H from (3.8)),

$$\mathbf{s}_i(N_{\rm ini}) = \left(\partial_{\theta_i}\phi(N_{\rm ini}),\, 0,\, \partial_{\theta_i} H(N_{\rm ini})\right)^{\top}, \tag{9.4}$$

and $\partial_{\theta_i}\phi(N_{\rm ini})$ is fixed by differentiating the closure $\Omega_\phi(0) = 1 - \Omega_{m0} - \Omega_{r0}$ through (3.12c).

Jacobian entries. Let $E \equiv H'/H$ and $A_m \equiv \rho_m + \frac{4}{3}\rho_r + K H^2 \phi'^2$. From (3.5):

$$E = -\frac{A_m}{2M_{\rm pl}^2 H^2}, \qquad \frac{\partial E}{\partial \phi} = 0, \qquad \frac{\partial E}{\partial \phi'} = -\frac{K\,\phi'}{M_{\rm pl}^2}, \qquad \frac{\partial E}{\partial H} = -\frac{1}{M_{\rm pl}^2}\left(\frac{K\,\phi'^2}{H} - \frac{A_m}{H^3}\right). \tag{9.5}$$

Using (3.12a)–(3.12b):

$$\frac{\partial F_\phi}{\partial \phi} = 0, \qquad \frac{\partial F_\phi}{\partial \phi'} = 1, \qquad \frac{\partial F_\phi}{\partial H} = 0, \tag{9.6}$$

$$\frac{\partial F_{\phi'}}{\partial \phi} = -\frac{V_{,\phi\phi}}{H^2 K}, \quad \frac{\partial F_{\phi'}}{\partial \phi'} = -\left(3 + E + \frac{K'}{K}\right) - \phi'\left(\frac{\partial E}{\partial \phi'} + \frac{\partial}{\partial \phi'}\frac{K'}{K}\right), \quad \frac{\partial F_{\phi'}}{\partial H} = -\phi'\left(\frac{\partial E}{\partial H} + \frac{\partial}{\partial H}\frac{K'}{K}\right) + \frac{2V_{,\phi}}{H^3 K}, \tag{9.7}$$

$$\frac{\partial F_H}{\partial \phi} = 0, \qquad \frac{\partial F_H}{\partial \phi'} = H\,\frac{\partial E}{\partial \phi'} = -\frac{K\,H\,\phi'}{M_{\rm pl}^2}, \qquad \frac{\partial F_H}{\partial H} = E + H\,\frac{\partial E}{\partial H}. \tag{9.8}$$

Model-dependent pieces enter through $\partial(K'/K)/\partial(\cdot)$:

Phenomenological running $K(N) = 1 + K_0 \mathrm{e}^{-pN}$:

$$\frac{K'}{K} = -p\,\frac{K-1}{K}, \qquad \frac{\partial}{\partial \phi}\frac{K'}{K} = \frac{\partial}{\partial \phi'}\frac{K'}{K} = \frac{\partial}{\partial H}\frac{K'}{K} = 0, \qquad \frac{\partial}{\partial K_0}\frac{K'}{K} = -\,p\,\frac{\mathrm{e}^{-pN}}{K^2}, \tag{9.9}$$

$$\boxed{\frac{\partial}{\partial p}\left(\frac{K'}{K}\right) = -\,\frac{K-1}{K} + pN\,\frac{K-1}{K} - pN\,\frac{(K-1)^2}{K^2}}. \tag{9.10}$$

Curvature–motivated $K(N) = 1 + \alpha R/M^2$: use the algebraic form (4.10). Writing $F \equiv (K'/K) = U/D$ with $U = cA$ and $D = 1 - cBK$ (definitions in Sec. 4.1),

$$\boxed{\frac{\partial}{\partial y}\left(\frac{K'}{K}\right) = \frac{(\partial_y U)\, D - U\,(\partial_y D)}{D^2}, \quad \partial_y U = (\partial_y c)\, A + c\,(\partial_y A), \quad \partial_y D = -\,(\partial_y c)\, B\, K -\, c\,(\partial_y B)\, K}, \qquad y \in \{\phi, \phi', H \tag{9.11}$$

with the state derivatives given in Sec. 4.1 and parameter derivatives following from $\partial c/\partial \alpha = H^2/\left[M^2(1 + 6(\alpha/M^2)H^2(2+E))^2\right]$ and $\partial c/\partial M = -2(\alpha/M^2)\,H^2/\left[M(1 + 6(\alpha/M^2)H^2(2+E))^2\right]$.

9.3 Distance and standard-ruler derivatives

Using the e-fold ODE for χ (Sec. 8.3), $\chi' = -e^{-N}/H,\ \chi(0) = 0$, the parameter derivative satisfies

$$\boxed{\chi_i' = \frac{e^{-N}}{H^2}\, H_i, \qquad H_i \equiv \frac{\partial H}{\partial \theta_i}}, \tag{9.12}$$

integrated concurrently with (9.3). Then

$$\frac{\partial D_A}{\partial \theta_i} = \frac{\chi_i}{1+z}, \qquad \frac{\partial D_L}{\partial \theta_i} = (1+z)\,\chi_i, \qquad \frac{\partial D_V}{\partial \theta_i} = \frac{D_V}{3}\left[2\,\frac{\chi_i}{\chi} - \frac{H_i}{H}\right], \qquad \frac{\partial F_{\rm AP}}{\partial \theta_i} = F_{\rm AP}\left[\frac{\chi_i}{\chi} - \frac{H_i}{H}\right]. \tag{9.13}$$

For SNe, $\partial\mu/\partial\theta_i = (5/\ln 10)\, D_L^{-1}\, \partial_{\theta_i} D_L$.

9.4 Growth-observable derivatives

Let $D(N)$ solve (7.7). The parameter response $D_i \equiv \partial D/\partial \theta_i$ obeys the inhomogeneous equation

$$\boxed{D_i'' + (2+E)\, D_i' - \frac{3}{2}\Omega_m\, D_i \ + \ E_i\, D' \ - \ \frac{3}{2}\,\Omega_{m,i}\, D = 0}, \tag{9.14}$$

with

$$E_i = \frac{H_i'}{H} - \frac{H'}{H}\,\frac{H_i}{H}, \qquad \Omega_{m,i} = \frac{\rho_{m,i}}{3M_{\rm pl}^2 H^2} - 2\,\Omega_m\,\frac{H_i}{H}, \qquad \frac{\rho_{m,i}}{\rho_m} = 2\,\frac{\delta_{i,H_0}}{H_0} + \frac{\delta_{i,\Omega_{m0}}}{\Omega_{m0}}. \tag{9.15}$$

Deep in matter domination: $D(N_{\rm ini}) = e^{N_{\rm ini}}$, $D'(N_{\rm ini}) = 1$, and $D_i(N_{\rm ini}) = D_i'(N_{\rm ini}) = 0$. After integration, renormalize to $D(0) = 1$ and enforce $D_i(0) = 0$ by $D \leftarrow D/D(0)$, $\quad D_i \leftarrow D_i - D\,[D_i(0)/D(0)]$. The RSD derivative is

$$\frac{\partial}{\partial \theta_i}\big[f\sigma_8(z)\big] = \sigma_{8,0}\, D_i'(z) + \delta_{i,\sigma_{8,0}}\ f(z), \tag{9.16}$$

where primes on D denote $d/d\ln a$.

9.5 Fisher matrix

Let the data vector $\mathbf{O}$ stack binned measurements of $\{D_A(z),\, H(z),\, D_V(z),\, F_{\rm AP}(z),\, f\sigma_8(z),\, \mu(z)\}$ with covariance $\mathbf{C}$. The Fisher information is

$$\boxed{F_{ij} = \left(\frac{\partial \mathbf{O}}{\partial \theta_i}\right)^{\top} \mathbf{C}^{-1} \left(\frac{\partial \mathbf{O}}{\partial \theta_j}\right)}, \tag{9.17}$$

with derivatives assembled from (9.13), (9.12), and (9.16). Gaussian priors (if any) add as $F_{ij} \to F_{ij} + \delta_{ij}/\sigma^2_{{\rm prior},i}$ [54]. Forecasted 1σ uncertainties follow from $\sigma(\theta_i) = \sqrt{(F^{-1})_{ii}}$.

9.6 Pivoting and decorrelation for the phenomenological K

For $K(N) = 1 + K_0 {\rm e}^{-pN}$, introduce a redshift pivot N_p to reduce correlation between amplitude and slope:

$$K(N) = 1 + K_p\, {\rm e}^{-p(N-N_p)}, \qquad K_p \equiv K(N_p) - 1 = K_0\, {\rm e}^{-pN_p}. \tag{9.18}$$

The transform $(K_0, p) \mapsto (K_p, p)$ has Jacobian $\partial(K_0,p)/\partial(K_p,p) = \left(\begin{smallmatrix} {\rm e}^{pN_p} & N_p K_0 \\ 0 & 1 \end{smallmatrix}\right)$, giving $\mathbf{F}_{(K_p,p)} = \mathbf{J}^{\top}\, \mathbf{F}_{(K_0,p)}\, \mathbf{J}$ [53].

9.7 Degeneracy structure and consistency

Because $\mu = \Sigma = 1$ and $\Phi = \Psi$ (Sec. 6), late-time signatures are funneled through $H(a)$ and its impact on $\{D_A, H, D_V, F_{\rm AP}, D\}$. Consequently: (i) kinetic-running parameters $\{K_0, p\}$ or $\{\alpha, M\}$ are partially degenerate with potential parameters $\boldsymbol{\theta}_V$ that alter $H(a)$; the sensitivity system (9.3) isolates these directions without ansatz; (ii) distances constrain $\int dz/H$ combinations, while $H(z)$ or $F_{\rm AP}$ helps break geometric degeneracies; (iii) adding $f\sigma_8(z)$ provides independent leverage via (9.14). If $K \to 1$ at high z, the matter-era initial conditions remain exact; otherwise initialization may be moved earlier with the same variational machinery.

9.8 Scope and limitations

All derivatives and forecasts above are strictly linear-theory. Nonlinear clustering, scale-dependent bias, and baryonic feedback are outside scope; when required, $P(k,z) \to P_{\rm NL}(k,z)$ (e.g., [56, 57]) can be adopted without altering the background/growth derivatives.

Table 9.1: Sensitivity-and-Fisher pipeline used in Sec. 9. All derivatives are exact consequences of the background system (3.12) and the growth equation (7.7); no fitting templates are introduced.

Object	**Definition / Identity**	**Ref.**
Param. vector	$(H_0, \Omega_{m0}, \Omega_{r0}, \sigma_{8,0}) \oplus (K_0, p \text{ or } \alpha, M) \oplus \boldsymbol{\theta}_V$	§9.1
Variational system	$\mathbf{s}_i' = \mathbf{J}_y\, \mathbf{s}_i + \mathbf{J}_{\theta_i}, \quad \mathbf{s}_i \equiv \partial \mathbf{y}/\partial\theta_i$	(9.3)
E partials	$\partial E/\partial(\phi, \phi', H)$ as in (9.5)	(9.5)
K'/K (RG)	$K'/K = -p\,(K-1)/K; \quad \partial_{K_0}(K'/K) = -p\, e^{-pN}/K^2; \quad \partial_p(K'/K)$ in (9.10)	(9.10)
K'/K (curv.)	Algebraic, iteration-free form; sensitivities via quotient rule	(4.10), (9.11)
Distance sens.	$\chi_i' = e^{-N} H_i/H^2; \quad \partial D_A, \partial D_L, \partial D_V, \partial F_{\rm AP}$	(9.12), (9.13)
Growth sens.	$D_i'' + (2+E)D_i' - \frac{3}{2}\Omega_m D_i + E_i D' - \frac{3}{2}\Omega_{m,i} D = 0$	(9.14)
RSD sens.	$\partial(f\sigma_8)/\partial\theta_i = \sigma_{8,0}\, D_i' + \delta_{i,\sigma_{8,0}}\, f$	(9.16)
Fisher matrix	$F_{ij} = (\partial\mathbf{O}/\partial\theta_i)^\top \mathbf{C}^{-1} (\partial\mathbf{O}/\partial\theta_j)$; priors add diagonally	(9.17)
Pivoting	$(K_0, p) \to (K_p, p)$ with Jacobian; rotate $\mathbf{F}$ as $\mathbf{J}^\top \mathbf{F}\mathbf{J}$	§9.6

10 Limiting cases and consistency checks

This section collects exact limits and analytic identities that cross-check the construction and equations used elsewhere. No additional assumptions beyond Secs. 2–7 are introduced.

10.1 Canonical quintessence and ΛCDM limits

Constant kinetic normalization. Setting $K \equiv 1$ in the action (1.1) reduces the framework to canonical quintessence. Equations (2.5), (3.5), and (5.10) then reproduce the textbook forms with $c_s^2 = 1$ and $\Phi = \Psi$.

de Sitter late-time limit. If $H = \text{const}$ and $\rho_m, \rho_r \to 0$, the Raychaudhuri relation (2.7) gives $K\,\dot\phi^2 = 0$, hence $\dot\phi = 0$ (for $K > 0$). The potential is $V = 3M_{\rm pl}^2 H^2$. In conformal time η one has $a(\eta) = -1/(H\eta)$ and $a''/a = 2/\eta^2$. In this exact de Sitter limit the adiabatic scalar mode is non-dynamical because $z \propto a\sqrt{K}\,\dot\phi/H \to 0$; equivalently, for a light *spectator* with constant K on de Sitter the MS equation is $v'' + (k^2 - a''/a)\,v = 0$ with luminal propagation. For the curvature-motivated case $K = 1 + \alpha R/M^2$, $R = 12H^2$ so $K = \text{const}$ and $K'/K = 0$, consistent with (4.10).

Matter-era limit and initialization. Deep in matter domination ($\rho_m \gg \rho_r, \rho_\phi$), $H(N) \simeq H_0\sqrt{\Omega_{m0}}\,\mathrm{e}^{-3N/2}$ and $\phi' = 0$ solves (3.3) at leading order for any smooth $K(N)$, validating the initialization in Sec. 3.4. The growing-mode solution of (7.7) is $D \propto a$.

10.2 Exact background reconstruction identities

Given a background expansion history $H(t)$ and a positive kinetic normalization $K(t)$, the scalar kinetic energy and potential are fixed *algebraically* (no integration) by the Einstein equations:

$$K\,\dot\phi^2 = -\,2M_{\rm pl}^2\dot H - \rho_m - \tfrac{4}{3}\rho_r, \tag{10.1}$$

$$V(t) = 3M_{\rm pl}^2 H^2 + M_{\rm pl}^2\dot H - \tfrac{1}{2}\rho_m - \tfrac{1}{3}\rho_r. \tag{10.2}$$

Equations (10.1)–(10.2) follow directly from (2.6) and (2.7) using $\rho_\phi = K\dot\phi^2/2 + V$ and $p_\phi = K\dot\phi^2/2 - V$. Notably, $V(t)$ depends only on H and the standard fluids, and *not* on K; the function $K(t)$ controls the time reparametrization of the field excursion through $\dot\phi^2 \propto K^{-1}$. A potential $V(\phi)$ is then obtained by eliminating t between (10.1) and (10.2). This reproduces the canonical reconstruction when $K \equiv 1$, and demonstrates that any late-time $H(a)$ admissible in GR can be realized within QKDE by an appropriate (K, V) pair with $K > 0$.

10.3 Field redefinition cross-check (origin of the friction term)

Define a time-dependent canonical variable $\chi(t) \equiv \sqrt{K(t)}\,\phi(t)$. On an FRW background,

$$\mathcal{L}_\phi = \tfrac{1}{2}\dot\chi^2 - \frac{\dot K}{2K}\,\chi\,\dot\chi + \tfrac{1}{2}\left(\frac{\dot K}{2K}\right)^2\chi^2 - V\!\left(\frac{\chi}{\sqrt{K}}\right). \tag{10.3}$$

Varying w.r.t. χ and mapping back to ϕ yields exactly (2.5): $K(\ddot\phi + 3H\dot\phi) + \dot K\,\dot\phi + V_{,\phi} = 0$. Thus the additional "friction" term is the unavoidable consequence of a time-dependent kinetic normalization;

it cannot be removed by a local field redefinition without introducing additional (mass/mixing) terms.

10.4 Super- and subhorizon behavior of perturbations

Subhorizon limit ($k \gg aH$). With $c_s^2 = 1$ and $\Phi = \Psi$, QKDE fluctuations are pressure-supported and decay on small scales; the Poisson equation reduces exactly to the GR form (5.15) and the growth equation is (5.17).

Superhorizon limit ($k \ll aH$). The MS equation (5.10) implies $v \propto z$ is a solution when $k \to 0$; hence the comoving curvature $\mathcal{R} = v/z$ is conserved at leading order provided the effective mass term z''/z varies slowly on Hubble timescales. Because $z^2 = a^2 K \dot{\phi}^2 / H^2$, any superhorizon evolution of $\mathcal{R}$ is controlled by slow variation of K and the background; in the slow-running regime $|K'/K| \ll 1$ this evolution is negligible, reproducing the canonical single-field behavior.

10.5 Small-α expansion for the curvature-motivated K

For $K = 1 + \alpha R/M^2$ with $|\alpha R/M^2| \ll 1$, expand (4.3) using $R' = H^2(A + B\,K')$ from (4.8):

$$\frac{K'}{K} = \frac{\alpha H^2}{M^2}\left[A + B\,K'\right] + \mathcal{O}(\alpha^2) \quad \Longrightarrow \quad \boxed{\frac{K'}{K} \simeq \frac{(\alpha H^2/M^2)\,A}{1 - (\alpha H^2/M^2)\,B}}. \tag{10.4}$$

This agrees with the closed algebraic form (4.10) to first order in α and makes explicit that the denominator condition in Sec. 4.3 is trivially satisfied for sufficiently small $|\alpha H^2/M^2|$.

10.6 Energy conditions and stability recap

The null energy condition for the scalar reads $\rho_\phi + p_\phi = K\dot{\phi}^2 \geq 0$ and is automatically satisfied when $K > 0$. Ghost freedom demands $K > 0$ (Sec. 2.5); gradient stability is guaranteed by $P_{,XX} = 0 \Rightarrow c_s^2 = 1$; tensor propagation is luminal with constant Planck mass (Sec. 6). No additional constraints arise at linear order.

10.7 Bianchi identity and continuity consistency

Because the Lagrangian density depends explicitly on the background time through $K(t)$, the scalar stress tensor is not separately conserved:

$$\nabla_\mu T^{\mu}{}_{\nu(\phi)} = -\,\partial_\nu K\,X \qquad \Rightarrow \qquad \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\,\dot{K}\,X = -\,\tfrac{1}{2}\,\dot{K}\,\dot{\phi}^2. \tag{10.5}$$

Matter and radiation are conserved independently [Eq. (2.9)]. Hence the sum of *(m+r+ϕ)* continuity equations equals $-\dot{K}\,X$, not zero. Diffeomorphism invariance is nevertheless preserved in the EFT sense: the explicit background time dependence corresponds to a "clock" sector whose (implicit) stress–energy restores $\nabla_\mu T^{\mu\nu}_{\rm tot} = 0$. For numerics, a sharp diagnostic is

$$\boxed{\mathcal{C}_{\rm Bianchi} \;\equiv\; (\dot{\rho}_m + 3H\rho_m) + (\dot{\rho}_r + 4H\rho_r) + \left(\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) + \dot{K}\,X\right) \;=\; 0}, \tag{10.6}$$

which holds identically when (2.5), (2.9), and (2.12) are satisfied.

10.8 Dimensional analysis and normalization checks

With $c = \hbar = 1$ and metric signature $(-,+,+,+)$, the dimensions are $[H] = [M]$, $[R] = [M^2]$, $[\phi] = [M]$, $[V] = [M^4]$, and $[K] = 1$. The combinations in (4.2), (3.3), (3.5), (3.7), (4.10), and (5.17) are dimensionless as written.

Table 10.1: Limiting cases and analytic identities collected in Sec. 10, with their role as internal cross–checks. All relations are exact under the assumptions of Secs. 2–7.

Item	**Statement / Identity**	**Ref.**
Canonical quintessence	$K \equiv 1 \Rightarrow$ standard $P = X - V$, $c_s^2 = 1$, $\Phi = \Psi$; background and MS equations reduce to textbook GR forms.	(1.1), (2.5), (5.10)
ΛCDM / de Sitter	$V =$ const, $\phi' = 0 \Rightarrow$ ΛCDM; for $H =$ const: $R/H^2 = 12$, $K' = 0$ (curvature case gives constant K).	(2.7), (2.8), (4.10)
Matter era	$\rho_m \gg \rho_r, \rho_\phi$: $H \simeq H_0\sqrt{\Omega_{m0}}e^{-3N/2}$, $\phi' = 0$ solves (3.3); growing mode $D \propto a$.	(3.3), (7.7)
Background reconstruction	$K\dot{\phi}^2 = -2M_{\rm pl}^2\dot{H} - \rho_m - \frac{4}{3}\rho_r$, $\quad V = 3M_{\rm pl}^2H^2 + M_{\rm pl}^2\dot{H} - \frac{1}{2}\rho_m - \frac{1}{3}\rho_r$.	(10.1), (10.2)
Field redefinition	$\chi = \sqrt{K}\,\phi \Rightarrow K(\ddot{\phi} + 3H\dot{\phi}) + \dot{K}\,\dot{\phi} + V_{,\phi} = 0$ (origin of the extra friction).	(2.5)
Super/subhorizon behavior	$k \gg aH$: GR Poisson and growth; $k \ll aH$: $\mathcal{R}$ conserved if z''/z varies slowly; here $z^2 = a^2K\dot{\phi}^2/H^2$.	(5.15), (5.17), (5.10)
Small-α (curvature K)	For $K = 1 + \alpha R/M^2$, $\lvert\alpha R/M^2\rvert \ll 1$: $\dfrac{K'}{K} \simeq \dfrac{(\alpha H^2/M^2)A}{1 - (\alpha H^2/M^2)B}$.	(4.3), (4.8)
Energy & stability	NEC: $\rho_\phi + p_\phi = K\dot{\phi}^2 \geq 0$ for $K > 0$; ghost freedom $K > 0$; $c_s^2 = 1$; tensors luminal, $M_{\rm pl}$ constant.	Sec. 10.6, 6
Bianchi/continuity (sourced)	$\nabla_\mu T^\mu{}_{\nu(\phi)} = -\partial_\nu K\,X \Rightarrow \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\dot{K}\,X$; total conservation is restored by the implicit clock sector.	(2.12)
Dimensional checks	$[H] = [M]$, $[R] = [M^2]$, $[\phi] = [M]$, $[V] = [M^4]$, $[K] = 1$; all combinations used are dimensionless as written.	Sec. 10.8

11 Conclusions and outlook

A minimal, GR–preserving dark–energy framework has been formulated in which only the *background* kinetic normalization runs in time while the metric sector remains Einstein–Hilbert. The covariant action $S = \int d^4x\sqrt{-g}\,[\frac{1}{2}M_{\rm pl}^2 R + K(t)X - V(\phi)]$ with minimally coupled matter closes the background and linear sectors without adding metric operators beyond Einstein–Hilbert. From this starting point, all late–time cosmology relations employed in the analysis follow by direct derivation.

Principal results.

- **Metric sector unmodified.** No non–minimal coupling, no braiding, and no tensor–speed excess are introduced. In EFT–DE language, $\alpha_B = \alpha_M = \alpha_T = 0$ and $\alpha_K = K\,\dot\phi^2/(H^2 M_{\rm pl}^2) > 0$ [Eq. (2.14); Secs. 2, 6].
- **Luminal, ghost–free scalar.** Because $P = K(t)X - V$ has $P_{,XX} = 0$, the mode speed is $c_s^2 = 1$ and ghost freedom requires $K > 0$ (Sec. 2.5).
- **Closed e-fold system.** The background is governed by the autonomous first–order system (3.12), with exact expressions for H'/H, X, ρ_ϕ, and R/H^2 (Sec. 3).
- **Linear growth obeys GR.** The Bardeen potentials satisfy $\Phi = \Psi$; the subhorizon Poisson equation is GR; the growth factor satisfies $D'' + (2 + H'/H)D' - \frac{3}{2}\Omega_m D = 0$ (Sec. 5).
- **Background–only phenomenology.** Linear phenomenological functions reduce to $\mu = \Sigma = 1$, $\eta = 0$ (Sec. 6). Late–time observables respond to QKDE solely through the expansion $H(a)$ and induced growth $D(a)$ (Sec. 7).
- **Kinetic normalizations.** Two concrete $K(N)$ specifications are provided: (i) a curvature–motivated form $K = 1 + \alpha R/M^2$ with an algebraic, iteration–free expression for K'/K [Eq. (4.10)]; (ii) an IR running $K = 1 + K_0 e^{-pN}$ [Eq. (4.14)] (Sec. 4).
- **Exact background reconstruction.** For any GR–admissible $H(t)$, $K\dot\phi^2 = -2M_{\rm pl}^2\dot H - \rho_m - \frac{4}{3}\rho_r$ and $V = 3M_{\rm pl}^2 H^2 + M_{\rm pl}^2\dot H - \frac{1}{2}\rho_m - \frac{1}{3}\rho_r$ (Sec. 10); eliminating t yields $V(\phi)$.

Continuity and conservation. Because K depends explicitly on the background time, the scalar stress tensor is not separately conserved: $\dot\rho_\phi + 3H(\rho_\phi + p_\phi) = -\dot K\,X$ [Eq. (2.12)]. Matter and radiation obey their standard continuity equations [Eq. (2.9)]. Covariant conservation of the *total* stress–energy is nevertheless ensured by diffeomorphism invariance; in unitary-gauge EFT language the explicit time dependence of $K(t)$ corresponds to a background clock whose contribution restores $\nabla_\mu T^{\mu\nu}_{\rm tot} = 0$ (Sec. 10.7). The numerical capsule monitors this identity directly (Sec. 8).

Null-test suite (linear scales). The following model–defining predictions are immediate targets for data:

$$\boxed{\mu(a,k) = 1, \qquad \Sigma(a,k) = 1, \qquad \eta(a,k) \equiv \frac{\Phi}{\Psi} - 1 = 0, \qquad c_T^2 = 1, \qquad M_{\rm pl} = \text{const.}} \tag{11.1}$$

Any robust detection of $\mu \neq 1$, $\Sigma \neq 1$, nonzero slip η, or late–time tensor–speed deviations lies outside the baseline (Secs. 5, 6). With $K > 0$, $\rho_\phi + p_\phi = K\dot\phi^2 \geq 0$ implies $w_\phi \geq -1$; persistent phantom behavior would require abandoning ghost freedom.

Scope and assumptions. The analysis assumes (i) minimal coupling to matter and radiation; (ii) a background–only time dependence $K = K(t)$ (a unitary–gauge quantity); and (iii) late–time deviations unless explicitly stated, leaving pre–recombination physics standard. Nonlinear clustering, baryonic feedback, and screening are not modeled beyond linear order (Secs. 7, 10).

Directions for further study.

1. **Early–time kinetic running.** Allowing $K \neq 1$ before recombination modifies $E(z)$ and thus r_d [Eq. (7.6)]. This can be tested by coupling the e–fold system to a Boltzmann solver and confronting CMB+BAO.
2. **Model selection with data.** Bayesian inference on (α, M) or (K_0, p) using SN, BAO, RSD, and chronometers quantifies any preference for kinetic running (pipeline in Secs. 8, 9).
3. **Nonlinear regime.** Although $\mu = \Sigma = 1$ at linear scales, background–driven growth histories alter halo statistics and lensing nonlinearity. Mapping QKDE backgrounds to emulators or response functions would enable calibrated small–scale predictions.
4. **Perturbative EFT embedding.** A decoupling–limit derivation from curvature–suppressed operators in QFT in curved spacetime can delineate the parameter range where a background–only $K(t)$ captures the leading low–energy effects.

Take–home message. Within its explicit assumptions, Quantum–Kinetic Dark Energy sits at the conservative edge of EFT–DE: a single running parameter $\alpha_K > 0$ driven by $K(t)$, luminal scalar and tensor propagation, and GR relations for linear growth and lensing. The phenomenology is intentionally simple—*all* late–time signatures flow through the background $H(a)$. This clarity renders the framework transparent, readily testable, and straightforward to falsify with distance and growth data from current and forthcoming surveys.

A Conventions and handy identities

This appendix fixes notation and sign choices used throughout. Units $c = \hbar = 1$ and metric signature $(-,+,+,+)$ are adopted. The reduced Planck mass is $M_{\rm pl} \equiv (8\pi G)^{-1/2}$. Greek indices run over spacetime $\{0,1,2,3\}$; spatial indices are Latin $\{1,2,3\}$.

Geometry and curvature (signs). The Levi–Civita connection is $\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$. Curvature follows the *Wald* sign convention [58]:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}, \quad R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \tag{A.1}$$

The Einstein tensor is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. The d'Alembertian is $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$.

Background spacetime and clocks. A spatially flat FRW line element is

$$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2, \qquad H \equiv \frac{\dot a}{a}, \qquad N \equiv \ln a, \qquad (\,)' \equiv \frac{d}{dN} = \frac{1}{H}\frac{d}{dt}. \tag{A.2}$$

Useful chain rules for any sufficiently smooth background scalar $f(t)$ are

$$\dot f = H f', \qquad \ddot f = H^2 f'' + H H' f'. \tag{A.3}$$

Redshift is related to the e–fold clock by $1+z = e^{-N}$, hence

$$\frac{d}{dz} = -\frac{1}{1+z}\frac{d}{dN}, \qquad \frac{d}{dN} = -(1+z)\frac{d}{dz}. \tag{A.4}$$

The FRW Ricci scalar and its Hubble–normalized form are

$$R = 6\left(2H^2 + \dot H\right) = 6H^2\left(2 + \frac{H'}{H}\right). \tag{A.5}$$

Matter, radiation, and closure. Minimal coupling implies the standard continuity equations

$$\dot\rho_m + 3H\rho_m = 0, \qquad \dot\rho_r + 4H\rho_r = 0 \implies \rho_m(N) = \rho_{m0}\, e^{-3N}, \quad \rho_r(N) = \rho_{r0}\, e^{-4N}. \tag{A.6}$$

Density fractions are $\Omega_i \equiv \rho_i/(3M_{\rm pl}^2 H^2)$ with flatness $\Omega_m + \Omega_r + \Omega_\phi = 1$. The effective equation of state and deceleration parameter are

$$w_{\rm eff} = -1 - \frac{2}{3}\frac{H'}{H}, \qquad q = -1 - \frac{H'}{H}. \tag{A.7}$$

Scalar-field sector (background). The kinetic invariant is $X \equiv -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$, giving on FRW

$$X = \tfrac{1}{2}\dot\phi^2 = \tfrac{1}{2}H^2\phi'^2, \qquad \rho_\phi = K(t)\,X + V(\phi), \qquad p_\phi = K(t)\,X - V(\phi), \qquad w_\phi = \frac{K\dot\phi^2/2 - V}{K\dot\phi^2/2 + V}. \tag{A.8}$$

Ghost freedom requires $K(t) > 0$; for $P(X,\phi,t) = K(t)X - V(\phi)$ one has $P_{,XX} = 0$ and the scalar sound speed $c_s^2 = 1$ [6]. The GR background equations are

$$H^2 = \frac{\rho_m + \rho_r + \rho_\phi}{3M_{\rm pl}^2}, \qquad \dot H = -\frac{1}{2M_{\rm pl}^2}\left(\rho_m + \tfrac{4}{3}\rho_r + K\,\dot\phi^2\right). \tag{A.9}$$

A time–dependent $K(t)$ implies a covariant source in the scalar continuity relation,

$$\nabla_\mu T^\mu{}_{\nu(\phi)} = -\,\partial_\nu K\, X \quad \Rightarrow \quad \dot\rho_\phi + 3H(\rho_\phi + p_\phi) = -\tfrac{1}{2}\dot K\,\dot\phi^2, \tag{A.10}$$

while total energy–momentum remains conserved by the Bianchi identity.

Fourier and perturbation conventions. The Fourier transform is $f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})$, so $\nabla^2 \to -k^2$. Newtonian–gauge scalar perturbations use $ds^2 = -(1+2\Psi)\,dt^2 + a^2(t)(1-2\Phi)\,d\vec{x}^2$. In the GR–preserving setup with no anisotropic stress from the scalar, $\Phi = \Psi$ at linear order on late–time, subhorizon scales [18].

Dimensional analysis (natural units). With $c = \hbar = 1$ and the above signature, $[H] = [M]$, $[R] = [M^2]$, $[\phi] = [M]$, $[V] = [M^4]$, $[X] = [M^4]$, and $[K] = 1$. All combinations appearing in the main text (§2–7) are dimensionless as written.

Primary references for conventions:

Wald [58]; Weinberg [34]; Ma and Bertschinger [18]; Garriga and Mukhanov [6].

B From covariant scalar dynamics to FRW relations

This appendix makes explicit the steps connecting the covariant scalar Lagrangian to the background (FRW) equations used throughout. Conventions are those in App. A; curvature signs follow Wald [58].

B.1 Euler–Lagrange equation for a time–dependent kinetic normalization

Start from

$$\mathcal{L}_\phi = K(t)\, X - V(\phi), \qquad X \equiv -\tfrac{1}{2}\, g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi. \tag{B.1}$$

The field variation yields

$$0 = \frac{\partial \mathcal{L}_\phi}{\partial \phi} - \nabla_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial(\nabla_\mu\phi)}\right) = -V_{,\phi} - \nabla_\mu[-K(t)\, g^{\mu\nu}\nabla_\nu\phi]\,, \tag{B.2}$$

hence

$$K\,\Box\phi + (\nabla_\mu K)\,\nabla^\mu\phi - V_{,\phi} = 0. \tag{B.3}$$

Because $K = K(t)$ depends only on the background clock, $(\nabla_\mu K)\nabla^\mu\phi = g^{00}\dot{K}\,\dot{\phi} = -\dot{K}\,\dot{\phi}$.

B.2 FRW evaluation and e–fold form

On spatially flat FRW, $\Box\phi = -(\ddot{\phi} + 3H\dot{\phi})$. Substituting into (B.3) gives the background Klein–Gordon equation

$$K(\ddot{\phi} + 3H\dot{\phi}) + \dot{K}\,\dot{\phi} + V_{,\phi} = 0. \tag{2.5}$$

Reparametrizing time by $N \equiv \ln a$ with the chain rules $\dot{f} = Hf'$ and $\ddot{f} = H^2 f'' + HH'f'$ leads to

$$\phi'' + \left(3 + \frac{H'}{H} + \frac{K'}{K}\right)\phi' + \frac{V_{,\phi}}{H^2 K} = 0. \tag{3.3}$$

B.3 Stress–energy tensor and fluid variables

Varying $\mathcal{L}_\phi$ with respect to $g^{\mu\nu}$ gives

$$T^{(\phi)}_{\mu\nu} = K\,\partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}(KX - V)\,. \tag{2.2}$$

For a homogeneous background $\phi = \phi(t)$,

$$X = \tfrac{1}{2}\dot\phi^2, \qquad \rho_\phi = K\,\frac{\dot\phi^2}{2} + V(\phi), \qquad p_\phi = K\,\frac{\dot\phi^2}{2} - V(\phi), \qquad w_\phi = \frac{K\dot\phi^2/2 - V}{K\dot\phi^2/2 + V}. \tag{2.3}$$

B.4 Energy exchange induced by $\dot K$: covariant identity and direct check

Because $\mathcal{L}_\phi$ has explicit coordinate dependence through $K(t)$, the scalar stress tensor is not separately conserved. A general identity for matter with explicit x^ν dependence implies

$$\nabla_\mu T^\mu{}_{\nu(\phi)} = -\,\partial_\nu K\,X. \tag{2.10}$$

On FRW ($\nu = 0$) this becomes

$$\dot\rho_\phi + 3H(\rho_\phi + p_\phi) = -\tfrac{1}{2}\,\dot K\,\dot\phi^2. \tag{2.12}$$

A direct check using (2.3) is instructive. One has $\dot\rho_\phi = \frac{1}{2}\dot K\,\dot\phi^2 + K\dot\phi\,\ddot\phi + V_{,\phi}\dot\phi$. Adding $3H(\rho_\phi + p_\phi) = 3HK\dot\phi^2$ and inserting the background equation $K(\ddot\phi + 3H\dot\phi) = -\dot K\,\dot\phi - V_{,\phi}$ from (2.5) gives $\dot\rho_\phi + 3H(\rho_\phi + p_\phi) = -\frac{1}{2}\dot K\,\dot\phi^2$, as stated. The total fluid remains conserved, $\nabla_\mu T^{\mu\nu}_{\rm tot} = 0$, once matter and radiation continuity equations are included (Bianchi identity).

B.5 Einstein equations, Raychaudhuri form, and curvature

With a constant Planck mass and minimal couplings,

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left(\rho_m + \rho_r + \rho_\phi\right), \qquad \dot H = -\frac{1}{2M_{\rm pl}^2}\Big(\rho_m + \tfrac{4}{3}\rho_r + \rho_\phi + p_\phi\Big) = -\frac{1}{2M_{\rm pl}^2}\Big(\rho_m + \tfrac{4}{3}\rho_r + K\,\dot\phi^2\Big), \tag{2.6–2.7}$$

and the FRW Ricci scalar is

$$R = 6\left(2H^2 + \dot H\right) = 6H^2\left(2 + \frac{H'}{H}\right). \tag{2.8, 3.7}$$

Differentiating the first Friedmann relation and using matter/radiation continuity together with (2.12) reproduces the Raychaudhuri equation exactly, providing an internal consistency check for numerical solutions.

B.6 Mukhanov–Sasaki inputs (background factors)

For linear perturbations the gauge–invariant MS variable v and pump field z are

$$v = a\left(\delta\phi + \frac{\dot\phi}{H}\Phi\right), \qquad z^2 = a^2\,\frac{K\,\dot\phi^2}{H^2}, \tag{5.8}$$

and the quadratic action is canonical with unit sound speed, $S^{(2)} = \frac{1}{2}\int d\eta\,d^3x\,\left[(v')^2 - (\nabla v)^2 + z''/z\,v^2\right]$, $c_s^2 = 1$ for $P = K(t)X - V(\phi)$ [6, 18].

All relations above are the ones used in the main text; no additional assumptions beyond spatial flatness, minimal couplings, and $K(t) > 0$ have been introduced.

C Closed algebraic form of K'/K for $K = 1 + \alpha R/M^2$

This appendix derives, step by step, the iteration–free expression used in Sec. 4 for K'/K when

$$K(N) = 1 + \alpha \frac{R(N)}{M^2}, \qquad E \equiv \frac{H'}{H}, \qquad \frac{R}{H^2} = 6(2+E). \tag{4.2, 3.7}$$

Dimensions: K is dimensionless; R has mass dimension 2; therefore α/M^2 is dimensionless.

C.1 Expressing R'/H^2 in terms of background variables

Let $r \equiv R/H^2 = 6(2+E)$, so $R = H^2 r$. Using $(H^2)' = 2EH^2$ gives

$$\frac{R'}{H^2} = r' + 2Er = 6\Big(E' + 4E + 2E^2\Big). \tag{C.1}$$

From the Raychaudhuri form (Sec. 3.2)

$$E = -\frac{1}{2M_{\rm pl}^2 H^2}\left(\rho_m + \tfrac{4}{3}\rho_r + KH^2\phi'^2\right), \tag{3.5}$$

one finds, after differentiating w.r.t. N and using $\rho_m' = -3\rho_m$, $\rho_r' = -4\rho_r$ and the e–fold Klein–Gordon equation (Sec. 3.1),

$$E' = -\frac{1}{2M_{\rm pl}^2 H^2}\left[-3\rho_m - \tfrac{16}{3}\rho_r + \left(KH^2\phi'^2\right)'\right] - 2E^2, \tag{C.2}$$

$$\left(KH^2\phi'^2\right)' = -\,H^2\Big(K'\phi'^2 + 6K\phi'^2\Big) - 2\,\phi' V_{,\phi}. \tag{C.3}$$

Substituting (C.3) into (C.1) makes R'/H^2 *affine* in K':

$$\frac{R'}{H^2} \;=\; A \;+\; B\,K', \qquad B = \frac{3\,\phi'^2}{M_{\rm pl}^2}, \qquad A = 24E + \frac{18K\,\phi'^2}{M_{\rm pl}^2} + \frac{9\rho_m + 16\rho_r + 6\phi' V_{,\phi}}{M_{\rm pl}^2 H^2}. \tag{4.8}$$

All quantities A and B are algebraic functions of the state vector (ϕ, ϕ', H) and known sources (ρ_m, ρ_r).

C.2 Solving algebraically for K'/K

From $K = 1 + \alpha R/M^2$,

$$\frac{K'}{K} \;=\; \frac{\alpha}{M^2}\,\frac{R'}{1+\alpha R/M^2} \;=\; \frac{\alpha H^2}{M^2}\,\frac{A + B\,K'}{1+\alpha R/M^2}. \tag{C.4}$$

Define

$$c \;\equiv\; \frac{\alpha H^2/M^2}{1+\alpha R/M^2}. \tag{C.5}$$

Equation (C.4) becomes $\frac{K'}{K} = c\,(A + B\,K')$. Bring the K' terms to one side and factor:

$$\left[\tfrac{1+\alpha R/M^2}{K} - \tfrac{\alpha H^2}{M^2}\,B\right] K' = \frac{\alpha H^2}{M^2}\,A \quad\Longrightarrow\quad \frac{K'}{K} = \frac{c\,A}{1 - c\,B\,K}. \tag{C.6}$$

Equivalently,

$$\boxed{\frac{K'}{K} = \frac{\dfrac{\alpha H^2}{M^2}\,\dfrac{A}{1+\alpha R/M^2}}{1-\dfrac{\alpha H^2}{M^2}\,\dfrac{3K\,\phi'^2}{M_{\rm pl}^2\,(1+\alpha R/M^2)}}}\,, \tag{4.10}$$

the algebraic, iteration–free result used in Sec. 4.1.

C.3 Regularity, limits, and implementation notes

Regularity condition. The denominator $1 - cBK$ must remain nonzero across the integration domain. For sufficiently small $|\alpha R/M^2|$ this is automatically satisfied and is cheaply monitored alongside $K > 0$ (ghost freedom).

Background limits (sanity checks). Radiation era: $R = 0 \Rightarrow K \simeq 1$ and $K'/K \simeq 0$. Matter era: $E \simeq -3/2 \Rightarrow R/H^2 \simeq 3$, so $K \simeq 1 + 3\alpha H^2/M^2$ and K'/K is slow. de Sitter: $E = 0 \Rightarrow R/H^2 = 12$, hence $K =$ const and $K'/K = 0$, consistent with (4.10).

Small-α expansion. When $|\alpha R/M^2| \ll 1$, insertion of $R'/H^2 = A + B\,K'$ into (C.4) and solving to first order gives

$$\boxed{\frac{K'}{K} \simeq \frac{(\alpha H^2/M^2)\,A}{1-(\alpha H^2/M^2)\,B}}\,, \tag{C.7}$$

which is the $\mathcal{O}(\alpha)$ truncation of (4.10).

Units and dependence. Every factor in (4.8) and (4.10) is dimensionless as written: $[H] = [M]$, $[R] = [M^2]$, $[\phi] = [M]$, $[V] = [M^4]$, $[K] = 1$. Numerically, all inputs are drawn from the background state (ϕ, ϕ', H) and the fixed sources (ρ_m, ρ_r); no higher derivatives are required.

Implementation checklist. At each e–fold step: (i) compute $E = H'/H$ from the background system; (ii) form $R/H^2 = 6(2 + E)$; (iii) assemble A and B in (4.8) and c; (iv) evaluate K'/K via (4.10). This procedure is algebraic and avoids recursion in R' or K'.

D Quadratic action and the Mukhanov–Sasaki system

This appendix derives the quadratic action for scalar perturbations and the associated Mukhanov–Sasaki (MS) equation starting from the covariant action in Eq. (1.1). Only linear order in perturbations is retained throughout.

D.1 Perturbation variables and gauge invariants

Consider scalar perturbations around a spatially flat FRW background. In Newtonian (longitudinal) gauge,

$$ds^2 = -(1+2\Psi)\,dt^2 + a^2(t)\,(1-2\Phi)\,d\vec{x}^2, \qquad \phi(t,\vec{x}) = \phi_0(t) + \delta\phi(t,\vec{x}), \tag{D.1}$$

with $\Phi = \Psi$ in the present framework at linear order (vanishing anisotropic stress; Sec. 5.1). A convenient gauge-invariant scalar fluctuation is the Mukhanov variable (sometimes called the Sasaki–Mukhanov or "Q" variable) [59, 60]

$$v \equiv a\left(\delta\phi + \frac{\dot\phi_0}{H}\,\Phi\right), \qquad \mathcal{R} \equiv \frac{v}{z}, \tag{D.2}$$

where $\mathcal{R}$ is the comoving curvature perturbation and the pump field z will be specified below. The combination in (D.2) is gauge-invariant at linear order [34, 61].

D.2 ADM expansion and constraint elimination

Starting from $S = \int d^4x\sqrt{-g}\,[\frac{1}{2}M_{\rm pl}^2 R + P(X,\phi,t)]$ with $P(X,\phi,t) = K(t)X - V(\phi)$ and $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$, expand the action to second order in scalar perturbations, solve the linearized Hamiltonian and momentum constraints (lapse and shift are nondynamical), and substitute back. For general $P(X,\phi)$ this procedure yields the well-known MS form [6, 33]

$$S^{(2)} = \frac{1}{2}\int d\eta\, d^3x\,\left[(v')^2 - c_s^2(\nabla v)^2 + \frac{z''}{z}\,v^2\right], \qquad v'' + \left(c_s^2k^2 - \frac{z''}{z}\right)v = 0, \tag{D.3}$$

where η is conformal time, a prime denotes $d/d\eta$, and

$$c_s^2 \;=\; \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}, \qquad z^2 \;=\; a^2\frac{\rho_\phi + p_\phi}{c_s^2H^2} \;=\; a^2\frac{2XP_{,X}}{c_s^2H^2}. \tag{D.4}$$

In the present case $P_{,X} = K(t)$ and $P_{,XX} = 0$, so

$$c_s^2 = 1, \qquad z^2 = a^2\frac{2XK}{H^2} = a^2\frac{K\,\dot\phi_0^2}{H^2}, \tag{D.5}$$

and (D.3) reduces to the canonical MS system with a rescaled pump field. Ghost freedom requires $P_{,X} > 0 \Rightarrow K(t) > 0$, which coincides with the background admissibility condition (Secs. 2.5 and 4.3).

Relation to Newtonian-gauge field equations. The perturbed Klein–Gordon equation in Newtonian gauge, is equivalent to (D.3) after using the linear Einstein constraints to eliminate Φ and Ψ and rewriting in terms of v [18]. The absence of braiding ($\alpha_B = 0$) and of any metric-sector modification ensures that the constraint structure is the GR one and that no additional operators enter the quadratic scalar action (Sec. 6).

D.3 Dynamics of $\mathcal{R}$, super/subhorizon limits, and stability

With $v = z\mathcal{R}$, Eq. (D.3) implies

$$\mathcal{R}'' + 2\,\frac{z'}{z}\,\mathcal{R}' + c_s^2k^2\mathcal{R} = 0. \tag{D.6}$$

On superhorizon scales ($k\to 0$), the solution $\mathcal{R}' \simeq 0$ is adiabatically conserved provided z'/z varies slowly on Hubble timescales. Using (D.5),

$$z^2 = a^2\frac{K\,\dot\phi_0^2}{H^2} \quad\Rightarrow\quad \frac{z'}{z} = \mathcal{H} + \frac{1}{2}\frac{(K\dot\phi_0^2/H^2)'}{(K\dot\phi_0^2/H^2)}, \qquad \mathcal{H} \equiv aH, \tag{D.7}$$

so slow variation of K and of the background ($|K'/K| \ll 1$, smooth H) ensures the standard single-field conservation of $\mathcal{R}$ [23]. On subhorizon scales ($k \gg aH$), $c_s^2 = 1$ implies luminal propagation and pressure support; scalar fluctuations oscillate and do not cluster appreciably, consistent with the GR Poisson equation used in the growth analysis (Sec. 5.5).

Stability summary. The kinetic term in (D.3) is canonically normalized; gradient stability requires $c_s^2 > 0$ and ghost freedom requires $P_{,X} > 0$. For $P = K(t)X - V(\phi)$ one has $c_s^2 = 1$ and $P_{,X} = K(t) > 0$. The tensor sector is Einstein–Hilbert with luminal propagation and constant Planck mass (Sec. 6), consistent with GW170817 constraints [1, 2].

D.4 Useful identities and mappings

For later reference, the following relations are used in the main text:

$$v \;=\; z\,\mathcal{R}, \qquad z^2 = a^2\frac{K\,\dot\phi_0^2}{H^2}, \qquad c_s^2 = 1, \qquad \Phi = \Psi, \tag{D.8}$$

and, in the EFT–DE language of Bellini and Sawicki [5], the scalar kinetic running appears as $\alpha_K = K\,\dot\phi_0^2/(H^2 M_{\rm pl}^2)$ with $\alpha_B = \alpha_M = \alpha_T = 0$ (Sec. 6.1). These equalities explain why linear observables in QKDE reduce to their GR forms apart from the background dependence through $H(a)$ (Secs. 5–7).

References for this appendix:

Sasaki [59]; Mukhanov [60]; Kodama and Sasaki [61]; Ma and Bertschinger [18]; Weinberg [34]; Garriga and Mukhanov [6]; De Felice and Tsujikawa [33]; Bellini and Sawicki [5]; Abbott et al. [1]; Sakstein and Jain [2].

E Einstein constraints and GR Poisson equation at subhorizon scales

This appendix makes explicit how the GR Poisson equation used in the text follows from the linearized Einstein constraints and why the QKDE scalar does not source it on subhorizon scales.

E.1 Newtonian gauge constraints and the comoving overdensity

In Newtonian gauge,

$$ds^2 = -(1+2\Psi)\,dt^2 + a^2(t)\,(1-2\Phi)\,d\vec{x}^2, \qquad \phi(t,\vec{x}) = \phi_0(t) + \delta\phi(t,\vec{x}), \tag{E.1}$$

and, in the present framework, the scalar sector carries no anisotropic stress so that $\Phi = \Psi$ at linear order (Secs. 5.1, 5.4). The 0–0 and 0–i Einstein equations in Fourier space are (see Ma and Bertschinger [18] for the conformal-time forms and Weinberg [34] for a textbook derivation)

$$-\,k^2\Phi - 3aH\Big(\dot\Phi + aH\Psi\Big) = 4\pi G\,a^2\sum_i \delta\rho_i, \tag{E.2}$$

$$k^2\Big(\dot\Phi + aH\Psi\Big) = 4\pi G\,a^2\sum_i (\rho_i + p_i)\,\theta_i, \tag{E.3}$$

where θ_i is the velocity-divergence potential of species i. Combining (E.2) and (E.3) eliminates the $\dot{\Phi}$ term and yields the GR Poisson equation written in terms of the *comoving* overdensity

$$-\frac{k^2}{a^2}\Phi = 4\pi G \sum_i \rho_i\, \Delta_i, \qquad \Delta_i \equiv \delta_i + 3\,\frac{aH}{k^2}\,(1+w_i)\,\theta_i, \tag{E.4}$$

valid for any mixture of minimally coupled fluids at linear order [18, 34]. The lensing (Weyl) potential obeys the analogous GR relation

$$-\frac{k^2}{a^2}(\Phi+\Psi) = 8\pi G \sum_i \rho_i\, \Delta_i. \tag{E.5}$$

In QKDE, $\Phi = \Psi$ identically at linear order, so (E.4)–(E.5) coincide up to the factor of two.

E.2 QKDE scalar perturbations and subhorizon suppression

For $P(X, \phi, t) = K(t)X - V(\phi)$, the scalar-fluid perturbations in Newtonian gauge are (Sec. 5.2)

$$\delta\rho_\phi = K\left(\dot{\phi}_0\,\delta\dot{\phi} - \dot{\phi}_0^2\Psi\right) + V_{,\phi}\,\delta\phi, \qquad \delta p_\phi = K\left(\dot{\phi}_0\,\delta\dot{\phi} - \dot{\phi}_0^2\Psi\right) - V_{,\phi}\,\delta\phi, \tag{E.6}$$

$$(\rho_\phi + p_\phi)\,\theta_\phi = K\,\dot{\phi}_0\,\frac{k^2}{a^2}\,\delta\phi, \qquad \rho_\phi + p_\phi = K\,\dot{\phi}_0^2. \tag{E.7}$$

The quadratic action analysis (App. D) gives $c_s^2 = 1$ and the MS equation $v'' + (k^2 - z''/z)v = 0$ with $v = a(\delta\phi + \dot{\phi}_0\Phi/H)$ and $z^2 = a^2K\dot{\phi}_0^2/H^2$. In the *subhorizon* regime $k \gg aH$,

$$v(\eta, \mathbf{k}) \simeq C_1(\mathbf{k})\,e^{+ik\eta} + C_2(\mathbf{k})\,e^{-ik\eta}, \qquad \Rightarrow \qquad \delta\phi = \mathcal{O}\left(\frac{v}{a}\right), \quad \delta\dot{\phi} = \mathcal{O}\left(\frac{k}{a}\,\delta\phi\right), \tag{E.8}$$

so pressure support prevents growth of $\delta\phi$ on scales well inside the sound horizon (which is luminal here). A standard scaling argument for smooth dark energy with $c_s^2 = \mathcal{O}(1)$ then gives

$$\Delta_\phi = \mathcal{O}\Big((1+w_\phi)\Big)\left(\frac{aH}{k}\right)^2 \Delta_m, \qquad (k \gg aH), \tag{E.9}$$

i.e. the dark-energy comoving overdensity is suppressed by $(aH/k)^2$ relative to matter on subhorizon scales [62–64]. Equation (E.9) holds for canonical quintessence and applies here unchanged because $c_s^2 = 1$ and the metric sector is GR.

E.3 GR Poisson equation with matter source only

Using (E.9) in (E.4) and neglecting late-time photon/neutrino shear (tiny on the scales of interest), the subhorizon Poisson equation reduces to

$$-\frac{k^2}{a^2}\Phi \;=\; 4\pi G\,\rho_m\,\Delta_m \quad \Big[1 + \mathcal{O}\big((aH/k)^2\big)\Big], \tag{E.10}$$

with $\Delta_m \equiv \delta_m + 3(aH/k^2)\,\theta_m$. The same reduction holds for the Weyl potential, so linear lensing is also unmodified at subhorizon scales. In the EFT–DE language of Sec. 6, $\alpha_B = \alpha_M = \alpha_T = \alpha_H = 0$ and $c_s^2 = 1$ imply the quasi-static relations

$$\mu(a,k) = 1, \qquad \Sigma(a,k) = 1, \qquad \eta(a,k) \equiv \frac{\Phi}{\Psi} - 1 = 0, \tag{E.11}$$

in agreement with the main-text Eq. (6.5). Departures from (E.10) would require either a modification of the metric sector (e.g. $\alpha_M \neq 0$ or $\alpha_B \neq 0$) or a subluminal sound speed generating clustering ($c_s^2 \ll 1$); neither occurs in QKDE.

Scope of the approximation. The suppression (E.9) applies for $k \gg aH$. Near the horizon ($k \sim aH$) or if early radiation shear is included, the full system (E.2)–(E.3) should be used; this does not affect the late-time, subhorizon observables employed in Sec. 7.

References for this appendix:

Ma and Bertschinger [18]; Weinberg [34]; Garriga and Mukhanov [6]; Bean and Doré [62]; Hu [63]; De Felice and Tsujikawa [33].

F Growth equation from continuity and Euler relations

This appendix derives the linear growth equation used in Secs. 5 and 7 directly from the dust (pressureless matter) continuity and Euler relations, together with the GR constraints summarized in App. E. Throughout this appendix a prime denotes a derivative with respect to e-fold time, $(\ )' \equiv d/d\ln a$, and Fourier conventions follow Ma and Bertschinger [18]. The Newtonian gauge metric is $ds^2 = -(1+2\Psi)\,dt^2 + a^2(t)(1-2\Phi)\,d\vec{x}^2$, and in the present framework $\Phi = \Psi$ at linear order.

F.1 Dust fluid: continuity and Euler in $d/d\ln a$ form

For nonrelativistic matter ($w = 0$) the linearized continuity and Euler equations in Newtonian gauge are (see, e.g., Ma and Bertschinger [18], Weinberg [34])

$$\dot{\delta}_m = -\frac{\theta_v}{a} + 3\,\dot{\Phi}, \qquad \dot{\theta}_v + H\,\theta_v = \frac{k^2}{a}\,\Psi, \tag{F.1}$$

where $\theta_v \equiv \nabla\cdot\vec{v}_m$ is the (proper) velocity divergence. It is convenient to work with the dimensionless velocity divergence

$$\theta_m \ \equiv\ \frac{\theta_v}{aH} = \frac{\nabla\cdot\vec{v}_m}{aH}, \qquad (\ )' = \frac{1}{H}\frac{d}{dt}. \tag{F.2}$$

Equations (F.1) then become

$$\boxed{\delta_m' = -\,\theta_m + 3\,\Phi', \qquad \theta_m' = -(2+\tfrac{H'}{H})\,\theta_m \ +\ \frac{k^2}{a^2H^2}\,\Psi\ .} \tag{F.3}$$

The coefficient $2 + H'/H$ follows from $d\ln(aH)/d\ln a = 1 + H'/H$ and the extra $-\theta_m$ term arising in the time derivative of a^{-1} [cf. App. A of 18].

F.2 Elimination of the velocity and use of the GR constraints

Differentiating the continuity relation in (F.3) and using the Euler relation to eliminate θ_m' gives

$$\begin{aligned}
\delta_m'' &= -\,\theta_m' + 3\,\Phi'' \\
&= (2+\tfrac{H'}{H})\theta_m - \frac{k^2}{a^2H^2}\Psi + 3\,\Phi'' \\
&= -(2+\tfrac{H'}{H})\delta_m' \ +\ 3(2+\tfrac{H'}{H})\Phi' \ -\ \frac{k^2}{a^2H^2}\Psi \ +\ 3\,\Phi'',
\end{aligned} \tag{F.4}$$

where the continuity equation was used again to replace θ_m by $-\delta_m' + 3\Phi'$.

On subhorizon scales $k \gg aH$ and for smooth late-time species ($c_s^2 = \mathcal{O}(1)$), the time derivatives of the potentials are suppressed relative to the gradient term by $\mathcal{O}[(aH/k)^2]$ [18, 62, 63]. Neglecting the Φ' and Φ'' terms at this order and using $\Psi = \Phi$, the GR Poisson equation in comoving form (App. E) reduces to

$$-\frac{k^2}{a^2}\,\Phi = 4\pi G\,\rho_m\,\delta_m \quad \Rightarrow \quad \frac{k^2}{a^2H^2}\,\Psi = -\,\frac{3}{2}\,\Omega_m(a)\,\delta_m. \tag{F.5}$$

Substituting into (F.4) yields the standard GR growth equation in $d/d\ln a$ form:

$$\boxed{\delta_m'' + \left(2 + \frac{H'}{H}\right)\delta_m' - \frac{3}{2}\,\Omega_m(a)\,\delta_m = 0,} \tag{F.6}$$

valid for $k \gg aH$ in the absence of anisotropic stress and fifth forces. Defining the growing–mode factor by $\delta_m(\mathbf{k}, a) = D(a)\,\delta_m(\mathbf{k}, a_{\rm ini})$ reproduces Eq. (5.17) in the main text; the customary normalization is $D(1) = 1$.

Riccati form for the logarithmic growth rate. Let $f \equiv D'/D = d\ln D/d\ln a$. Dividing (F.6) by D gives

$$\boxed{f' + f^2 + \left(2 + \frac{H'}{H}\right) f - \frac{3}{2}\,\Omega_m(a) = 0,} \tag{F.7}$$

the form integrated alongside the background in Sec. 8.4.

Checks and scope. In exact matter domination, $H'/H = -3/2$ and $\Omega_m \to 1$, so (F.6) has the growing solution $D \propto a$ as expected. If one keeps the Φ' terms in (F.4), small $\mathcal{O}\big[(aH/k)^2\big]$ relativistic corrections appear on very large scales; these are irrelevant for the late-time, subhorizon observables considered in Sec. 7 but can be retained straightforwardly when needed (see Ma and Bertschinger [18], Weinberg [34] for the general formulas).

References for this appendix:

Ma and Bertschinger [18]; Weinberg [34]; Bean and Doré [62]; Hu [63].

G Variational (sensitivity) system: detailed steps

This appendix records the tangent–linear (variational) equations used in Sec. 9 and gives all Jacobians needed to evolve exact parameter derivatives concurrently with the background. Throughout, a prime denotes d/dN with $N \equiv \ln a$ and the state vector is

$$\mathbf{y} \equiv (\phi, \phi', H)^\top, \qquad \mathbf{y}' = \mathbf{F}(N, \mathbf{y}; \boldsymbol{\theta}), \tag{G.1}$$

with components read directly from the closed system (3.12):

$$F_\phi = \phi', \tag{G.2}$$

$$F_{\phi'} = -\left(3 + \frac{H'}{H} + \frac{K'}{K}\right)\phi' - \frac{V_{,\phi}}{H^2K}, \tag{G.3}$$

$$F_H = H\,\frac{H'}{H} \equiv H\,E, \qquad E \;=\; -\frac{\rho_m + \frac{4}{3}\rho_r + K\,H^2\phi'^2}{2M_{\rm pl}^2H^2}. \tag{G.4}$$

The sources are $\rho_m(N) = \rho_{m0}e^{-3N}$ and $\rho_r(N) = \rho_{r0}e^{-4N}$.

G.1 Tangent–linear system and state Jacobian

For any parameter θ_i, define the sensitivity vector $\mathbf{s}_i \equiv \partial\mathbf{y}/\partial\theta_i$. Differentiating (G.1) at fixed N gives the variational system

$$\boxed{\mathbf{s}_i' = \mathbf{J}_y\,\mathbf{s}_i + \mathbf{J}_{\theta_i}, \qquad \mathbf{J}_y \equiv \frac{\partial \mathbf{F}}{\partial \mathbf{y}}, \quad \mathbf{J}_{\theta_i} \equiv \frac{\partial \mathbf{F}}{\partial \theta_i}\,,} \tag{G.5}$$

with explicit entries[2]:

$$\frac{\partial F_\phi}{\partial \phi} = 0, \quad \frac{\partial F_\phi}{\partial \phi'} = 1, \quad \frac{\partial F_\phi}{\partial H} = 0, \tag{G.6}$$

$$\frac{\partial F_{\phi'}}{\partial \phi} = -\frac{V_{,\phi\phi}}{H^2 K}, \qquad \frac{\partial F_{\phi'}}{\partial \phi'} = -\left(3 + E + \frac{K'}{K}\right) - \phi'\left(\frac{\partial E}{\partial \phi'} + \frac{\partial}{\partial \phi'}\frac{K'}{K}\right), \tag{G.7}$$

$$\frac{\partial F_{\phi'}}{\partial H} = -\phi'\left(\frac{\partial E}{\partial H} + \frac{\partial}{\partial H}\frac{K'}{K}\right) + \frac{2V_{,\phi}}{H^3 K}, \tag{G.8}$$

$$\frac{\partial F_H}{\partial \phi} = H\,\frac{\partial E}{\partial \phi} = 0, \qquad \frac{\partial F_H}{\partial \phi'} = H\,\frac{\partial E}{\partial \phi'} = -\,\frac{K\,H\,\phi'}{M_{\rm pl}^2}, \qquad \frac{\partial F_H}{\partial H} = E + H\,\frac{\partial E}{\partial H}. \tag{G.9}$$

G.2 Model-dependent pieces: K'/K

All appearances of $\partial(\,K'/K\,)/\partial(\cdot)$ and $\partial(\,K'/K\,)/\partial\theta_i$ are supplied by the chosen $K(N)$.

Phenomenological running $K(N) = 1 + K_0 e^{-pN}$.

$$\frac{K'}{K} = -p\,\frac{K-1}{K}, \qquad \frac{\partial}{\partial\phi}\frac{K'}{K} = \frac{\partial}{\partial\phi'}\frac{K'}{K} = \frac{\partial}{\partial H}\frac{K'}{K} = 0, \tag{G.10}$$

and the parameter derivatives used in $\mathbf{J}_{\theta_i}$ are

$$\boxed{\frac{\partial}{\partial K_0}\left(\frac{K'}{K}\right) = -\,p\,\frac{e^{-pN}}{K^2}, \qquad \frac{\partial}{\partial p}\left(\frac{K'}{K}\right) = -\,\frac{K-1}{K} + pN\,\frac{K-1}{K^2}\ .} \tag{G.11}$$

Curvature-motivated $K = 1 + \alpha R/M^2$. Use the closed algebraic form (4.10):

$$F \equiv \frac{K'}{K} = \frac{U}{D}, \qquad U \equiv c\,A, \quad D \equiv 1 - cBK, \quad B = \frac{3\phi'^2}{M_{\rm pl}^2}, \quad A = 24E + \frac{18K\phi'^2}{M_{\rm pl}^2} + \frac{9\rho_m + 16\rho_r + 6\phi' V_{,\phi}}{M_{\rm pl}^2 H^2}, \tag{G.12}$$

with

$$c = \frac{\xi H^2}{1 + 6\xi H^2(2+E)}, \qquad \xi \equiv \frac{\alpha}{M^2}. \tag{G.13}$$

State derivatives (for $\partial(\,K'/K\,)/\partial y$ with $y \in \{\phi, \phi', H\}$) follow from the quotient rule:

$$\boxed{\frac{\partial}{\partial y}\left(\frac{K'}{K}\right) = \frac{(\partial_y U)\,D - U\,(\partial_y D)}{D^2}, \quad \partial_y U = (\partial_y c)\,A + c\,(\partial_y A), \quad \partial_y D = -\,(\partial_y c)\,BK - c\,(\partial_y B)\,K\ ,} \tag{G.14}$$

[2] Partial derivatives of E are evaluated at fixed $(N, \mathbf{y})$: $\partial E/\partial\phi = 0$, $\partial E/\partial\phi' = -K\,\phi'/M_{\rm pl}^2$, $\partial E/\partial H = -\frac{1}{M_{\rm pl}^2}\left(\frac{K\,\phi'^2}{H} - \frac{\rho_m + \frac{4}{3}\rho_r + K\,H^2\phi'^2}{H^3}\right)$ (cf. (9.5)).

with

$$\partial_\phi B = 0, \quad \partial_{\phi'} B = \tfrac{6\phi'}{M_{\rm pl}^2}, \quad \partial_H B = 0, \tag{G.15}$$

$$\partial_\phi A = \frac{6\phi' V_{,\phi\phi}}{M_{\rm pl}^2 H^2}, \quad \partial_{\phi'} A = 24\,\partial_{\phi'} E + \frac{36K\phi'}{M_{\rm pl}^2} + \frac{6V_{,\phi}}{M_{\rm pl}^2 H^2}, \quad \partial_H A = 24\,\partial_H E - \frac{2\,(9\rho_m + 16\rho_r + 6\phi' V_{,\phi})}{M_{\rm pl}^2 H^3}, \tag{G.16}$$

$$\frac{\partial c}{\partial H} = \frac{2\xi H}{\left[1 + 6\xi H^2(2+E)\right]^2}, \qquad \frac{\partial c}{\partial E} = -\frac{6\xi^2 H^4}{\left[1 + 6\xi H^2(2+E)\right]^2}. \tag{G.17}$$

Parameter derivatives entering $\mathbf{J}_{\theta_i}$ (holding $\mathbf{y}$ fixed) are

$$\boxed{\begin{aligned} \frac{\partial c}{\partial \alpha} &= \frac{H^2}{M^2\left[1 + 6\xi H^2(2+E)\right]^2}, \\ \frac{\partial c}{\partial M} &= -\frac{2\,\xi H^2}{M\left[1 + 6\xi H^2(2+E)\right]^2}, \\ \frac{\partial A}{\partial \alpha} &= \frac{18\,(\partial K/\partial\alpha)\,\phi'^2}{M_{\rm pl}^2}, \qquad \frac{\partial K}{\partial \alpha} = \frac{R}{M^2}. \end{aligned}} \tag{G.18}$$

and similarly for $\partial/\partial M$ via $\partial K/\partial M = -2\alpha R/M^3$. Derivatives with respect to potential parameters $\theta \in \boldsymbol{\theta}_V$ enter only through $V_{,\phi}$ and $V_{,\phi\phi}$ below.

G.3 Parameter-source vectors $\mathbf{J}_{\theta_i}$

With the pieces above,

$$J_{\theta_i,\phi} = \frac{\partial F_\phi}{\partial \theta_i} = 0, \tag{G.19}$$

$$J_{\theta_i,\phi'} = \frac{\partial F_{\phi'}}{\partial \theta_i} = -\,\phi' \frac{\partial}{\partial \theta_i}\left(\frac{K'}{K}\right) - \frac{1}{H^2 K}\frac{\partial V_{,\phi}}{\partial \theta_i} + \frac{V_{,\phi}}{H^2 K^2}\frac{\partial K}{\partial \theta_i}, \tag{G.20}$$

$$J_{\theta_i,H} = \frac{\partial F_H}{\partial \theta_i} = H\,\frac{\partial E}{\partial \theta_i}. \tag{G.21}$$

For late-time cosmology parameters that appear only in the sources, $\rho_m = 3M_{\rm pl}^2 H_0^2 \Omega_{m0} e^{-3N}$ and $\rho_r = 3M_{\rm pl}^2 H_0^2 \Omega_{r0} e^{-4N}$ imply

$$\frac{\partial E}{\partial H_0} = -\frac{2\,\rho_{\rm tot}}{2M_{\rm pl}^2 H^2}\frac{1}{H_0}, \qquad \frac{\partial E}{\partial \Omega_{m0}} = -\frac{\rho_m}{2M_{\rm pl}^2 H^2\,\Omega_{m0}}, \qquad \frac{\partial E}{\partial \Omega_{r0}} = -\frac{2\,\rho_r}{3M_{\rm pl}^2 H^2\,\Omega_{r0}}, \tag{G.22}$$

with $\rho_{\rm tot} \equiv \rho_m + \rho_r$. The amplitude $\sigma_{8,0}$ does not enter the background ($J_{\theta_i} = 0$ for $\theta_i = \sigma_{8,0}$).

G.4 Initialization: differentiating the shoot condition

Initial data are set in the matter era by

$$\phi'(N_{\rm ini}) = 0, \qquad H(N_{\rm ini}) = H_0\sqrt{\Omega_{m0}e^{-3N_{\rm ini}} + \Omega_{r0}e^{-4N_{\rm ini}}}, \tag{G.23}$$

and by choosing $\phi(N_{\rm ini})$ so that the $N = 0$ closure holds:

$$g(\phi(N_{\rm ini}), \boldsymbol{\theta}) \;\equiv\; \Omega_\phi(0) - (1 - \Omega_{m0} - \Omega_{r0}) = 0, \qquad \Omega_\phi = \frac{K\,\phi'^2}{6M_{\rm pl}^2} + \frac{V}{3M_{\rm pl}^2 H^2}. \tag{G.24}$$

The sensitivity of the unknown $\phi(N_{\rm ini})$ follows from the implicit–function theorem. Let $\mathbf{s}_i^{(0)}$ denote the solution of (G.5) with *provisional* initial data

$$\mathbf{s}_i(N_{\rm ini}) = (0,\,0,\,\partial_{\theta_i} H_{\rm ini})^\top, \quad \partial_{\theta_i} H_{\rm ini} \equiv \frac{\partial H(N_{\rm ini})}{\partial \theta_i}, \tag{G.25}$$

and let $\mathbf{e}$ be the solution of the homogeneous variational system with unit shift in the unknown initial field,

$$\mathbf{e}(N_{\rm ini}) = (1,\,0,\,0)^\top, \qquad \mathbf{e}' = \mathbf{J}_y\,\mathbf{e}. \tag{G.26}$$

Evolving both to $N = 0$, the linear response of the constraint is

$$\partial_{\theta_i} g = \left.\frac{\partial g}{\partial \mathbf{y}}\right|_0 \cdot \mathbf{s}_i^{(0)}(0) + \left.\frac{\partial g}{\partial \phi_{\rm ini}}\right|_0 \partial_{\theta_i}\phi_{\rm ini}, \qquad \frac{\partial g}{\partial \phi_{\rm ini}} = \left.\frac{\partial g}{\partial \mathbf{y}}\right|_0 \cdot \mathbf{e}(0), \tag{G.27}$$

and the condition $\partial_{\theta_i} g = 0$ yields the sought initial sensitivity

$$\boxed{\partial_{\theta_i}\phi(N_{\rm ini}) = -\,\frac{\left.\dfrac{\partial g}{\partial \mathbf{y}}\right|_0 \cdot \mathbf{s}_i^{(0)}(0)}{\left.\dfrac{\partial g}{\partial \mathbf{y}}\right|_0 \cdot \mathbf{e}(0)}\,.} \tag{G.28}$$

The full initial vector is then $\mathbf{s}_i(N_{\rm ini}) = \big(\partial_{\theta_i}\phi(N_{\rm ini}),\,0,\,\partial_{\theta_i}H_{\rm ini}\big)^\top$, and a single forward pass produces $\mathbf{s}_i(N)$ for all N.

G.5 Distance sensitivities (for completeness)

The comoving distance ODE $\chi' = -e^{-N}/H$ implies the exact sensitivity

$$\boxed{\chi_i' = \frac{e^{-N}}{H^2}\,H_i, \qquad H_i \equiv \frac{\partial H}{\partial \theta_i}, \qquad \chi(0) = 0,\;\; \chi_i(0) = 0,} \tag{G.29}$$

which is integrated alongside (G.5). Algebraic propagation to $D_A, D_L, D_V, F_{\rm AP}$ is given in (9.13).

G.6 Consistency and numerical remarks

All expressions above are algebraic in $(N, \mathbf{y})$ and previously defined sources; no finite differencing is required. The tangent–linear integration inherits the stability of the background scheme; standard references for sensitivity ODEs and embedded RK integrators include Dormand and Prince [50], Hairer et al. [51]. Identity checks listed in Sec. 8.5 (e.g., Friedmann closure and Raychaudhuri residual) must remain at or below the requested tolerances when computed from either the background or the sensitivities.

References for this appendix: Dormand and Prince [50], Hairer et al. [51].

H Numerical algorithm (pseudocode, guards, and reproducibility)

This appendix specifies an end–to–end, reproducible algorithm for integrating the closed background system (3.12), evaluating observables (Sec. 7), and, when requested, co–evolving exact parameter sensitivities (Sec. 9). All steps are direct evaluations of equations already derived; no fitting templates or phenomenological shortcuts are introduced.

H.1 Inputs and preprocessing

Required inputs

- Late–time cosmology at $N=0$: $(H_0, \Omega_{m0}, \Omega_{r0})$; if growth observables are reported, also $\sigma_{8,0}$.
- Scalar sector: a twice–differentiable potential $V(\phi)$ providing $V_{,\phi}$ and $V_{,\phi\phi}$; a kinetic normalization specification either (i) phenomenological $K(N) = 1 + K_0 e^{-pN}$ or (ii) curvature–motivated $K(N) = 1 + \alpha R/M^2$.
- Integration domain: $N \in [N_{\rm ini}, 0]$ with $N_{\rm ini}$ chosen deep in matter domination.

Derived quantities (evaluated pointwise)

- Sources $\rho_m(N) = \rho_{m0} e^{-3N}$, $\rho_r(N) = \rho_{r0} e^{-4N}$ from (3.8).
- Raychaudhuri ratio $E \equiv H'/H$ from (3.5); Ricci ratio $R/H^2 = 6(2+E)$ from (3.7).

H.2 Initialization (matter era) and shoot to closure

Initialization

1. Set $\phi'(N_{\rm ini}) = 0$.
2. Set $H(N_{\rm ini}) = H_0 \sqrt{\Omega_{m0} e^{-3N_{\rm ini}} + \Omega_{r0} e^{-4N_{\rm ini}}}$ (matter+rad. limit of (3.8)).
3. Determine $\phi(N_{\rm ini})$ by a one–dimensional shoot such that the closure at $N=0$ holds:

$$g\big(\phi(N_{\rm ini})\big) \;\equiv\; \Omega_\phi(0) - \big(1 - \Omega_{m0} - \Omega_{r0}\big) = 0, \quad \Omega_\phi = \frac{K\,\phi'^2}{6M_{\rm pl}^2} + \frac{V}{3M_{\rm pl}^2 H^2}.$$

Robust shoot (bracketed Newton)

1. Establish a bracket $[\phi_{\min}, \phi_{\max}]$ with opposite–sign residuals $g(\phi_{\min})\,g(\phi_{\max}) < 0$ (expand geometrically if needed).
2. Iterate with safeguarded Newton/bisection: $\phi \leftarrow \phi - \lambda\, g/g'$ if the Newton step remains inside the bracket; otherwise bisect. The derivative g' is computed by reusing the variational method (App. G) with the homogeneous seed $\mathbf{e}(N_{\rm ini}) = (1,0,0)^\top$ carried to $N=0$.
3. Stop when $|g| < \varepsilon_{\rm shoot}$ and $|\Delta\phi| < \varepsilon_\phi$ (e.g. 10^{-12} in double precision).

Note. For thawing–like histories $\partial\Omega_\phi(0)/\partial\phi(N_{\rm ini}) > 0$ is typical, which accelerates convergence. The bracketed variant guarantees termination even when this monotonicity is weak.

H.3 Core integration loop (embedded RK with adaptive steps)

State and auxiliaries

$$\mathbf{y} \equiv (\phi,\ s,\ y_H) = (\phi,\ \phi',\ \ln H), \qquad E = y_H'.$$

At each accepted step

1. Evaluate sources ρ_m, ρ_r at current N.
2. Compute E from (3.12b); set $R/H^2 = 6(2+E)$.
3. Evaluate K and K'/K:
   - Phenomenological case: use the identity (4.14).
   - Curvature case: use the closed algebraic form (4.10) with A, B, c from (4.8).
4. Advance the state with DOPRI5 (Dormand–Prince 4(5) [50, 51]); adjust stepsize by standard local–error control.
5. Update diagnostics $\mathcal{C}_F, \mathcal{C}_R, \mathcal{C}_\phi, \mathcal{C}_{R/H^2}, \mathcal{C}_{\nabla\cdot T_\phi}$ (Sec. 8.5); retain running maxima.
6. (Optional) co–integrate the distance ODE $\chi' = -e^{-N-y_H}$ and, if forecasts are required, the sensitivity system (9.3) and the growth equations (Sec. 8.4).

Stepsize policy Set $(\mathrm{rtol}, \mathrm{atol})$ componentwise on (ϕ, s, y_H); impose $h_{\max}$ (e.g. 10^{-2} in N) and $h_{\min}$ (e.g. 10^{-8}); if $h < h_{\min}$ is requested repeatedly, trigger the adaptive–recovery rule in Sec. H.4.

H.4 Guards, event handling, and early termination

Physical and algebraic guards (checked every stage)

- *Ghost freedom:* enforce $K(N) > 0$; on violation, terminate with a descriptive flag.
- *Positivity of H:* because $y_H = \ln H$, positivity is built in; nonetheless assert $H > 0$ on output.
- *Curvature case denominator:* require $|1 - cBK| > \epsilon_{\mathrm{den}}$ in (4.10); default $\epsilon_{\mathrm{den}} \sim 10^{-12}$ in double precision. If violated, declare (α, M) inadmissible for that trajectory.
- *Early–time safety (optional prior):* if standard pre–drag physics is desired, assert $|\alpha R/M^2| \ll 1$ for $N \leq N_{\mathrm{drag}}$ (Sec. 4.1).

Adaptive recovery

- If diagnostics saturate near machine precision while the stepsize underflows, set $(\mathrm{rtol}, \mathrm{atol}) \to (\mathrm{rtol}/10, \mathrm{atol}/10)$ and restart from the last checkpoint (equations unchanged), as in Table 8.1.

H.5 Growth and sensitivities (co–integration)

Growth Integrate

$$D'' + (2+E)D' - \tfrac{3}{2}\Omega_m(N)D = 0, \quad D(N_{\rm ini}) = e^{N_{\rm ini}},\ D'(N_{\rm ini}) = 1,$$

then renormalize $D \leftarrow D/D(0)$ to enforce $D(0)=1$; compute $f = D'/D$ and $f\sigma_8(z) = f(z)\,\sigma_{8,0}\,D(z)$.

Sensitivities For each parameter θ_i, integrate the variational system

$$\mathbf{s}_i' = \mathbf{J}_y\,\mathbf{s}_i + \mathbf{J}_{\theta_i},$$

with $\mathbf{J}_y$ from (G.6)–(G.9) and model–specific $\mathbf{J}_{\theta_i}$ (App. G); initialize $\mathbf{s}_i$ by differentiating the matter–era rules and the closure (Eq. (G.28)). Co–integrate the distance sensitivity $\chi_i' = e^{-N}H_i/H^2$ (Eq. (9.12)). After D–renormalization, apply $D_i \leftarrow D_i - D\,[D_i(0)/D(0)]$ so that $D_i(0)=0$.

H.6 Postprocessing, diagnostics, and reproducibility capsule

Geometric outputs From $\chi(N)$ obtain D_A and D_L using (7.2); compute D_V and $F_{\rm AP}$ from (7.4); report SNe modulus $\mu(z) = 5\log_{10}(D_L/{\rm Mpc}) + 25$.

Identity checks Report $\max_N |\mathcal{C}_F|$, $|\mathcal{C}_R|$, $|\mathcal{C}_\phi|$, $|\mathcal{C}_{R/H^2}|$, $|\mathcal{C}_{\nabla\cdot T_\phi}|$ (Sec. 8.5); target tolerances are summarized in Table 8.1.

Reproducibility metadata Save: input parameter file; $({\rm rtol}, {\rm atol}, h_{\max}, h_{\min})$; guard thresholds $(\epsilon_{\rm den}, \varepsilon_{\rm shoot}, \varepsilon_\phi)$; integrator scheme (DOPRI5); checkpoint cadence; and the full diagnostics log. No pseudo–random sampling is used, so runs are deterministic for identical inputs.

H.7 Recommended tolerances

Embedded Runge–Kutta with adaptive steps (DOPRI5 [50, 51]) and

$${\rm rtol} \in [10^{-10}, 10^{-8}], \qquad {\rm atol} \in [10^{-12}, 10^{-10}]$$

is sufficient for all tables and figures, consistent with the targets in Table 8.1. Tighter tolerances may be advisable if $|V_{,\phi\phi}|/H^2$ is large (incipient stiffness); switching to a stiff BDF method is permissible *without* modifying any equation.

References for this appendix: embedded Runge–Kutta (DOPRI5; [50]), adaptive nonstiff ODE solvers and error control ([51]), and Gauss–Kronrod/QUADPACK quadrature ([52]).

I Analytic limits: radiation, matter, and de Sitter

This appendix records explicit background and linear–response limits of the construction, using the identities $E \equiv H'/H$ and $R/H^2 = 6(2+E)$ [Eq. (3.7)], together with the two $K(N)$ specifications of Sec. 4. The effective equation of state and deceleration parameter follow from $w_{\rm eff} = -1 - \frac{2}{3}E$ and $q = -1 - E$ [Eq. (3.10)]. All statements are exact for the corresponding perfect–fluid limit and receive only subleading corrections from any subdominant component.

Radiation era ($w = 1/3$)

For a radiation–dominated background,

$$H(N) = H_0\sqrt{\Omega_{r0}}\, e^{-2N}, \qquad E \equiv \frac{H'}{H} = -2, \qquad \frac{R}{H^2} = 6(2+E) = 0, \qquad w_{\rm eff} = \tfrac{1}{3}, \quad q = 1. \quad \text{(I.1)}$$

Curvature–motivated K: with $K = 1 + \alpha R/M^2$ one has

$$K \equiv 1, \qquad \frac{K'}{K} \equiv 0, \tag{I.2}$$

exactly in the radiation limit (since $R = 0$). Thus the BAO/CMB anchor r_d is unmodified by construction when the early–time prior $|\alpha R/M^2| \ll 1$ is enforced (Sec. 4.1). *Phenomenological K:* $K(N) = 1 + K_0 e^{-pN}$ with $K'/K = -p(K-1)/K$ [Eq. (4.14)]. Imposing $K \to 1$ at high z (e.g., $p > 0$ with K_0 small) preserves the standard ruler.

Linear response: $c_s^2 = 1$ and $\Phi = \Psi$ (Secs. 5, 6) imply the GR Poisson relation on subhorizon scales; matter clustering is suppressed during radiation domination as in GR (subhorizon growth $\propto \ln a$ for CDM once modes enter the horizon), since $\mu = \Sigma = 1$ [Eq. (6.5)].

Matter era ($w = 0$)

For a matter–dominated background,

$$H(N) = H_0\sqrt{\Omega_{m0}}\, e^{-3N/2}, \qquad E = -\tfrac{3}{2}, \qquad \frac{R}{H^2} = 6(2+E) = 3, \qquad w_{\rm eff} = 0, \quad q = \tfrac{1}{2}. \quad \text{(I.3)}$$

Curvature–motivated K: with $R = 3H^2$ and $R' = -9H^2$,

$$K = 1 + \frac{3\alpha H^2}{M^2}, \qquad K' = \frac{\alpha R'}{M^2} = -\frac{9\alpha H^2}{M^2}, \qquad \boxed{\frac{K'}{K} = \frac{-9(\alpha H^2/M^2)}{1 + 3(\alpha H^2/M^2)}}, \tag{I.4}$$

which is slowly varying whenever $|\alpha H^2/M^2| \ll 1$ (the same condition that ensures early–time safety). *Phenomenological K:* $K'/K = -p(K-1)/K$ as above.

Linear response: the growing mode satisfies $D'' + (2+E)D' - \frac{3}{2}\Omega_m D = 0$ [Eq. (7.7)] with solution $D \propto a$ in the pure matter limit; $\Phi = \Psi =$ const. on subhorizon scales, exactly as in GR.

de Sitter (late–time Λ limit)

For a de Sitter background with constant H,

$$E = 0, \qquad \frac{R}{H^2} = 12, \qquad w_{\text{eff}} = -1, \quad q = -1. \tag{I.5}$$

Curvature–motivated K: $K = 1 + 12\alpha H^2/M^2$ is constant and $\frac{K'}{K} = 0$ (consistent with Eq. (4.10)). The Raychaudhuri relation (2.7) enforces $K\dot{\phi}^2 = 0$, so $\dot{\phi} = 0$ for $K > 0$ and the field is at rest. The MS pump field $z^2 = a^2 K\dot{\phi}^2/H^2$ vanishes, yielding $v'' + (k^2 - a''/a)v = 0$ with luminal propagation (App. D). *Phenomenological K:* any finite K with $K' \to 0$ as $E \to 0$ is admissible; the background dynamics fix $\dot{\phi} \to 0$ as above.

At–a–glance summary

Regime	$E \equiv H'/H$	R/H^2	K (curvature case)	K'/K (curvature case)
Radiation	-2	0	1	0
Matter	$-3/2$	3	$1 + 3\,\alpha H^2/M^2$	$-9(\alpha H^2/M^2)/[1 + 3(\alpha H^2/M^2)]$
de Sitter	0	12	$1 + 12\,\alpha H^2/M^2$	0

In all three limits $c_s^2 = 1$, $\Phi = \Psi$, and the linear phenomenology remains $\mu = \Sigma = 1$ [Secs. 5–6]. The curvature–motivated specification reproduces $K \to 1$ in the radiation era (hence preserves r_d), drifts slowly during matter domination, and freezes in de Sitter; the phenomenological running remains governed by the exact identity $K'/K = -p(K-1)/K$ [Eq. (4.14)] and can be pivoted to suppress any early–time deviation if desired (Sec. 4.2).

J Symbol index

Table J.1: Key symbols and definitions used throughout the text. Units are in natural conventions $c = \hbar = 1$. Mass dimensions are indicated by M^n.

Symbol	Units	Definition / Role
$a(t)$	1	Scale factor; $N \equiv \ln a$ is the e–fold time.
z	1	Redshift; $z(N) = e^{-N} - 1$.
$H,\ H_0$	M, M	Hubble rate and its value today.
E	1	Hubble e–fold derivative: $E \equiv H'/H$.
R	M^2	FRW Ricci scalar: $R = 6(2H^2 + \dot{H}) = 6H^2(2 + E)$.
$M_{\rm pl}$	M	Reduced Planck mass, $(8\pi G)^{-1/2}$.
$\rho_i,\ p_i$	M^4	Energy density and pressure for $i \in \{m, r, \phi\}$.
Ω_i	1	$\Omega_i \equiv \rho_i/(3M_{\rm pl}^2 H^2)$.
$w_{\rm eff},\ q$	1	$w_{\rm eff} = -1 - \frac{2}{3}E$, $q = -1 - E$.
χ	M^{-1}	$\chi(z) = \int_0^z dz'/H(z')$.
$D_A,\ D_L$	M^{-1}	$D_A = \chi/(1+z)$, $D_L = (1+z)\chi$.
D_V	M^{-1}	$D_V = [(1+z)^2 D_A^2\, z/H]^{1/3}$.
$F_{\rm AP}$	1	$(1+z)D_A H^{-1}$.
r_d	M^{-1}	$r_d = \int_{z_d}^{\infty} c_s/H\, dz$.
η (conf. time)	M^{-1}	Conformal time, $dt = a\, d\eta$.
ϕ	M	Scalar field.
X	M^4	$X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = \frac{1}{2}\dot{\phi}^2$ (FRW).
$V(\phi)$	M^4	Scalar potential.
$K(t)$	1	Background kinetic normalization; $K > 0$ for ghost freedom.
$\alpha,\ M$	1, M	Curvature $K = 1 + \alpha R/M^2$.
$K_0,\ p,\ N_p$	1, 1, 1	$K(N) = 1 + K_0 e^{-pN} = 1 + K_p e^{-p(N-N_p)}$.
K'/K	1	E–fold derivative; closed form in Eq. (4.10).
$\rho_\phi,\ p_\phi$	M^4	$\rho_\phi = KX + V$, $p_\phi = KX - V$.
w_ϕ	1	$(KX - V)/(KX + V)$.

Symbol	Units	Definition / Role
$\Phi,\ \Psi$	1	Bardeen potentials (Newtonian gauge); here $\Phi = \Psi$.
$v,\ z$	M, M	$v = a(\delta\phi + \dot{\phi}\,\Phi/H)$; $z^2 = a^2 K\dot{\phi}^2/H^2$.
$\mathcal{R}$	1	Comoving curvature: $\mathcal{R} = v/z$.
c_s^2	1	Scalar sound speed; here $c_s^2 = 1$.
c_T^2	1	Tensor speed; here $c_T^2 = 1$.
$\alpha_{K,B,M,T},\ \alpha_H$	1	QKDE: $\alpha_K = K\dot{\phi}^2/(H^2 M_{\rm pl}^2) \geq 0$, others 0.
$\mu(a,k),\ \Sigma(a,k)$	1	Here $\mu = \Sigma = 1$ (linear, subhorizon).
$\eta(a,k)$ (slip)	1	$\eta \equiv \Phi/\Psi - 1$; here $\eta = 0$.
$k,\ \ell$	M, 1	Comoving wavenumber and multipole.
$P(k,z)$	M^{-3}	Linear scaling $P(k,z) = D^2(z)P(k,0)$.
$D(a)$	1	$D'' + (2 + H'/H)D' - \frac{3}{2}\Omega_m D = 0$.
f	1	$f \equiv d\ln D/d\ln a$.
$\sigma_{8,0}$	1	Present-day rms in $8\,h^{-1}$Mpc.
$f\sigma_8(z)$	1	$f(z)\,\sigma_{8,0}\,D(z)/D(0)$.
s	M	Background velocity: $s \equiv \phi'$.
y_H	1	$y_H \equiv \ln H$.
$\mathcal{C}_F$	1	$1 - \Omega_m - \Omega_r - \Omega_\phi$.
$\mathcal{C}_R$	1	Raychaudhuri residual (e–fold form).
$\mathcal{C}_\phi$	1	Background Klein–Gordon residual.
$\mathcal{C}_{R/H^2}$	1	$R/H^2 - 6(2 + H'/H)$.
$\mathcal{C}_{\nabla\cdot T_\phi}$	1	Non-conservation due to $K'(N)$.
$\mu_{\rm SN}(z)$	1	$5\log_{10}(D_L/\mathrm{Mpc}) + 25$.
Q^μ	M^3	Interaction 4-vector (absent; minimal coupling).

K Reproducibility checklist (at a glance)

Inputs & priors

1. Select a kinetic specification and parameters: curvature–motivated (α, M) or phenomenological (K_0, p). Enforce $K > 0$ (Sec. 4.3); for the curvature case apply the early–time prior in Eq. (4.11).
2. Fix the baseline cosmology $\{H_0, \Omega_{m0}, \Omega_{r0}\}$ (and $\sigma_{8,0}$ if reporting $f\sigma_8$). Assume spatial flatness unless stated.
3. Early–time choice: adopt $K \to 1$ for $z \geq z_{\rm drag}$ (baseline). If exploring early running, recompute r_d from Eq. (7.6).

Initialization

4. Choose $N_{\rm ini}$ deep in matter domination; set $\phi'(N_{\rm ini}) = 0$. Initialize $H(N_{\rm ini})$ using the fluid scalings in Eq. (3.8). Determine $\phi(N_{\rm ini})$ by a one–dimensional shoot so that $\Omega_\phi(0) = 1 - \Omega_{m0} - \Omega_{r0}$ via Eq. (3.9).

Integration

5. Evolve the state $\mathbf{y} = (\phi,\, s,\, y_H) = (\phi,\, \phi',\, \ln H)$ using the autonomous system (3.12) with adaptive RK(4,5). Use tolerances in Table 8.1. Evaluate K'/K exactly: Eq. (4.10) (curvature) or Eq. (4.14) (phenomenological). Enforce at each step: $H > 0$, $K > 0$, and a non-vanishing denominator in Eq. (4.10).

Diagnostics (record maxima over the run)

6. Log $\max|\mathcal{C}_F|$ (Friedmann closure), $\max|\mathcal{C}_R|$ (Raychaudhuri), $\max|\mathcal{C}_\phi|$ (Klein–Gordon), $\max|\mathcal{C}_{R/H^2}|$ (Ricci identity), and $\max|\mathcal{C}_{\nabla\cdot T_\phi}|$ (source identity). Target levels are given in Table 8.1 (Sec. 8).

Derived observables

7. Distances: compute χ (Eq. (7.1)), then D_A, D_L (Eq. (7.2)); BAO summaries: $D_V, F_{\rm AP}$ (Eq. (7.4)). Growth: solve Eq. (7.7) for $D(a)$, then obtain f and $f\sigma_8$ (Eq. (7.8)). If BAO set the absolute scale, include r_d from Eq. (7.6).

Artifacts to report

8. Parameter values and priors; integrator and tolerances; maximum diagnostics; redshift grid. For bitwise reproducibility, include platform/precision, compiler/interpreter version, and any quasirandom settings (if used).

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