On the stability of discrete-time Markov jump linear systems

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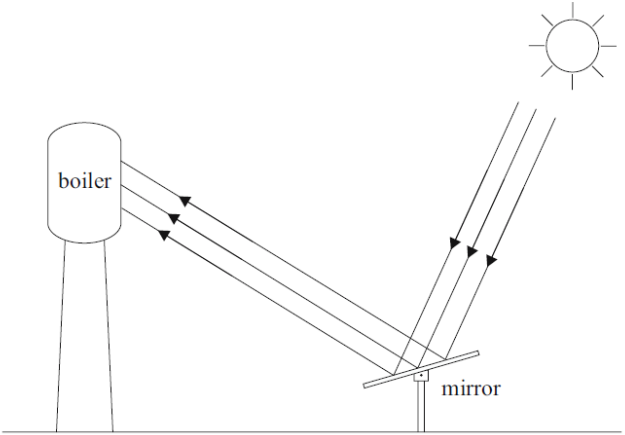
Contents

The solar energy plant



A solar energy plant (Costa et al. 2006).

The solar energy plant

Atmospheric agents:

- Sunny day
- Cloudy day

The solar energy plant

Atmospheric agents:

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Three types of system design:

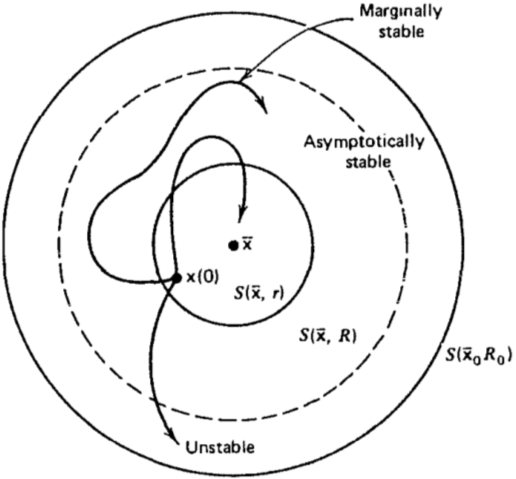
1. A single control law with pertubation.
2. Two control laws, one for each operation mode, independent.
3. Two control laws, one for each operation mode, dependent.

Discrete-time systems

Let the system of equations in differences

$$\begin{cases} x(k+1) = Ax(k) \\ x(0) = x_0 \end{cases} . \tag{1}$$

Discrete-time systems



Types of stability (Luenberger 1979).

Discrete-time systems

Theorem

They are equivalent:

1. $\overline{x} = 0$ *is the only global asintolly stable equilibrium point for System* (1)*.*
2. $\rho(A) < 1$*.*

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4. *For some* $V > 0$, *we have* $V - A^* VA > 0$.
5. *For some* $0 < \alpha < 1 \leq \beta$, *we have for all* x_0, $\|x(k)\|^2 \leq \beta\alpha^k \|x_0\|^2, \quad k = 0, 1, \ldots$
6. *For all* x_0, *we have* $\sum_{k=0}^{\infty} \|x(k)\|^2 < \infty$.

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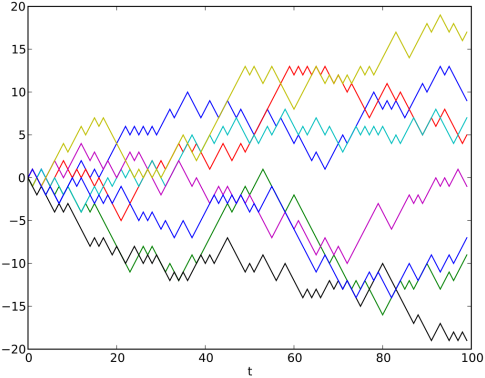
6. *For all* x_0, *we have* $\sum_{k=0}^{\infty} \|x(k)\|^2 < \infty$.
7. $\lim_{k\to\infty} \frac{1}{k} \ln \|A^k\| < 0$.

Markov chains



Random walk on $\mathbb{Z}$.

Markov chains

Transition probability $P = [P(i,j)]$, initial distribution ν and

$$\begin{cases} \theta(0) \ \sim \ \nu \\ \theta(k+1)|\theta(k) = i, \theta(k-1) = i_{k-1}, \ldots, \theta(0) = i_0 \ \sim \ P(i,\cdot) \end{cases}$$

define a Markov chain $\{\theta(k)\}_{k\geq 0}$.

The stochastic model for the solar energy plant

- $x(k) \in \mathbb{C}^d$: state variable
- $\{A_1, \ldots, A_S\}$: operation mode
- $\theta(k) \in \mathbb{S} = \{1, \ldots, S\}$: transition dynamics, Markov.
- x_0, θ_0: initial conditions, independent.

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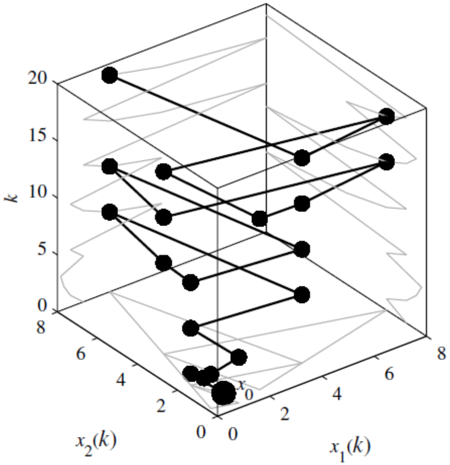
$$\begin{cases} x(k+1) = A_{\theta(k)} x(k) \\ x(0) = x_0, \ \theta(0) = \theta_0 \end{cases} \quad . \tag{2}$$

The stability



Temporal evolution of the system, $d = 3$ (Costa et al. 2006).

The stability

We say that System (2) is mean square stable (MSS) if for any initial condition x_0, θ_0 we have

$$\mathbb{E}[x(k)x(k)^*] \to 0$$

as $k \to \infty$.

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Theorem (Costa & Fragoso 1993)

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3. *For any* $U \in \mathbb{H}^{d+}$, $U > 0$, *there exists a unique* $V \in \mathbb{H}^{d+}$, $V > 0$, *such that* $V - \mathcal{T}(V) = U$.
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5. *For some* $0 < \alpha < 1 \leq \beta$, *we have for all* x_0 *and all* θ_0,
$$\mathbb{E}\left[\|x(k)\|^2\right] \leq \beta\alpha^k \mathbb{E}\left[\|x_0\|^2\right], \quad k = 0, 1, \ldots$$
6. *For all* x_0 *and all* θ_0, *we have* $\sum_{k=0}^{\infty} \mathbb{E}\left[\|x(k)\|^2\right] < \infty$.

The stability

Consider that $\{\theta(k)\}_{k\geq 0}$ is an Markov chain with $\mathbb{S} = \{1, 2\}$ and transition probability

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

The possible operations mode

$$A_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0.5 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.5 & 0 \\ 2 & 0 \end{pmatrix}.$$

This System (2) is not **MSS**.

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This System (2) is not **MSS**. Note that $\rho(A_1) = \rho(A_2) = 0.5 < 1$, i.e.,

Stability for each mode operation $\not\Rightarrow$ MSS.

The stability

We say that System (2) is stochastic stable (SS) if for any initial condition x_0, θ_0 we have

$$\sum_{k=0}^{\infty} \mathbb{E}\left[\|x(k)\|^2\right] < \infty.$$

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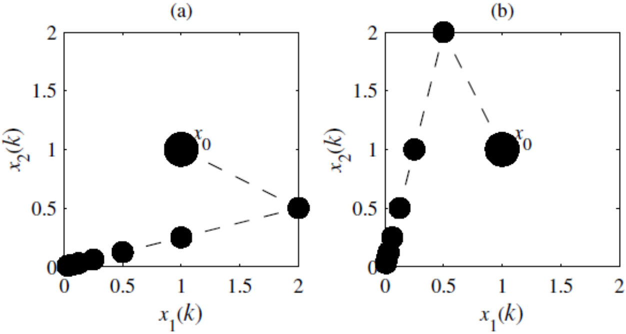
Theorem (Costa & Fragoso 1993)

If System (2) is MSS (SS) then we have for all x_0 and all θ_0, $x(k) \to 0$ with probability one (ASS).

The almost sure stability



Almost sure stability, $d = 2$ (Costa et al. 2006).

The almost sure stability

Ergodic properties: If P irreducible then there exists stationary distribution

$$\pi P = \pi.$$

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The top Lyapunov exponent of System (2) is

$$\lambda = \limsup_{k} \frac{1}{k} \ln \|A_{\theta(k-1)} \dots A_{\theta(0)}\|.$$

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$$\lambda = \limsup_{k} \frac{1}{k} \ln \|A_{\theta(k-1)} \dots A_{\theta(0)}\|.$$

Theorem (Fang et al. 1995)

If $\mathbb{E}[\lambda] < 0$ *then System* (2) *is ASS.*

The almost sure stability

Consider that $\{\theta(k)\}_{k\geq 0}$ is an irreducible Markov chain with $\mathbb{S} = \{1, 2\}$, stationary distribution $\pi = (0.5, 0.5)$ and transition probability

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

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This System (2) is ASS. Note that $\rho(A_1) = \rho(A_2) = 2 \geq 1$, i.e.,

ASS $\not\Rightarrow$ stability for each mode operation.

When the Markov chain has state space contable or Borel

- $x(k) \in \mathbb{C}^d$: state variable.
- $\{A_i : i \in \mathbb{S}\}$: operation mode.
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$$\begin{cases} x(k+1) = A_{\theta(k)} x(k) \\ x(0) = x_0,\ \theta(0) = \theta_0 \end{cases} \quad . \tag{3}$$

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¡Thanks!