FINM3405 Derivatives and Risk Management

Week 12: Value at risk (VaR) and expected shortfall (ES)

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Contents

Introduction

In this course we've discussed some basic notions relating to risk management or hedging using derivatives. This week we present some methods of measuring or quantifying the total or aggregate market risk exposure that a portfolio or indeed entire bank may face, namely value at risk (VaR) and expected shortfall (ES). These metrics are central to bank capital management and regulation, since they effectively quantify how much money a bank could potentially lose from its aggregate financial market trading activities and positions, and thus how much capital the bank should hold to cover these potential losses and remain solvent.

- Readings: Chapter 22 of Hull.

Introduction

- Also consider Chapter 44 Value at Risk, and Chapter 46 VaR: Alternative Measures, of Cuthbertson, Nitzsche and O'Sullivan, Derivatives: Theory and Practice.
- To read about the Basel regulatory framework overseen by the Bank for International Settlements I recommend reading:
  - History of the Basel Committee.
  - The Basel Framework.
  - Explanatory note on minimum capital requirements for market risk.

Many banks have portfolios of traded instruments for short-term profits. These portfolios – referred to as trading books – are exposed to market risk, or the risk of losses resulting from changes in the prices of instruments such as bonds, shares and currencies. Banks are required to maintain a minimum amount of capital to account for this risk.

Measures of market risk

Value-at-risk (VaR)	Expected shortfall (ES)
A measure of the worst expected loss on a portfolio of instruments resulting from market movements over a given time horizon and a pre-defined confidence level.	A measure of the average of all potential losses exceeding the VaR at a given confidence level, which makes up for VaR's shortcomings in capturing the risk of extreme losses (ie tail risk).

Risk concepts

We first discuss some basic concepts relating to risk, which we use later:

- What is risk?
- Individual security risk.
- The normal distribution.
- Portfolio risk.

What is risk?

FINM3405 is titled "*Derivative securities and risk management*" and we've discussed some hedging concepts in an "ad hoc" fashion.

- But what do we really mean by "risk" and "risk management"?

Define **risk** as the dependence of a portfolio's or a company's value, solvency, or profits, etc, on external factors that are unpredictable and out of the control of the portfolio or business manager.

Derivative securities are tools in larger toolkits used in more general risk management frameworks within businesses and financial institutions, which involves the identification, measurement and control of risks.

What is risk?

We could classify risk into the following 4 broad categories:

1. **Market risks**: These are the risks we've mostly been discussing this semester, namely risks due to movements in market variables such as interest rates, exchange rates, stock prices, commodity prices, etc.
2. **Liquidity risk**: The inability to sell or liquidate assets and financial securities quickly and at prices close to fair market values.
3. **Credit risk**: The risk of loss due to borrowers and counterparties failing to meet, and thus defaulting on, their payment obligations.
4. **Operational risks**: "All others" including human error, fraud and theft, model risk, technology failure, legal risk, weather events, etc.

Of course these categories are not "water tight".

What is risk?

- This semester we've discussed using derivative securities to *control* or *manage* (hedge against) various sources of risk we may face.
- We also discussed ways of *measuring* or *quantifying* market risks:
  - Standard deviation σ (volatility) and beta β (systematic risk).
  - Delta Δ, gamma Γ, vega ν, rho ρ, theta θ of options positions.
- We also discussed survival probabilities, loss given default and recovery rates, probability of default, etc, which are ways of measuring and quantifying to credit risk.
- You may also recall the notions of duration and convexity for interest rate securities and portfolios, particularly bonds.

What is risk?

There is a very large range of individual risk measures, metrics and techniques for quantifying different kinds of market, liquidity and credit risk, including and not limited to those mentioned above.

- Furthermore, financial institutions will hold a very large number and variety of positions across all asset classes, from stocks, interest rates, currencies, loans, derivatives, property, etc - you name it.
- And financial institutions will individually measure and quantify in the best way they can all the risks they face across all their positions using all available metrics and techniques specific to their positions.

What is risk?

Consequently, you could imagine that the situation would get quite complicated and complex, with all these different traders and departments in a big bank all measuring their unique exposures and risks in their own unique ways relating to the particular positions they hold.

- However, managers, the board of directors, regulators, etc, require simple to understand and simple to calculate measures of "aggregate" or "institution-wide" or "total" risk.
- In other words, they want a financial institution's risks "all packaged up" into single, simple to understand metrics or measures.

Value at risk (VaR) and expected shortfall (ES) seek to do this for a bank's <u>market risks</u>: package them all up into a single total risk measure.

Individual security risk

We first refresh some introductory calculations about individual security and portfolio risk or volatility (standard deviation) which we use later.

- Let $\{R_1, \ldots, R_N\}$ be N time period returns for a financial asset.

Recall that the **mean return** is given by

$$\mu = \frac{1}{N} \sum_{i=1}^{N} R_i.$$

Also recall that the **volatility** (standard deviation) in returns is

$$\sigma = \sqrt{\sum_{i=1}^{N} \frac{(R_i - \mu)^2}{N-1}}.$$

Individual security risk

How do we calculate returns?

Remark

We could use discrete or continuous compounding, respectively:

$$R_i = \frac{P_i}{P_{i-1}} - 1 \qquad \text{or} \qquad R_i = \log \frac{P_i}{P_{i-1}},$$

where $\{P_0, P_1, \ldots, P_N\}$ are the $N + 1$ security price observations.

Later, we'll also use some notions relating to the normal distribution:

The normal distribution

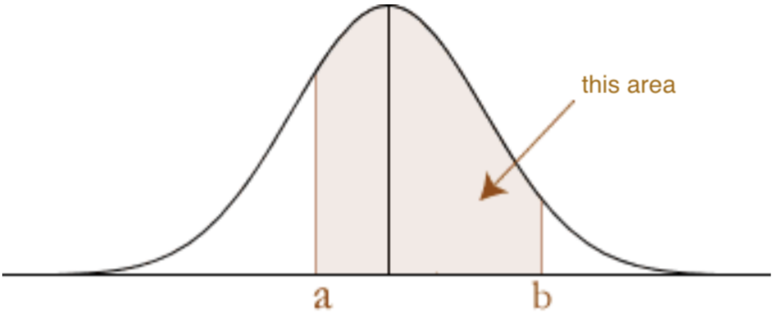
If we're willing to believe that returns are normally distributed (and we will make this dubious assumption later) then the mean μ and variance σ^2 completely characterise the probability distribution of the returns:

$$\mathbb{P}(a \leq R \leq b) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x,$$

where R is the random variable representing the security's returns.

- In words, $\mathbb{P}(a \leq R \leq b)$ is the probability of the asset's returns R falling between a and b, that is, in the interval $[a, b]$.
- It is the area under the normal distribution PDF over $[a, b]$, namely:

The normal distribution



Later we will use the normal distribution to model the distribution of changes $\mathrm{d}V$ in the value V of a portfolio.

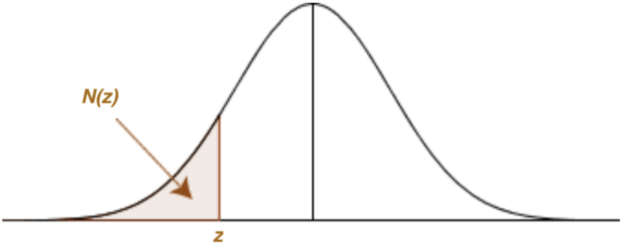
- And here, we're mostly interested in the "left tail" probabilities:

The normal distribution

The CDF of the standard normal distribution ($\mu = 0$ and $\sigma^2 = 1$) is

$$\mathcal{N}(z) = \mathbb{P}(R \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx.$$

For negative z-values, it gives the following “left tail” area or probability:



We will also denote these “left tail” areas or probabilities $\alpha = \mathcal{N}(z)$.

The normal distribution

Later, when doing VaR and ES calculations, we're interested in the following "left tail" areas $\alpha = \mathcal{N}(z)$ and z-values z that give them:

- $\alpha = \mathcal{N}(-1.645) = 0.05.$
- $\alpha = \mathcal{N}(-1.96) = 0.025.$
- $\alpha = \mathcal{N}(-2.326) = 0.01.$

We can find these z-values using `norm.ppf()` in Python:

```
from scipy.stats import norm
In [1]: norm.ppf(0.05)
Out[1]: -1.6448536269514729
In [1]: norm.ppf(0.025)
Out[1]: -1.959963984540054
In [1]: norm.ppf(0.01)
Out[1]: -2.3263478740408408
```

The normal distribution

Above we're using the standard normal distribution, which has a mean of $\mu = 0$ and variance of $\sigma^2 = 1$. But what about for a general normal distribution with non-zero mean μ and variance σ^2 different to 1?

- To find the z-value z_α corresponding to a given left tail probability or area α for a normal distribution with mean μ and variance σ^2 we set

$$z_\alpha = \mu + z\sigma,$$

where z is the z-value for a standard normal distribution corresponding to the left tail probability $\alpha = \mathcal{N}(z)$.

An example should clarify:

The normal distribution

Example

We hold a $V = \$1\text{m}$ portfolio whose daily return has mean $\mu = 0$ and standard deviation $\sigma = 1.25\%$. The standard deviation of the daily change $\mathrm{d}V$ in portfolio value V is $\sigma_{\mathrm{d}V} = \sigma V = \$12,500$.

- **Question**: What is the value v_α that corresponds to a tail probability of $\alpha = 1\%$ for the daily changes $\mathrm{d}V$ in the value V of our portfolio, given $\mu_{\mathrm{d}V} = \$0$ and $\sigma_{\mathrm{d}V} = \$12,500$?

Answer: We calculate that

$$v_{0.01} = \mu_{\mathrm{d}V} + z\sigma_{\mathrm{d}V} = 0 - 2.326 \times 12500 = -\$29,079.35,$$

where z is the z-value corresponding to the probability $\alpha = \mathcal{N}(z)$.

The normal distribution

Example (Continued)

We can check this with a computer, here Python:

```
from scipy.stats import norm
In [1]: norm.ppf(0.01, loc=0, scale=12500)
Out[1]: -29079.34842551051
```

Remark

We note upfront that when doing value at risk (VaR) and expected shortfall (ES) calculations, we're interested in the time period (say daily) <u>changes $\mathrm{d}V$ in the portfolio value V</u>, which we think of as dollar value profits/losses over the given time period.

The normal distribution

Remark

To answer a student's question, and elaborate on something I used above and repeatedly below, I show that $\sigma_{\mathrm{d}V} = V\sigma$.

- Let R be the portfolio daily returns random variable, with mean $\mu = \mathbb{E}[R]$ and standard deviation $\sigma = \sqrt{\mathbb{E}\left[(R-\mu)^2\right]}$.
- Let $\mathrm{d}V = VR$ be the random variable representing the daily portfolio value change (or profits/losses). We calculate:

$$\mu_{\mathrm{d}V} = \mathbb{E}[\mathrm{d}V] = \mathbb{E}[VR] = V\mathbb{E}[R],$$

$$\sigma_{\mathrm{d}V} = \sqrt{\mathbb{E}\left[(\mathrm{d}V - \mu_{\mathrm{d}V})^2\right]} = \sqrt{\mathbb{E}\left[(VR - V\mu)^2\right]} = V\sqrt{\mathbb{E}\left[(R-\mu)^2\right]}.$$

Portfolio risk

- We now refresh calculating portfolio return means and volatilities:

Suppose we hold a portfolio of 2 assets with mean returns μ_1 and μ_2, and standard deviations (volatilities) in returns σ_1 and σ_2. If we have w_1 and w_2 weights (percent) invested in asset 1 and 2, then we recall that:

- Our **portfolio mean return** is

$$\mu = w_1\mu_1 + w_2\mu_2$$

- Our **portfolio standard deviation in returns** is

$$\sigma = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}},$$

with $\sigma_{12} = \sigma_1\sigma_2\rho$ and ρ the covariance and correlation in returns.

Portfolio risk

Remark

Writing $\sigma_{11} = \sigma_1^2$ and $\sigma_{22} = \sigma_2^2$ for the variances, $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ for the variance-covariance matrix, and $\boldsymbol{w} = \binom{w_1}{w_2}$ for the weights vector, then the portfolio standard deviation is given by

$$\sigma = \sqrt{\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}}.$$

This enables us to generalise to portfolios of more than 2 assets.

VaR and ES

We now have all the basic concepts we need to discuss VaR and ES.

VaR and ES

Using tail probability $\alpha = 1 - p$, **value at risk** (**VaR**) answers:

What is the maximum dollar loss VaR_α that would be incurred over a given time period with probability p?

Ie: "*We will lose at most* $\$VaR_\alpha$ *over the time period in p% of cases.*"

- From the "opposite" perspective:

What is the minimum dollar loss VaR_α that would be incurred over a given time period with probability α?

Ie: "*We will lose at least* $\$VaR_\alpha$ *over the time period in* α*% of cases.*"

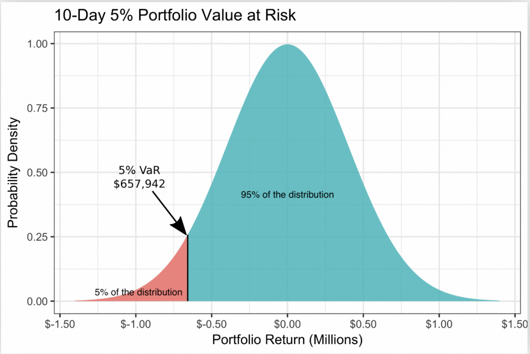
VaR and ES

For some terminology and notation:

- VaR_α is the **value at risk** (what we risk losing).
  - It's the most we'd lose with probability $p = 1 - \alpha$
  - Or the least we'd lose with probability $\alpha = 1 - p$.
- p is the **level of confidence**.
  - It's typically set to $p = 95\%$, or $p = 97.5\%$, or $p = 99\%$.
- $\alpha = 1 - p$ is called the **tail probability**.
  - It's typically small at $\alpha = 5\%$, $\alpha = 2.5\%$, $\alpha = 1\%$, etc.
- Also, the time period is usually "small" at 1 or 10 days.

VaR_α is possibly best conceptualised graphically:

VaR and ES



This is the distribution of a portfolio's value changes $\mathrm{d}V$ over a 10-day period. The $\alpha = 5\%$ VaR is $\mathrm{VaR}_{0.05} = \$657,942$, which in words is:

- $\mathrm{VaR}_{0.05}$ is the worst outcome over 10 days with $p = 95\%$ confidence.
- Or, $\alpha = 5\%$ of the time we lose at least $\mathrm{VaR}_{0.05}$ dollars over 10 days.

VaR and ES

Remark

In other words, using the previous notation, VaR_α is the negative of the value v_α corresponding to a left tail probability of $\alpha\%$.

So we can "rephrase" the previous example:

Example

The $\alpha = 1\%$ VaR of our $V = \$1\text{m}$ portfolio with zero mean and $\sigma_{\mathrm{d}V} = \$12,500$ volatility in the daily changes $\mathrm{d}V$ in the portfolio value is $\text{VaR}_{0.01} = \$29,079.35$.

- We lose at least \$29,079.35 with probability $\alpha = 1\%$ in 1 day.

VaR and ES

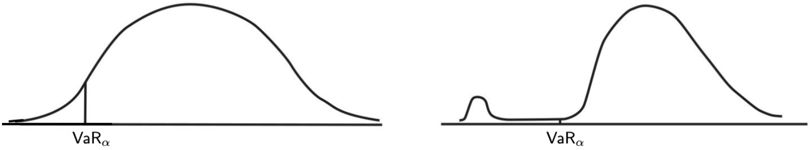
Remark

VaR_α tells us the least amount we expect to lose with tail probability α% in a given time period.

So VaR has the shortcoming that it does not tell us what our expected tail risk or loss or shortfall (ES) is, that is, how much we expect to lose if our portfolio outcomes fall in the α% left tail area of the distribution.

- Two distributions may have the same VaR_α but different expected losses or shortfalls ES_α for a given left tail probability α, as in:

VaR and ES



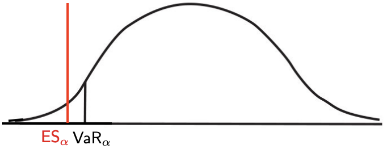
Both distributions have the same VaR$_\alpha$ but you expect larger losses (much larger than VaR$_\alpha$) in the RHS distribution than the LHS, if the outcomes fell in the α% left tail area of the distribution (left of VaR$_\alpha$).

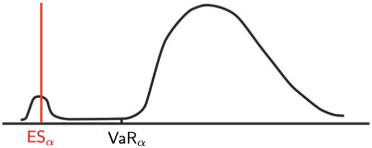
- The **expected shortfall** (**ES**) answers the question:

> How much ES$_\alpha$ do we expect to lose if our outcomes fall in the α% left tail area of the distribution, so to the left of VaR$_\alpha$?

VaR and ES

Probabilistically, ES_α is the expected value or mean in the case that our outcomes are worse than (left of) VaR_α for a given tail probability α:





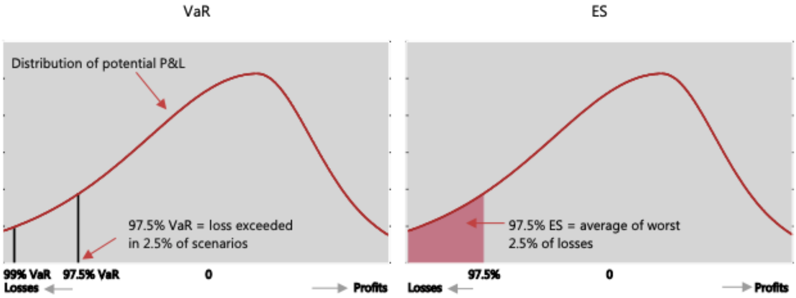
Remark

Regulators use VaR_α and ES_α to calculate the amount of capital a bank needs to hold to remain solvent: $\geq 3 \times \mathrm{ES}_\alpha$ I believe.

VaR and ES

From the Bank for International Settlements' Explanatory note on the minimum capital requirements for market risk:

Graph 1: Expected shortfall compared to value-at-risk



Question: Given tail probability α, how do we calculate VaR_α and ES_α?

VaR and ES estimation methods

We cover two contrasting approaches to VaR_α and ES_α calculation:

1. **Parametric**: Assumes the distribution of changes $\mathrm{d}V$ in our portfolio value V can be described or modelled by a known "parametric family" of probability distributions.
   - We assume the normal distribution.
   - Uses historical data to estimate the parameters μ and σ^2 of the best fitting normal distribution to calculate VaR_α and ES_α.
2. **Nonparametric**: We don't assume a "parametric family", but instead use historical data directly to construct histograms and "rank" or "order" or manually "sort" the changes $\mathrm{d}V$ in our portfolio value V in order to calculate VaR_α and ES_α for a given α.

We start with the parametric method:

Parametric

When asset returns are normally distributed with mean μ and variance σ^2, for a given left tail probability α we already know that the **value at risk** VaR_α of the change $\mathrm{d}V$ in the portfolio value V is given by

$$\boxed{\mathrm{VaR}_\alpha = -(\mu_{\mathrm{d}V} + z\sigma_{\mathrm{d}V}),}$$

with z the z-value of the standard normal distribution corresponding to the left tail probability $\alpha = \mathcal{N}(z)$. The **expected shortfall** ES_α is

$$\boxed{\mathrm{ES}_\alpha = -\mu_{\mathrm{d}V} + \frac{\sigma_{\mathrm{d}V}}{\alpha} f(z),}$$

where $f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$ is the standard normal PDF.

Parametric

Example

Suppose you hold a $V = \$5$m portfolio in CBA shares, whose daily returns have mean $\mu = 0$ and standard deviation $\sigma = 1.45\%$.

- The standard deviation in the daily value changes $\mathrm{d}V$ is

$$\sigma_{\mathrm{d}V} = \sigma V = 0.0145 \times 5000000 = \$72,500.$$

Question: What are the daily $\alpha = 1\%$ VaR and ES?

- We calculate the daily $\alpha = 1\%$ VaR to be

$$\mathrm{VaR}_{0.01} = 0 - z\sigma_{\mathrm{d}V} = 2.326 \times 72500 = \$168,660.22.$$

Parametric

Example (Continued)

So we lose at least $\text{VaR}_{0.01} = \$168,660.22$ $\alpha = 1\%$ of the time.

- We calculate the daily $\alpha = 1\%$ ES to be

$$\text{ES}_{0.01} = \frac{\sigma_{\text{d}V}}{\alpha} f(z) = \frac{72500}{0.01} \times 0.0267 = \$193,228.03.$$

So if our daily portfolio value change $\text{d}V$ falls in the left $\alpha = 1\%$ of the tail of the distribution, we expect to lose \$193,228.03.

Parametric

Remark

We can use Python to check our above $ES_{0.01}$ calculation:

```
from scipy.stats import norm
z = norm.ppf(0.01)
In [1]: (72500/0.01)*norm.pdf(z)
Out[1]: 193228.03097507107
```

Parametric portfolio VaR

We now want to write the portfolio VaR_α in terms of the values at risk $\text{VaR}_{\alpha,1}$ and $\text{VaR}_{\alpha,2}$ of asset 1 and 2 in the portfolio:

- Let V_1 and V_2 be the value of assets 1 and 2 in the portfolio.
- Then the portfolio value is given by $V = V_1 + V_2$.
- And the portfolio weights are given by

$$w_1 = \frac{V_1}{V} \qquad \text{and} \qquad w_2 = \frac{V_2}{V}.$$

- Assuming zero mean returns, the α% VaRs of each asset are

$$\text{VaR}_{\alpha,1} = -z\sigma_{\text{d}V_1} = -z\sigma_1 V_1,$$

$$\text{VaR}_{\alpha,2} = -z\sigma_{\text{d}V_2} = -z\sigma_2 V_2.$$

Parametric portfolio VaR

Then we rearrange the portfolio VaR_α to see that

$$\begin{aligned}
\mathrm{VaR}_\alpha &= -z\sigma_{\mathrm{d}V} = -zV\sigma \\
&= -zV\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2} \\
&= \sqrt{(-zV_1\sigma_1)^2 + (-zV_2\sigma_2)^2 + 2\rho(-zV_1\sigma_1)(-zV_2\sigma_2)} \\
&= \sqrt{\mathrm{VaR}_{\alpha,1}^2 + \mathrm{VaR}_{\alpha,2}^2 + 2\rho\mathrm{VaR}_{\alpha,1}\mathrm{VaR}_{\alpha,2}}.
\end{aligned}$$

The result we want to remember is the **portfolio VaR**$_\alpha$ is given by

$$\mathrm{VaR}_\alpha = \sqrt{\mathrm{VaR}_{\alpha,1}^2 + \mathrm{VaR}_{\alpha,2}^2 + 2\rho\mathrm{VaR}_{\alpha,1}\mathrm{VaR}_{\alpha,2}}.$$

Parametric portfolio VaR

Question: Do we get portfolio VaR_α diversification benefits?

Answer: Yes provided that the assets are not perfectly correlated.

- First observe that when $\rho = 1$ (perfect correlation), we get

$$\begin{aligned}\mathrm{VaR}_\alpha &= \sqrt{\mathrm{VaR}_{\alpha,1}^2 + \mathrm{VaR}_{\alpha,2}^2 + 2\mathrm{VaR}_{\alpha,1}\mathrm{VaR}_{\alpha,2}} \\ &= \mathrm{VaR}_{\alpha,1} + \mathrm{VaR}_{\alpha,2},\end{aligned}$$

  which follows from the simple identity $(a+b)^2 = a^2 + b^2 + 2ab$.

The **worst case portfolio VaR_α** is thus defined as

$$\text{worst case } \mathrm{VaR}_\alpha = \mathrm{VaR}_{\alpha,1} + \mathrm{VaR}_{\alpha,2}.$$

It represents no VaR_α diversification benefits due to perfect correlation.

Parametric portfolio VaR

So the worst case portfolio VaR_α is when the assets are perfectly correlated ($\rho = 1$), and is just the sum of the individual VaRs.

- From the portfolio VaR_α formula, when $\rho < 1$, the portfolio VaR_α is less than the worst case VaR_α.

We define the **benefits from diversification** to be

$$\begin{aligned} \text{diversification benefits} &= \text{worst case VaR}_\alpha - \text{VaR}_\alpha \\ &= \text{VaR}_{\alpha,1} + \text{VaR}_{\alpha,2} - \text{VaR}_\alpha. \end{aligned}$$

Let's consider an example.

Parametric portfolio VaR

Example

Consider a portfolio of $V_1 = \$5\text{m}$ in Microsoft and $V_2 = \$7.5\text{m}$ in Tesla (go Elon!). Suppose their daily expected returns are zero, their volatilities are $\sigma_1 = 0.01$ and $\sigma_2 = 0.015$, and their correlation is $\rho = 0.6$. The individual daily $\alpha = 1\%$ VaRs are:

- Microsoft: $\text{VaR}_{0.01,1} = -z\sigma_{\text{d}V_1} = -z\sigma_1 V_1 = \$116,317.39$.
- Tesla: $\text{VaR}_{0.01,2} = -z\sigma_{\text{d}V_2} = -z\sigma_2 V_2 = \$261,714.14$.

Hence, the worst case daily portfolio $\alpha = 1\%$ VaR is

$$\text{worst case VaR}_\alpha = \text{VaR}_{0.01,1} + \text{VaR}_{0.01,2} = \$378,031.53.$$

Parametric portfolio VaR

Example (Continued)

The true daily portfolio $\alpha = 1\%$ VaR is

$$\begin{aligned}\mathrm{VaR}_{0.01} &= \sqrt{\mathrm{VaR}^2_{0.01,1} + \mathrm{VaR}^2_{0.01,2} + 2\rho \mathrm{VaR}_{0.01,1}\mathrm{VaR}_{0.01,2}} \\ &= \$344,317.17.\end{aligned}$$

Hence, here the benefits from diversification are

$$\begin{aligned}\text{diversification benefits} &= \text{worst case } \mathrm{VaR}_{0.01} - \mathrm{VaR}_{0.01} \\ &= 378031.53 - 344317.17 \\ &= \$33,714.36.\end{aligned}$$

Nonparametric

But wait, there's just one problem with the above parametric method:

- Financial asset returns are not normally distributed!

Consequently, we could consider fitting a different family of distributions to asset returns, such as a Student's *t*-distribution, which even has a closed-form solution for the expected shortfall:

Expected shortfall	$\mu + s\left(\dfrac{\nu + T^{-1}(1-p)^2 \;\times\; \tau\left(T^{-1}(1-p)^2\right)}{(\nu-1)(1-p)}\right)$ Where $T^{-1}(\,)$ is the inverse standardized Student *t* CDF, and $\tau(\,)$ is the standardized Student t PDF.[2]

Here μ is the mean, s is the standard deviation, ν is the degrees of freedom, τ and T are defined in the image, and p is the level of confidence as above, so $1-p$ is the tail probability. But we don't do this.

Nonparametric

We present the nonparametric (historical simulation) method for estimating VaR_α and ES_α, as follows:

- Collect a sample $\{P_0, P_1, \ldots, P_N\}$ of $N+1$ historical asset prices.
- Calculate the historical returns $\{R_1, \ldots, R_N\}$ given by

$$R_i = \frac{P_i}{P_{i-1}} - 1 \qquad \text{or} \qquad R_i = \log \frac{P_i}{P_{i-1}} \qquad \text{for } i = 1, \ldots, N.$$

Now suppose we hold a portfolio of Q units in the (single) asset whose current price is P. The current portfolio value is therefore

$$V = QP.$$

Nonparametric

Using the returns sample $\{R_1, \ldots, R_N\}$, we get a sample $\{\mathrm{d}V_1, \ldots, \mathrm{d}V_N\}$ of portfolio value changes starting at the current value $V = QP$ given by

$$\mathrm{d}V_i = VR_i \qquad \text{for } i = 1, \ldots, N.$$

We can then plot a histogram of the portfolio value changes $\{\mathrm{d}V_1, \ldots, \mathrm{d}V_N\}$ to visually inspect the distribution.

- And we calculate the VaR_α and ES_α as follows:

Nonparametric

To calculate the VaR_α and ES_α we order or rank the portfolio value changes $\mathrm{d}V_i$ from smallest to largest (largest loss to largest profit):

- By definition, the $\alpha\%$ VaR is the portfolio value change $\mathrm{d}V_i$ that leaves $\alpha\%$ of the portfolio value changes "to the left" of it.
- Also, by definition the $\alpha\%$ ES is the average (or mean) of these $\alpha\%$ portfolio value changes that are "to the left" of VaR_α.

Remark

Due to the sample size N, there may not be a portfolio value change $\mathrm{d}V_i$ that leaves *exactly* $\alpha\%$ of the portfolio value changes to the left of it, so VaR_α and ES_α will be slight, but usually quite accurate, approximations, which improve as N increases.

Nonparametric

Example

You hold $V = \$10$m in a S&P 500 ETF. We use the `yfinance` Python module to download the historical daily index values and calculate the daily $\alpha = 1\%$ VaR and ES. We get:

- $N + 1 = 24312$ daily index values and $N = 24,311$ returns.
- Mean return and volatility: $\mu = 0.0238\%$ and $\sigma = 1.196\%$.
- Mean and standard deviation of portfolio value change: $\mu_{\mathrm{d}V} = \$2,384.48$ and $\sigma_{\mathrm{d}V} = \$119,604.22$.
  - We can now calculate the <u>parametric</u> $\mathrm{VaR}_{0.01}$ and $\mathrm{ES}_{0.01}$:

Nonparametric

Example (Continued)

Recall that the parametric $\text{VaR}_{0.01}$ is given by

$$\begin{aligned}\text{VaR}_{0.01} &= -(\mu_{\text{d}V} + z\sigma_{\text{d}V}) \\ &= -(2384.48 - 2.3263 \times 119604.22) = \$275,856.54.\end{aligned}$$

The parametric $\text{ES}_{0.01}$ is calculated to be

$$\begin{aligned}\text{ES}_{0.01} &= -\mu_{\text{d}V} + \frac{\sigma_{\text{d}V}}{\alpha} f(z) \\ &= -2384.48 + \frac{119,604.22}{0.01} \times 0.0267 = \$316,386.38.\end{aligned}$$

Let's see how these compare to the nonparametric estimates.

Nonparametric

Example (Continued)

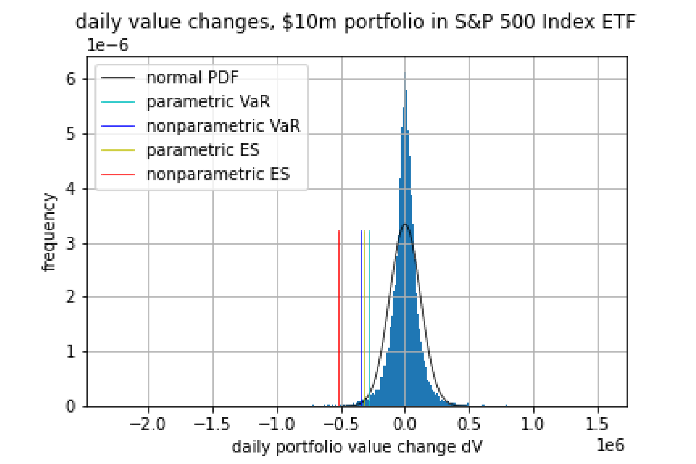
The below code gives the nonparametric estimates as

$$\text{VaR}_{0.01} = \$349,550.12 \qquad \text{and} \qquad \text{ES}_{0.01} = \$523,830.03.$$

The histogram with both the parametric and nonparametric estimates is on the next slide. From it, we see that the parametric estimates underestimate the true $\text{VaR}_{0.01}$ and $\text{ES}_{0.01}$, due to the index returns not being normal, particularly having "fat tails".

Nonparametric

Example (Continued)



Nonparametric

Example (Continued)

```
from scipy.stats import norm
import numpy as np, matplotlib.pyplot as plt, yfinance as yf
S = yf.download("^GSPC")["Adj Close"]
ret = np.log(S).diff(1).dropna()
mu, sigma = norm.fit(ret) # mean and std daily returns
alpha = 0.01 # tail probability
z = norm.ppf(alpha) # z-value
V = 10000000 # portfolio value
dV = V*ret # daily changes in portfolio value
m, std = norm.fit(dV) # mean and standard deviation of dV
# estimates
VaR_p = -(m + z*std) # parametric VaR
ES_p = -m + (std/alpha)*norm.pdf(z) # parametric ES
dV_sorted = dV.sort_values() # sort dV for nonparametric estimates
N = len(ret) # number of returns
idx = round(alpha*N) # 1% of dV observations are "to the left" of this array index
VaR_np = -dV_sorted.iloc[idx] # nonparametric VaR
ES_np = -np.mean(dV_sorted.iloc[0:idx]) # nonparametric ES
```

Nonparametric

Example (Continued)

```
# normal distribution PDF:
x = np.linspace(-4*std, 4*std, 100)
npdf = norm.pdf(x, m, std)
# plots:
plt.hist(dV, density=True, bins="auto")
plt.plot(x, npdf, color="k", linewidth=0.75, label="normal PDF")
plt.axvline(x=-VaR_p, color='c', linewidth=0.75, ymax=0.50, label='parametric VaR')
plt.axvline(x=-VaR_np, color='b', linewidth=0.75, ymax=0.50,label='nonparametric VaR')
plt.axvline(x=-ES_p, color='y', linewidth=0.75, ymax=0.50, label='parametric ES')
plt.axvline(x=-ES_np, color='r', linewidth=0.75, ymax=0.50,label='nonparametric ES')
plt.title("daily value changes, $10m portfolio in S&P 500 Index ETF")
plt.xlabel("daily portfolio value change dV")
plt.ylabel("frequency")
plt.grid()
plt.legend()
```

Nonparametric

Remark

It is just as easy to calculate nonparametric portfolio VaRs and ESs. Suppose we hold Q_1 and Q_2 units of assets 1 and 2, whose prices are P_1 and P_2. So we have values $V_1 = Q_1 P_1$ and $V_2 = Q_2 P_2$ invested in assets 1 and 2, and $V = V_1 + V_2$ in total in the portfolio. Then calculate the N daily returns $\{R_1^1, \ldots, R_N^1\}$ and $\{R_1^2, \ldots, R_N^2\}$ in assets 1 and 2 and thus their N daily value changes $\{\mathrm{d}V_1^1, \ldots, \mathrm{d}V_N^1\}$ and $\{\mathrm{d}V_1^2, \ldots, \mathrm{d}V_N^2\}$ as above (here superscripts denote assets 1 and 2). The i^{th} portfolio value change is $\mathrm{d}V_i = \mathrm{d}V_i^1 + \mathrm{d}V_i^2$ for $i = 1, \ldots, N$, yielding the N portfolio value changes $\{\mathrm{d}V_1, \ldots, \mathrm{d}V_N\}$. The rest is as above.

Summary