Time Varying Correlation Matrices with an Application to Financial Crises

Daniel Andrew Coulson[1] and David S. Matteson[2]

[1,2]Department of Statistics and Data Science, Cornell University, Ithaca, NY

Abstract

In this paper we propose a novel approach to quantifying the risk from a co-movement of stocks in a portfolio through time, derived from time varying model-based correlation matrices. The correlation matrices are estimated in a Bayesian fashion utilizing a dynamic shrinkage prior process for the state variables to be estimated and a multivariate factor stochastic volatility process for the observation error covariance matrices. To summarize the information in correlation matrices we create an intuitive and simple scalar score. Through a simulation study we demonstrate our estimation approach achieves superior performance in terms of several metrics and has an ability to rapidly adapt to changing market conditions compared to competing methods. Through real world examples we demonstrate the new insights provided by our proposed framework in identifying known periods of financial instability and the additional information it provides beyond existing measures such as the VIX index. We subsequently compare the static minimum variance portfolio to that of a dynamically changing minimum variance portfolio in times of financial crisis.

Key words: Bayesian methods, Time Series, Financial risk management, Financial crisis, Correlation matrices, and Portfolio allocation

1 Introduction

Crises such as the U.S. subprime mortgage crisis or the Covid-19 pandemic can have a large detrimental impact on an investor's portfolio especially through increased market volatility, as discussed in Karunanayake et al. [2009],Chen et al. [2022], and Foo and Witkowska [2024]. Thus, understanding the impact of these market instabilities is essential for investors to have a fuller insight into the risks of their portfolio.

During a financial crisis correlations both within, and across markets increase. Lin et al. [1994], Solnik et al. [1996], and Junior and Franca [2012] find that the correlations between international stock markets increase during periods of high volatility. Silvennoinen and Teräsvirta [2005] observe that correlations amongst S&P 500 stocks are greater during periods of more volatility.

Therefore, understanding the correlations within a portfolio is important in order to mitigate the impact of time periods of greater market volatility. The benefits of diversification have been exalted in the literature, such as Wagner and Lau [1971] and Lumby and Jones [1998]. In this paper we propose a methodology for quantifying the time varying correlation in a stock portfolio and demonstrate that diversification does not help in times of financial crisis.

The standard method for computing time varying correlation matrices of a portfolio rely on rolling correlation estimators. Rolling correlation estimators suffer from several problems. Lag effects induced by using a rolling window, which contains several data points unrelated to the current data point, can lead to previous data points having an undue impact on estimated correlation matrices. These lag effects further induce a pronounced inability of the estimator to respond to changes in market conditions in a sufficiently responsive manner. Furthermore, rolling estimators are inherently high rank estimators due to over parameterization, which naturally leads to estimators with a larger variance than is usually desired. Rolling estimators also have an inability to estimate instantaneous correlation matrices which is undesirable, especially when alternative methods exist that can do this.

Bayesian methods can provide us with the desired uncertainty quantification, such as the construction of credible intervals. Multivariate factor stochastic volatility models such as Chib et al. [2006] and Hosszejni and Kastner [2021] utilize a latent factor model approach. However, they do not utilize time dependent shrinkage which leads to very loose credible intervals.

To account for these difficulties, we propose a novel Bayesian approach to estimating correlation matrices from multivariate factor models. We utilize dynamic shrinkage processes (DSP) (Kowal et al. [2019]) for Bayesian estimation of the idiosyncratic parameter variances and a multivariate factor stochastic volatility model for the observation error covariances. That is, we assume the state variables across assets are independent, with dependence across series through the unobserved observation error covariances. Through the DSP priors this allows the model to locally adapt to changing market conditions and gives tighter credible intervals.

A problem with estimating time series of correlation matrices is how do we summarize this information? The most popular solution is to plot the estimated pair wise correlations through time; however this can become cumbersome, and ultimately uninformative in even moderate dimension. Therefore, we propose a novel scalar score which ranges between -1 to +1, to summarize individual correlation matrices. We then derive posterior samples of this score and track this through time. This score is based upon the concept of scalar projection and possesses several desirable properties, most importantly simplicity and interpretability. The score along with the proposed estimation framework provides both industry practitioners and researchers with a novel tool for quantifying risk in a portfolio linked to a co-movement of its constituent stocks. This portfolio specific measure can provide portfolio managers with a clearer understanding of the structure of their portfolio and help them decide on future portfolio allocation.

This paper will proceed as follows: in section two we will introduce the methodology including specification of priors and observation equations. Section three will discuss the estimation of correlation matrices and state how we summarize these correlation matrices through our novel score and some properties of this score. Section four will discuss the computational details involved with estimation such as the details of our Gibbs sampling algorithm. Section five will discuss results of our proposed method, both in a simulation study and real-world examples where we demonstrate that the impacts of financial crises on an investor's portfolio cannot be diversified away. Section six will conclude and suggest future research directions.

2 Methodology

2.1 Notation and definitions

In this section we present our modelling methodology based on multivariate time varying parameter linear factor models with an application focus on asset return. Throughout the paper we will denote vectors with an underlined letter, unless we directly specify the components of the vector, and matrices with bold faced letters.

Definition 1. *The net return of an asset adjusted for dividends at time t is given by $R_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}}$, where P_t is the price of the asset at time t and D_t is the dividend paid before time t.*

We will also be utilizing the concept of risk-free rate. This refers to the return someone can earn on an asset, where the variance of this return is zero, for example some fixed income securities such as U.S. treasury bills, although such assets are not truly risk free.

Definition 2. *The excess market return is the return you can make by investing in the market portfolio (theoretical collection of all investable assets) minus the risk free rate. Similarly, the excess return of an asset is the return of the asset minus the risk-free rate.*

Definition 3. *A multivariate linear factor model for n assets is given by:*

$$\underline{r}_t = \underline{\alpha}_t + \underline{\beta}_{1,t} F_{1,t} + ... + \underline{\beta}_{z,t} F_{z,t} + \underline{\epsilon}_t. \tag{1}$$

Where $t \in \mathbb{N}$, $\underline{r}_t = (r_{1,t}, ..., r_{n,t})^T$ *is the vector of n excess asset returns at time t.* $\underline{\alpha}_t = (\alpha_{1,t}, ..., \alpha_{n,t})^T$ *is the vector of intercept terms.* $F_{1,t}, ..., F_{z,t}$ *are the factors in our model which in this paper are observed, and* $\underline{\beta}_{1,t}, ..., \underline{\beta}_{z,t}$ *are the factor loading vectors.* $\underline{\epsilon}_t$ *is the vector of idiosyncratic observation errors. In section 2.2 we place further constraints on this model such as prior distributional assumptions.*

More information on financial time series,and fundamental quantitative finance methods can be found in several texts including Ruppert and Matteson [2011] and Tsay [2005].

2.2 Model

We are proposing a Bayesian time series model utilizing dynamic shrinkage prior processes put forward in Kowal et al. [2019]. DSP priors build on the Horseshoe prior of Carvalho et al. [2009] which is a global-local shrinkage prior using normal scale mixtures. Global-local shrinkage priors are continuous priors which impose a global level of shrinkage on all the state variables in a model, but also allow for parameter specific levels of shrinkage and are an alternative to exact sparsity inducing priors such as the spike and slab prior.

DSP priors extend this idea to the four parameter Z-distribution which provide a natural extension as they can be written as Normal mean-scale mixtures (Barndorff-Nielsen et al. [1982]) and therefore can provide additional flexibility in the shape of the shrinkage, which includes horseshoe shaped shrinkage as a special case (the shrinkage type utilized in this paper). However, when applied to time series analysis such a prior suffers from a lack of temporal adaptability, that is the shrinkage is constant with respect to time.

Kowal et al. [2019] instead proposes a prior processes for the amount of shrinkage which has the advantage of having the shrinkage be locally adaptive with respect to time. This is very helpful from the perspective of time series analysis; for example suppose we are moving from one time point to the next time point in a random walk fashion, if there is little change in the signal we would desire the innovation of the process to be shrunk strongly towards zero, alternatively if there is suddenly a large innovation then we would prefer very little shrinkage. By modelling the shrinkage through a prior stochastic process, DSP priors utilize the previous observations to determine a good amount of shrinkage but can also adapt the shrinkage to sudden large innovations.

The second component of our model relies on multivariate factor stochastic volatility (MFSV) processes. As highlighted in section one, estimation of covariance matrices can suffer from the curse of dimensionality, since we have several free parameters but only one data point for a single moment in discrete time. Therefore, to make estimation feasible we need to make some low rank inducing assumptions. To do this the model assumes that the time series of covariances is driven by a small set of common latent factors. This results in a computationally tractable model which we utilize for the time varying observation errors covariances between the assets in a portfolio.

Throughout the rest of this article we will refer to the model which combines DSP with MFSV as DSP-MFSV with observation equation and prior distribution specification discussed in sections 2.2.1 and 2.2.2 respectively. We discuss the computation of drawing samples from the posterior distribution of our model in section four.

2.2.1 Observation equation

For ease of explanation we will focus on rank one factor models, but our methodology works for higher rank factor models too.

Consider the capital asset pricing model of Sharpe [1964], Lintner [1965a], Lintner [1965b], and Mossin [1966].

$$r_{a,t} = \alpha_{a,t} + \beta_{a,t} r_{M,t} + \epsilon_{a,t} \tag{2}$$

$r_{a,t}$ is the excess return of asset $a \in 1, ..., N$ in our portfolio at time $t \in \mathbb{N}$, $r_{M,t}$ is the excess market return at time t, $\epsilon_{a,t}$ is the idiosyncratic observation error, with $\alpha_{a,t}$ and $\beta_{a,t}$ being unobserved state variables. We assume the state variables are independent across assets, but allow the observation errors to exhibit dependence. That is, we assume that the observation errors follow a MFSV process as discussed in Hosszejni and Kastner [2021].In this model we assume the process of time-varying covariance matrices is driven by a small number of latent factors, which is appropriate for modelling the observation errors due to their unobserved nature and the fact that there are attributes in the wider economy which cannot be accounted for in our model but are leading to co-movement of stock returns. Particularly,

$$\underline{\epsilon}_t^T = (\epsilon_{1,t}, ..., \epsilon_{N,t})^T | \mathbf{\Lambda}, \underline{f}_t, \bar{\mathbf{\Sigma}}_t \sim N_N(\mathbf{\Lambda} \underline{f}_t, \bar{\mathbf{\Sigma}}_t) \text{with} \underline{f}_t | \tilde{\mathbf{\Sigma}}_t \sim N_m(\underline{0}_m, \tilde{\mathbf{\Sigma}}_t) \tag{3}$$

$\underline{f}_t$ is a vector of length m consisting of the values of m latent factors, and $\mathbf{\Lambda}$ is a static matrix of factor loadings. $\bar{\mathbf{\Sigma}}_t$ and $\tilde{\mathbf{\Sigma}}_t$ are diagonal matrices whose entries we discuss in section 2.2.2. Then, we may write that at a given time point the covariance matrix of $\underline{\epsilon}_t$ is given by $\mathbf{\Lambda \tilde{\Sigma}_t \Lambda^T} + \mathbf{\bar{\Sigma}_t}$.

2.2.2 Priors

In this section we discuss our prior distribution specification for the observation equation in section 2.2.1.

We assume the state variables between the assets in our model evolve independently of each other, and for each state variable in a given assets observation equation we further assume that they too are independent and evolve according to their own specific dynamic shrinkage prior process. These assumptions seem valid as the alpha and beta of one stock should not influence the alpha and beta of another stock, and for any asset its alpha and beta are distinct qualities and therefore independent. Then for a given asset a in our portfolio we have:

$$\beta_{a,t+1} = \beta_{a,t} + \omega_{\beta_a,t} \tag{4}$$

$$\omega_{\beta_a,t} | \tau_{a,0} \tau_{\beta_a}, \{\lambda_{\beta_a,s}\} \sim N(0, \tau^2_{a,0} \tau^2_{\beta_a} \lambda^2_{\beta_a,t}) \tag{5}$$

$$h_{\beta_a,t} = log(\tau^2_{a,0} \tau^2_{\beta_a} \lambda^2_{\beta_a,t}) \tag{6}$$

$$h_{\beta_a,t} = \mu_{\beta_a} + \phi_{\beta_a}(h_{\beta_a,t-1} - \mu_{\beta_a}) + \eta_{\beta_a,t} \tag{7}$$

$$\tau_{a,0} \sim C^+(0, \frac{1}{\sqrt{T}}), \tau_{\beta_a} \sim C^+(0,1), \eta_{\beta_a,t} \sim Z(\frac{1}{2}, \frac{1}{2}, 0, 1) \tag{8}$$

$$\frac{\phi_{\beta_a} + 1}{2} \sim Beta(10, 2). \tag{9}$$

The prior specification for the other state variables in a given observation equation are the same as above. By stating our state variables evolve according to a normal random walk in equation four of our prior specification the problem becomes tractable and still allows us to capture a rich collection of dynamics while maintaining algebraic simplicity.

We then allow the innovation of the process to follow a global local shrinkage prior, where $\tau^2_{a,0}$ determines the global level of shrinkage across all the state variables, and across all of time. Similarly, $\tau^2_{\beta_a}$ determines the amount of shrinkage over all time of the parameter β of asset a in our portfolio. Finally $\lambda^2_{\beta_a,t}$, the local shrinkage parameter, determines the amount of shrinkage of the β parameter of asset a, at a particular point in time, that is it determines the amount of temporally local shrinkage.

As is common practice, the model specifies the prior distribution in terms of the log conditional variance of the innovation from equation five. We assume the log-variance process evolves according to an autoregressive one model (equation seven), but with Z-distributed errors (equation eight), rather than the typical normally distributed errors, due to their ability to induce shrinkage. Particularly, we utilize horeshoe like shrinkage by specifying $Z(\frac{1}{2}, \frac{1}{2}, 0, 1)$ due to its symmetric level of shrinkage, which mostly applies either a lot of shrinkage or a little shrinkage. See Kowal et al. [2019] for a discussion on some of the other shrinkage types available.

$$\bar{\mathbf{\Sigma}}_t = diag(exp(\bar{h}_{t,1}), ..., exp(\bar{h}_{t,m})), \tilde{\mathbf{\Sigma}}_t = diag(exp(\tilde{h}_{t,1}), ..., exp(\tilde{h}_{t,r})) \tag{10}$$

$$\bar{h}_{t,i} \sim N(\bar{\mu}_i + \bar{\psi}_i(\bar{h}_{t-1,i} - \bar{\mu}_i), \bar{\sigma}_i^2), i = 1, ..., m \tag{11}$$

$$\tilde{h}_{t,j} \sim N(\tilde{\mu}_j + \tilde{\psi}_j(\tilde{h}_{t-1,j} - \tilde{\mu}_j), \tilde{\sigma}_j^2), j = 1, ..., r. \tag{12}$$

$$\mathbf{\Lambda}_{i,j}|\tau^2_{i,j} \sim N(0, \tau^2_{i,j}), \tau^2_{i,j}|\lambda^2_i \sim Ga(0.1, \frac{0.1\lambda^2_i}{2}), \lambda^2_i \sim Ga(1, 1) \tag{13}$$

$$\log(\bar{\sigma}_i) \sim Ga(\frac{1}{2}, \frac{5}{2}), \log(\tilde{\sigma}_i) \sim Ga(\frac{1}{2}, \frac{5}{2}) \tag{14}$$

$$\bar{\mu}_i \sim N(0, 10), \tilde{\mu} \sim N(0, 10), i = 1, ..., m, j = 1, ..., r. \tag{15}$$

$$\frac{\bar{\psi}_i + 1}{2} \sim Beta(10, 3), \frac{\tilde{\psi}_i + 1}{2} \sim Beta(10, 3), i = 1, .., m, j = 1, ..., r. \tag{16}$$

As can been seen in equations 10-12 the entries of the conditional covariance matrices from section 2.2.1 follow independent stochastic volatility processes of order one which allows the model to capture time-varying covariances of the observation errors of our observation equations. The prior utilizes normal gamma shrinkage priors (Griffin and

Brown [2010]) for the entries of the matrix of factor loadings $\mathbf{\Lambda}$. Finally, we place a beta prior on a function of the persistence parameters to ensure they remain between 0 and 1 for stationarity.

3 Estimation and summary of correlation matrices

We utilize a Gibbs sampling algorithm (section four) to perform posterior inference. For each MCMC sample we compute an estimated model based covariance matrix, where there is one covariance matrix for each observed time point. We then standardize these covariance matrices to obtain correlation matrices, and finally we use our proposed scalar summary to summarize the correlation matrices in the series. After this procedure we will then have posterior draws consisting of times series of the estimated scalar summaries of correlation matrices.

3.1 Construction of Covariance matrices

To obtain good estimators of covariance matrices we propose to use those derived from the estimated multivariate linear factor model which allows us to utilize the dynamic shrinkage prior processes and take advantage of the time dependent shrinkage. By parameterizing our covariance estimators using low rank linear factor models (section 2.2.1) this induces approximate low rank structure. This low rankness combined with dynamic shrinkage provide low variance covariance estimators. Furthermore, by reserving the use of latent factor models for the unobserved observation error, our covariance estimators are more explainable and easier to interpret.

Proceeding from the CAPM where $\underline{r}_t = \underline{\alpha}_t + X_t\underline{\beta}_t + \epsilon_t$ we obtain the formula for the model-based covariance matrix of the returns as

$$var(\underline{r}_t) = var(\underline{\alpha}_t) + var(X_t)E[\underline{\beta}_t\underline{\beta}_t^T] + var(\epsilon_t). \tag{17}$$

3.1.1 Derivation

$\underline{r}_t = \underline{\alpha}_t + X_t \underline{\beta}_t + \underline{\epsilon}_t$

$\implies var(\underline{r}_t) = var(\underline{\alpha}_t) + var(X_t \underline{\beta}_t) + var(\underline{\epsilon}_t)$, by assuming mutual independence

$var(X_t \underline{\beta}_t) = E[X_t^2]E[\underline{\beta}_t \underline{\beta}_t^T] - E[X_t]^2 E[\underline{\beta}_t]E[\underline{\beta}_t]^T$, by mutual independence

$\implies var(X_t \underline{\beta}_t) = (var(X_t) + E[X_t]^2)E[\underline{\beta}_t \underline{\beta}_t^T] - E[X_t]^2 E[\underline{\beta}_t]E[\underline{\beta}_t]^T$

$\implies var(X_t \underline{\beta}_t) = var(X_t)E[\underline{\beta}_t \underline{\beta}_t^T]$, since X_t follows a mean zero stochastic process.

$\implies var(\underline{r}_t) = var(\underline{\alpha}_t) + var(X_t)E[\underline{\beta}_t \underline{\beta}_t^T] + var(\underline{\epsilon}_t)$.

3.2 Summary of Correlation matrices

In a given MCMC sample, once we have an estimated covariance matrix we can derive the associated correlation matrix by standardization. However, for correlation matrices of even moderate dimension, say 5 or 10 assets, how do we gain valuable insights from our correlation matrices to aid portfolio allocation? The common practice is to plot the time series of the estimated pairwise correlations. However, having plots of multiple estimated pairwise correlation series can become cumbersome and ultimately uninformative, particularly in understanding the overall amount of correlation in a portfolio. Therefore, we propose a simple scalar summary of correlation matrices based on scalar projection.

The scalar projection of the vector $\underline{a}$ onto the vector $\underline{b}$ says how much of the vector $\underline{a}$ is in the direction of vector $\underline{b}$ and is given by the dot product of the two vectors divided by the vector norm of $\underline{b}$. Basic algebra shows that scalar projection takes into account both the length of the vector $\underline{a}$ but also the cosine of the angle between the vectors $\underline{a}$ and $\underline{b}$.

In our application we are interested in how close a correlation matrix is to being perfectly correlated in accordance with the literature which demonstrates that during times of financial crisis market correlations converge towards one. A correlation

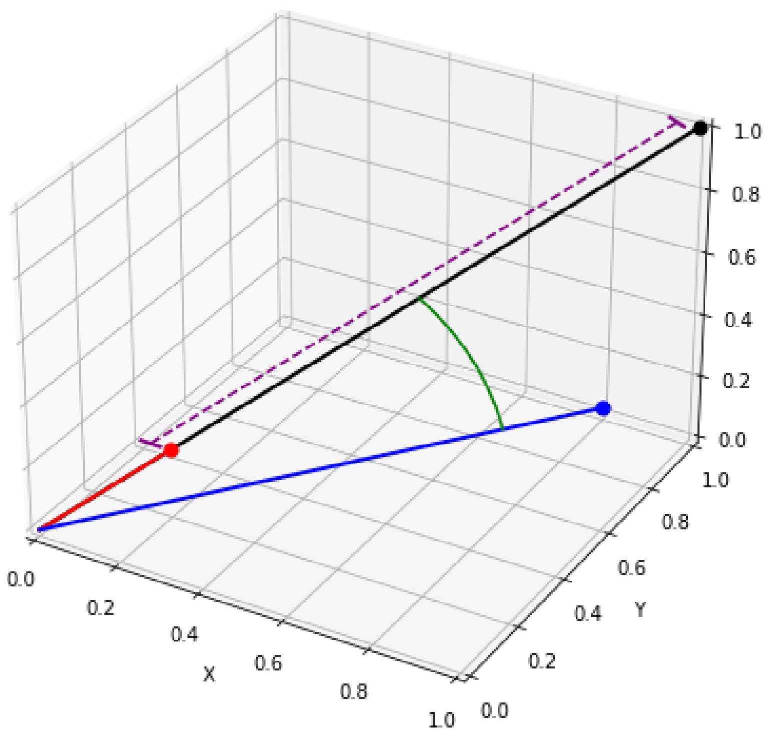

Figure 1: Plot of three vectors. The vector $(1, 1, 1)^T$ is in black, the vector $(0.2, 0.2, 0.2)^T$ is in red, and the vector $(0.9, 0.8, 0.2)^T$ is in blue. The difference in the length between $(0.2, 0.2, 0.2)^T$ and $(1, 1, 1)^T$ is represented by the purple dashed line, and the angle between the vectors $(0.9, 0.8, 0.2)^T$ and $(1, 1, 1)^T$ is given by the green arc.

matrix is perfectly correlated when all the pairwise correlations are one and therefore all the columns are one vectors. Therefore, to measure how close a matrix is to the all one matrix we propose to see how close each column vector of the matrix is to the all one vector. In figure one we can see that the $(0.2, 0.2, 0.2)^T$ vector is perfectly aligned with the $(1, 1, 1)^T$ vector but it is noticeably shorter. Consequently our proposed summary should take into account the length of the column vector. We also have the $(0.9, 0.8, 0.2)^T$ vector. This vector is longer than the $(0.2, 0.2, 0.2)^T$ vector but it has a non-zero angle with the all one vector. Thus to see how close a given column of a correlation matrix is to the all one vector we also need to consider the angle between the column vector and $(1, 1, 1)^T$. A quantity that takes both of these aspects into account is the scalar projection of the respective column of the correlation matrix onto the all one vector. Based upon this we propose the following score to summarize correlation matrices.

Definition 4. *For an $N \times N$ correlation matrix $\boldsymbol{A}$ with column vectors $\underline{x}_1, ..., \underline{x}_N$ our proposed scalar score is given by*

$$score(\mathbf{A}) = \frac{\sum_{j=1}^{N} \underline{x}_j^T \underline{1}_N - N}{N(N-1)}. \tag{18}$$

This score is the sum of the scalar projections of the columns of the correlation matrix onto the all one vector scaled to the range $[-1, 1]$ to allow the measure to not depend on the dimension of the matrix. The score also has some simple but desirable properties.

Property 1. *If all the pairwise correlations are equal to c, the score for the correlation matrix is c.*

$score = \frac{1}{N(N-1)} \sum_{j=1}^{N} [c(N-1)+1] - N = \frac{1}{N(N-1)}(cN(N-1) + N - N)$

$= \frac{1}{N(N-1)} cN(N-1) = c.$

Property 2. *The score is invariant to the dimension of the matrix. For example if all*

the pairwise correlations in an $N \times N$ *matrix are equal to c, then from property 1, the score is c. That is* $\frac{1}{N(N-1)}(\sum_{j=1}^{N}[c(N-1)+1]-N)=c$*. Then if we add an additional row and column where all the pairwise correlations are c, then*
$score = \frac{1}{N(N+1)}(\sum_{j=1}^{N+1}[cN+1]-(N+1))=c$.

Beyond these properties, this score summarizes different pairwise correlations in an intuitive manner. For example if there is a 3×3 correlation matrix with pairwise correlations equal to 0.9,0.9,0.7 then the score is 0.833 (3.d.p.); for the 4×4 correlation matrix with pairwise correlations 0.9,0.2,-0.1,0.87,0.5,0.52 the score is 0.481 (3.d.p.). Hence, once we have the estimated correlation matrix time series for a given MCMC sample we can then summarize this matrix time series as a scalar time series using the above score function.

4 MCMC sampler

All our computation is through the R programming language (R Core Team [2023]). Much of our computation utilizes the R code of Kowal et al. [2019] and Hosszejni and Kastner [2021], and we highlight the key methods discussed in these papers.

The first component of our MCMC algorithm is to draw samples of the posterior variance of the factors in our factor model, such as the excess market return in the CAPM, which is independent of the other components of our MCMC sampling algorithm. For example, we assume the excess market return follows a stochastic volatility process (Taylor [1982]) of order one. To obtain posterior samples we utilize the R package stochvol (Hosszejni and Kastner [2021]). When performing sampling we utilize the same number of MCMC samples, size of the burn in period and level of thinning as the other components of our sampling algorithm in order to maintain cohesion. The sampling algorithm of Hosszejni and Kastner [2021] utilizes the ancilitary interweaving strategy proposed in Yu and Meng [2011]. The application of this sampling technique to stochastic volatility models was then discussed in Kastner and Frühwirth-Schnatter [2014]. To improve computation time Hosszejni and Kastner [2021] utilize C++ and then interface this with R.

A technique utilized in both sampling from stochastic volatility models and dynamic shrinkage models is approximating a χ_1^2 distribution with the mean variance mixture of normal distributions proposed in Omori et al. [2007], which can allow for a Gibbs sampling algorithm rather than a more computationally expensive Metropolis-Hastings algorithm.

The next step of our sampling algorithm is to sample the state variables in the linear factor models for each asset in the portfolio of interest. For example, in the time varying parameter CAPM for a given asset in a portfolio, this would be α and β. We sample the time series of the state variables independently for each asset, with the only dependence across assets coming from the observation error covariances which is discussed in the next paragraph. For sampling the state variables, we utilize the sampler of Kowal et al. [2019]. However, we make some important changes. Firstly, since we have several assets which have a joint model (through the observation error) we sample several times series of tuples of variables, where each time series of tuples would correspond to a particular asset. Secondly, since we assume the assets are dependent through the observation error, we replace the observation error variance for a single asset which was assumed to follow a stochastic volatility process of order one in the original DSP model, with the sampled observation error variance for the respective asset from the multivariate stochastic volatility process (discussed next). In addition to the χ_1^2 approximation, Kowal et al. [2019] also utilize parameter expansion (Liu and Wu [1999]) which is known in the literature for improving the efficiency in MCMC computation, particularly for Gibbs sampling algorithms. Specifically, Kowal et al. [2019] use a Pólya-gamma parameter expansion for sampling from the four parameter

Z distribution which improves the computational efficiency due to the computational ease of sampling from Pólya-gamma distributions. This is based upon combing the work of Polson et al. [2013] with Barndorff-Nielsen et al. [1982] using the fact that a four parameter Z distribution can be written as a Normal mean scale mixture.

For drawing samples of the observation error covariances from the MFSV model we utilize the R package factorstochvol (Hosszejni and Kastner [2021]). For computational tractability the authors assume that the covariances are driven by a small number of unobserved latent factors. The sampling of the idiosyncratic variances utilize the same sampling procedure as univariate stochastic volatility processes. The authors also utilize the ancillarity interweaving strategy of Yu and Meng [2011] and offer alternative interweaving strategies. For our sampling scheme we use deep interweaving across the largest absolute entries in each column of the observation error covariance matrices.

To compute our covariance matrices (equation 17) for a given MCMC sample we need to estimate the variance of the intercept term ($\underline{\alpha}$), and the expected outer product of the parameters associated with the excess market return ($\underline{\beta}$). For this calculation we utilize Welford's online algorithm (Welford [1962]) for computing mean and variance estimates. From this we can then compute the covariance matrices, and then standardize to obtain correlation matrices. We can then apply the score function to the derived correlation matrix series to obtain a univariate time series consisting of the summary of the respective correlation matrix at each time point.

5 Results

To assess our proposed methodology, we performed a simulation study to assess its performance in section 5.1. We also apply our proposed methodology to two real world examples in section 5.2 of financial crises to observe the impact, if any, that portfolio diversification has on mitigating the impact of such crises. We also construct

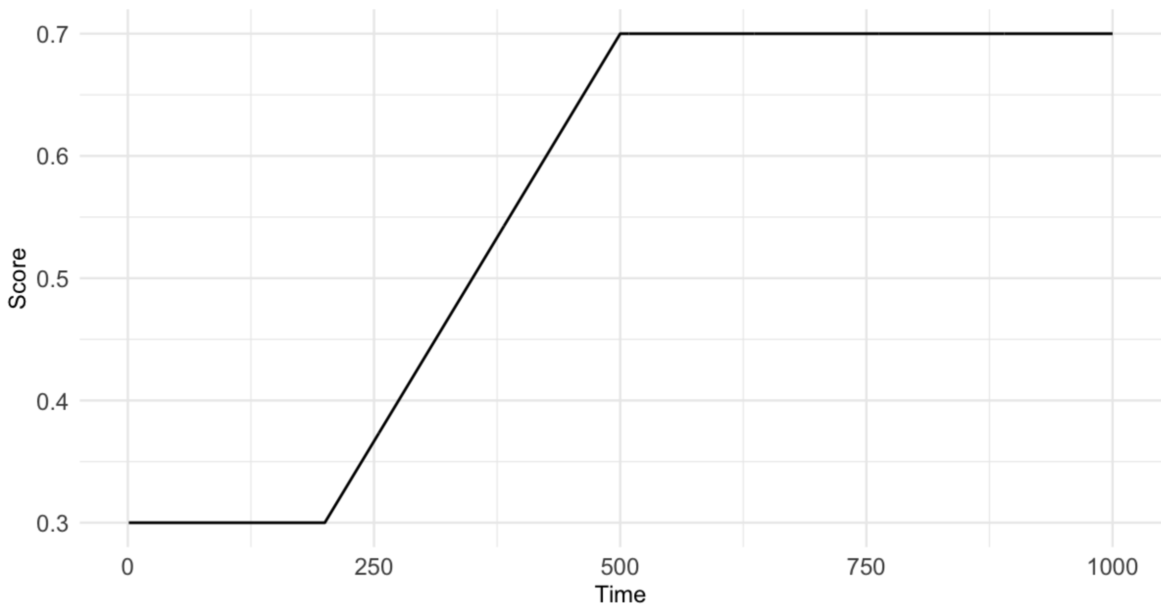

Figure 2: Plot of the time series of the simulated score used in our simulation study.

the minimum variance portfolios and assess how the dynamically estimated portfolio compares to the statically estimated portfolios.

5.1 Simulation study

In this section we discuss our simulation study including the details of how we formed the simulations and the results from our simulation study based on 100 simulations.

For a given simulation we fix the length of all time series to 1000 time points and the number of simulated assets to 30. We also fix all the pairwise correlations to be equal. We then construct the time series of the scores, where we use the same time series of scores in each simulation as displayed in figure two. Since all our pairwise correlations are equal they are equal to the constructed scalar score at the respective time point. We then construct the associated model-based covariance matrices by fixing the variances of the 30 excess asset returns to two. We build the rest of our simulation from the time varying parameter CAPM.

We simulate the time series of the α's and β's for each asset from independent multivariate normal distributions with mean vector $(0, 1)^T$ and diagonal covariance matrix with entries equal to 0.1. The time series of the excess market returns are simulated from a standard normal white noise process. Subsequently we construct the observation error covariance matrices such that the overall model-based covariance matrices are equal to those defined in the earlier step. Following this we simulate the observation errors from a multivariate normal distribution with mean vector equal to the zero vector, and covariance matrix given by the computed covariance matrix at the respective time point from the previous step.

Now we have simulated α, β, and observation error time series for each of the 30 assets. In addition, we also have the simulated excess market returns. We then combine these according to the CAPM to obtain simulated excess asset returns for 30 assets. Then we fit an exponentially weighted rolling correlation estimate, the proposed DSP-MFSV CAPM, and an MFSV model.

For each simulation we computed the root mean squared error (RMSE) for each of the fitted models and for the Bayesian models the empirical coverage and mean empirical credible interval width. The root mean squared error is given by:

$$RMSE = \sqrt{\frac{\sum_{t=1}^{T}(x_t - \hat{x}_t)^2}{T}}. \tag{19}$$

Where, $\underline{x}$ is the vector containing the observations of the true time series, and $\hat{\underline{x}}$ is the estimate of the time series $\underline{x}$, and T is the length of the time series . The RMSE gives a measure of the accuracy of estimates with a RMSE of zero corresponding to perfect estimation. The empirical coverage and mean empirical credible interval width are given by:

$$\text{empirical coverage} = \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}\{\text{lower(t)}, \text{upper(t)}\}(x(t)) \tag{20}$$

$$\text{mean credible interval width} = \frac{1}{T}\sum_{t=1}^{T} \text{lower(t)} - \text{upper(t)}. \tag{21}$$

Where, lower(t) and upper (t) refer to the lower and upper bounds of the estimated 95 % highest density interval (HDI) and $\mathbb{1}\{\text{lower(t)}, \text{upper(t)}\}(x(t))$ is 1 if the observed value of the time series is within the interval $[\text{lower(t)}, \text{upper(t)}]$ and zero otherwise. The empirical coverage gives us the empirical probability that the estimated HDIs contain the true time series. The mean credible interval informs us the average width of the estimated HDIs across the time series.

The results of our 100 simulations are as follows, where we saved 3000 MCMC samples from the Bayesian models with a burn in period of 1500 samples and a thinning rate of 4. The root mean squared error to three decimal places is 0.039, 0.071, 0.048 for the posterior mean estimate of the DSP-MFSV CAPM, the rolling correlation estimator, and the posterior mean estimate of the MFSV model respectively. The mean empirical coverage to three decimal place of DSP-MFSV CAPM is 0.936 and the mean empirical coverage from the MFSV model is 1.00. Finally, the mean credible interval widths to three decimal places are 0.114 and 0.332 for the DSP-MFSV CAPM and MFSV model respectively.

Our model achieves the best performance in terms of RMSE, and achieves close to 95% coverage as desired, and with noticeably tighter HDIs when compared to the normal approximation from the MFSV model which are approximately three times wider.

5.2 Real world examples

To demonstrate the applicability of our proposed methodology to practitioners and researchers we investigated two questions. Firstly, we show that in two major financial crises this century, the U.S subprime mortgage crisis (section 5.2.1-section 5.2.3), and the 2020 covid-19 pandemic (section 5.2.4- section 5.2.6), that portfolio diversification does not avail investors from correlation risks induced by such high volatility events. Secondly, we explore how the standard static minimum variance portfolio differs from the dynamically estimated minimum variance portfolio at the peak of these financial crises. In performing this analysis we drew 13,500 samples from the posterior distribution,with a burn-in period of 1500 samples, and a thinning rate of 4, which gave us a total of 3000 saved MCMC samples.

5.2.1 U.S. subprime mortgage crisis

The U.S. subprime mortgage crisis occurred from 2007 to 2010. This was a global financial crisis which originated from the U.S. housing bubble. Particularly, the securitization of mortgages including the infamous collateralized debt obligations which although highly rated were of a vastly higher risk than advertised. When the U.S housing bubble burst this triggered a global financial crisis. We consider two portfolios with 30 stocks. The first portfolio consists of 30 large technology stocks from the period. The second portfolio is a more diversified portfolio in which we include 10 of the technology stocks from the first portfolio with the remainder consisting of large stocks from a range of industries from the S&P 500 stock index. We compute the excess asset returns by using data downloaded from Yahoo Finance and the Fama-French data library (Fama [2023]). We then consider daily data from the 4th of January 2006 to the 31st of December 2009.We then fit DSP-MFSV CAPM to the data and obtain 3000 posterior samples of the score time series, where the score is the scalar summary of the estimated correlation matrix at a given time point.

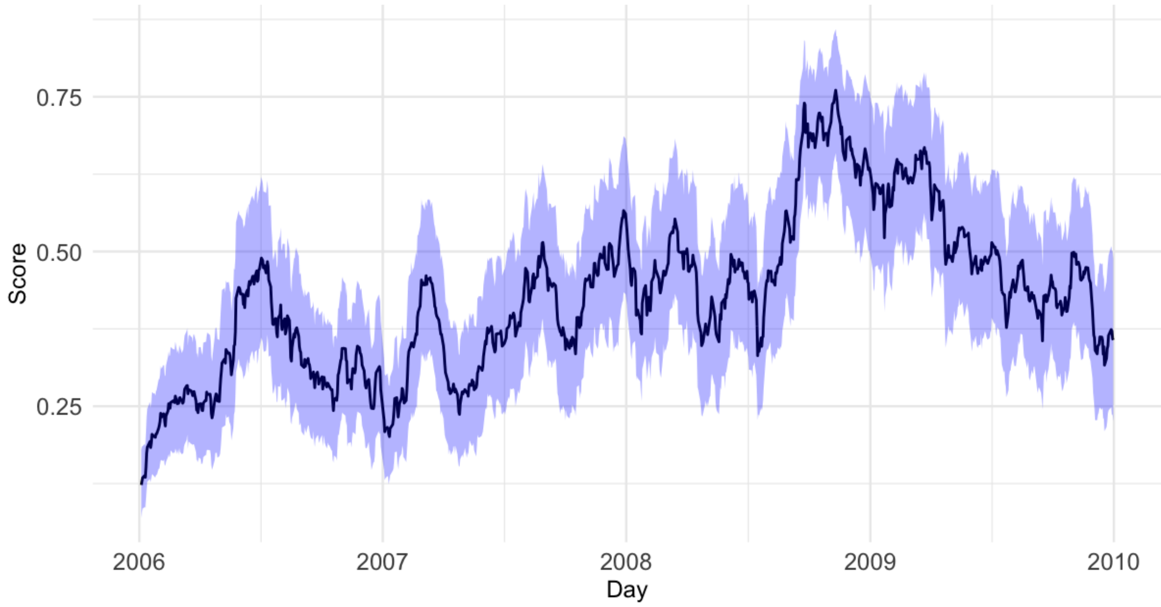

Figure 3: Plot of the estimated posterior score time series for our technology portfolio. The posterior mean score time series is represented by the black line, and the 95% HDIs are represented by the boundary of the purple area.

5.2.2 Diversification risk

In this section we discuss the extent to which portfolio diversification could have helped in shielding an investor's portfolio from such a crisis.

Having fitted our DSP-MFSV CAPM we plot the time series of the estimated scores along with 95% HDIs. In figure three we plot the posterior score of our technology portfolio and in figure four we plot the posterior score of our diversified portfolio. By inspecting these plots, we can see that despite diversifying two thirds of our portfolio, both in our purely technology portfolio and diversified portfolio we still see spikes in the latter half of 2008 which corresponds with the occurrence of some of the most dramatic events of the crisis. Interestingly our proposed approach also predates the spiking of the VIX index which is a measure of the expected volatility in the financial markets and is derived from S&P 500 index options. As can be seen in figure five, the increase in our scalar score starts to increase noticeably before the VIX index does,

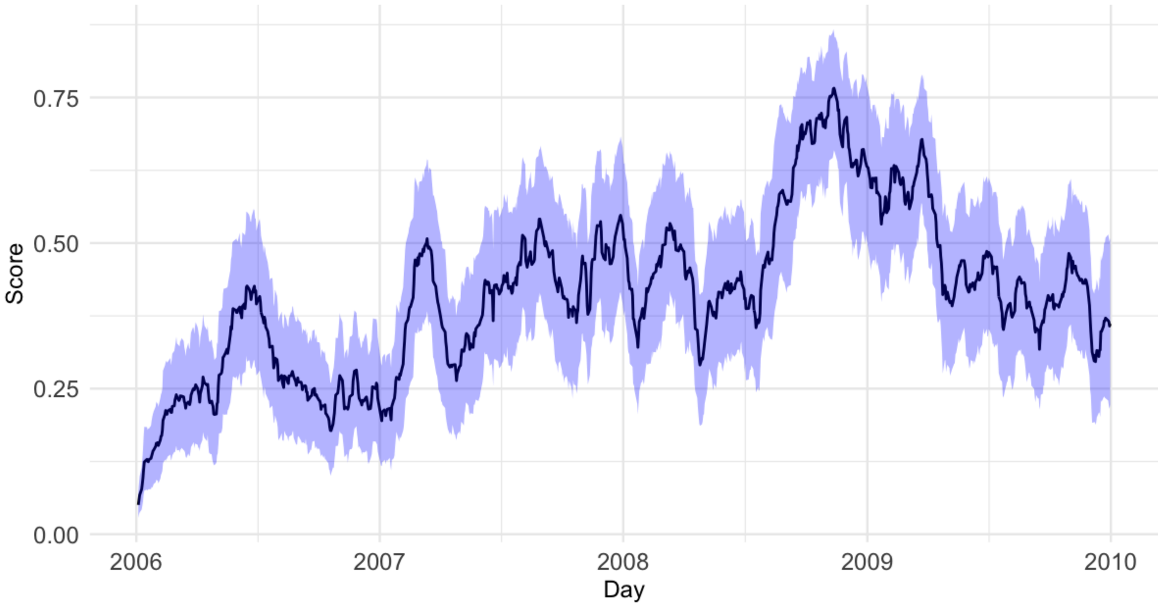

Figure 4: Plot of the estimated posterior score time series for our diversified portfolio. The posterior mean score time series is represented by the black line, and the 95% HDIs are represented by the boundary of the purple area.

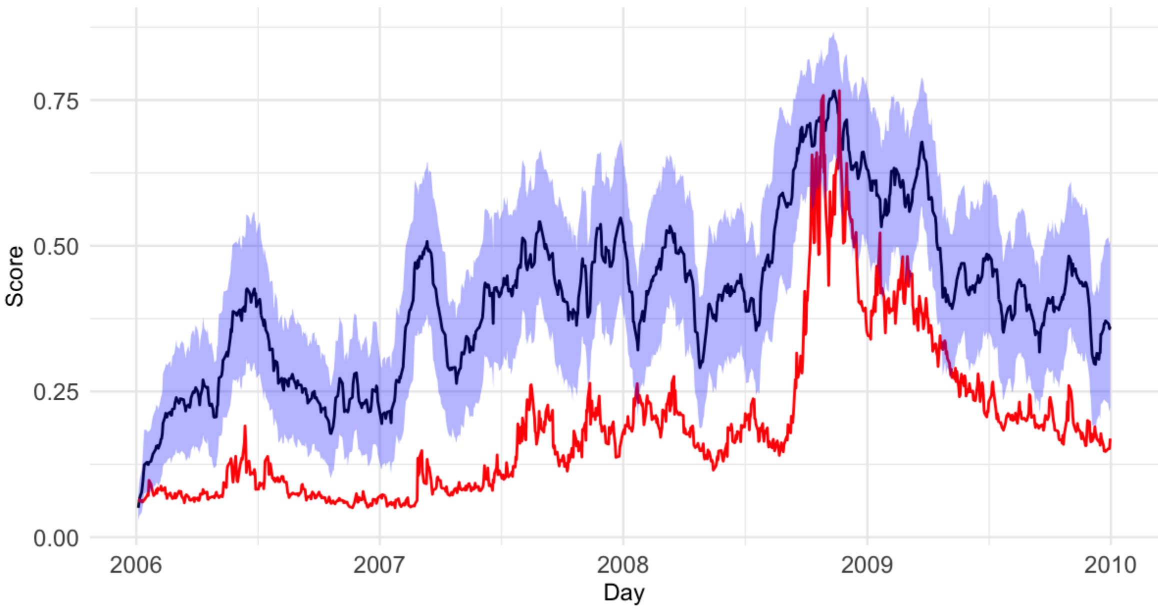

Figure 5: Plot of the estimated posterior score time series for our diversified portfolio in black and the VIX index in red. The 95% HDIs of our scores are represented by the boundary of the purple area. The VIX index is scaled to be between the smallest and largest posterior mean of the estimated score time series.

demonstrating that additional information about market conditions can be found by considering correlation information in an investor's portfolio.

5.2.3 Minimum variance portfolio

The minimum variance portfolio is the portfolio which solves the problem $\min_{\underline{\omega}} = \underline{\omega}^T \mathbf{\Sigma} \underline{\omega}$, where $\underline{\omega}$ is a vector which sums to one and gives us the proportion of each stock we should hold a long or short position in, and $\mathbf{\Sigma}$ is the covariance matrix of the excess asset returns (Tsay [2005]). The solution to this problem is given by:

$$\underline{\omega} = \frac{\mathbf{\Sigma}^{-1}\underline{1}}{\underline{1}^T \mathbf{\Sigma}^{-1}\underline{1}}. \tag{22}$$

The minimum variance portfolio is the portfolio allocation which minimizes volatility irrespective of expected return. Usually this is estimated statically by either using the covariance derived from a static CAPM or using the sample covariance matrix. We will compare how the static global minimum variance portfolios compare to the dynamic global minimum variance portfolio with respect to our diversified portfolio at the time of maximum correlation as identified by the largest value of the posterior mean of the score from the fitted DSP-MFSV CAPM.

In this case the maximum correlation point occurs on the 12th of November 2008. The minimum variance portfolios are given in table one. As expected, the static methods place positive weights on stocks which would generally be considered historically low volatility stocks. Interestingly, the dynamic minimum variance portfolio doubles (compared to the static methods) the weights of PG, MCD, and WMT on the worst day of the financial crisis which reflects that these companies intuitively would seem the lowest volatility stocks and have fairly stable business operations, such that consumers will still likely buy their products during financial crises. The static portfolios are more balanced and don't have as extreme weights.

5.2.4 2020 COVID-19 pandemic

The COVID-19 pandemic started in Wuhan, China in December 2019. It quickly spread triggering governments across the world to issue nationwide lock downs in an attempt to slow down the spread of the virus, which had a large negative economic impact. We consider two portfolios with 30 stocks. The first portfolio consists of 30 large technology companies listed on the NASDAQ. The second portfolio is a diversified portfolio where we have 12 of the stocks from our first portfolio and the rest are large stocks from other industries included in the S&P 500. We compute the excess asset returns by using data downloaded from Yahoo Finance and the Fama-French data library (Fama [2023]).We consider daily data from the 3rd of January 2019 to the 29th of December 2023. We then fit DSP-MFSV CAPM to the data and obtain 3000 posterior samples of the score time series.

5.2.5 Diversification risk

We now discuss to what extent diversifying our first portfolio to obtain our second portfolio helped in protecting our portfolio from the economic impact of the COVID-19 pandemic.

Having fitted our DSP-MFSV model we plot the time series of the estimated scores along with 95% HDIs. In figure six we plot the posterior score of our technology portfolio and in figure seven we plot the posterior score of our diversified portfolio. In both figure six and figure seven we see a large spike in our portfolios correlation on the 10th of March 2020 with nationwide lock downs across the globe starting soon afterwards such as in the U.K, U.S, and Europe. As can be seen in figure eight our proposed approach increases and spikes prior to the VIX index which reaches its peak on the 16th of March 2020 which marked the start of several nationwide lock downs. This again demonstrates the additional information about the current market conditions that can be inferred by considering the correlation information in portfolios such as the one we

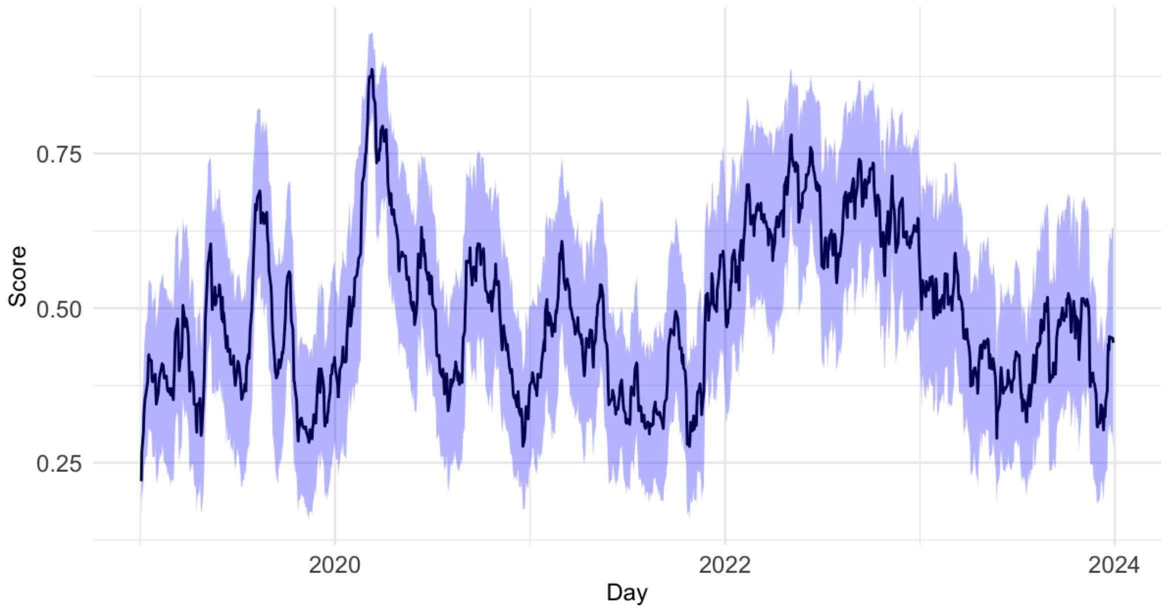

Figure 6: Plot of the estimated posterior score time series for our technology portfolio.The posterior mean score time series is represented by the black line, and the 95% HDIs are represented by the boundary of the purple area.

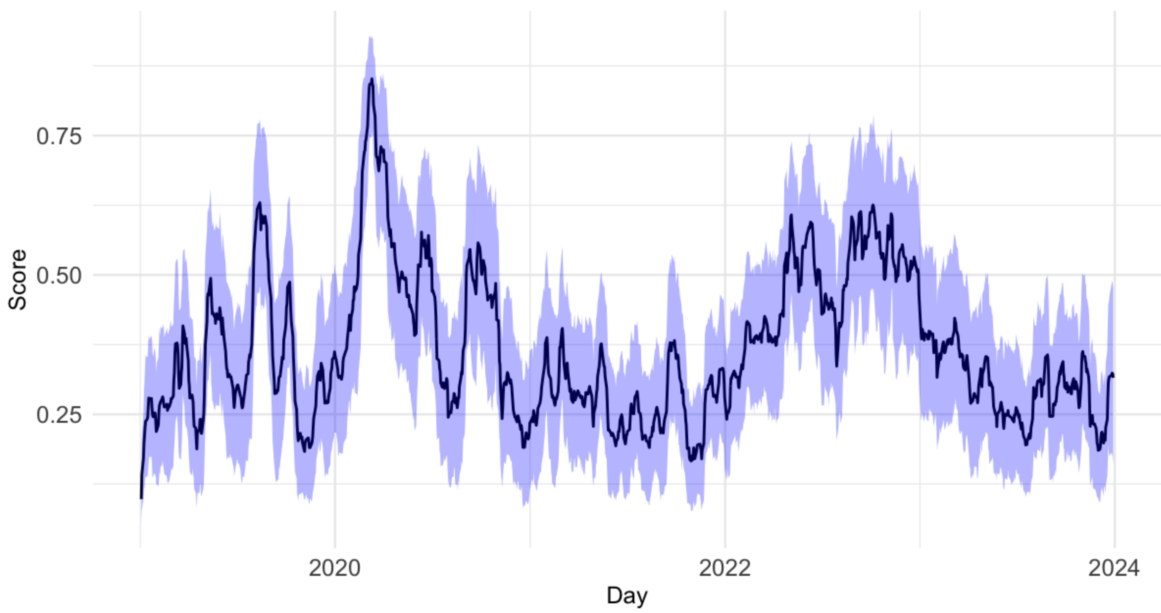

Figure 7: Plot of the estimated posterior score time series for our diversified portfolio.The posterior mean score time series is represented by the black line, and the 95% HDIs are represented by the boundary of the purple area.

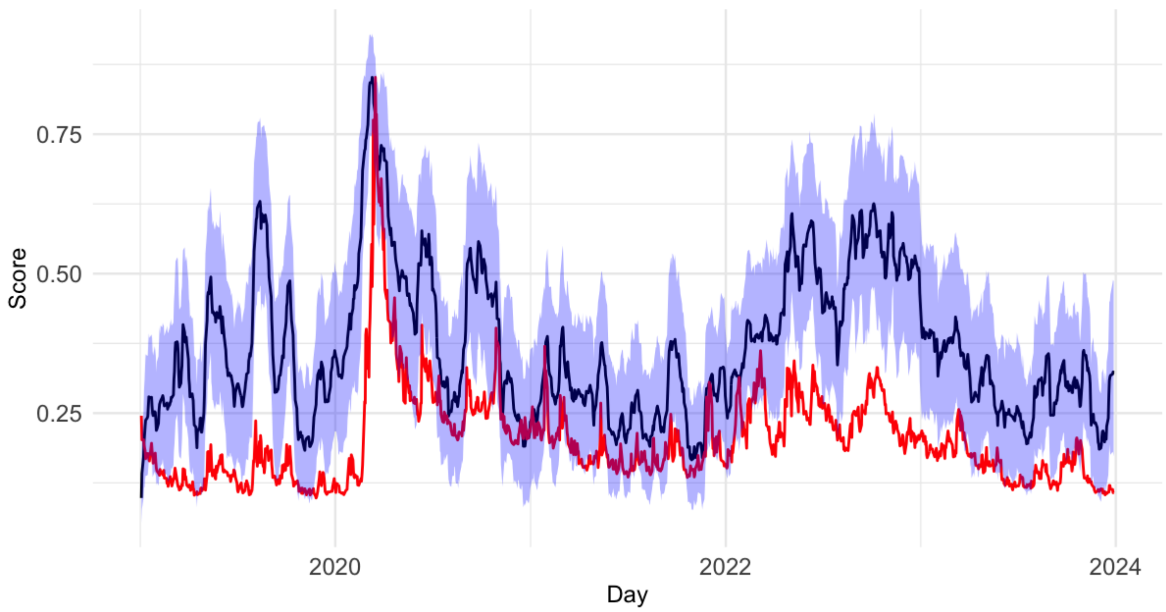

Figure 8: Plot of the estimated posterior score time series for our diversified portfolio in black and the VIX index in red. The 95% HDIs of our scores are represented by the boundary of the purple area. The VIX index is scaled to be between the smallest and largest posterior mean of the estimated score time series.

constructed.

5.2.6 Minimum variance portfolio

See section 5.2.3 for a brief introduction to minimum variance portfolios.

We will compare the minimum variance portfolios estimated by traditional static methods with a dynamically estimated minimum variance portfolio with respect to our diversified portfolio on the 10th of March 2020, which is the day with the largest correlation in our diversified portfolio.

The minimum variance portfolios computed using two static methods and our DSP-MFSV CAPM are presented in table two. We can see that the DSP-MFSV CAPM places a very large weight on AMZN stock and slightly smaller weights on GOOG, META and NFLX stocks. This reflects the demand for service-based stocks and how these companies' services were seeing increased demand during the Covid-19 pandemic. Other technology stocks such as AAPL,MSFT and NVDA have quite pronounced negative weights which could reflect fears about the chip shortage crisis. Interestingly the DSP-MFSV CAPM places some large weights on JNJ and MRK representing the fact that there will be an increased demand for pharmaceutical products as we enter a severe global pandemic and increased pressures on health services. The two static techniques do not have as extreme weights and give a more balanced portfolio.

6 Conclusion

In summary we have proposed a novel approach for estimating time varying correlation matrices in a Bayesian fashion based upon dynamic shrinkage processes. To allow practitioners to derive meaningful information from time series consisting of even moderate dimension correlation matrices we propose a scalar score to summarize a given correlation matrix. Through a simulation study we show that our proposed model achieves desirable results in terms of RMSE and tight highest density intervals when compared to the competing method. Through two real world examples we demonstrated the applicability of our model especially in providing novel insights into an investor's portfolio and established that portfolio diversification does not avail the problems caused by financial crises on an investor's portfolio. Finally, we compared the dynamically estimated minimum variance portfolio at the peak of each crisis with the traditional statically estimated minimum variance portfolios. Through this we observed that the dynamically estimated minimum variance portfolio has more extreme weights in a few companies compared to the more balanced statically estimated portfolios. Future work could include studying the theoretical properties of the proposed scalar score and expanding the framework to provide further insights into portfolio allocation.

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Stock	Static CAPM covariance	Static Sample covariance	DSP-MFSV CAPM covariance
AAPL	-0.012	0.026	0.133
MSFT	-0.004	-0.023	-0.095
AMZN	-0.020	-0.026	-0.049
GOOGL	-0.003	0.037	0.02
BRK-B	0.088	0.205	0.114
JNJ	0.313	0.349	0.162
PG	0.191	0.117	0.328
JPM	-0.075	-0.057	-0.040
XOM	-0.017	-0.027	0.048
INTC	-0.038	-0.005	-0.053
CSCO	-0.045	-0.045	-0.071
CVX	-0.046	-0.064	-0.114
PFE	0.057	0.025	-0.015
KO	0.170	0.066	0.177
DIS	-0.059	-0.103	-0.121
VZ	0.052	0.034	-0.025
IBM	0.083	0.188	0.204
PEP	0.198	0.199	0.215
MRK	0.025	-0.053	-0.171
HD	-0.026	-0.043	-0.014
BAC	-0.052	-0.023	-0.056
UNH	-0.006	-0.037	-0.054
T	0.024	0.022	-0.089
CMCSA	-0.044	-0.062	-0.169
NVDA	-0.032	-0.015	-0.033
ADBE	-0.040	-0.027	-0.097
MCD	0.136	0.162	0.356
MMM	0.057	0.070	0.111
WMT	0.141	0.112	0.348
BB	-0.015	-0.002	0.050

Table 1: Global minimum variance portfolios using three different approaches. The first approach fits a static CAPM and then uses the model implied covariance matrix. The second approach uses the static sample covariance matrix. The third approach uses the model based covariance matrix from the 12th of November 2008 from fitting a DSP-MFSV CAPM model.

Stock	Static CAPM covariance	Static Sample covariance	DSP-MFSV CAPM covariance
AAPL	-0.055	-0.062	-0.142
MSFT	-0.054	-0.049	-0.246
AMZN	-0.010	0.095	0.530
NFLX	-0.004	0.023	0.220
GOOG	-0.026	0.032	0.290
NVDA	-0.068	-0.037	-0.164
META	-0.027	0.010	0.299
BRK-B	0.141	0.201	0.194
TSLA	-0.022	0.022	0.017
V	-0.002	0.142	-0.165
JPM	-0.014	0.050	-0.041
JNJ	0.216	0.204	0.356
UNH	0.04	-0.034	-0.061
PG	0.176	0.122	-0.008
MA	-0.042	-0.162	-0.225
XOM	0.020	0.118	0.072
HD	0.005	0.005	-0.095
PFE	0.095	0.038	0.143
ABBV	0.101	0.094	0.200
MRK	0.145	0.117	0.259
KO	0.162	0.167	-0.002
PEP	0.153	-0.114	-0.072
BAC	-0.036	-0.151	-0.073
WMT	0.165	0.224	0.188
ADBE	-0.046	-0.016	-0.110
CSCO	0.028	-0.011	-0.177
CVX	0.002	-0.076	-0.027
INTC	-0.025	-0.021	-0.158
AVGO	-0.062	0.026	-0.146
CMCSA	0.045	0.040	0.147

Table 2: Global minimum variance portfolios to two decimal places using three different approaches. The first approach fits a static CAPM and then uses the model implied covariance matrix. The second approach uses the static sample covariance matrix. The third approach uses the model based covariance matrix from the 10th of March 2020 from fitting a DSP-MFSV CAPM model.