Problems

Algebra

A1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

A2. Let $u_1, u_2, \ldots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \ldots, u_{2019})$ and $b = \max(u_1, u_2, \ldots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

A3. Let $n \geqslant 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left|1 - \sum_{i \in X} a_i\right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

A4. Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set A by

$$A = \left\{(i, j) \,\middle|\, 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\right\}.$$

Prove that, if A is not empty, then

$$\sum_{(i,j) \in A} a_i a_j < 0.$$

A5. Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1 \leqslant i \leqslant n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

A6. A polynomial $P(x, y, z)$ in three variables with real coefficients satisfies the identities

$$P(x, y, z) = P(x, y, xy - z) = P(x, zx - y, z) = P(yz - x, y, z).$$

Prove that there exists a polynomial $F(t)$ in one variable such that

$$P(x, y, z) = F(x^2 + y^2 + z^2 - xyz).$$

A7. Let $\mathbb{Z}$ be the set of integers. We consider functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(x + y) + y\big) = f\big(f(x) + y\big)$$

for all integers x and y. For such a function, we say that an integer v is *f-rare* if the set

$$X_v = \{x \in \mathbb{Z}\colon f(x) = v\}$$

is finite and nonempty.

(a) Prove that there exists such a function f for which there is an f-rare integer.

(b) Prove that no such function f can have more than one f-rare integer.

Combinatorics

C1. The infinite sequence a_0, a_1, a_2, ... of (not necessarily different) integers has the following properties: $0 \leqslant a_i \leqslant i$ for all integers $i \geqslant 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geqslant 0$.

Prove that all integers $N \geqslant 0$ occur in the sequence (that is, for all $N \geqslant 0$, there exists $i \geqslant 0$ with $a_i = N$).

C2. You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leqslant r \leqslant 2n-2$ you can choose a subset of the blocks whose total weight is at least r but at most $r+2$.

C3. Let n be a positive integer. Harry has n coins lined up on his desk, each showing heads or tails. He repeatedly does the following operation: if there are k coins showing heads and $k > 0$, then he flips the k^{th} coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \to HHT \to HTT \to TTT$, which takes three steps.)

Letting C denote the initial configuration (a sequence of n H's and T's), write $\ell(C)$ for the number of steps needed before all coins show T. Show that this number $\ell(C)$ is finite, and determine its average value over all 2^n possible initial configurations C.

C4. On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each n, what are all possible values for $k(\mathfrak{L})$, where $\mathfrak{L}$ is a labyrinth with n walls?

C5. On a certain social network, there are 2019 users, some pairs of which are friends, where friendship is a symmetric relation. Initially, there are 1010 people with 1009 friends each and 1009 people with 1010 friends each. However, the friendships are rather unstable, so events of the following kind may happen repeatedly, one at a time:

> Let A, B, and C be people such that A is friends with both B and C, but B and C are not friends; then B and C become friends, but A is no longer friends with them.

Prove that, regardless of the initial friendships, there exists a sequence of such events after which each user is friends with at most one other user.

C6. Let $n > 1$ be an integer. Suppose we are given $2n$ points in a plane such that no three of them are collinear. The points are to be labelled $A_1, A_2, \ldots, A_{2n}$ in some order. We then consider the $2n$ angles $\angle A_1A_2A_3$, $\angle A_2A_3A_4$, $\ldots$, $\angle A_{2n-2}A_{2n-1}A_{2n}$, $\angle A_{2n-1}A_{2n}A_1$, $\angle A_{2n}A_1A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between $0°$ and $180°$). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

C7. There are 60 empty boxes $B_1, \ldots, B_{60}$ in a row on a table and an unlimited supply of pebbles. Given a positive integer n, Alice and Bob play the following game.

In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes. Each subsequent round consists of two steps:

(a) Bob chooses an integer k with $1 \leqslant k \leqslant 59$ and splits the boxes into the two groups $B_1, \ldots, B_k$ and $B_{k+1}, \ldots, B_{60}$.

(b) Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group.

Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

C8. Alice has a map of Wonderland, a country consisting of $n \geqslant 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

Comment. This problem could be posed with an explicit statement about points being awarded for weaker bounds cn for some $c > 4$, in the style of IMO 2014 Problem 6.

C9. For any two different real numbers x and y, we define $D(x, y)$ to be the unique integer d satisfying $2^d \leqslant |x - y| < 2^{d+1}$. Given a set of reals $\mathcal{F}$, and an element $x \in \mathcal{F}$, we say that the *scales* of x in $\mathcal{F}$ are the values of $D(x, y)$ for $y \in \mathcal{F}$ with $x \neq y$.

Let k be a given positive integer. Suppose that each member x of $\mathcal{F}$ has at most k different scales in $\mathcal{F}$ (note that these scales may depend on x). What is the maximum possible size of $\mathcal{F}$?

Geometry

G1. Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

G2. Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

G3. In triangle ABC, let A_1 and B_1 be two points on sides BC and AC, and let P and Q be two points on segments AA_1 and BB_1, respectively, so that line PQ is parallel to AB. On ray PB_1, beyond B_1, let P_1 be a point so that $\angle PP_1C = \angle BAC$. Similarly, on ray QA_1, beyond A_1, let Q_1 be a point so that $\angle CQ_1Q = \angle CBA$. Show that points P, Q, P_1, and Q_1 are concyclic.

G4. Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

G5. Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if $\text{area}(BCD) + \text{area}(ADE) = \text{area}(ABD) + \text{area}(ABP)$.

G6. Let I be the incentre of acute-angled triangle ABC. Let the incircle meet BC, CA, and AB at D, E, and F, respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P. Prove that $\angle DPA + \angle AQD = \angle QIP$.

G7. The incircle ω of acute-angled scalene triangle ABC has centre I and meets sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at $Q \neq P$. Prove that lines DI and PQ meet on the external bisector of angle BAC.

G8. Let $\mathcal{L}$ be the set of all lines in the plane and let f be a function that assigns to each line $\ell \in \mathcal{L}$ a point $f(\ell)$ on ℓ. Suppose that for any point X, and for any three lines ℓ_1, ℓ_2, ℓ_3 passing through X, the points $f(\ell_1)$, $f(\ell_2)$, $f(\ell_3)$ and X lie on a circle.

Prove that there is a unique point P such that $f(\ell) = P$ for any line ℓ passing through P.

Number Theory

N1. Find all pairs (m, n) of positive integers satisfying the equation

$$(2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}) = m!$$

N2. Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

N3. We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \ldots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

N4. Let $\mathbb{Z}_{>0}$ be the set of positive integers. A positive integer constant C is given. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that, for all positive integers a and b satisfying $a + b > C$,

$$a + f(b) \mid a^2 + b\, f(a).$$

N5. Let a be a positive integer. We say that a positive integer b is *a-good* if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geqslant b$. Suppose b is a positive integer such that b is a-good, but $b + 2$ is not a-good. Prove that $b + 1$ is prime.

N6. Let $H = \left\{ \lfloor i\sqrt{2} \rfloor : i \in \mathbb{Z}_{>0} \right\} = \{1, 2, 4, 5, 7, \ldots\}$, and let n be a positive integer. Prove that there exists a constant C such that, if $A \subset \{1, 2, \ldots, n\}$ satisfies $|A| \geqslant C\sqrt{n}$, then there exist $a, b \in A$ such that $a - b \in H$. (Here $\mathbb{Z}_{>0}$ is the set of positive integers, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

N7. Prove that there is a constant $c > 0$ and infinitely many positive integers n with the following property: there are infinitely many positive integers that cannot be expressed as the sum of fewer than $cn\log(n)$ pairwise coprime n^{th} powers.

N8. Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z.)