Project for The Open University of Israel
Done at The Weizmann Institute of Science

Controlling a Superconducting Quantum Computer

Daniel Cohen Hillel

Supervisor: Dr. Serge Rosenblum

Advanced Project in Physics A (20382)

The Open University of Israel

The Weizmann Institute

Contents

1 Introduction

1.1 What is a Quantum Computer?

We'll assume the reader understands the basics of computing and quantum mechanics, here's a brief overview. A classical computer is, essentially, a calculator, not of "Regular" numbers but of *binary numbers* [1]. A *binary digit*("*bits*" from now on) can be in one of two states, usually represented by 0 and 1. We can use *logic gates* to control and manipulate bits to do all kinds of calculations[2]. This is the building blocks of the classical computer, with the ability to do calculation with bits, and the ability to store bits in the memory we are able to construct a computer.

So what is a quantum computer then? Well, if the classical computer uses bits to do calculations, a quantum computer uses *quantum bits*("*qubits*" from now on) for it's calculations. A qubit, much the same as a bit, has 2 states, a 0 state and a 1 state(notated $|0\rangle$ and $|1\rangle$ for reasons we'll see later), the difference is that a qubit can be in a *superposition* of the 2 states, so the qubit has essentially and infinite amount of possible states

1.2 Qubits and Quantum Gates

1.3 Algorithms and Further motivation

1.4 Superconducting Quantum Computers

[1] add further reading about binary numbers
[2] Additional information about bit calculation

2 Our System

2.1 The cavity

2.2 The Transmon

2.3 The Oscillator

2.4 Describing the System Mathematically

To describe the system mathematically we are going to calculate it's Hamiltonian, that way we can run simulations of the system with the Schrodinger equation.

We can partition the system into different parts and analyse each part individually, so the total system Hamiltonian will be:

$$H(t) = H_{Oscillator} + H_{Transmon} + H_{Interaction} + H_{Drive}(t) \tag{2.1}$$

We'll begin by the easy to characterize, Transmon And Oscillator, they are a simple quantum system that can be described by the uppering and lowering operators [3]. We can write:

$$H_{Oscillator} = \omega_C a^\dagger a \tag{2.2}$$

$$H_{Transmon} = \omega_T b^\dagger b \tag{2.3}$$

We can find ω_C and ω_T experementally.

The next part to characterize is the interaction(See appendix A), again by the Jaynes–Cummings model, we can write the interaction hamiltonian as:

$$H_{Interaction} = \chi a^\dagger a b^\dagger b \tag{2.4}$$

Again, we can measure χ experimentally.

Finally, we can characterize the driven and most important part of the system. It can be written as:

$$H_{Drive} = \epsilon_C(t) a + \epsilon_T(t) b + h.c. \tag{2.5}$$

[3] See appendix A

Or as(expanding the hermitian conjurgate):

$$H_{Drive} = \epsilon_{I_C}(t)(a + a^\dagger) + \epsilon_{Q_C}(t)(a - a^\dagger)i + \epsilon_{I_T}(t)(b + b^\dagger) + \epsilon_{Q_T}(t)(a - a^\dagger)i \tag{2.6}$$

3 Optimal Control

3.1 What's GRAPE

As explained in previous sections, to control our qubit and manipulate it, we need to send microwave pulses in the cavity. The problem is, what pulse do we send? How does it look? sin? cos? In what frequency? or maybe even some arbitrary wave.

To answer this question, we can model our system on a normal computer and simulate what happens when you send a pulse, then, we can try to change the pulse(in a smart way) until we get the desired effect. So for example, let's say we want to find the wave pulse that corresponds to the NOT gate, we can start by guessing some random wave(constant zero, sin and so on), the random wave probably won't act as a NOT gate, then, we change the wave a little bit many times and on each iteration the pulse acts more and more as a NOT gate.

So what's GRAPE than? The *GR*adient *A*scent *P*ulse *E*ngineering was first proposed in [2]. When we model our system, we treat the wave as a step-wise constant function, so the wave is just an array with many variables and we want to find the best values that give the result that we want. then we set a cost function [4] that tells us how are the values of the wave to give the wanted result. This cost function is a many dimensional function(each step of the wave is a dimension of the cost function), and we can find its gradient. Using the cost function and it's gradient we can use some optimization algorithm(mainly, the L-BFGS-B method) to find the maximum of the cost function that gives us the optimal wave to send to the cavity.

3.2 The Cost Function

So what is this cost function really? ***Fidelity***. It measures the "closeness" of two quantum states and varies between 0 and 1. So for example, the fidelity between state $|0\rangle$ and state $|1\rangle$ is equal to 0, because they are the most different two quantum states can be, while for example the fidelity between state $|0\rangle$ and state $|0\rangle$ is equal to 1, because they are the closest two states can be to each other(for a matter of fact, every state has fidelity 1 with itself and end fidelity 0 with an opposite state). But still, how do we calculate the fidelity between two states? well, turns out its very simple, it's just their product [5]. We can write:

$$F(\psi_1, \psi_2) = |\langle\psi_1|\psi_2\rangle|^2 \tag{3.1}$$

We want to maximize the fidelity with GRAPE.

In a previous chapter we characterized the Hamiltonian of the system(equation (num)), so now we can use

[4] discussed in details in the next section
[5] Assuming both states are pure states

the good old *time-dependent Schrodinger equation*:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle \tag{3.2}$$

And also, the Hamiltonian of the system is in the form[6]:

$$H(t) = H_0 + \sum_k \epsilon_k(t)H_k \tag{3.3}$$

Because of the way this Hamiltonian is built, on each constant step of the wave function, the Hamiltonian is constant, and luckily for us, the solution of the Schrodinger equation for a constant Hamiltonian is pretty simple and given by[7]

$$U(t) = e^{-\frac{i}{\hbar}\int_{T_0}^{T_1} H(t)dt} \tag{3.4}$$

And because we chose T_0 and T_1 as the end points of a step of the functions, the total Hamiltonian of the system is constant so the integral is just a simple multiplication by $T_1 - T_0$ which we'll write as δt. So the solution is

$$U(t) = e^{-\frac{i\cdot\delta t}{\hbar}H(t)} \tag{3.5}$$

More then that, to solution over all time is the product of all the solutions for each constant piece. So

$$U(\epsilon(t)) = \prod_{k=1}^{N} U_k \tag{3.6}$$

Where N is the number of time steps in the simulation(larger N is a more precise simulation).

This way the state of the qubit after the drive is given by:

$$|\Psi_{final}\rangle = U|\Psi_{initial}\rangle \tag{3.7}$$

This way if we want to calculate the fidelity after applying the drives we can simply calculate the fidelity between the wanted state and the final state,

$$F(\epsilon(\vec{t})) = F(\Psi_{target}, \Psi_{final}) = |\langle\Psi_{target}|U|\Psi_{initial}\rangle|^2 \tag{3.8}$$

m Now, theoretically we can use an algorithm to try different waves until we find a wave that does what we want(brute force for example), but this will take to much time and the computation won't finish in any

[6] Further details were given in section 2.4
[7] Add a simple justification

reasonable amount of time. Because of this, we'll want to use a smart search algorithm(such as L-BFGS-B) but to do so we need the gradient of the cost function(the variables of the cost function are the values of the steps of the drive pulses). We can obviously use the finite difference method to calculate the gradient but this method is heavy on the computation and has a lot of over head. We'll take a smarter approach to calculating the gradient.

We can look at the expression,

$$c = \langle \Psi_{target} | \Psi_{final} \rangle = \langle \Psi_{target} | \, U \, | \Psi_{initial} \rangle \tag{3.9}$$

We want to differentiate this expression by each control parameter. U is defined as:

$$U = U_N U_{n-1} ... U_2 U_1$$

And when differentiating by a control parameter only one U_k is affected, so we can write,

$$\frac{dc}{d\epsilon_k} = \langle \Psi_{target} | \, U_N U_{N-1} ... \frac{dU}{d\epsilon_k} ... U_2 U_1 \, | \Psi_{initial} \rangle$$

Assuming very small time steps we can derive U_k over ϵ_k,

$$U_k = e^{-\frac{i \cdot \delta t}{\hbar} H(t)}$$

$$\frac{dU_k}{d\epsilon_k} = \frac{i\delta t}{\hbar} \frac{dH}{d\epsilon_k} U_k$$

We can use this expression as the gradient values but it's still rather complex computationally(N^2 complexity).

We can use a bit different method to calculate the gradient to save on the computation by reducing the overhead.

3.3 Constraints

4 Controlling the FPGA(wave generator)

5 Conclusion

A The Jaynes–Cummings Model

Our goal is to mathematically model the Hamiltonian of a system of a two-level atom interacting with a single quantized mode of an optical cavity's electromagnetic field.

First we'll divide the system into 3 parts, The atom(it can be other two-level quantum systems), the cavity(electromagnetic field with quantized modes) and the interaction between the atom and the cavity(an atom can emit a photon to the cavity and change it's electromagnetic field, or catch a photon from the cavity and go up an energy level).

Let's start with the cavity(we'll consider a one dimensional cavity for now).

A.1 The Homogeneous Electromagnetic Wave Equations

Maxwell's equations in free space are:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \tag{A.1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{A.2}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{A.3}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{A.4}$$

Taking the curl of A.3 and A.4 we get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla} \times (-\frac{\partial \mathbf{B}}{\partial t}) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \tag{A.5}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla} \times (\mu_0 \epsilon_0 \frac{\partial E}{\partial t}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{\nabla} \times E) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{A.6}$$

We can use the vector identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} \tag{A.7}$$

And obtain from A.5 and A.6

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{A.8}$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{A.9}$$

Now, we can use A.1 and A.2 To cancel the left most term and get

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{A.10}$$

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{A.11}$$

Now because we know that $v_{ph} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ and that the phase velocity of electromagnetic waves in a vacuum is the speed of light, c_0, we get

$$\begin{aligned} \nabla^2 \mathbf{E} &= \frac{1}{c_0^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla^2 \mathbf{B} &= \frac{1}{c_0^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \end{aligned} \tag{A.12}$$

These equation are called *the homogeneous electromagnetic wave equations.* We'll pick a polarization arbitrarily to be in the x direction(that way we get only the component of the electric field and the y component of the magnetic field, E_x and B_y) so now we get,

$$\begin{aligned} \frac{\partial^2 E_x}{\partial x^2} &= \frac{1}{c_0^2} \frac{\partial^2 E_x}{\partial t^2} \\ \frac{\partial^2 B_y}{\partial y^2} &= \frac{1}{c_0^2} \frac{\partial^2 B_y}{\partial t^2} \end{aligned} \tag{A.13}$$

A.2 The Hamiltonians

We can easily solve A.13 using separation of variables,

$$E_x(z,t) = Z(z)T(t)$$

Yielding the solution,

$$\begin{aligned} E_x(z,t) &= \sqrt{\frac{2\omega_c^2}{V\epsilon_0}} q(t) \sin kz \\ B_y(z,t) &= \sqrt{\frac{2\mu_0}{V}} \dot{q}(t) \cos kz \end{aligned} \tag{A.14}$$

where V is the effective volume of the cavity, q is a time-dependent amplitude with units of length, and $k = m\pi/L$ for an integer $m > 0$

The Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2}\int \epsilon_0 \mathbf{E}^2 + \frac{\mathbf{B}^2}{\mu_0} dV \\ &= \frac{1}{2}\int \epsilon_0 E_x^2(z,t) + \frac{B_y^2(z,t)}{\mu_0} dz \\ &= \frac{1}{2}[\dot{q}^2(t) + \omega_c^2 q^2(t)] \end{aligned} \tag{A.15}$$

This looks like the Hamiltonian of an harmonic oscillator.

Now, going from dynamical variables to operators(considering $\dot{q} \equiv p$) we get,

$$\begin{aligned} \hat{E}_x(z,t) &= \sqrt{\frac{2\omega_c^2}{V\epsilon_0}}\hat{q}(t)\sin kz \\ \hat{B}_y(z,t) &= \sqrt{\frac{2\mu_0}{V}}\hat{p}(t)\cos kz \\ \hat{H} &= \frac{1}{2}[\hat{p}^2(t) + \omega_c^2\hat{q}^2(t)] \end{aligned} \tag{A.16}$$

Let's introduce creation and annihilation operators,

$$\hat{a}(t) = \frac{1}{\sqrt{2\hbar\omega_c}}[\omega_c\hat{q}(t) + i\hat{p}(t)]$$

$$\hat{a}^\dagger(t) = \frac{1}{\sqrt{2\hbar\omega_c}}[\omega_c\hat{q}(t) - i\hat{p}(t)]$$

We can write the electric and magnetic field as,

$$\begin{aligned} \hat{E}_x(z,t) &= E_0[\hat{a}(t) + \hat{a}^\dagger(t)]\sin kz \\ \hat{B}_y(z,t) &= \frac{E_0}{c}[\hat{a}(t) - \hat{a}^\dagger(t)]\cos kz \end{aligned} \tag{A.17}$$

And we can write the Hamiltonian as,

$$\hat{H} = \hat{H}_{cavity} = \hbar\omega_c[\hat{a}\hat{a}^\dagger + \frac{1}{2}] \approx \hbar\omega_c\hat{a}\hat{a}^\dagger \tag{A.18}$$

We can ignore the zero-point energy $\frac{\hbar\omega_c}{2}$ if we define it as the zero energy point.

Now that we have the cavity's Hamiltonian, we can go on to calculate the atom(qubit) Hamiltonian. Remember that the qubit is a 2-level system, meaning we can define it has a superposition of the ground, $|g\rangle$, and excited, $|e\rangle$, states. The energy of the atom is the sum of the energy of each state times it's

energy($\sum E_s P(|s\rangle)$). The probabilty to be in a state $|s\rangle$ is given by $|s\rangle\langle s|$ so we can write,

$$\hat{H}_{atom} = E_g |g\rangle\langle g| + E_e |e\rangle\langle e| \tag{A.19}$$

Using the vector representation of these states we'll write,

$$\begin{aligned}
\hat{H}_{atom} &= E_e \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + E_g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} E_e & 0 \\ 0 & E_g \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} E_g + E_e & 0 \\ 0 & E_g + E_e \end{bmatrix} + \frac{1}{2}\begin{bmatrix} E_e - E_g & 0 \\ 0 & -(E_e - E_g) \end{bmatrix} \\
&= \frac{1}{2}(E_g + E_e)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}(E_e - E_g)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \frac{1}{2}(E_g + E_e)\mathbb{I} + \frac{1}{2}(E_e - E_g)\hat{\sigma}_z
\end{aligned}$$

Again, we can define the zero point energy so that the first term becomes 0. We know the difference between the excited state energy and the ground state energy because it's approximately an harmonic osscilator so $E_e - E_g = \hbar\omega_a$ where ω_a is the atom frequancy. Now we can write,

$$\hat{H}_{atom} = \frac{1}{2}\hbar\omega_a\hat{\sigma}_z \tag{A.20}$$

A.3 The Dispersive Limit