FINM3405 Derivatives and Risk Management

Week 3: Futures and forwards - Pricing and optimal hedging

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Contents

Contract pricing

Last week we looked at the main classes of futures and forward contracts in terms of the underlying asset. In the tutorial questions we even introduced CME Group's Bitcoin futures. We also discussed the important concepts of leverage in futures trading, as well as simple hedging and speculating scenarios with futures and forwards. Finally, we introduced some basic interest rate contracts, namely forward rate agreements (FRA) and Australian 90 day bank accepted bill futures. In fact, the world of interest rate futures and forwards, and more generally swaps (FRA are really just 1-period interest rate swaps), is very large and we devote two weeks to it towards the end of the course.

Contract pricing

Last week we saw that at the time of writing, the S&P 500 spot price was 5,633.91, but the CME Group E-Mini futures September contract price was 5.682.5 and the December contract price was 5,748.

- This week we cover futures pricing, which means calculating the theoretically correct contract prices.
- We also cover optimal hedging, since most hedging scenarios in practice are rarely as "neat and clean" as those from last week.

Readings: Chapters 3 and 5 of Hull.

Remark

Last week we didn't finish the material on FRA and BAB futures, so this week repeat some of it.

Contract pricing

Recall that the *value* at time t of a long futures or forward position entered into at an earlier time $t = 0$ is $V_t^{\text{long}} = e^{-r(T-t)}(K_t - K)$.

Pricing a futures or forward contract at time t involves using no-arbitrage arguments to calculate the contract price K_t that yields 0 value to short and long positions entered into at time t.

Remark

The time t at which we price contracts is arbitrary so in what follows we simply let $t = 0$ in order to reduce notation, and in which case there is T years to maturity or the delivery date.

Contract pricing

Recall that we use the following general notation:

- We work on a hypothetical time interval $[0, T]$.
  - Time $t = 0$ is the date we enter into a contract.
  - Time T is a contract's maturity or delivery date.
  - Time t is some intermediate date: $0 \leq t \leq T$.
- S_t is the underlying asset's spot price at time t.
- K_t is the contract price at time t.
  - We write $S = S_0$ and $K = K_0$ to reduce notation.
- m is the number of assets in 1 contract (the multiplier).
- h is the number of contracts we enter into.

Contract pricing

In order to present the basic cost-of-carry arbitrage argument for pricing futures and forwards, we use the following additional notation:

- I is the time T capitalised **interest** paid on a loan used to buy the underlying asset at time $t = 0$.
- J is the time T capitalised **storage cost** of owning and holding the underlying asset from time $t = 0$ to time T.
  - Transport, warehouse storage, insurance, maintenance, etc.
- D is the time T capitalised **dividends** or other **income** or benefits received from owning the asset up to time T.
  - Share dividends, foreign interest, convenience/benefit from holding the asset in stock and/or using or consuming it, etc.

Contract pricing

Remark

By "time T capitalised" we mean that any cashflows (loan or coupon payments, storage costs, dividends or other income, etc) received or paid between times $t = 0$ and T are capitalised or *compounded* forward to time T.

Quite simply, the contract price K must be given by the following, otherwise there is an arbitrage opportunity:

$$K = S + I + J - D.$$

Contract pricing

- To show why $K = S + I + J - D$ must hold, consider the following arbitrage arguments:

Suppose $K > S + I + J - D$ and consider the following short trade:

Transactions at time $t = 0$:

- Borrow S to buy 1 unit of the underlying asset spot.
- Short 1 contract to sell the asset for K at maturity T.

Note that your net cashflow at time $t = 0$ is 0, since the money you received from borrowing was used to buy the asset.

Contract pricing

Transactions at maturity, time T:

- Receive K for selling the asset in the contract.
- Pay off the loan S with interest I.
- Pay J for owning, holding and storing the asset.
- Receive any other income D from owning the asset.

Then your net cashflow at maturity is positive:

$$K - S - I - J + D > 0.$$

Keep borrowing to buy the asset and short the contract for no initial net cashflow, but receive the positive net cashflow at maturity: An arbitrage.

Contract pricing

Remark

Another way to look at it is in terms of contract *value*. If $K > S + I + J - D$ then your short trade locks in the positive cashflow of $K - S - I - J + D$ at maturity.

- This cashflow is thus risk free and its value

$$V^{\text{short}} = e^{-rT}(K - S - I - J + D) > 0$$

  is positive to you and negative to the long position.
- You'd need to compensate the long position by this amount.

But K is set so the contract has 0 initial value to both parties.

Contract pricing

Also, if $K < S + J + I - D$ and if you currently own the underlying asset, then you can consider taking the following long position:

Transactions at time $t = 0$:

- Sell the asset spot for S and invest the proceeds.
- Go long to buy back the asset for K at maturity T.

Again, your net cashflow at initiation is 0.

Contract pricing

Transactions at maturity, time T:

- Pay K for buying back the asset in the contract.
- Receive S plus interest I from investing.
- Receive (more accurately, not have to pay) J for no longer having to own, hold and store the asset.
- Pay (more accurately, no longer receive) any other income D from no longer owning the asset.

Then your net cashflow at maturity is positive since

$$-K + S + I + J - D > 0.$$

Contract pricing

Remark

Again, in this case the long position has positive value given by

$$V^{\text{long}} = e^{-rT}(-K + S + I + J - D) > 0,$$

which represents negative value to the short party who would demand this amount in compensation upfront or they wouldn't be interested in entering into the contract.

Contract pricing

We thus get the **cost of carry** model for pricing forwards and futures:

$$K = S + I + J - D.$$

Here $I + J - D$ is the net cost of carrying (holding, storing) the asset.

Remark

The cost of carry model is usually written with annual borrowing rates r, storage rates s and dividend or convenience yields q, and is called spot-forward parity since it defines the relation between spot and forward prices:

Contract pricing

Let r be a *simple* annual **interest rate**, s be a *simple* annual **storage rate**, and q be a *simple* annual **dividend** (or later **convenience**) **yield**.

- Then the cost of carrying the asset is $I + J - D = SrT + SsT - SqT$.
- The cost of carry relation $K = S + I + J - D$ becomes

$$K = S\left[1 + (r + s - q)T\right]$$

and is called **spot-forward parity**. Under compound interest it becomes

$$K = S(1 + r + s - q)^T \qquad \text{or} \qquad K = Se^{(r+s-q)T}.$$

Contract pricing

What is the relation between spot S and forward K prices?

Remark

From

$$K = Se^{(r+s-q)T},$$

if the **cost of carry** $r + s - q$ (in rates) is:

- Positive, so $r + s - q > 0$ then $K > S$ (forward prices are greater than spot prices) and we say the market is **normal**.
- Negative, so $r + s - q < 0$ then $K < S$ (forward prices are less than spot prices) and we say the market is **inverted**.

Commodity contract pricing

We now give some pricing examples in terms of the underlying asset:

- Commodities.
- Equities.
- FX.
- FRA.

Commodity contract pricing

In the case of commodity futures and forwards, we have the:

- Interest rate r on borrowing and lending.
- Carrying cost s representing storage and insurance, etc, costs.
- **Convenience yield** q representing benefits of owning the asset:
  - Wholesalers maintain adequate stock levels to supply their customers in a timely and reliable manner.
  - Manufacturers maintain adequate stock in order to use/consume and keep their production processes running and not get held up.

So the cost-of-carry model and spot-forward parity relation is as above.

Commodity contract pricing

Example

Consider the CME Gold futures contract.

GOLD FUTURES - QUOTES

VENUE: GLOBEX

AUTO-REFRESH IS OFF   *Last Updated 19 Jul 2024 09:11:05 PM CT. Market data is delayed by at least 10 minutes.*

MONTH	OPTIONS	CHART	LAST	CHANGE	PRIOR SETTLE	OPEN	HIGH	LOW	VOLUME	UPDATED
JUL 2024 GCN4	OPT		2398.9	-52.9 (-2.16%)	-	2418.8	2419.2	2398.9	15	16:37:38 CT 19 Jul 2024
AUG 2024 GCQ4	OPT		2402.8	-53.6 (-2.18%)	-	2448.0	2448.4	2395.7	238,616	16:38:08 CT 19 Jul 2024
SEP 2024 GCU4	OPT		2413.7	-56.1 (-2.27%)	-	2460.5	2460.5	2409.0	676	16:38:32 CT 19 Jul 2024
OCT 2024 GCV4	OPT		2427.2	-53.6 (-2.16%)	-	2470.4	2471.6	2419.6	11,691	16:38:30 CT 19 Jul 2024
DEC 2024 GCZ4	OPT		2450.4	-55.0 (-2.20%)	-	2496.5	2496.9	2443.6	44,435	16:37:20 CT 19 Jul 2024

Commodity contract pricing

Example

Compare this to say the October futures delivery price of $2,427.2.

Commodity contract pricing

Example

DATE	CME TERM SOFR (%)			
	1 MONTH	3 MONTH	6 MONTH	12 MONTH
19 Jul 2024	5.34675	5.28299	5.13468	4.80026
18 Jul 2024	5.34513	5.28201	5.13575	4.80556
17 Jul 2024	5.34119	5.27947	5.13081	4.79975
16 Jul 2024	5.33399	5.27924	5.12855	4.80059
15 Jul 2024	5.33459	5.28577	5.15058	4.83577

Last Updated 19 Jul 2024 05:00:00 AM CT

Commodity contract pricing

Example

Let's try pricing the October contract. The 3 month and 6 month CME Term SOFA rates of 5.283% and 5.135% are simple interest rates that use the actual/360 day count convention of $T = \dfrac{d}{360}$, where d is the days to maturity, here 90 and 180. CME gold futures mature on the 3$^{\text{rd}}$ last business day of October, giving 101 days from now, so we need a 101 day interest rate. I'll linearly interpolate that the "101 day Term SOFR rate" is 5.265%.

Commodity contract pricing

Example

The gold spot price is $2,398.84. I calculate that

$$\begin{aligned} K &= S\left[1 + (r + s - q)\,T\right] \\ &= 2398.84\left[1 + (0.05265 + 0 - 0)\frac{101}{360}\right] = \$2,434.27, \end{aligned}$$

noting that the spot/futures/SOFR quote times don't quite align, and I assumed negligible storage costs and convenience yield.

Equity contract pricing

In the case of share and index futures, we have an interest rate r and a dividend yield of q. Hence the spot-forward parity relations become

$$\boxed{K = S\big[1+(r-q)T\big], \quad K = S(1+r-q)^T \quad \text{and} \quad K = Se^{(r-q)T},}$$

depending on the interest rate and dividend yield conventions.

Equity contract pricing

Example

Consider the December CME E-mini S&P500 futures contract.

E-MINI S&P 500 FUTURES - QUOTES

VENUE: GLOBEX

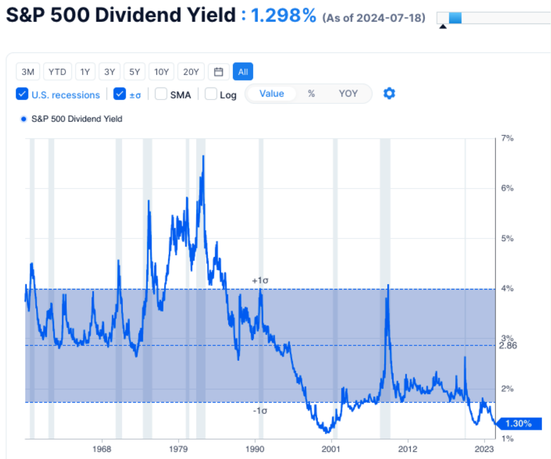
AUTO-REFRESH IS OFF   *Last Updated 19 Jul 2024 09:50:33 PM CT. Market data is delayed by at least 10 minutes.*

MONTH	OPTIONS	CHART	LAST	CHANGE	PRIOR SETTLE	OPEN	HIGH	LOW	VOLUME	UPDATED
SEP 2024 ESU4	OPT		5552.75	-41.75 (-0.75%)	-	5602.50	5607.50	5542.00	2,020,989	16:37:20 CT 19 Jul 2024
DEC 2024 ESZ4	OPT		$\boxed{5614.00}$	-43.00 (-0.76%)	-	5665.75	5670.00	5604.75	3,332	16:38:35 CT 19 Jul 2024
MAR 2025 ESH5	OPT		5670.50	-45.50 (-0.80%)	-	5724.00	5724.00	5665.25	231	16:38:19 CT 19 Jul 2024

The dividend yield of the S&P500 index is say 1.3%, and we'll assume it's an annual simple interest rate:

Equity contract pricing

Example (Continued)



Equity contract pricing

Example (Continued)

Equity contract pricing

Example (Continued)

From the above commodity futures example, we'll linearly interpolate that the 5 month Term SOFA rate is 5.16%. The spot S&P 500 value is 5,505 and there is 153 days until maturity, the Thursday before the 3rd Friday of December. We calculate that

$$\begin{aligned} K &= S\left[1 + (r - d)T\right] \\ &= 5505\left[1 + (0.0516 - 0.013)\frac{153}{360}\right] = 5,595.31, \end{aligned}$$

again noting that our quote times don't necessarily align and the dividend yield should be an expected yield.

FX contract pricing

Spot-forward parity for FX contracts is best derived separately. The main complications are that we have to consider the interest rates in each country and we have to be careful about exchange rate quoting.

- Let r_{d} be domestic interest rate and r_{f} be the foreign rate.
- Let $S_{\mathrm{d:f}}$ be the domestic:foreign spot exchange rate.
  - 1 unit of the domestic currency exchanges spot for $S_{\mathrm{d:f}}$ units of the foreign currency.
- Let $K_{\mathrm{d:f}}$ be the domestic:foreign forward exchange rate.

We'll first use simple interest, and the resulting spot-forward parity relation is often called **covered interest rate parity**:

FX contract pricing

Two ways of earning interest over a time period T, namely:

1. Invest domestically at r_d to get $1 + r_\mathrm{d} T$ per unit invested.
2. Invest internationally:
   - Buy $S_{\mathrm{d:f}}$ units of foreign currency spot and simultaneously lock in the futures exchange rate of $K_{\mathrm{f:d}}$ at time T.
   - Invest $S_{\mathrm{d:f}}$ at r_f to get $S_{\mathrm{d:f}}(1 + r_\mathrm{f} T)$.
   - Exchange this amount back into the domestic currency via the futures contract to end up with $S_{\mathrm{d:f}}(1 + r_\mathrm{f} T) K_{\mathrm{f:d}}$

Both avenues must have the same final value or else there's an arbitrage opportunity: Invest in the best avenue fund it with a reverse transaction (borrowing) in the other avenue.

FX contract pricing

Hence, the no arbitrage relation is $1 + r_d T = S_{d:f}(1 + r_f T)K_{f:d}$ which we rearrange to get the **covered interest rate parity** relation

$$K_{f:d} = S_{f:d} \frac{1 + r_d T}{1 + r_f T}.$$

Note that in the above we used $1/S_{d:f} = S_{f:d}$. Under compound interest:

$$K_{f:d} = S_{f:d} \left(\frac{1 + r_d}{1 + r_f} \right)^T \qquad \text{or} \qquad K_{f:d} = S_{f:d} e^{(r_d - r_f)T}.$$

FX contract pricing

Example

Let's price the October CME Group Euro FX futures, which are futures contracts between the Euro and USD, the two most actively traded currencies in the world.

- Quote is $K_{\text{Euro:USD}}$.

The foreign currency is being viewed as the "underlying asset".

FX contract pricing

Example (Continued)

EURO FX FUTURES - QUOTES

AUTO-REFRESH IS OFF Last Updated 19 Jul 2024 11:47:58 PM CT. Market data is delayed by at l

MONTH	OPTIONS	CHART	LAST	CHANGE
AUG 2024 6EQ4	OPT		1.0897	-0.0018 (-0.16%)
SEP 2024 6EU4	OPT		1.0913	-0.0016 (-0.15%)
OCT 2024 6EV4	OPT		1.0926	-0.0017 (-0.16%)

FX contract pricing

Example (Continued)

FX contract pricing

Example (Continued)

Euribor

Euribor is short for Euro Interbank Offered Rate. The Euribor rates are based or
from one another. There are different maturities, ranging from one week to one

7/18/2024

Euribor 1 week	3.626 %
Euribor 1 month	3.633 %
Euribor 3 months	3.688 %
Euribor 6 months	3.624 %
Euribor 12 months	3.503 %

FX contract pricing

Example (Continued)

There is 86 days until maturity, being two business days before the 3rd Wednesday of October. This is "close enough" to 90 days to use 3 month interest rates. From a previous example, the 3 month Term SOFA rate is 5.28% and from above the 3 month EURIBOR (also see global-rates.com) rate is 3.688%. We get

$$K_{\text{Euro:USD}} = S_{\text{Euro:USD}} \frac{1 + r_{\text{USD}}\, T}{1 + r_{\text{Euro}}\, T}$$

$$= 1.09 \frac{1 + 0.0528 \frac{86}{360}}{1 + 0.03688 \frac{86}{360}} = 1.0941.$$

FRA Pricing

Recall that a $T_1 \times T_2$ **FRA** is an OTC agreement to lock in a future borrowing or lending **fixed rate** k starting at time T_1, and finishing at time T_2 over a notional principal or face value F.

- Time T_1 is the FRA's maturity date.
- The length of the time period is given by $T = T_2 - T_1$.
- The fixed rate receiver hypothetically agrees to invest F at time T_1 and receive the cashflow $F(1 + kT)$ at time T_1.
- The fixed rate payer hypothetically agrees to borrow F at time T_1 and pay off the loan amount of $F(1 + kT)$ at time T_2.
- FRA are cash settled at maturity, time T_1.
- FRA are written over reference rates such as SOFR or EURIBOR.

FRA Pricing

Also recall that the payoffs at maturity to each party are given by

$$\text{fixed rate receiver payoff} = \frac{F(k-r)T}{1+rT},$$
$$\text{fixed rate payer payoff} = \frac{F(r-k)T}{1+rT},$$

where r is the spot rate at maturity over the period $[T_1, T_2]$ of length T.

- We derived these payoff equations last week.

This week we want to price an FRA, which involves calculating the theoretically correct fixed rate k.

FRA Pricing

The key insight is that the fixed rate k is set so that the time $t = 0$ *value* of a FRA is 0 to both the fixed rate receiver and payer.

So we want to calculate the time $t = 0$ values of a FRA to both parties:

- The fixed rate receiver of a $T_1 \times T_2$ FRA hypothetically agrees to pay F at time T_1 and receive $F(1 + kT)$ at time T_2.
- These cashflows are risk free so their present value is

$$V = -\frac{F}{1 + r_1 T_1} + \frac{F(1 + kT)}{1 + r_2 T_2},$$

where r_1 and r_2 are the time $t = 0$ spot reference rates for the period from time $t = 0$ to times T_1 and T_2, respectively.

FRA Pricing

The value to the fixed rate payer is simply the negative of this.

- In either case, we find k by setting $V = 0$ and hence solving

$$\frac{F}{1 + r_1 T_1} = \frac{F(1 + kT)}{1 + r_2 T_2}.$$

This easily rearranges to

$$1 + r_2 T_2 = (1 + r_1 T_1)(1 + kT).$$

Hence, since k is an interest rate starting at time T_1 and ending at time T_2, it must be given by $k = r_{1,2}$, the implied forward rate over the time period $[T_1, T_2]$ embedded in the reference rate's yield curve.

FRA Pricing

We easily rearrange the above to get

$$k = \left(\frac{1 + r_2 T_2}{1 + r_1 T_1} - 1 \right) \frac{1}{T}.$$

Example

The above EURIBOR yield curve from the FX contracts pricing example had $r_3 = 3.688\%$ and $r_6 = 3.624\%$.

- Hence, we calculate the fixed rate k in a 3×6 FRA to be

$$k = \left(\frac{1 + r_6 T_6}{1 + r_3 T_3} - 1 \right) \frac{360}{30} = \left(\frac{1 + 0.03624 \frac{180}{360}}{1 + 0.03688 \frac{90}{360}} - 1 \right) \frac{360}{30} = 3.528\%.$$

Optimal hedging

Last week we presented "perfect" hedging scenarios in which the:

- Underlying asset of a futures/forward contract is precisely the asset we want to hedge an exposure to,
- Number of assets we held was divisible by the contract multiplier m, enabling us to use an integer number of contracts.
- Contract maturity date is precisely our desired hedging date.

Perfect hedging scenarios are rare in practice:

Optimal hedging

Example

- An Australian wheat farmer may want to hedge their exposure to their wheat price, but there may be no contracts over the exact type or grade/quality of their wheat.
- You hold a fairly diversified equities portfolio of USA stocks, but its composition and weights do not exactly match those of the S&P500 or NASDAQ indices, etc.
- You might want to hedge on a date falling in a month not covered by a say quarterly contract maturity date cycle.

Basis risk

We saw above that the contract price K is usually different to the spot price S of the underlying asset. Our pricing equations, such as

$$K = Se^{(r+s-q)T}$$

for commodity futures, tell us that the difference between K and S, which we call the **basis**, is due to the cost of carry $r + s - q$.

- As the cost of carry changes over time, the basis changes, introducing **basis risk**: K may not be perfectly correlated with S.

Remark

The situation is even more complicated if the asset we hold and want to hedge is not the same as the contract's underlying asset.

Basis risk

The time t **basis** B_t is the difference between K_t and S_t, namely:

$$B_t = K_t - S_t.$$

Note that the basis approaches 0 over time, and $K_T = S_T$ at maturity.

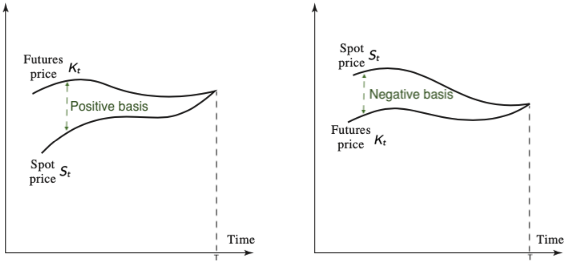


Figure: The contract price K_t and spot price S_t converge as $t \to T$.

Optimal hedging

Optimal hedging basically means minimising basis risk. Suppose:

- We hold Q units in an asset at time $t = 0$.
  - Its price at time t is denoted by A_t.
- We want to sell our holding at time t.
  - We want to hedge our exposure to a fall in the asset price.
- There is a futures contract maturing at time T, with $t < T$.
  - The underlying asset of the futures contract S may not necessarily be the same as the asset A we hold.
- We short h units of the futures contract at time $t = 0$ for K.
- We sell our asset holding and close out the futures position at time t.

How many contracts h should we short?

Optimal hedging

At time t, the **liquidation value** (net cashflow) of our position is

$$L_t = QA_t + h(K - K_t)m,$$

where:

- Q is the number of units we hold in our asset.
- A_t is the time t price of our asset (we let $A = A_0$).
- h is the number of futures contracts we shorted.
- K is the contract price at time $t = 0$ when we went short.
- K_t is the contract price at time t when we went long to close out.
- m is the futures contract multiplier.

Optimal hedging

In an imperfect hedging scenario, in which there is basis risk, we choose h that minimises the variance in the liquidation value L_t.

Then h is called the **minimum variance** or **optimal hedge quantity**.

- At time $t = 0$, A_t and K_t are unknown, thus random variables.

We use the following notation:

- σ_{A_t} is the standard deviation (volatility) in $A_t - A$.
- σ_{K_t} is the standard deviation (volatility) in $K_t - K$.
- ρ is the correlation between $A_t - A$ and $K_t - K$.

Optimal hedging

We can prove that the **optimal** or **minimum variance hedge quantity** h, which minimises the variance in the above liquidation value, or equivalently minimises basis risk, is given by

$$h = \rho \frac{\sigma_{A_t} Q}{\sigma_{K_t} m}.$$

Remark

The number $\rho \frac{\sigma_{A_t}}{\sigma_{K_t}}$ is called the **minimum variance** or **optimal hedge ratio**.

As it stands here, this formula for h is not very useful, but we can use it to calculate h from historical data as follows:

Optimal hedging

- Collect samples of daily historical asset $\{A_0, A_1, \ldots, A_N\}$ and futures $\{K_0, K_1, \ldots, K_N\}$ prices.
- Calculate the sample standard deviations $\bar{\sigma}_A$ and $\bar{\sigma}_K$ and sample correlation $\bar{\rho}$ of the daily <u>price changes</u>:
  - $\{\Delta A_1, \ldots, \Delta A_N\}$, where $\Delta A_n = A_n - A_{n-1}$.
  - $\{\Delta K_1, \ldots, \Delta K_N\}$, where $\Delta K_n = K_n - K_{n-1}$

The optimal hedge quantity is then given by

$$h = \bar{\rho} \frac{\bar{\sigma}_A Q}{\bar{\sigma}_K m}.$$

Optimal hedging

Also, suppose that in the above we used daily returns $(A_n - A_{n-1})/A_{n-1}$ and $(K_n - K_{n-1})/K_{n-1}$ to calculate $\bar{\sigma}_A$, $\bar{\sigma}_K$ and $\bar{\rho}$.

- Then the optimal hedge quantity is given by

$$h = \bar{\rho}\frac{\bar{\sigma}_A V}{\bar{\sigma}_K F}, \qquad \text{where } V = AQ \text{ and } F = Km.$$

Remark

The CAPM beta β of a share is calculated from historical returns, and is given by $\beta = \bar{\rho}\frac{\bar{\sigma}_A}{\bar{\sigma}_K}$. Hence β is the optimal hedge ratio and the optimal hedge quantity is $h = \beta\frac{V}{F}$, where $V = AQ$ is our portfolio value and $F = Km$ is the notional value of 1 contract.

Optimal hedging

We calculate the optimal hedge quantity of a single holding.

Example

You hold $Q = 4,600$ Tesla shares and are worried that Tesla has been very volatile lately and will fall back to its recent lows.

Optimal hedging

Example (Continued)

Suppose you're not aware of any individual share futures on Tesla, so you decide that the "next best thing" is MICRO NYSE FANG+ Index Futures, since the NYSE FANG+ Index contains only 10 stocks and surely they must all be fairly positively correlated.

Meta (META)	Apple (AAPL)	Amazon (AMZN)	Netflix (NFLX)	Microsoft (MSFT)
Google (GOOGL)	Tesla (TSLA)	NVIDIA (NVDA)	Snowflake (SNOW)	Broadcom Inc. (AVGO)

Optimal hedging

Example (Continued)

Optimal hedging

Example (Continued)

Here $m = 5$ and below $K = 10,937.2$, so the notional value of 1 September contract is $F = K \times m = \$54,686$. The following Python code downloads the daily closing prices of Tesla and the futures contract, and calculates h.

ICE Futures U.S.

MICRO NYSE FANG+™ Index Futures

Description

NYSE FANG+™ index provides exposure to a select group of highly-traded growth stocks of next generation technology and tech-enabled companies. The MICRO futures contract on the index is designed to offer the ability to gain or reduce exposure to this key group of growth stocks in a capital efficient manner.

Market Specifications

Trading Screen Product Name

Micro FANG+ Index Futures

Trading Screen Hub Name

ICUS

Contract Symbol

FNG

Contract Size

$5 times the NYSE FANG+ Index

Contract Series

4 contracts in the March, June, September and December cycle

Optimal hedging

Example (Continued)

```python
import numpy as np
import yfinance as yf
all_prices=yf.download("TSLA FNG=F",start="2020-09-16")["Adj
    Close"].dropna()
all_returns=np.log(all_prices).diff(1).dropna()
# contract historical prices and returns
K_prices=all_prices["FNG=F"]
K_returns=all_returns["FNG=F"]
# contract price
K=K_prices.iloc[-1]
# Tesla historical prices and returns
tesla_prices=all_prices["TSLA"]
tesla_returns=all_returns["TSLA"]
```

Optimal hedging

Example (Continued)

```
# Tesla portfolio value
A=tesla_prices.iloc[-1]
Q=4600
V=A*Q
# standard deviations and correlation
sigma_K=np.std(K_returns)
sigma_A=np.std(tesla_returns)
rho=np.corrcoef(K_returns,tesla_returns)[0,1]
# portfolio beta
beta=rho*sigma_A/sigma_K
# optimal hedge quantity
m=5
F=K*m
h=beta*V/F
```

Optimal hedging

Example (Continued)

In this example we have:

- $K = 10,937.2$.
- $V = \$997,556$.
- $\bar{\sigma}_A = 0.0371$, $\bar{\sigma}_K = 0.0197$ and $\bar{\rho} = 0.663$.
- $\beta = 1.248$.
- $h = 22.7692$.

Hence, we short $h = 23$ contracts. Note that "naively" shorting $V/F \approx 18$ contracts is probably not enough due to Tesla's β.

Optimal hedging

We now calculate the optimal hedge quantity of a portfolio of shares.

Example

You own a portfolio of 1,000 shares in each of the NYSE FANG+ Index constituents.

Meta (META)	Apple (AAPL)	Amazon (AMZN)	Netflix (NFLX)	Microsoft (MSFT)
Google (GOOGL)	Tesla (TSLA)	NVIDIA (NVDA)	Snowflake (SNOW)	Broadcom Inc. (AVGO)

You're worried that the index is in bubble territory, which might burst by September, and want to hedge this risk.

Optimal hedging

Example (Continued)

You decide to use MICRO NYSE FANG+ Index Futures again.

Optimal hedging

Example (Continued)

```python
import numpy as np, yfinance as yf
# download historical prices and calculate historical returns
all_prices=yf.download("META AAPL AMZN NFLX MSFT GOOGL TSLA NVDA SNOW AVGO FNG=F", start="2020-09-16")["Adj Close"].dropna()
all_returns=np.log(all_prices).diff(1).dropna()
# contract historical prices and returns
K_prices=all_prices["FNG=F"]
K_returns=all_returns["FNG=F"]
K=K_prices.iloc[-1] # the September contract price we use
# share (asset) historical prices and returns
A_prices=all_prices.drop("FNG=F",axis=1)
A_returns=all_returns.drop("FNG=F",axis=1)
A_spot_prices=A_prices.iloc[-1]
```

Optimal hedging

Example (Continued)

```
# portfolio value, weights and historical returns
V=np.sum(A_spot_prices)*1000
P_weights=1000*A_spot_prices/V
P_returns=np.sum(A_returns*P_weights,axis=1)
# standard deviations and correlation
sigma_K=np.std(K_returns)
sigma_P=np.std(P_returns)
rho=np.corrcoef(K_returns,P_returns)[0,1]
# portfolio beta
beta=rho*sigma_P/sigma_K
# optimal hedge ratio
m=5
F=K*m
h=beta*V/F
```

Optimal hedging

Example (Continued)

In this example we have:

- $K = 10,937.2$.
- $V = \$2,710,179.96$.
- $\bar{\sigma}_P = 0.0195$, $\bar{\sigma}_K = 0.0197$ and $\bar{\rho} = 0.9318$.
- $\beta = 0.922$.
- $h = 45.69$.

Hence, we short $h = 46$ contracts. This is not too different to the "naive" $V/F \approx 50$ contracts, which would be too many given the portfolio beta (the NYSE FANG+ index is equally weighted).

Summary