Derivation of beam and 1-DOF moving vehicle coupled formulation

Derivation of the equations of motion (in terms of coupled system matrices) for 1-DOF vehicle moving with variable speed over beam.

The derivation below includes:

- Road profile
- Vehicle position
    - Uniformly accelerated vehicle motion
- Vehicle equation
    - Represented as a 1-DOF oscillator, a sprung mass with a viscous damper
- Beam
    - Simply supported without damping

Figure 1 shows the schematic overview of the model and the variables used in the derivation. This figure presents a example of a road profile with a ramp on the beam. However, the derivation below is for any generic road profile.

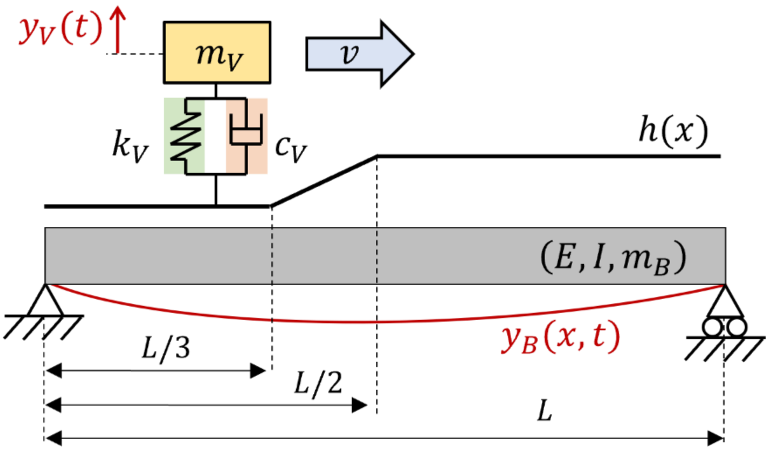

Figure 1: Overview of beam a vehicle model

Road profile

A generic profile is defined a $h(x)$.

Vehicle position

The vehicle moves along the ground with uniform acceleration. Its position x_V can be writen as:

$$x_V = x_{V0} + v_0\, t + \frac{1}{2} a\, t^2 \qquad \text{Eq. (1)}$$

where, x_{V0} is the initial position of the vehicle, v_0 the initial value of the vehicle speed, a the uniform rate of acceleration and t the time instant. The first time-derivative of the vehicle position is:

$$v(t) = v_0 + a\, t = v_t \qquad \text{Eq. (2)}$$

Which is the instantaneous velocity of the vehicle, which is termed v_t in the rest of the document.

Vehicle equations

The vertical position of the mass of the 1-DOF system is y_V measured positive upwards. The corresponding equation of motion of the vehicle is:

$$m_V \ddot{y}_V + c_V(\dot{y}_V - \dot{y}_B - \dot{h}) + k_V(y_V - y_B - h) = 0 \qquad \text{Eq. (3)}$$

where m_V, c_V and k_V are the mass, viscous damping, and spring stiffness of the vehicle.

Beam equations

The deflection of the beam (y_B) is measured positive upwards. The vertical vibration of the beam is described by following differential equation:

$$EI \frac{\partial^4 y_B}{\partial x^4} + m_B \frac{\partial^2 y_B}{\partial t^2} = f(x,t) \qquad \text{Eq. (4)}$$

where *EI* is the beam's bending stiffness (elastic modulus and second moment of area), and m_B the mass per unit length. Where $f(x,t)$ is the impressed force on the beam by the vehicle, thus for the single point wheel contact this is:

$$f(x,t) = [c_V(\dot{y}_V - \dot{y}_B - \dot{h}) + k_V(y_V - y_B - h)]\delta(x - x_V) + m_V g \qquad \text{Eq. (5)}$$

Where g is the value of gravity. The same sign convention is used as the rest, so a negative g value should be used.

Together, the beam equation can be written as:

$$\begin{aligned} &EI \frac{\partial^4 y_B}{\partial x^4} + m_B \frac{\partial^2 y_B}{\partial t^2} \\ &= [c_V(\dot{y}_V - \dot{y}_B - \dot{h}) + k_V(y_V - y_B - h)]\delta(x - x_V) + m_V g \end{aligned} \qquad \text{Eq. (6)}$$

This partial differential equation can be expressed as a set of ordinary differential equations by using normal coordinates, i.e. as a superposition of mode shapes. Therefore, the variable $y_B(x,t)$ can be expressed as:

$$y_B(x,t) = \sum_{i=1}^{\infty} \Phi_i(x)\, \eta_i(t) \approx \sum_{i=1}^{n} \Phi_i(x)\, \eta_i(t) \qquad \text{Eq. (7)}$$

where η_i are the normal coordinates and $\Phi_\mathrm{i}(x)$ are the associated deflection functions. The infinite series can be approximated by truncating it using the first *n* modes. For the rest of the derivation, it is assumed that the number of modes used is sufficient. In the rest of the document, it is assumed that this approximation is sufficiently accurate, thus replacing the "≈" simply by "=".

Its first derivative in time is then:

$$\dot{y}_B(x,t) = \sum_{i=1}^{n} \Phi_i(x)\, \dot{\eta}_i(t) \qquad \text{Eq. (8)}$$

However, to establish the coupling between beam and vehicle, later in the derivation it is necessary to know the first time-derivative at location x_V. This is a time dependent variable; thus it is necessary to apply the chain rule to find the first derivative, which gives:

$$\dot{y}_B(x_V,t) \approx \sum_{i=1}^{n} \Phi_i(x_V)\, \dot{\eta}_i(t) + \sum_{i=1}^{n} v_t\, \Phi_i'(x_V)\, \eta_i(t) \qquad \text{Eq. (9)}$$

where the dot notation indicates derivation in time and prime notation indicates derivation in space.

For the case of a simply supported beam the mode shapes are of the form:

$$\Phi_i(x) = \sqrt{2/L}\,\sin\left(\frac{i\,\pi\,x_V}{L}\right) \qquad \text{Eq. (10)}$$

These mode shapes have been normalized. The orthonormal mode shapes have the property:

$$\int_L \Phi_i(x)\Phi_j(x)dx = \delta_{ij} \qquad \text{Eq. (11)}$$

For δ_{ij} the delta-Kroenecker function. As a result, we have.

$$\int_L \Phi_i^2(x)dx = 1 \qquad \text{Eq. (12)}$$

Now substituting the expression of $y_B(x,t)$ in Eq. (8) into the differential equation Eq. (4) gives:

$$EI\sum_{i=1}^{n} \eta_i(t)\frac{\partial^4\Phi_i(x)}{\partial x^4} + m_B\sum_{i=1}^{n}\Phi_i(x)\frac{\partial^2\eta_i(t)}{\partial t^2} = f(x,t) \qquad \text{Eq. (13)}$$

Following the definition of $\Phi_i(x)$ we know that:

$$\frac{\partial^4\Phi_i(x)}{\partial x^4} = \sqrt{2/L}\left(\frac{i\,\pi}{L}\right)^4 \sin\left(\frac{i\,\pi\,x_V}{L}\right) = \beta_i^4\Phi_i(x) \qquad \text{Eq. (14)}$$

So that the differential equation is:

$$EI\sum_{i=1}^{n}\eta_i(t)\beta_i^4\Phi_i(x)+m_B\sum_{i=1}^{n}\Phi_i(x)\frac{\partial^2\eta_i(t)}{\partial t^2}= f(x,t) \qquad \text{Eq. (15)}$$

Taking the constants out of the sums and using the dot notation, gives:

$$m_B\sum_{i=1}^{n}\Phi_i(x)\ddot{\eta}_i(t)+\beta_i^4 EI\sum_{i=1}^{n}\Phi_i(x)\eta_i(t)=f(x,t) \qquad \text{Eq. (16)}$$

Dividing over m_B:

$$\sum_{i=1}^{n}\Phi_i(x)\ddot{\eta}_i(t)+\beta_i^4\frac{EI}{m_B}\sum_{i=1}^{n}\Phi_i(x)\eta_i(t)=\frac{f(x,t)}{m_B} \qquad \text{Eq. (17)}$$

It is known that the natural frequencies of the beam are:

$$\omega_{B,i}=\left(\frac{\mu_i}{L}\right)^2\sqrt{\frac{EI}{m_B}} \qquad \text{Eq. (18)}$$

where μ_i are the roots of the characteristic equation for the particular boundary conditions of the beam. In this case, the roots for the simply supported beam are $\mu_i = i\,\pi$. Therefore, we can write:

$$\omega_{B,i}=\left(\frac{k\pi}{L}\right)^2\sqrt{\frac{EI}{m_B}} \qquad \text{Eq. (19)}$$

Squaring the expression is:

$$\omega_{B,i}^2=\left(\frac{k\pi}{L}\right)^4\frac{EI}{m_B}=\beta_i^4\frac{EI}{m_B} \qquad \text{Eq. (20)}$$

Using this in the differential equation, gives:

$$\sum_{i=1}^{n}\Phi_i(x)\ddot{\eta}_i(t)+\omega_{B,i}^2\sum_{i=1}^{n}\Phi_i(x)\eta_i(t)=\frac{f(x,t)}{m_B} \qquad \text{Eq. (21)}$$

Multiplying the equation by a mode *j*, this is $\Phi_j(x)$, gives:

$$\Phi_j(x)\sum_{i=1}^{n}\Phi_i(x)\ddot{\eta}_i(t)+\omega_{B,i}^2\Phi_j(x)\sum_{i=1}^{n}\Phi_i(x)\eta_i(t)=\frac{f(x,t)}{m_B}\Phi_j(x) \qquad \text{Eq. (22)}$$

Integrating over *x*, gives:

$$\int_L \Phi_j(x)\sum_{i=1}^{n}\Phi_i(x)\ddot{\eta}_i(t)\,dx+\int_L \omega_{B,i}^2\Phi_j(x)\sum_{i=1}^{n}\Phi_i(x)\eta_i(t)\,dx = \int_L \frac{f(x,t)}{m_B}\Phi_j(x)dx \qquad \text{Eq. (23)}$$

Using the orthogonality condition of the normalized modes, gives:

$$\ddot{\eta}_j(t)\int_L \Phi_j^2(x)dx+\omega_{B,j}^2\eta_j(t)\int_L \Phi_j^2(x)dx=\int_L \frac{f(x,t)}{m_B}\Phi_j(x)dx \qquad \text{Eq. (24)}$$

Which simplifies to:

$$m_B\ddot{\eta}_j(t)+m_B\omega_{B,j}^2\eta_j(t)=\int_L f(x,t)\Phi_j(x)dx \qquad \text{Eq. (25)}$$

<u>Expressions of the coupled vehicle-beam system</u>

The integral on the right-hand side in Eq. (25) is:

$$I=\int_L f(x,t)\Phi_j(x)dx = = \int_L \Big[\big[c_V\big(\dot{y}_V-\dot{y}_B-\dot{h}\big)+k_V(y_V-y_B-h)\big]\delta(x-x_V)+m_V g\Big]\,\Phi_j(x)dx \qquad \text{Eq. (26)}$$

The delta function property, integration of translated delta function, that states that:

$$\int_{-\infty}^{+\infty} p(t)\delta(t-T)dt=p(T) \qquad \text{Eq. (27)}$$

Using this property on the expression of the integral gives:

$$I=\big[c_V\big(\dot{y}_V(t)-\dot{y}_B(x_V,t)-\dot{h}(t)\big)+k_V\big(y_V(t)-y_B(x_V,t)-h(t)\big) + m_V g\big]\Phi_j(x_V) \qquad \text{Eq. (28)}$$

Which, based on previous results, is:

$$I = \left[c_V \left(\dot{y}_V(t) - \dot{h}(x_V) - \sum_{i=1}^{n} \Phi_i(x_V)\, \dot{\eta}_i(t) - \sum_{i=1}^{n} v_t\, \Phi_i'(x_V)\, \eta_i(t) \right) + k_V \left(y_V(t) - h(x_V) - \sum_{i=1}^{n} \Phi_i(x_V)\, \eta_i(t) \right) + m_V g \right] \Phi_j(x_V) \quad \text{Eq. (29)}$$

To simplify the expression, the variables x_V and *t* will not be show, leading to:

$$I = \left[c_V \left(\dot{y}_V - \dot{h} - \sum_{i=1}^{n} \Phi_i\, \dot{\eta}_i - \sum_{i=1}^{n} v_t\, \Phi_i'\, \eta_i \right) + k_V \left(y_V - h - \sum_{i=1}^{n} \Phi_i\, \eta_i \right) + m_V g \right] \Phi_j \quad \text{Eq. (30)}$$

Expanding the sums into its elements gives:

$$I = c_V \Phi_j \big(\dot{y}_V - \dot{h} - \Phi_1 \dot{\eta}_1 - \Phi_2 \dot{\eta}_2 - \Phi_3 \dot{\eta}_3 + \cdots - v_t \Phi_1'\, \eta_1 - v_t \Phi_2'\, \eta_2 - v_t \Phi_3'\, \eta_3 + \cdots \big) + k_V \Phi_j (y_V - h - \Phi_1 \eta_1 - \Phi_2 \eta_2 - \Phi_3 \eta_3 + \cdots) + m_V g \Phi_j \quad \text{Eq. (31)}$$

So, the equation associated to mode *j* of the beam is:

$$m_B \ddot{\eta}_j + m_B \omega_{B,j}^2 \eta_j = c_V \Phi_j \big(\dot{y}_V - \dot{h} - \Phi_1\, \dot{\eta}_1 - \Phi_2\, \dot{\eta}_2 - \Phi_3\, \dot{\eta}_3 + \cdots - v_t \Phi_1'\, \eta_1 - v_t \Phi_2'\, \eta_2 - v_t \Phi_3'\, \eta_3 + \cdots \big) + k_V \Phi_j (y_V - h - \Phi_1\, \eta_1 - \Phi_2\, \eta_2 - \Phi_3\, \eta_3 + \cdots) + m_V g \Phi_j \quad \text{Eq. (32)}$$

This can be rearranged to:

$$m_B \ddot{\eta}_j + c_V \phi_j \phi_1 \dot{\eta}_1 + c_V \phi_j \phi_2 \dot{\eta}_2 + c_V \phi_j \phi_3 \dot{\eta}_3 + \cdots - c_V \Phi_j \dot{y}_V + m_B \omega_{B,j}^2 \eta_j + v_t c_V \phi_j \phi_1' \eta_1 + v_t c_V \phi_j \phi_2' \eta_2 + v_t c_V \phi_j \phi_3' \eta_3 + \cdots + k_V \Phi_j \Phi_1\, \eta_1 + k_V \Phi_j \Phi_2\, \eta_2 + k_V \Phi_j \Phi_3\, \eta_3 + \cdots - k_V \Phi_j y_V = -c_V \dot{h} \Phi_j - k_V h \Phi_j + m_V g \Phi_j \quad \text{Eq. (33)}$$

On the other hand, the equation of the sprung mass can be written as:

$$m_V \ddot{y}_V + c_V \left(\dot{y}_V - \dot{h} - \sum_{i=1}^{n} \Phi_i(x_V)\, \dot{\eta}_i(t) - \sum_{i=1}^{n} v_t\, \Phi_i'(x_V)\, \eta_i(t) \right) + k_V \left(y_V - h - \sum_{i=1}^{n} \Phi_i(x_V)\, \eta_i(t) \right) = 0 \quad \text{Eq. (34)}$$

This can be rearranged to:

$$m_V \ddot{y}_V + c_V \dot{y}_V + k_V y_V - c_V \sum_{i=1}^{n} \Phi_i(x_V)\, \dot{\eta}_i(t) - c_V \sum_{i=1}^{n} v_t\, \Phi_i'(x_V)\, \eta_i(t) - k_V \sum_{i=1}^{n} \Phi_i(x_V)\, \eta_i(t) = c_V \dot{h} + k_V h \qquad \text{Eq. (35)}$$

That can be expanded to:

$$\begin{aligned} m_V \ddot{y}_V &+ c_V \dot{y}_V + k_V y_V - c_V \Phi_1 \dot{\eta}_1 - c_V \Phi_2 \dot{\eta}_2 - c_V \Phi_3 \dot{\eta}_3 + \cdots - v_t c_V \Phi_1' \eta_1 \\ &- v_t c_V \Phi_2' \eta_2 - v_t c_V \Phi_3' \eta_3 + \cdots - k_V \Phi_1 \eta_1 - k_V \Phi_2 \eta_2 - k_V \Phi_3 \eta_3 \\ &+ \cdots = c_V \dot{h} + k_V h \end{aligned} \qquad \text{Eq. (36)}$$

And rearranged to:

$$\begin{aligned} m_V \ddot{y}_V &+ c_V \dot{y}_V - c_V \Phi_1 \dot{\eta}_1 - c_V \Phi_2 \dot{\eta}_2 - c_V \Phi_3 \dot{\eta}_3 + \cdots + k_V y_V \\ &+ (-v_t c_V \Phi_1' - k_V \Phi_1)\eta_1 + (-v_t c_V \Phi_2' - k_V \Phi_2)\eta_2 \\ &+ (-v_t c_V \Phi_3' - k_V \Phi_3)\eta_3 + \cdots = c_V \dot{h} + k_V h \end{aligned} \qquad \text{Eq. (37)}$$

Matrix formulation

The equation for the vehicle (Eq. (37)) and the n equations for the beam (Eq. (33)), can be expressed in matrix form. For this, all the unknows are collected into the vector:

$$q = (y_V \quad \eta_1 \quad \eta_2 \quad \ldots \quad \eta_n)^T \qquad \text{Eq. (38)}$$

We can rearrange previous $N = n + 1$ equations and present them in matrix form, as:

$$M\ddot{q} + C\dot{q} + Kq = F \qquad \text{Eq. (39)}$$

where:

$$M = \begin{pmatrix} m_V & 0 & 0 & \ldots & 0 \\ 0 & m_B & 0 & \ldots & 0 \\ 0 & 0 & m_B & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & m_B \end{pmatrix} \qquad \text{Eq. (40)}$$

$$C = c_V \begin{pmatrix} 1 & -\Phi_1 & -\Phi_2 & \ldots & -\Phi_n \\ -\Phi_1 & \Phi_1^2 & \Phi_1\Phi_2 & \ldots & \Phi_1\Phi_n \\ -\Phi_2 & \Phi_2\Phi_1 & \Phi_2^2 & \ldots & \Phi_2\Phi_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Phi_n & \Phi_n\Phi_1 & \Phi_n\Phi_2 & \ldots & \Phi_n^2 \end{pmatrix} \qquad \text{Eq. (41)}$$

$$K = K_k + K_c \quad \text{Eq. (42)}$$

$$K_k = k_V \begin{pmatrix} 1 & -\Phi_1 & -\Phi_2 & \dots & -\Phi_n \\ -\Phi_1 & \frac{m_B \omega_{B,1}^2}{k_V} + \Phi_1^2 & \Phi_1 \Phi_2 & \dots & \Phi_1 \Phi_n \\ -\Phi_2 & \Phi_2 \Phi_1 & \frac{m_B \omega_{B,2}^2}{k_V} + \Phi_2^2 & \dots & \Phi_2 \Phi_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Phi_n & \Phi_n \Phi_1 & \Phi_n \Phi_2 & \dots & \frac{m_B \omega_{B,n}^2}{k_V} + \Phi_n^2 \end{pmatrix} \quad \text{Eq. (43)}$$

$$K_c = v_t c_V \begin{pmatrix} 0 & -\Phi_1' & -\Phi_2' & \dots & -\Phi_n' \\ 0 & \phi_1 \phi_1' & \phi_1 \phi_2' & \dots & \phi_1 \phi_n' \\ 0 & \phi_2 \phi_1' & \phi_2 \phi_2' & \dots & \phi_2 \phi_n' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \phi_n \phi_1' & \phi_n \phi_2' & \dots & \phi_n \phi_n' \end{pmatrix} \quad \text{Eq. (44)}$$

$$F = \begin{pmatrix} k_V h + c_V \dot{h} \\ \left(m_V g - k_V h - c_V \dot{h}\right) \Phi_1 \\ \left(m_V g - k_V h - c_V \dot{h}\right) \Phi_2 \\ \vdots \\ \left(m_V g - k_V h - c_V \dot{h}\right) \Phi_n \end{pmatrix} \quad \text{Eq. (45)}$$

For $\Phi_i = \Phi_i(x_V, t)$ being the i-th mode of vibration value at the location of the vehicle x_V and Φ_i' its first spatial derivative, g the gravity acceleration and $\dot{h}$ the first time-derivative of the road profile.