Well-Balanced Schemes for Shallow Water Equations

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Multiscale Numerical Methods
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Shallow Water Equations

Shallow water equations (SWE):

$$h_t + (hu)_x = 0$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = -ghB_x$$

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Goal: Derive a numerical scheme for the SWE that preserves steady-states.

System of Balance Laws

General Form:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}),$$

where $\mathbf{U} = \mathbf{U}(x, t)$ is a vector of unknowns, $\mathbf{F}$ is a flux function and $\mathbf{S}$ is a source term.

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For SWE, we set $\mathbf{U} = (h, q)$ with $q = hu$, and

$$\mathbf{F}(h, q) = \left(q, \frac{q^2}{h} + \frac{gh^2}{2}\right)^T, \quad \text{and} \quad \mathbf{S}(h, q) = (0, -ghB_x)^T.$$

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The system is hyperbolic since the eigenvalues of the Jacobian are given by $q/h \pm \sqrt{gh}$.

Semi-Discrete ODE via Finite Volume

Consider a uniform mesh (x_j) with mesh-size Δx and cells $C_j = [x_{j-1/2}, x_{j+1/2}]$.

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We aim to solve

$$\frac{d}{dt}\overline{\mathbf{U}}_j(t) = -\frac{1}{\Delta x}\left[\boldsymbol{\mathcal{F}}_{j+1/2}(t) - \boldsymbol{\mathcal{F}}_{j-1/2}(t)\right] + \ \overline{\mathbf{S}}_j(t).$$

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Note: We require $\mathbf{U}^{\pm}_{j+1/2}$ to compute $\boldsymbol{\mathcal{F}}_{j+1/2}$.

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Solver of choice for semi-discrete ODE: SSPRK(2,2):

$$u^{(1)} = u^n + \Delta t f(u^n), \quad u^{n+1} = \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t f(u^{(1)}).$$

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Method A:

- Reconstruct $\mathbf{U}^{\pm}_{j+1/2} = \left(h^{\pm}_{j+1/2}, q^{\pm}_{j+1/2}\right)$ using linear numerical derivatives of $\left(\overline{\mathbf{U}}_j\right) = \left(\overline{h}_j, \overline{q}_j\right)$.

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- Approximate $\overline{\mathbf{S}}_j$ using the midpoint rule as

$$\overline{\mathbf{S}}_j^{(2)} \approx -g \cdot \overline{h_j} \cdot B_x(x_j).$$

Method A Applied to Still-Water Steady-State

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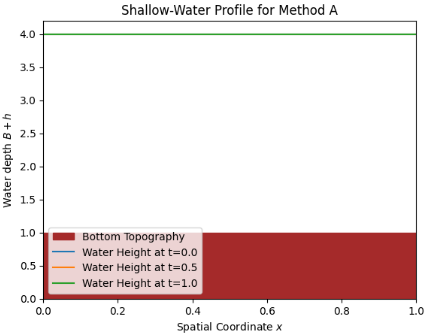

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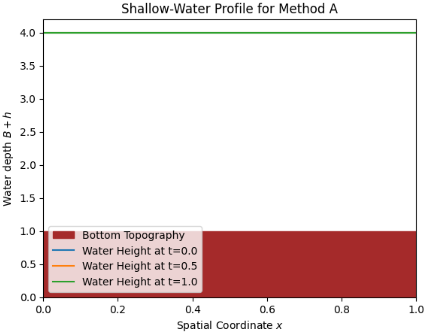

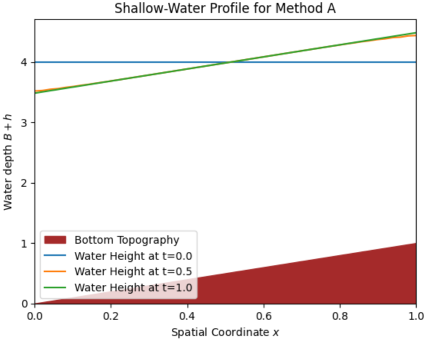

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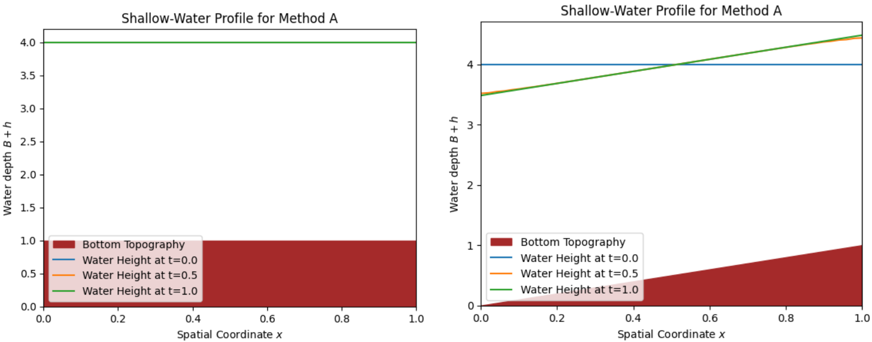

Problem: Cannot preserve still-water equilibrium on tilted bottom.

Preserving Still-Water Steady-State

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Method B:

Preserving Still-Water Steady-State

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- Define $\overline{w}_j = B(x_j) + \overline{h}_j$. Reconstruct $w^{\pm}_{j+1/2}$ and $q^{\pm}_{j+1/2}$ using linear numerical derivatives from $(\overline{w}_j)$ and $\left(\overline{q}_j\right)$. Define $h^{\pm}_{j+1/2} = w^{\pm}_{j+1/2} - B_{j+1/2}$, and set $\mathbf{U}^{\pm}_{j+1/2} = \left(h^{\pm}_{j+1/2}, q^{\pm}_{j+1/2}\right)$.

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- Approximate $\overline{\mathbf{S}}_j$ using the midpoint rule combined with finite difference approximation of B_x as

$$\overline{\mathbf{S}}_j^{(2)} \approx -g \cdot \overline{h}_j \cdot \frac{B_{j+1/2} - B_{j-1/2}}{\Delta x}.$$

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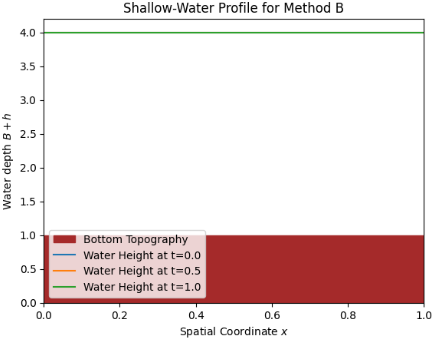

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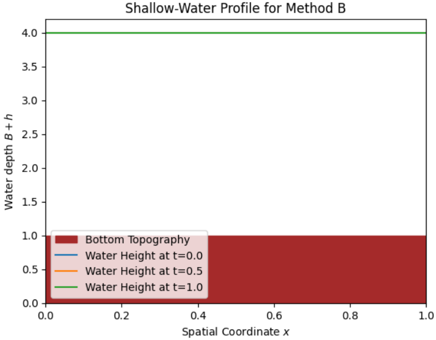

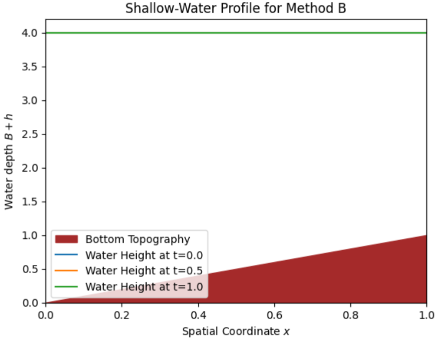

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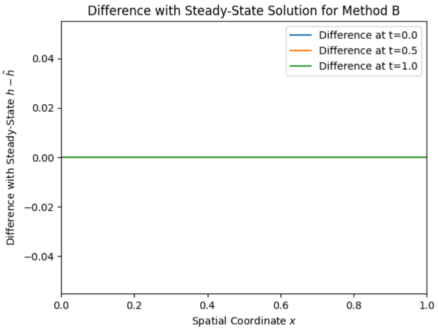

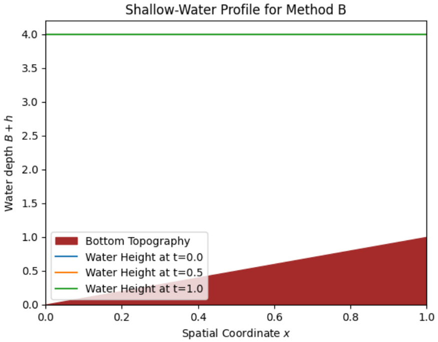

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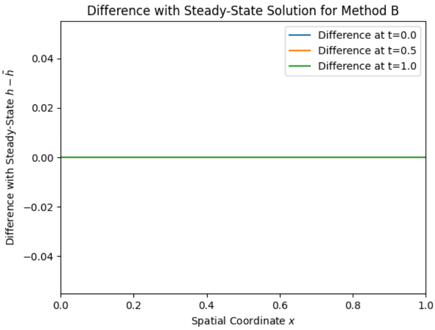

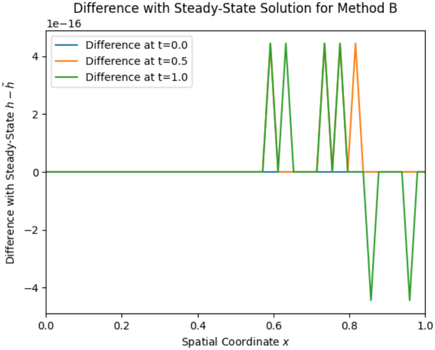

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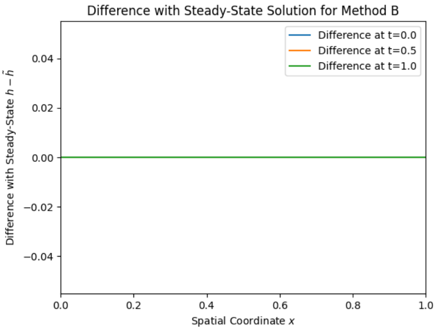

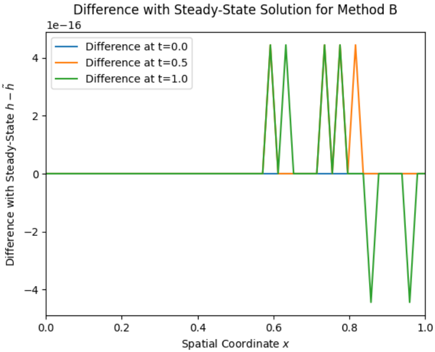

Is capable of preserving the “lake at rest” steady-state.

Method B Applied to Perturbed Still-Water Steady-State

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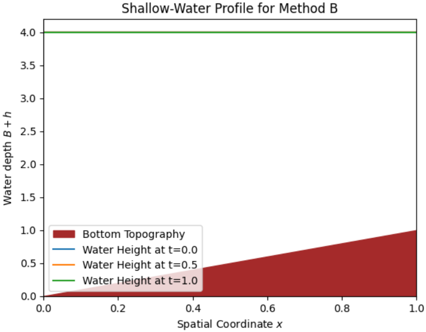

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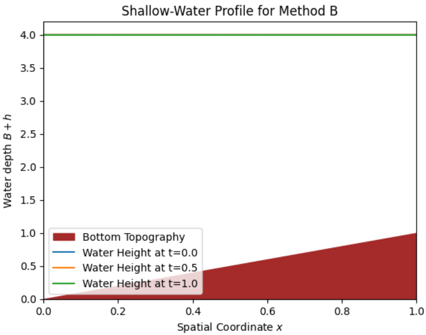

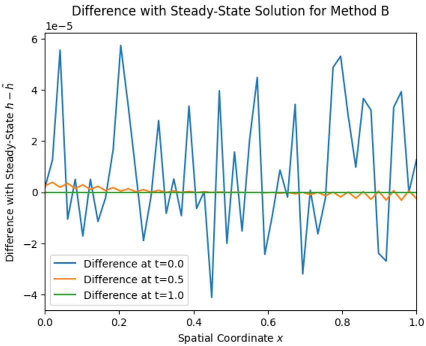

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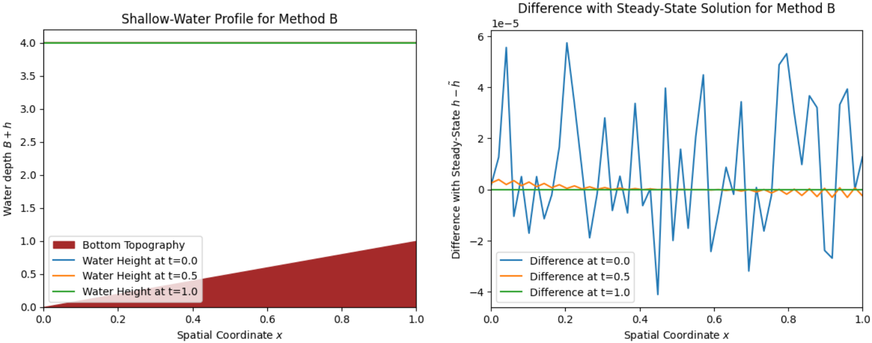

Is capable of recovering the "lake at rest" steady-state.

Method B Applied to Moving-Water Steady-State

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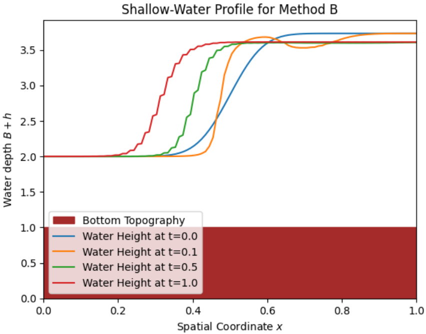

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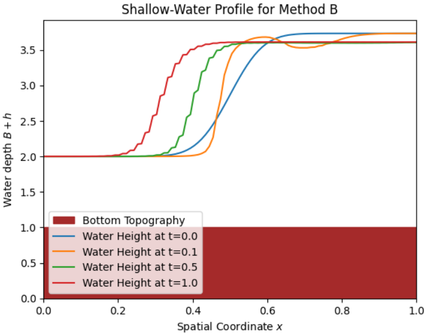

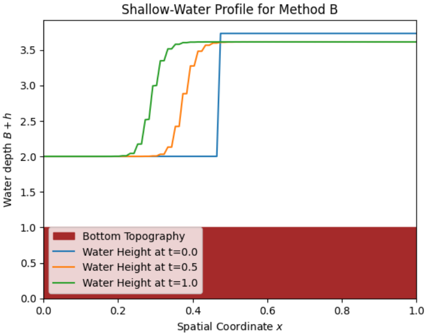

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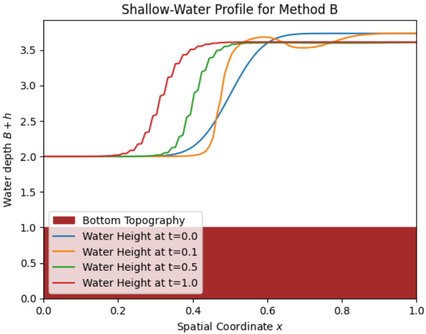

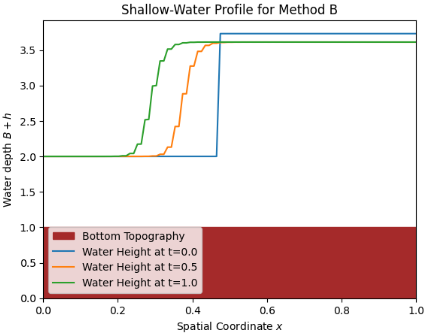

Problem: Is not capable of preserving general steady-states!

Preserving Moving-Water Steady-State

Method C:

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- Define $\overline{E}_j = \frac{\overline{q}_j^2}{2\overline{h}_j^2} + g\left(\overline{h}_j + B(x_j)\right)$. Reconstruct $E^{\pm}_{j+1/2}$ and $q^{\pm}_{j+1/2}$ using linear derivatives from $\left(\overline{E}_j\right)$ and $\left(\overline{q}_j\right)$. Define $h^{\pm}_{j+1/2}$ as the solution to

$$\phi(h) = \left(q^{\pm}_{j+1/2}\right)^2 / \left(2 \cdot h^2\right) + g \cdot (h + B_{j+1/2}) - E^{\pm}_{j+1/2} = 0,$$

and set $\mathbf{U}^{\pm}_{j+1/2} = \left(h^{\pm}_{j+1/2}, q^{\pm}_{j+1/2}\right)$.

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  and set $\mathbf{U}_{j+1/2}^{\pm} = \left(h_{j+1/2}^{\pm}, q_{j+1/2}^{\pm}\right)$.
- Approximate $\overline{\mathbf{S}}_j$ using

$$\overline{\mathbf{S}}_j^{(2)} \approx -g \cdot \frac{h_{j+1/2}^{-} + h_{j-1/2}^{+}}{2} \cdot \frac{B_{j+1/2} - B_{j-1/2}}{\Delta x} + \frac{\left(u_{j+1/2}^{-} - u_{j-1/2}^{+}\right)^2}{4\Delta x} \cdot (h_{j+1/2}^{-} - h_{j-1/2}^{+}).$$

Method C Applied to Moving-Water Steady-State 1/2

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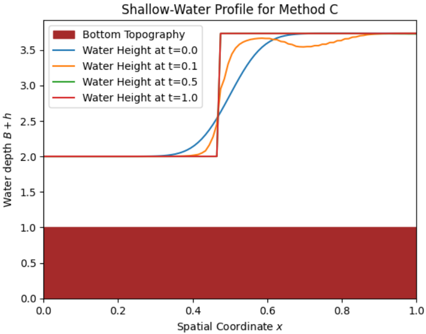
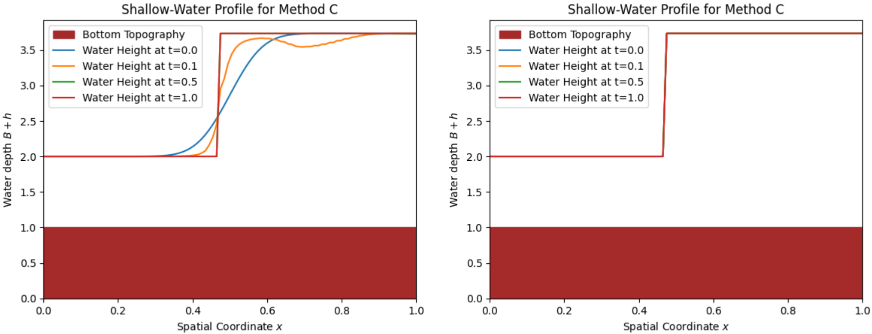
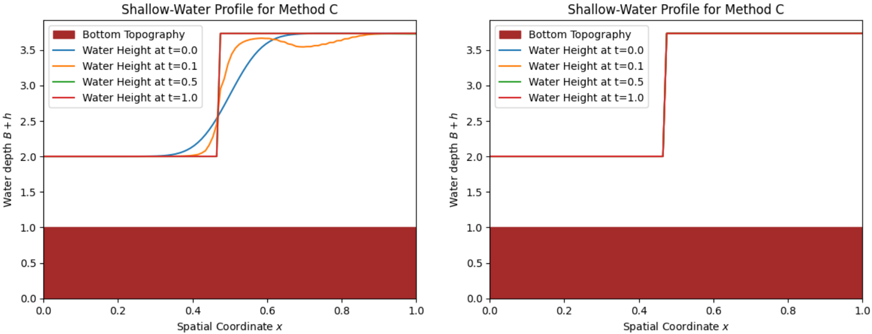


Method C Applied to Moving-Water Steady-State 1/2



Method C Applied to Moving-Water Steady-State 1/2



It is capable of preserving the moving-water steady-states!

Method C Applied to Moving-Water Steady-State 2/2

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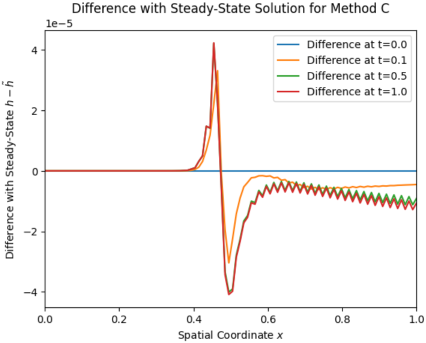
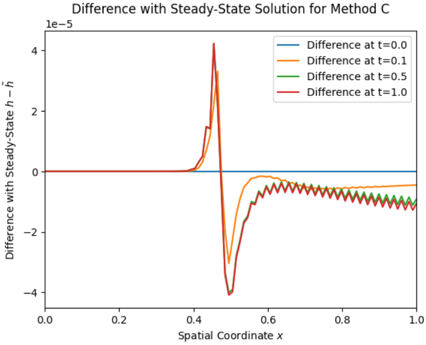
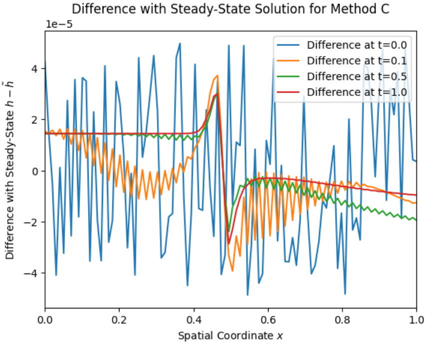
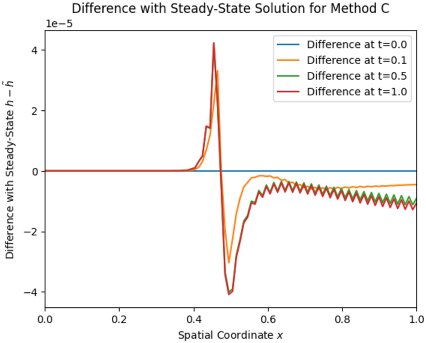


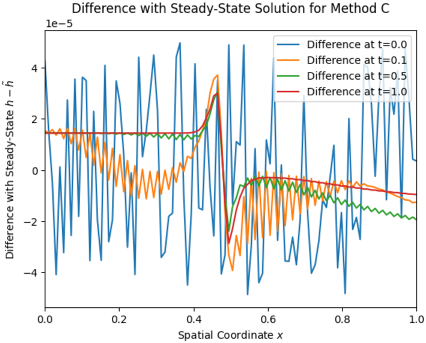
Method C Applied to Moving-Water Steady-State 2/2





Method C Applied to Moving-Water Steady-State 2/2





It is capable of recovering the moving-water steady-states!

Method C Applied to Non-Equilibrium Situations

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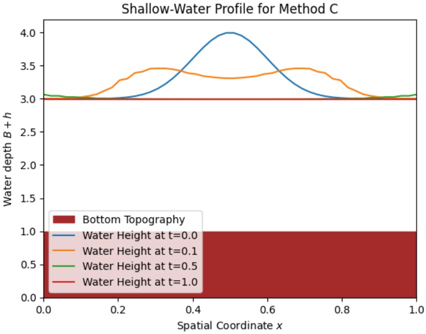

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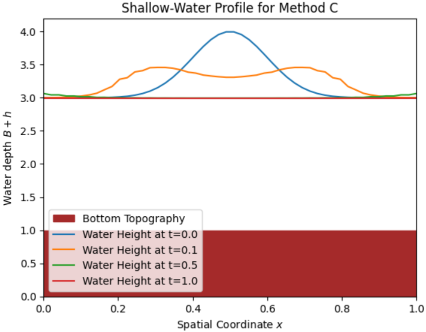

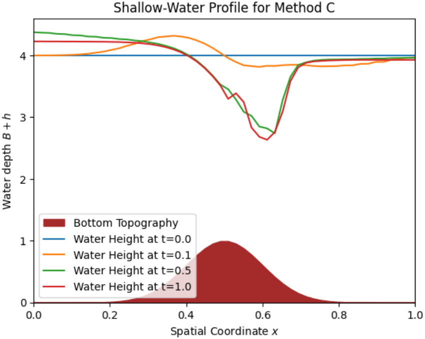

Thank you!